Constrained implementation*

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Abstract

Consider a society with two sectors (issues or objects) that faces a design problem. Suppose that the sector-2 dimension of the design problem is fixed and represented by a mechanism Γ^2, and that the designer operates under this constraint for institutional reasons. A sector-1 mechanism Γ^1 constrained implements a social choice rule φ in Nash equilibrium if for each profile of agents' preferences, the set of (pure) Nash equilibrium outcomes of the mechanism $\Gamma^1 \times \Gamma^2$ played by agents with those preferences always coincides with the recommendations made by φ for that profile. If this mechanism design exercise could be accomplished, φ would be constrained implementable. We show that *constrained monotonicity*, a strengthening of (Maskin) monotonicity, is a necessary condition for constrained implementation. When there are more than two agents, and when the designer can use the private information elicited from agents via Γ^2 to make a socially optimal decision for sector 1, constrained monotonicity, combined with an auxiliary condition, is sufficient. This sufficiency result does *not* rule out any kind of complementarity between the two sectors.

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1. Introduction

The challenge of implementation lies in designing a mechanism (i.e., game form) where the equilibrium behavior of agents always coincides with the recommendations given by a *social choice rule* (SCR) φ. If such a mechanism exists, φ is said to be implementable. The fundamental study on implementation in (pure strategies) Nash equilibrium is thanks to Maskin (1999; circulated since 1977).[1]

Since this seminal work, the method used in the literature to understand how to solve an implementation problem is partial equilibrium analysis. This method isolates outcomes to be allocated as well as agents' preferences for those outcomes from the rest of the world, under a ceteris paribus (all else equal) assumption. When there is more than one decision problem and the practice dictated by the partial equilibrium analysis is taken as a given institutional constraint, Hayashi and Lombardi (2017) quantify the effect of the practice by showing that the scope of implementation has to be confined essentially to separable preferences. A centralized allocation mechanism may be better equipped to deal with issues arising from non-separability of preferences. However, this mechanism is not available or feasible in real life.

This leaves the question of whether there are ways to broaden the scope of implementation. The main objective of this study is to examine this question with only a minimal departure from the standard practice. The departure consists of two elements.

First, we consider a society with two sectors (issues or objects) that faces a design problem. We suppose that the sector-2 dimension of the design problem is fixed and represented by a mechanism $\Gamma^2 = \left((M_i^2)_{i\in I}, g^2\right)$, and that the designer operates under this constraint for institutional reasons.[2] This departure is motivated by the

[1] Moore and Repullo (1990), Dutta and Sen (1991), Sjöström (1991) and Lombardi and Yoshihara (2013) refine Maskin's theorem by providing necessary and sufficient conditions for an SCR to be implementable in (pure strategies) Nash equilibrium. For an introduction to the theory of implementation see Jackson (2001) and Maskin and Sjöström (2002).

[2] M_i^2 is agent i's sector-2 message space and $g^2 : M^2 \to A^2$ is the sector-2 outcome function.

observation that when, for example, a society faces the issue of whether to supply a public good, it must solve this problem by taking as given how private goods markets work. When a school authority faces the issue of allocating students to schools, the authority must solve it by taking as given how the housing market works.

Second, we introduce incomplete, yet not negligible, communication between the designer and other mechanisms, in the sense that the designer bases his allocation decision not only on information directly elicited from the agents but also on information elicited from them via Γ^2. Suppose that Γ^2 is a market mechanism for private goods and that a society faces the issue of sharing the cost of providing a public good under the constraint of Γ^2. Then, the society needs to use the private information elicited from the agents via Γ^2 if it wants to make a socially optimal decision. For the same reason, when Γ^2 is a housing market and a school authority faces the issue of allocating students to schools under the constraint of Γ^2, it needs to use the private information elicited from the agents via the housing market Γ^2.[3]

Note that this communication creates incentive problems. Indeed, given that agents are aware that the designer also bases his sector-1 decision on the information that they transmit to sector-2 mechanism Γ^2, agents may have incentives to lie for manipulating not only outcomes determined by Γ^2 but also sector-1 decisions. For example, once families know that the school authority bases its school allocation on information that they transmit to the housing market, families have incentives to use this information for manipulating not only the housing market outcome but also the school allocation outcome. Therefore, the change described above makes a significant departure from the standard literature as well as from the problem studied by Hayashi and Lombardi (2017). Recall that in Hayashi and Lombardi (2017), every

Sector 2 can itself be a collection of many sectors.

[3]When the designer does not have access to information elicited via other mechanisms, we basically go back to implementation in partial equilibrium, as studied by Lombardi and Hayashi (2017), in which there is no communication among sector authorities about the information elicited from the agents.

sector-designer is under the assumption of partial equilibrium analysis, and so each allocation problem is isolated from others.

In this paper, the designer faces a constrained implementation problem. It consists of designing a mechanism for sector 1, Γ^1, with the property that for any type of agents' preferences, the set of (pure) Nash equilibrium outcomes of the mechanism $\Gamma^1 \times \Gamma^2$ coincides with the set of outcomes that φ would select for those preferences. If this design exercise can be accomplished, the SCR is said to be *constrained implementable*.

Within this set-up, we investigate the theory of implementation pioneered by Maskin (1999) under the constraint Γ^2. Our conclusion is that a small departure from the standard practice dramatically increases the scope for implementation, in the sense that our sufficiency results do not rely on any domain restriction of agents' preferences. Thus, unlike the negative result of Hayashi and Lombardi (2017), our sufficiency result does *not* rule out any kind of complementarity between the two sectors.

We also show that an SCR that can be constrained implemented satisfies an invariance condition, named *constrained monotonicity*. This condition is a strengthening of monotonicity. Monotonicity means that if an outcome (a^1, a^2) is recommended by the SCR φ in state θ but φ does not recommend it when the state is changed to θ', then the outcome (a^1, a^2) must have fallen strictly in someone's ordering at the state θ'. To introduce our condition, suppose that the sector-2 outcome a^2 is supported by a profile of sector-2 strategy choices m^2—that is, $g^2(m^2) = a^2$. Constrained monotonicity requires that if an outcome $(a^1, g^2(m^2))$ is recommended by φ in state θ but the SCR φ does not recommend it when the state is changed to θ', then, to break the Nash equilibrium via some deviation, there exists an agent i who can generate a sector-2 outcome by varying his own sector-2 strategy choice, $\hat{m}_i^2$, while keeping the other agents' sector-2 strategy choices fixed at m_{-i}^2, such that an outcome $\left(b^1, g^2\left(\hat{m}_i^2, m_{-i}^2\right)\right)$, which is less preferred than $(a^1, g^2(m^2))$ at θ, is strictly preferred to $(a^1, g^2(m^2))$ at θ'.

Section 2 outlines the basic model. Section 3 defines constrained monotonicity and shows that it is necessary for constrained implementation. Section 4 provides our characterization result. Sections 5 provides an account of welfare implications of constrained implementability. Section 6 studies the relationships between constrained implementability and the standard unconstrained implementability. Section 7 concludes.

2. Setting

We consider a finite set of agents indexed by $i \in I = \{1, \cdots, n\}$, which we refer to as a society. The set of outcomes available to agents is $A^1 \times A^2$, where A^s is the set of sector-s outcomes, for $s = 1, 2$. The information held by the agents is summarized in the concept of a state. Write Θ for the domain of possible states, with θ as a typical state. In the usual fashion, agent i's preferences in state θ are given by a complete and transitive binary relation, subsequently an ordering, $R_i(\theta)$ over the set $A^1 \times A^2$. The corresponding strict relation is denoted by $P_i(\theta)$. We assume that the true state is common knowledge among the n agents but is unknown to the designer. To avoid trivialities, we assume that $|\Theta| \geq 2$ and $|A^1 \times A^2| \geq 2$.[4]

The goal of the designer is to implement an SCR φ, which is a correspondence $\varphi : \Theta \twoheadrightarrow A^1 \times A^2$ such that for each state $\theta \in \Theta$, $\varnothing \neq \varphi(\theta) \subseteq A^1 \times A^2$. We refer to $(a^1, a^2) \in \varphi(\theta)$ as a φ-optimal outcome at θ.

A sector-2 mechanism is a collection $\Gamma^2 = \left((M_i^2)_{i \in I}, g^2\right)$ where $M_i^2 \neq \varnothing$ is agent i's sector-2 message space and $g^2 : M^2 \to A^2$ is a sector-2 outcome function. Given a sector-2 mechanism Γ^2, a sector-1 mechanism is a collection $\Gamma^1 = \left((M_i^1 \times M_i^2)_{i \in I}, g^1\right)$ where $M_i^1 \times M_i^2 \neq \varnothing$ is agent i's message space and $g^1 : M^1 \times M^2 \to A^1$ is a sector-1 outcome function. Thus, the sector-1 outcome function is defined on $M^1 \times M^2$, rather

[4]Given a set Y, $|Y|$ denotes its cardinality.

than on M^1, and for each profile of (m^1, m^2), $g^1(m^1; m^2)$ represents the resulting sector-1 outcome. This property of Γ^1 captures the idea that the designer bases his sector-1 outcome decision not only on information supplied directly to him by agents but also on information elicited from agents via Γ^2.

A mechanism $\Gamma^1 \times \Gamma^2$ together with a state θ defines a strategic game $(\Gamma^1 \times \Gamma^2, \theta)$, in which each agent chooses his message and all agents' strategy choices are made simultaneously (i.e., when making a strategy choice, no agent is informed of the strategy choice made by any other agent). A strategy profile $(m^1, m^2) \in M^1 \times M^2$ is a Nash equilibrium (in pure strategies) of $(\Gamma^1 \times \Gamma^2, \theta)$ if for all $i \in I$, it holds that

$$\left(g^1 \times g^2\right)\left(m^1, m^2\right) R_i(\theta) \left(g^1 \times g^2\right)\left(\left(\hat{m}_i^1, m_{-i}^1\right), \left(\hat{m}_{-i}^2, m_{-i}^2\right)\right) \ \forall \left(\hat{m}_i^1, \hat{m}_i^2\right) \in M_i^1 \times M_i^2,$$

where, as usual, m_{-i}^s is the message profile of all agents except i such that $\left(m_i^s, m_{-i}^s\right) = m^s$, for $s = 1, 2$. We write $NE\left(\Gamma^1 \times \Gamma^2, \theta\right)$ for the set of Nash equilibrium profiles of $(\Gamma^1 \times \Gamma^2, \theta)$, and $(g^1 \times g^2)\left(NE\left(\Gamma^1 \times \Gamma^2, \theta\right)\right)$ for the set of Nash equilibrium outcomes of $(\Gamma^1 \times \Gamma^2, \theta)$.

Definition 1 *Let Γ^2 be given. A sector-1 mechanism, Γ^1,* constrained (Nash) implements *the SCR $\varphi : \Theta \twoheadrightarrow A^1 \times A^2$ if*

$$\varphi(\theta) = (g^1 \times g^2)(NE(\Gamma^1 \times \Gamma^2, \theta)) \ \forall \theta \in \Theta.$$

If such a sector-1 mechanism exists, we say that φ is constrained implementable*.*

Let (Γ^2, θ) be given. For any agent i, any m_{-i}^2 and any outcome (a^1, a^2), the lower contour set of $R_i(\theta)$ at (a^1, a^2) is

$$L_i\left(\theta, \left(a^1, a^2\right)\right) = \{\left(b^1, b^2\right) \in A^1 \times A^2 | \left(a^1, a^2\right) R_i(\theta) \left(b^1, b^2\right)\},$$

whereas the $\left(g^2, m_{-i}^2\right)$-constrained lower contour set of $R_i(\theta)$ at (a^1, a^2) is defined by

$$L_{i, m_{-i}^2}\left(\theta, \left(a^1, a^2\right)\right) = \left\{\left(b^1, g^2\left(m_i^2, m_{-i}^2\right)\right) \in A^1 \times g^2\left(M_i^2, m_{-i}^2\right) : \left(a^1, a^2\right) R_i(\theta) \left(b^1, g^2\left(m_i^2, m_{-i}^2\right)\right)\right\}.$$

It is clear that

$$L_{i,m^2_{-i}}\left(\theta,\left(a^1,a^2\right)\right)\subseteq L_i\left(\theta,\left(a^1,a^2\right)\right).$$

Finally, for any (nonempty) subset $B^1\times B^2\subseteq A^1\times A^2$, let $proj_2\left\{B^1\times B^2\right\}$ denote the projection of $B^1\times B^2$ onto A^2.

3. Constrained monotonicity

In this section, we introduce a new condition, called *constrained monotonicity*, which we show to be necessary for our notion of implementation.

A condition that is central to the implementation of SCRs in Nash equilibrium is (Maskin) monotonicity. According to this condition, if an outcome (a^1,a^2) is φ-optimal at state θ, and this (a^1,a^2) does not strictly fall in preference for anyone when the state is changed to θ', then (a^1,a^2) must remain an φ-optimal outcome at θ'. Formally, we present the following definition.

Definition 2 (Maskin, 1999) *An SCR* $\varphi:\Theta\twoheadrightarrow A^1\times A^2$ *is* monotonic *provided that* $\forall\left(a^1,a^2\right)\in A^1\times A^2$ *and* $\forall\theta,\theta'\in\Theta$, *if* $\left(a^1,a^2\right)\in\varphi\left(\theta\right)$ *and* $L_i(\theta,(a^1,a^2))\subseteq L_i(\theta',(a^1,a^2))$ $\forall i\in I$, *then* $\left(a^1,a^2\right)\in\varphi\left(\theta'\right)$.

The key to our analysis is identifying the appropriate notion of monotonicity in the present setting, which we call constrained monotonicity. To introduce this condition, we need additional notation. For any message profile of sector 2, $\bar{m}^2$, let $g^2\left(M^2_i,\bar{m}^2_{-i}\right)$ represent the set of outcomes that agent i can generate in sector 2 by varying his own strategy, keeping the other agents' strategy choices fixed at $\bar{m}^2_{-i}$. Therefore, we provide the following definition.

Definition 3 *Let* Γ^2 *be given. The SCR* $\varphi:\Theta\twoheadrightarrow A^1\times A^2$ *is* constrained monotonic *provided that* $\forall\left(a^1,a^2\right)\in A^1\times A^2$ *and* $\forall\theta\in\Theta$, *if* $\left(a^1,a^2\right)\in\varphi\left(\theta\right)$, *then there exists* $m^2\equiv m^2_\varphi\left(\theta,(a^1,a^2)\right)\in M^2$ *such that:*

(i) $g^2(m^2) = a^2$ *and* $proj_2\left\{L_{i,m^2_{-i}}(\theta,(a^1,a^2))\right\} = g^2\left(M^2_i, m^2_{-i}\right)$ $\forall i \in I$, *and*

(ii) $\forall \theta' \in \Theta$, $L_{i,m^2_{-i}}(\theta,(a^1,a^2)) \subseteq L_{i,m^2_{-i}}(\theta',(a^1,a^2))$ $\forall i \in I \implies (a^1,a^2) \in \varphi(\theta')$.

Suppose that the designer wants to implement the outcome (a^1, a^2) in state θ. Part (i) requires the existence of a sector-2 message profile, m^2, such that it induces the sector-2 outcome a^2. Moreover, since agents can deviate and misreport in sector 2, the second component of part (i) allows the designer to find a suitable sector-1 punishment outcome for any unilateral deviation from m^2. Part (ii) is a constrained version of monotonicity. With a fixed Γ^2 and the assumption that a sector-1 outcome also depends on the strategies played by agents in sector 2, the set of outcomes that agent i can generate by unilaterally deviating from an equilibrium profile is smaller than the set of outcomes that he can induce in the standard Nash implementation framework. Thus, part (ii) applies the monotonicity condition to this smaller set of outcomes.

The importance of part (i) is twofold. First, it allows us to construct a sector-1 mechanism, Γ^1, that guarantees that every φ-optimal outcome results in an equilibrium outcome. Second, as in the case of rule 2 of Maskin's canonical mechanism, part (i) allows us to devise a Γ^1 that questions the credibility of agent i's report when all agents except i make exactly the same announcement. Finally, part (ii) is used to incentivize agents to report the true environment when everyone is lying about it—that is, when there is a unanimous false announcement.

The condition is a strengthening of monotonicity. To this end, our first result shows that constrained monotonicity implies monotonicity.

Theorem 1 *Let* Γ^2 *be given. If* $\varphi : \Theta \twoheadrightarrow A^1 \times A^2$ *is* constrained monotonic*, then it is* monotonic*.*

Proof. Assume that the hypotheses of the theorem are met. Take any (a^1, a^2), any θ, and any θ' such that $(a^1, a^2) \in \varphi(\theta)$ and $L_i(\theta,(a^1,a^2)) \subseteq L_i(\theta',(a^1,a^2))$, $\forall i \in I$. We show that $(a^1, a^2) \in \varphi(\theta')$.

Since $(a^1, a^2) \in \varphi(\theta)$, part (i) of constrained monotonicity implies that there is $m^2 \equiv m^2_\varphi(\theta, (a^1, a^2))$ such that $g^2(m^2) = a^2$. Fix any $i \in I$ and any $\left(b^1, g^2\left(\bar{m}^2_i, m^2_{-i}\right)\right) \in L_{i,m^2_{-i}}(\theta, (a^1, a^2))$. Since, by assumption, $L_i(\theta, (a^1, a^2)) \subseteq L_i(\theta', (a^1, a^2))$ and since, by the definitions of the lower contour sets, $L_{i,m^2_{-i}}(\theta, (a^1, a^2)) \subseteq L_i(\theta, (a^1, a^2))$, it follows that $\left(b^1, g^2\left(\bar{m}^2_i, m^2_{-i}\right)\right) \in L_i(\theta', (a^1, a^2))$. The definition of $\left(g^2, m^2_{-i}\right)$-constrained lower contour set of $R_i(\theta)$ at (a^1, a^2) implies that $\left(b^1, g^2\left(\bar{m}^2_i, m^2_{-i}\right)\right) \in L_{i,m^2_{-i}}(\theta', (a^1, a^2))$. Since the choices of $\left(b^1, g^2\left(\bar{m}^2_i, m^2_{-i}\right)\right)$ and agent i are arbitrary, we have $L_{i,m^2_{-i}}(\theta, (a^1, a^2)) \subseteq L_{i,m^2_{-i}}(\theta', (a^1, a^2))$, $\forall i \in I$. Part (ii) of constrained monotonicity implies that $(a^1, a^2) \in \varphi(\theta')$, as we aimed to achieve. Thus, φ is monotonic. ■

One can easily check that monotonicity is equivalent to constrained monotonicity when $A^2 = \{a^2\}$. However, these two conditions are generally not equivalent. This point is illustrated in Example 1.

Example 1 *Monotonicity does not imply constrained monotonicity.* There are two agents, 1 and 2, two states, θ and θ', and three outcomes, (a^1, a^2), (a^1, b^2), and (a^1, c^2). Individuals have state-dependent preferences represented in the table below, where x^2 is understood as (a^1, x^2), and, as usual, $\substack{x^2 \\ y^2}$ for agent i means that i strictly prefers x^2 to y^2, while x^2, y^2 means that i is indifferent between x^2 and y^2.

θ		θ'	
Agent 1	*Agent 2*	*Agent 1*	*Agent 2*
a^2,b^2	c^2	c^2	c^2
c^2	a^2	a^2, b^2	a^2
	b^2		b^2

The designer aims to implement the SCR φ, with $\varphi(\theta) = \{(a^1, a^2), (a^1, b^2)\}$ and $\varphi(\theta') = \{(a^1, a^2)\}$. One can check that φ is monotonic.

Suppose that the designer faces a Γ^2 in which $M^2_i = \{m^2_i, \hat{m}^2_i\}$ for each agent i

and the outcome function g^2 is defined as follows:

	m_2^2	$\hat{m}_2^2$
m_1^2	a^2	b^2
$\hat{m}_1^2$	b^2	b^2

The outcome in each box is the outcome of the action profile to which the box corresponds. Note that, by definition of g^2, $g^2\left(m_1^2, m_2^2\right) = a^2$ and $g^2\left(\hat{m}_1^2, m_2^2\right) = b^2$.

Let us first check that part (i) of constrained monotonicity is satisfied. Let us define $m_\varphi^2\left(\theta, \left(a^1, a^2\right)\right)$, $m_\varphi^2\left(\theta, \left(a^1, b^2\right)\right)$ and $m_\varphi^2\left(\theta', \left(a^1, a^2\right)\right)$ as follows:

$$
\begin{aligned}
m_\varphi^2\left(\theta, \left(a^1, a^2\right)\right) &= m_\varphi^2\left(\theta', \left(a^1, a^2\right)\right) = \left(m_1^2, m_2^2\right), \\
m_\varphi^2\left(\theta, \left(a^1, b^2\right)\right) &= \left(\hat{m}_1^2, m_2^2\right).
\end{aligned}
$$

Let us first consider the case in which $\left(a^1, a^2\right) \in \varphi\left(\bar{\theta}\right)$ for each $\bar{\theta} \in \{\theta, \theta'\}$. Then, by definition, there exists $m_\varphi^2\left(\theta, \left(a^1, a^2\right)\right) = \left(m_1^2, m_2^2\right) \in M^2$ such that $g^2\left(m_1^2, m_2^2\right) = a^2$ and

$$
\begin{aligned}
proj_2\left\{L_{1,m_2^2}\left(\bar{\theta}, \left(a^1, g^2\left(m_1^2, m_2^2\right)\right)\right)\right\} &= g^2\left(M_1^2, m_2^2\right) = \left\{a^2, b^2\right\}, \\
proj_2\left\{L_{2,m_1^2}\left(\bar{\theta}, \left(a^1, g^2\left(m_1^2, m_2^2\right)\right)\right)\right\} &= g^2\left(m_1^2, M_2^2\right) = \left\{a^2, b^2\right\}.
\end{aligned}
$$

Thus, part (i) is satisfied for the case in which $\left(a^1, a^2\right) \in \varphi\left(\bar{\theta}\right)$ for each $\bar{\theta} \in \{\theta, \theta'\}$.

Next, let us consider the case in which $\left(a^1, b^2\right) \in \varphi(\theta)$. Then, by definition, there exists $m_\varphi^2\left(\bar{\theta}, \left(a^1, b^2\right)\right) = \left(\hat{m}_1^2, m_2^2\right) \in M^2$ such that $g^2\left(\hat{m}_1^2, m_2^2\right) = b^2$ and

$$
\begin{aligned}
proj_2\left\{L_{1,m_2^2}\left(\theta, \left(a^1, g^2\left(\hat{m}_1^2, m_2^2\right)\right)\right)\right\} &= g^2\left(M_1^2, m_2^2\right) = \left\{a^2, b^2\right\}, \\
proj_2\left\{L_{2,\hat{m}_1^2}\left(\theta, \left(a^1, g^2\left(\hat{m}_1^2, m_2^2\right)\right)\right)\right\} &= g^2\left(\hat{m}_1^2, M_2^2\right) = \left\{b^2\right\}.
\end{aligned}
$$

It follows that part (i) of constrained monotonicity is satisfied.

Finally, let us show that part (ii) of constrained monotonicity is violated. By definition of the outcome function g^2, one can observe that

$$
\begin{aligned}
L_{1,m_2^2}\left(\theta, \left(a^1, g^2\left(\hat{m}_1^2, m_2^2\right)\right)\right) &= L_{1,m_2^2}\left(\theta', \left(a^1, g^2\left(\hat{m}_1^2, m_2^2\right)\right)\right) \\
L_{2,\hat{m}_1^2}\left(\theta, \left(a^1, g^2\left(\hat{m}_1^2, m_2^2\right)\right)\right) &= L_{2,\hat{m}_1^2}\left(\theta', \left(a^1, g^2\left(\hat{m}_1^2, m_2^2\right)\right)\right).
\end{aligned}
$$

Since (a^1, b^2) is φ-optimal at state θ, it follows that the premises of part (ii) of constrained monotonicity are satisfied. However, (a^1, b^2) is not φ-optimal at state θ', in violation of constrained monotonicity.

The next result shows that only constrained monotonic SCRs are implementable.

Theorem 2 *If $\varphi : \Theta \twoheadrightarrow A^1 \times A^2$ is an SCR that is* constrained implementable, *then it is* constrained monotonic.

Proof. Suppose that $\varphi : \Theta \twoheadrightarrow A^1 \times A^2$ is constrained implementable. Thus, there exists a Γ^1 such that $\varphi(\bar{\theta}) = (g^1 \times g^2)(NE(\Gamma^1 \times \Gamma^2, \bar{\theta}))$ for all $\bar{\theta} \in \Theta$. Take any $\theta \in \Theta$ and any $(a^1, a^2) \in \varphi(\theta)$. By constrained implementability, there exists $(m^1, m^2) \in NE(\Gamma^1 \times \Gamma^2, \theta)$ such that $(g^1 \times g^2)(m^1, m^2) = (a^1, a^2)$ and that

$$\left(g^1 \times g^2\right)\left(\left(M_i^1, m_{-i}^1\right), \left(M_i^2, m_{-i}^2\right)\right) \subseteq L_{i,m_{-i}^2}\left(\theta, \left(a^1, a^2\right)\right), \forall i \in I. \tag{1}$$

Let $m^2 \equiv m_{\varphi}^2 (\theta, (a^1, a^2)) \in M^2$.

Let us show part (i). Since $(m^1, m^2) \in NE(\Gamma^1 \times \Gamma^2, \theta)$, it is clear that $g^2(m^2) = a^2$. Moreover, by (1), one can easily observe that $proj_2 \left\{L_{i,m_{-i}^2}(\theta, (a^1, a^2))\right\} = g^2\left(M_i^2, m_{-i}^2\right)$, for each $i \in I$. Thus, part (i) is satisfied.

Let us show part (ii). Take any $\theta' \in \Theta$ such that

$$L_{i,m_{-i}^2}\left(\theta, \left(a^1, a^2\right)\right) \subseteq L_{i,m_{-i}^2}\left(\theta', \left(a^1, a^2\right)\right), \forall i \in I. \tag{2}$$

Then, given (1) and (2), it follows that

$$\left(g^1 \times g^2\right)\left(\left(M_i^1, m_{-i}^1\right), \left(M_i^2, m_{-i}^2\right)\right) \subseteq L_{i,m_{-i}^2}\left(\theta', \left(a^1, a^2\right)\right), \forall i \in I,$$

and thus, $(m^1, m^2) \in NE(\Gamma^1 \times \Gamma^2, \theta')$. By constrained implementability of φ, we have that $(g^1 \times g^2)(m^1, m^2) \in \varphi(\theta')$, as we aimed to achieve. Thus, φ is constrained monotonic. ■

4. A characterization theorem

While constrained monotonicity is necessary for constrained implementation, it is not sufficient.[5] We need an extra condition for the sufficiency result.

As a part of sufficiency, we introduce *constrained no veto power*, which is analogous to the condition of no veto power (Maskin, 1999),[6] adjusted for the fact of a fixed mechanism in sector 2. According to this condition, if there is an outcome $(a^1, g^2(m^2))$ that is maximal under state θ for at least $n-1$ agents over the set $A^1 \times g^2\left(M_i^2, m_{-i}^2\right)$, then this $(a^1, g^2(m^2))$ must be a φ-optimal outcome at this θ. Formally, we provide the following definition.

Definition 4 *Let* Γ^2 *be given. The SCR* $\varphi : \Theta \twoheadrightarrow A^1 \times A^2$ *satisfies* constrained no veto power *provided that* $\forall\theta \in \Theta$, $\forall a^1 \in A^1$ *and* $\forall m^2 \in M^2$,

$$\left|\left\{i \in I : A^1 \times g^2\left(M_i^2, m_{-i}^2\right) \subseteq L_{i,m_{-i}^2}\left(\theta, \left(a^1, g^2\left(m^2\right)\right)\right)\right\}\right| \geq n-1 \implies \left(a^1, g^2\left(m^2\right)\right) \in \varphi(\theta).$$

The condition is weak in some contexts, such as in environments in which there are private goods in sector 1 and there are at least three (non-satiated) agents. However, in other environments, constrained no veto power might not be a trivial condition.

We are now ready to present our characterization theorem.[7]

Theorem 3 *Assume that* $n \geq 3$. *Any* constrained monotonic *SCR* $\varphi : \Theta \twoheadrightarrow A^1 \times A^2$ *satisfying* constrained no veto power *is* constrained implementable.

Proof. See the Appendix. ■

[5] For example, suppose that $A^2 = \{a^2\}$. As we noted in Section 3, constrained monotonicity is equivalent to monotonicity when $A^2 = \{a^2\}$. Maskin (1999, pp. 33–34)'s Example 2 shows that although an SCR is monotonic, it is not implementable.

[6] No veto power requires that if an outcome is top-ranked by at least $n-1$ agents, then this outcome should be chosen irrespective of the preferences of the remaining agent.

[7] We thank a referee for proposing a much simpler proof, which has clear parallels to Maskin's theorem (Maskin, 1999, Theorem 3).

The proof of Theorem 3 relies on the construction of a canonical mechanism that is similar to that used, for example, in Maskin (1999), but it is modified to deal with our objective. First, it is constructed to capture the designer's constraint represented by Γ^2: The designer cannot design the outcome function g^2. Second, to capture the notion that there is a unidirectional flow of information from sector 2 to sector 1, we modify the mechanism so that its outcomes depend on both m^1 and m^2.

Heuristically, the SCR is obtained by the following mechanism. Agents simultaneously report a state, θ, an outcome for sector 1, a^1, and an outcome for sector 2, a^2. Let $m^2 \in M^2$ be the information elicited from agents via Γ^2. If all reports coincide, and the jth coordinate of m^2 coincides with the jth coordinate of $m^2_\varphi\left(\theta, \left(a^1, a^2\right)\right)$ specified by part (i) of constrained monotonicity, then the designer chooses the sector-1 outcome according to agents' reports. However, if there is a single agent i whose report is not consistent with other reports, the ith coordinate of $m^2_\varphi\left(\theta, \left(a^1, a^2\right)\right)$, or both, then, the sector-1 outcome is the one announced by agent i if the pair $\left(a^{1i}, g^2\left(m^2\right)\right)$ is not better than the pair $\left(a^1, a^2\right)$ announced by the others under the preference for i announced by the other agents. Otherwise, the designer chooses a sector-1 outcome b^1 such that the pair $\left(b^1, g^2\left(m^2\right)\right)$ is not better than the pair $\left(a^1, a^2\right)$ announced by the others under the preference for i announced by the other agents—the outcome b^1 exists because $m^2 \in M^2$ and $proj_2\left\{L_{i,m^2_{-i}}\left(\theta, \left(a^1, a^2\right)\right)\right\} = g^2\left(M^2_i, m^2_{-i}\right)$. Finally, if neither of the abovementioned two cases holds, then agents play a modulo game, and the winner determines the sector-1 outcome.

5. Applications

This section provides two applications. In one, we design a school choice mechanism under a constraint that we cannot change, and we take how the housing market works as given. In the other application, we design a public goods provision mechanism under a constraint that we cannot change, and we take how the private goods market works as given.

Note that in these applications, the given sector-2 mechanisms are "satisfactory" enough to implement Walrasian equilibria. This leaves out the question of what we can do when we are constrained by a "poor" sector-2 mechanism. We return to this question in the next section.

5.1 School choice problems constrained by a competitive housing market

There are three disjoint sets C, I, and H of schools, agents, and houses, respectively. Agents do not own any house and each needs one house. Each school $c \in C$ has a quota, q_c, of places. Moreover, each school c has a (strict) priority ranking $>_c$ over I, where agent i has a priority higher than agent j if $i >_c j$. Note that the priority ranking is state independent. Although it would be more realistic to have the priority ranking depend on housing allocation, it is difficult to know what this ranking is when agent i takes over agent j's house by deviating in the housing market: how should agent j be ranked? Because of this, we assume that school admissions depend on housing allocations through catchment areas. Thus, let H^c be the set of houses in the catchment area of school $c \in C$. Note that for any $c, c' \in C$, H^c and $H^{c'}$ may or may not be disjoint sets.

In what follows, we assume that Π is the set of all school matchings. A school matching π is a function from I to C such that each agent is assigned to at most one school or remains unmatched (being unmatched is denoted by $\emptyset$), and each school is matched to at most its quota of agents. The set Σ denotes the set of all house matchings. A house matching σ is a one-to-one and surjective function from I to H such that each agent receives one and only one house.

Let $A^1 \equiv \Pi$ be the set of sector-1 outcomes and let $A^2 \equiv \Sigma \times \mathcal{R}^{|I|}$ be the set of sector-2 outcomes. Then, $(\pi, \sigma, z) \in A^1 \times A^2$ is an outcome, which specifies a school matching, a house matching, and income transfers. Individual i's (self-regarding) preferences in state $\theta \in \Theta$ are given by an ordering $R_i(\theta)$ over the set $A^1 \times A^2$. For each $\theta \in \Theta$, we assume that agent i's ordering $R_i(\theta)$ is numerically represented by $u_i^\theta : A^1 \times A^2 \to \mathbb{R}$, which is strictly increasing in money. For each $\theta \in \Theta$, each $i \in I$, and each $\pi \in A^1$, the sector-2 marginal ordering, $R_i^2[\theta, \pi]$, on A^2 induced by (θ, π) is defined by

$$\forall (\sigma, z), (\sigma', z') \in A^2 : (\sigma, z) R_i^2[\theta, \pi] (\sigma', z') \iff (\pi, \sigma, z) R_i(\theta) (\pi, \sigma', z').$$

We assume that agent i's sector-2 marginal ordering $R_i^2[\theta, \pi]$ is numerically represented by $\upsilon_i^{R_i^2[\theta,\pi]} : A^2 \to \mathbb{R}$, which is strictly increasing in money. Let $R^2[\theta, \pi] \equiv (R_i^2[\theta, \pi])_{i \in I}$ denote a profile of sector-2 marginal orderings induced by (θ, π). Let $\mathcal{R}(\Theta, \Pi)$ be the set of profiles of sector-2 marginal orderings induced by (Θ, Π), with R^2 as a typical element.

The set of feasible outcomes is denoted by $\mathcal{F}$ and is defined by

$$\mathcal{F} = \left\{ (\pi, \sigma, z) \in A^1 \times A^2 | \sigma(i) \in H^{\pi(i)}, \forall i \in I \right\}.$$

In other words, the allocation (π, σ, z) is feasible if each agent $i \in I$ is assigned to a school $\pi(i)$ that is compatible with his allocated house $\sigma(i)$, in that this house $\sigma(i)$ is in the catchment area of the assigned school $\pi(i)$. We assume that $\mathcal{F}$ is not empty.

Definition 5 *Let $\Gamma^2 = (M^2, g^2)$ be a sector-2 mechanism such that $\forall R^2 \in \mathcal{R}(\Theta, \Pi)$; it holds that*

1. $(\sigma, z) \in g^2(NE(\Gamma^2, R^2))$ *if and only if* $(\sigma, z) \in A^2$ *is a* housing competitive equilibrium *for R^2, that is, there exists a house price vector $p \in \mathbb{R}^{|H|}$ such that $\forall i \in I$, $z_i = -p^{\sigma(i)}$ and*

$$\upsilon_i^{R_i^2}\left(\sigma, -p^{\sigma(i)}\right) \geq \upsilon_i^{R_i^2}\left(h, -p^h\right), \forall h \in H.$$

2. $\forall(\sigma,z)\in g^2(NE(\Gamma^2,R^2))$, *there is a unique* $m^2\in NE(\Gamma^2,R^2)$ *such that* $(\sigma,z)=g^2(m^2)$ *and* $L_i\left(R^2,(\sigma,z)\right)=g^2(M_i^2,m_{-i}^2)$ $\forall i\in I$.[8]

Sector-2 mechanisms that satisfy Definition 5 are found, for example, in Svensson (1991) and Hayashi and Sakai (2009).

We adopt the following notion of stability.

Definition 6 *The outcome* $(\pi,\sigma,z)\in\mathcal{F}$ *is* stable under $\theta\in\Theta$ *if there exists a house price vector* $p\in\mathbb{R}^{|H|}$ *such that* $\forall i\in I$, $z_i=-p^{\sigma(i)}$ *and*

(a) $\forall c\in C$ *with* $c\neq\pi(i)$, *and* $\forall h\in H^c$,

$$u_i^\theta(\pi(i),\sigma(i),-p^{\sigma(i)})\geq u_i^\theta(c,h,-p^h)$$

or

$$\left|\pi^{-1}(c)\right|=q_c \text{ and } j>_c i\ \forall j\in\pi^{-1}(c).$$

(b) $\forall c\in C$ *with* $c\neq\pi(i)$ *and* $\sigma(i)\in H^c$,

$$u_i^\theta(\pi(i),\sigma(i),-p^{\sigma(i)})\geq u_i^\theta(c,\sigma(i),-p^{\sigma(i)})$$

or

$$\left|\pi^{-1}(c)\right|=q_c \text{ and } j>_c i\ \forall j\in\pi^{-1}(c).$$

(c) $\forall h\in H^{\pi(i)}$,

$$u_i^\theta(\pi(i),\sigma(i),-p^{\sigma(i)})\geq u_i^\theta(\pi(i),h,-p^h).$$

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[8] For any R_i^2 and any $(\sigma,z)\in A^2$, the *lower contour set of* R_i^2 *at* (σ,z) is defined by $L_i\left(R^2,(\sigma,z)\right)=\{(\sigma',z')\in A^2|\,(\sigma,z)\,R_i^2\,(\sigma',z')\}$. A sector-2 mechanism Γ^2 and a profile R^2 induce a strategic game $\left(\Gamma^2,R^2\right)$. A (pure strategy) Nash equilibrium of $\left(\Gamma^2,R^2\right)$ is a strategy profile m^2 such that $\forall i\in I$, $g^2\left(m^2\right)R_i^2 g^2\left(\bar{m}_i^2,m_{-i}^2\right)$ $\forall\bar{m}_i^2\in M_i^2$. We write $NE\left(\Gamma^2,R^2\right)$ for the sector-2 Nash equilibrium strategies of $\left(\Gamma^2,R^2\right)$ and $g^2\left(NE\left(\Gamma^2,R^2\right)\right)$ for its corresponding set of Nash equilibrium outcomes.

Parts (a)–(b) require that each agent i either has no incentives to move from $(\pi(i), \sigma(i), -p^{\sigma(i)})$ to $(c, h, -p^h)$, or cannot move because under (π, σ, z) school $c \neq \pi(i)$ has already filled its quota and each agent j assigned to c has a priority higher than agent i. Part (c) requires that no agent i wants to change $\sigma(i)$ with a house h that is in the same catchment area of school $\pi(i)$, and at the same time remain at the school $\pi(i)$ assigned to him by (π, σ, z).

Definition 7 *The* constrained stable solution *of* $\left(I, C, H, (>_c)_{c\in C}, (H^c)_{c\in C}, \theta; \Gamma^2\right)$, *denoted by* $\varphi^S(\theta)$, *is the collection of all feasible outcomes that are stable under* θ,

$$\varphi^S(\theta) = \{(\pi, \sigma, z) \in \mathcal{F} | (\pi, \sigma, z) \text{ is stable under } \theta\}.$$

The following theorem establishes that the constrained stable solution is constrained implementable when the domain of possible states, Θ, satisfies the following condition.

Condition 1 *The domain* Θ *satisfies* condition 1 *if* $\forall \theta \in \Theta$, $\forall i \in I$, *and* $\forall (\pi, \sigma, z) \in A^1 \times A^2$ *with* $\sigma(i) \notin H^{\pi(i)}$, *it holds that*

$$u_i^\theta(\pi', \sigma', z') > u_i^\theta(\pi, \sigma, z) \;\; \forall (\pi', \sigma', z') \in A^1 \times A^2 \text{ such that } \sigma'(i) \in H^{\pi'(i)}.$$

Condition 1 requires that each agent i strictly prefers living in rather than outside the catchment area of the assigned school. This domain requirement seems to be a natural one for the problem at hand.

Theorem 4 *Suppose that* Θ *satisfies Condition 1. Suppose that* Γ^2 *satisfies Definition 5. Let* $\left(I, C, H, (>_c)_{c\in C}, (H^c)_{c\in C}\right)$ *be given such that* $|I| = |H| \geq 3$. *Then, the* constrained stable solution φ^S, *defined over* Θ, *is* constrained implementable.

Proof. Let the premises hold. To prove the statement, it suffices to show that φ^S satisfies constrained monotonicity and constrained no veto power. Since the preferences of agents are strictly increasing in money, φ^S satisfies constrained no veto power.[9] Thus, we need only show that φ^S is constrained monotonic. To this end, from now on, we fix any θ and any $(\pi, \sigma, z) \in \varphi^S(\theta)$.

Since by part (c) of Definition 6 no agent wants to move within the catchment area of the school assigned to him under (π, σ, z), and since by Condition 1 no agent wants to move outside the catchment area of the school assigned to him under (π, σ, z) while remaining at the school $\pi(i)$, it follows that (σ, z) is a housing competitive equilibrium at $R^2[\theta, \pi]$. Definition 5 implies that $(\sigma, z) \in g^2(NE(\Gamma^2, R^2[\theta, \pi]))$, that there exists a unique $m^2 \in M^2$ such that $(\sigma, z) = g^2(m^2)$, and that $L_i(R^2, (\sigma, z)) = g^2(M_i^2, m_{-i}^2)$ $\forall i \in I$. Thus, one can see that part (i) of constrained monotonicity is satisfied.

Finally, let us show that φ^S satisfies part (ii) of constrained monotonicity. Fix any $\theta' \in \Theta$ and suppose that the premises of part (ii) of constrained monotonicity are satisfied. We show that $(\pi, \sigma, z) \in \varphi^S(\theta')$. Since $(\pi, \sigma, z) \in \varphi^S(\theta)$, it follows that (π, σ, z) is stable under θ, that is, it meets the requirements (a)-(c) of Definition 6 and $z_i = -p^{\sigma(i)}$ for all $i \in I$.

Take any $(c, h, -p^h)$ such that $c \neq \pi(i)$ and either $h \in H^c$, or $\sigma(i) \in H^c$ and $h = \sigma(i)$. From parts (a)-(b), it follows that each agent $i \in I$ either has no incentives to move from $(\pi(i), \sigma(i), -p^{\sigma(i)})$ to $(c, h, -p^h)$, or if he has incentives to move, he cannot move because at (π, σ, z) school c has already filled its quota and each agent j assigned to c has a priority higher than agent i. Since this does not change when the state moves from θ to θ', it follows that parts (a)-(b) of Definition 6 are satisfied at the new state θ'.

Since (π, σ, z) is stable under θ, by part (c) of Definition 6, it holds that no agent i wants to change the house $\sigma(i)$ with a house that is in the same catchment area

[9] By introducing infinitesimal monetary transfers in the sector-1 mechanism. An alternative way to see that the constrained stable solution is constrained implementable is to observe that in Theorem 3 constrained no veto power can be replaced with two other auxiliary conditions. These two conditions are generalizations of unanimity and weak no veto power (see Takashi and Lombardi, 2017).

while remaining at the school assigned to him by (π, σ, z). Since, by assumption, $L_{i,m^2_{-i}}(\theta, (\pi, \sigma, z)) \subseteq L_{i,m^2_{-i}}(\theta', (\pi, \sigma, z))$ for all $i \in I$, it follows that part (c) of the definition continues to hold at θ'.

Since parts (a)–(c) hold at θ' and since $p^\sigma = z$ is still a competitive price vector at θ', we conclude that (π, σ, z) is stable under θ', and therefore, $(\pi, \sigma, z) \in \varphi^S(\theta')$. Thus, φ^S satisfies constrained monotonicity. ■

5.2 A public goods provision problem constrained by a competitive market of private goods

Let us consider an economy with a finite number $\ell \geq 1$ of private goods and one public good.[10] The consumption set of each agent i is $\mathbb{R}^{\ell+1}_+$. Each agent i has an endowment $\omega_i \in \mathbb{R}^\ell_+$. The consumption of agent i is denoted by $(v_i, y) \in \mathbb{R}^{\ell+1}_+$, where $v_i \in \mathbb{R}^\ell_+$ is the vector of private goods and $y \in \mathbb{R}_+$ is the public good outcome consumed by agent i.

The public good is produced in sector 1. The production technology is described by a production function $f : \mathbb{R}^\ell_+ \to \mathbb{R}_+$, which is continuous, strictly increasing, and strictly quasi-concave. $f(x)$ denotes the maximum level of public good associated with the vector of factors of production $x \in \mathbb{R}^\ell_+$. The outcome space of sector 1 is given by $A^1 = \left\{(x, y) \in \mathbb{R}^{\ell \times n+1}_+ | f\left(\sum_{i \in I} x_i\right) = y\right\}$, where $x \equiv (x_i)_{i \in I}$ is the profile of individual contributions to the production of y.

The outcome space of sector 2 is the set of closed net trade vectors of private goods, which is defined by $A^2 = \left\{z \in \mathbb{R}^{\ell \times n} | \sum_{i \in I} z_i = 0\right\}$. Then, $(x, y, z) \in A^1 \times A^2$ is a typical allocation, which specifies a technological feasible scale of the public good, y, a technological feasible profile of individual contributions to the production of y, and a net trade vector.

[10] For the sake of simplicity, and to save space, we assume there is one public good. However, the application can easily be extended to the case in which there are $k > 1$ public goods.

We make the following assumption about the set Θ. Agent i's (self-regarding) preferences in state $\theta \in \Theta$ are given by an ordering $R_i(\theta)$ over the set $A^1 \times A^2$. For each $\theta \in \Theta$, we assume that agent i's ordering over the consumption space is numerically represented by $u_i^\theta : \mathbb{R}_+^{\ell+1} \to \mathbb{R}$, which is continuous, quasi-concave, and strictly increasing. For each $\theta \in \Theta$, each agent i, and each sector-1 allocation $(x, y) \in A^1$, the sector-2 marginal ordering, $R_i^2[\theta, x, y]$, on A^2 induced by (θ, x, y) is defined by

$$\forall z, z' \in A^2 : zR_i^2[\theta, x, y]\, z' \iff (x, y, z)\, R_i(\theta)\, (x, y, z').$$

We assume that agent i's sector-2 marginal ordering $R_i^2[\theta, x, y]$ is numerically represented over the consumption space of private goods $\upsilon_i^{R_i^2[\theta,x,y]} : A^2 + \{\omega_i\} \to \mathbb{R}$. Following Schmeidler (1980), we assume that $\upsilon_i^{R_i^2[\theta,x,y]}$ is strictly increasing in the consumption of private goods and satisfies the following assumptions: $\upsilon_i^{R_i^2[\theta,x,y]}(z_i + \omega_i) > -\infty$ if $z_i+\omega_i \in \mathbb{R}_+^\ell$; otherwise, $\upsilon_i^{R_i^2[\theta,x,y]}(z_i + \omega_i) = -\infty$.[11] Let $R^2[\theta, x, y] \equiv (R_i^2[\theta, x, y])_{i\in I}$ denote the profile of sector-2 marginal orderings induced by (θ, x, y). Let $\mathcal{R}(\Theta, A^1)$ be the set of profiles of sector-2 marginal orderings induced by (Θ, A^1), with R^2 as a typical element.

The set of feasible outcomes is denoted by $\mathcal{F}$ and is defined by

$$\mathcal{F} = \left\{(x, y, z) \in \mathbb{R}_+^{\ell\times n} \times \mathbb{R}_+ \times \mathbb{R}^{\ell\times n} \middle| \, (x, y) \in A^1,\, z \in A^2\right\}.$$

Given a vector $p \in \mathbb{R}_+^\ell$ of prices of factors of production and a level $y \in \mathbb{R}_+$ of public good, the cost-minimization problem of sector 1 is

$$c(p, y) = \min_{a\in\mathbb{R}_+^\ell : y=f(a)} p \cdot a.$$

Definition 8 *Let $\Gamma^2 = (M^2, g^2)$ be a sector-2 mechanism such that $\forall R^2 \in \mathcal{R}(\Theta, A^1)$; it holds that*

[11] Note that agent i's endowment may vary, depending on the production plan chosen by sector 1. However, this is not problematic, because for the sector-2 mechanism of Definition 8 to work, we need only agents' marginal preferences over the set of net trade vectors.

1. $z \in g^2(NE(\Gamma^2, R^2))$ *if and only if* z *is a* competitive equilibrium *for* R^2, *that is, there exists a price vector* $p \in \mathbb{R}^\ell$ *such that* $\forall i \in I$, $p \cdot z_i = 0$ *and*

$$\forall \bar{z}_i \in \mathbb{R}^\ell : p \cdot \bar{z}_i = 0 \implies \upsilon_i^{R_i^2}(z_i + \omega_i) \geq \upsilon_i^{R_i^2}(\bar{z}_i + \omega_i),$$

and

$$\textstyle\sum_{i \in I} z_i = 0.$$

2. $\forall z \in g^2(NE(\Gamma^2, R^2))$, *there is a unique* $m^2 \in NE(\Gamma^2, R^2)$ *such that* $z = g^2(m^2)$ *and* $L_i(R^2, z) = g^2(M_i^2, m_{-i}^2)$ $\forall i \in I$.

A sector-2 mechanism that satisfies Definition 8 is found in Schmeidler (1980).

In this subsection, we are interested in constrained implementing the ratio solution of Kaneko (1977) and Diamantaras and Wilkie (1994). To this end, the ratio r_i is the portion of the cost that agent $i \in I$ is required to contribute. Thus, ratio allocations can be defined as follows.

Definition 9 *A feasible allocation* $(x, y, z) \in \mathcal{F}$ *is a* ratio allocation *for* $\theta \in \Theta$ *if there exist a price vector* $p \in \mathbb{R}_+^\ell$ *and a ratio vector* $r \in \mathbb{R}_+^n$, *with* $\sum_{i \in I} r_i = 1$, *such that*

$$\sum_{i \in I} z_i = 0 \text{ and } y = f\left(\textstyle\sum_{i \in I} x_i\right),$$

and $\forall i \in I$; *it holds that*

$$p \cdot z_i = 0 \text{ and } p \cdot x_i = r_i c(p, y),$$

and

$$u_i^\theta(z_i + \omega_i - x_i, y) \geq u_i^\theta(\bar{z}_i + \omega_i - \bar{x}_i, \bar{y})$$

$\forall \bar{z}_i \in \mathbb{R}^\ell$ *such that* $p \cdot \bar{z}_i = 0$, *and* $\forall \bar{y} \in \mathbb{R}_+$ *and* $\forall \bar{x}_i \in \mathbb{R}_+^\ell$ *such that* $p \cdot \bar{x}_i = r_i c(p, \bar{y})$.

Thus, a ratio allocation (x, y, z) is generated by a ratio vector r and a price vector p such that the allocation yields each agent the largest level of utility given the generalized budget constraints, the market for the private goods clears, and $\sum_{i \in I} x_i$ is the vector of factors of production that minimizes the cost of production of y. Note that if (x, y, z) is a ratio allocation for θ generated by (r, p), then taking the pair (x, y) as given, the net trade vector z is a competitive allocation for $R^2[\theta, x, y]$ generated by the competitive price vector p. We let $\varphi^r(\theta)$ denote the set of ratio allocations for θ.

Definition 10 *The* constrained ratio solution*, φ^r, for $\left(I, \theta, (\omega_i)_{i \in I}; \Gamma^2\right)$ selects the set of ratio allocations for θ,*

$$\varphi^r(\theta) = \{(x, y, z) \in \mathcal{F} | (x, y, z) \text{ is a ratio allocation for } \theta\}.$$

The following theorem establishes that the constrained ratio solution is constrained implementable.

Theorem 5 *Let $n \geq 3$. Suppose that Γ^2 satisfies Definition 8. Then, the* constrained ratio solution *φ^r, defined over Θ, is* constrained implementable*.*

Proof. Let the premises hold. To prove the statement, it suffices to show that φ^r satisfies constrained monotonicity and constrained no veto power. Since agents' preferences are strictly monotonic, the ratio solution satisfies constrained no veto power. Thus, let us show that it is constrained monotonic. To this end, fix any θ and any $(x, y, z) \in \varphi^r(\theta)$.

Then, the ratio allocation (x, y, z) for θ is generated by a price vector p and a ratio vector r. Then, z is a competitive equilibrium for the marginal preferences over the consumption space induced by (x, y), and thus, it is a competitive equilibrium for $R^2[\theta, x, y]$. By Definition 8, it follows that $z \in g^2(NE(\Gamma^2, R^2[\theta, x, y]))$, that there is a unique $m^2 \in NE(\Gamma^2, R^2)$ such that $z = g^2(m^2)$, and that $L_i(R^2, z) = g^2(M_i^2, m_{-i}^2)$ $\forall i \in I$. It can be observed that part (i) of constrained monotonicity is satisfied.

Next, fix any θ' so that the premises of part (ii) of constrained monotonicity are satisfied when the state changes from θ to θ'. We show that $(x, y, z) \in \varphi^r(\theta')$. Given that no agent i finds it profitable to choose any different $(\bar{z}_i, \bar{y}, \bar{x}_i)$ such that $p \cdot z_i = 0$ and that $\bar{x}_i = r_i c(p, \bar{y})$ in state θ, it follows that no agent i wants to choose differently when the state moves to θ'. Thus, (x, y, z) is a ratio allocation for θ' generated by the price vector p and the ratio vector r. By definition of the ratio solution, we conclude that $(x, y, z) \in \varphi^r(\theta')$. Thus, the ratio solution is constrained monotonic. ■

6. Comparative statics

In this section, we examine the relationships between constrained implementability and the standard unconstrained implementability. The broad question is whether constrained implementation is easier or more difficult to achieve than unconstrained implementability? Put differently, when does constrained implementability reduce to unconstrained implementability? The answer to this broad question depends on what the given mechanism in sector 2 is, what our aim is, and what the preference domain is. To obtain answers, we pose specific questions and provide answers to them in this section.

6.1 Implementation when constrained by a "poor" sector-2 mechanism

First, we ask if we can implement something when an arbitrary mechanism is given as fixed in sector 2. There is a trivial yes answer here. Provided that Γ^2 always allows Nash equilibria under any marginal ordering, one can always constraint implement a 'trivial' SCR. It is simply that we do "nothing" in sector 1 and let agents play the sector-2 game as it is.

Example 2 Fix any $\hat{a}^1 \in A^1$. For each $\theta \in \Theta$, agent i's sector-2 marginal ordering, $R_i^2[\theta, \hat{a}^1]$, on A^2, is defined by

$$\forall a^2, b^2 \in A^2 : a^2 R_i^2 \left[\theta, \hat{a}^1\right] b^2 \iff \left(\hat{a}^1, a^2\right) R_i(\theta) \left(\hat{a}^1, b^2\right).$$

Let $R^2[\theta, \hat{a}^1] = (R_i^2[\theta, \hat{a}^1])_{i \in I}$ denote the profile of sector-2 marginal orderings induced by $\hat{a}^1$ under θ. Let Γ^2 be given. Let us suppose that the set of (pure) Nash equilibrium strategy profiles of $(\Gamma^2, R^2[\theta, \hat{a}^1])$, denoted by $NE(\Gamma^2, R_2[\theta, \hat{a}^1])$, is not empty for every state $\theta \in \Theta$. Define $\varphi : \Theta \to A^1 \times A^2$ by $\varphi(\theta) = \{(\hat{a}^1, a^2) : a^2 \in g^2 \circ NE(\Gamma^2, R^2[\theta, \hat{a}^1])\}$. One can implement φ by designing $\Gamma^1 = (M^1, g^1)$ such that $M_i^1 = \{m_i^1\}$, $\forall i \in I$, and $g^1(m^1) = \hat{a}^1$. One can easily verify that φ is constrained monotonic. Under a suitable restriction on the set Θ, one can observe that φ is constrained implementable.

Observe that the above example is consistent with Example 1. The reason is that in Example 1, not all equilibria of the sector-2 mechanism are selected by the SCR under the same profile of marginal orderings. Instead, in the above example, we include all sector-2 Nash equilibria in the definition of φ. This entails that the set of sector-2 Nash equilibrium outcomes is the same for states inducing the same marginal orderings over the sector-2 outcome space.

In light of the above example, the question is whether there always exists a "satisfactory" constrained implementable SCR or not. The answer depends on the quality of Γ^2. For example, the answer is no when Γ^2 is a "poor" sector-2 mechanism.

Example 3 Let Γ^2 be a constant mechanism, that is, there exists $\hat{a}^2 \in A^2$ such that $g^2(m^2) = \hat{a}^2$ for all $m^2 \in M^2$. Let us suppose that $A^1 = A^2 = A$. Moreover, let Θ be a domain of preferences such that $(a, a)P_i(\theta)(b, c)$ for all $a, b, c \in A$ such that $b \neq c$. Thus, at best, what we can constrained implement is represented by the pair $(\hat{a}^2, \hat{a}^2)$, which is clearly inefficient.

6.2 When does constrained implementability reduce to unconstrained implementability?

The analysis so far proves that the design problem of the sector-1 designer is rather complicated owing to the interplay between the sector-2 mechanism Γ^2, the preference domain Θ, and the SCR φ. However, it is noteworthy that the constrained implementation exercise of the sector-1 designer reduces to the classical sector-1 implementation exercise when the preference domain Θ is fully separable, that is, $\Theta = \Theta^{SEP}$, the SCR is decomposable, and Γ^2 implements the sector-2 SCR in Nash equilibrium or in one of its refinements.[12] Decomposability is not a necessary condition for constrained implementation.

We pose the following question: what are the necessary and almost sufficient conditions in the special (but still somewhat general) case in which the solution concept adopted by sector-2 designer is that of Nash equilibrium?[13] The answer is that the conditions of Theorem *3* do not change. To observe this, we introduce below a new condition, called *constrained monotonicity** using additional notation, and show that it is equivalent to constrained monotonicity.

We say that the ordering $R_i(\theta)$ is sector-2 separable if $R_i^2[\theta, a^1] = R_i^2[\theta, b^1]$ $\forall a^1, b^1 \in A^1$. To save space, for any sector-2 separable ordering $R_i(\theta)$, write $R_i^2(\theta)$ for the sector-2 marginal ordering induced by $R_i(\theta)$. Let $R^2(\theta) = (R_i^2(\theta))_{i \in I}$ denote the profile of sector-2 marginal orderings induced by θ. The preference domain Θ^* is defined by

$$\Theta^* = \{\theta \in \Theta | \forall i \in I, R_i(\theta) \text{ is a sector-2 separable ordering}\}.$$

We assume that $\Theta^{SEP} \subseteq \Theta^*$.

Let Γ^2 be given. Define the SCR φ, over Θ^*, as follows: $\forall \theta \in \Theta^*$, $\varphi(\theta) = \varphi^1(\theta) \times g^2(NE(\Gamma^2, R^2(\theta)))$, where $NE(\Gamma^2, R^2(\theta))$ denotes the set of (pure) Nash equilibrium strategy profiles of $(\Gamma^2, R^2(\theta))$, and $\varphi^1 : \Theta^* \twoheadrightarrow A^1$ is a correspondence.

[12] The preference domain is *fully separable*, denoted by Θ^{SEP}, when each state $\theta \in \Theta^{SEP}$ induces, for each agent and each sector s, a sector-s marginal ordering over A^s that is independent of outcomes chosen from the other component set of the outcome space. The SCR $\varphi : \Theta^{SEP} \twoheadrightarrow A^2 \times A^2$ is *decomposable* provided that for each sector $s \in \{1, 2\}$, there exists a (nonempty) correspondence $\varphi^s : \mathcal{D}^s \twoheadrightarrow A^s$ such that $\varphi(\theta) = \varphi^1(R^1(\theta)) \times \varphi^2(R^2(\theta))$ for each state $\theta \in \Theta^{SEP}$, where $\mathcal{D}^s$ is set of profiles of sector-s marginal orderings induced by Θ^{SEP}.

[13] The result of Theorem *6* holds for any suitable refinement of the Nash equilibrium.

Since $\varphi(\theta)$ is not an empty set, by definition, it follows that $NE(\Gamma^2, R^2(\theta))$ and $\varphi^1(\theta)$ are not empty. Constrained monotonicity* is defined as follows.

Definition 11 *Let Γ^2 and Θ^* be given. The SCR $\varphi : \Theta^* \twoheadrightarrow A^1 \times A^2$ is* constrained monotonic* *provided that $\forall (a^1, a^2) \in A^1 \times A^2$ and $\forall \theta \in \Theta^*$, if $(a^1, a^2) \in \varphi(\theta)$, then there exists $m^2 \equiv m^2_\varphi(\theta, (a^1, a^2)) \in NE(\Gamma^2, R^2(\theta))$ such that parts (i)–(ii) of constrained monotonicity are satisfied with respect to this m^2.*

Constrained monotonicity* differs from constrained monotonicity in only one aspect: the sector-2 message profile $m^2_\varphi(\theta, (a^1, a^2))$ is a Nash equilibrium strategy profile for $(\Gamma^2, R^2(\theta))$. Nevertheless, constrained monotonicity* is equivalent to constrained monotonicity.

Theorem 6 *Let Γ^2 be given. The SCR $\varphi : \Theta^* \twoheadrightarrow A^1 \times A^2$ is* constrained monotonic* *if and only if it is* constrained monotonic*.*

Proof. Let Γ^2 be given. Clearly, constrained monotonicity* implies constrained monotonicity. Thus, suppose that φ is constrained monotonic. We show that it is constrained monotonic*. Fix any $\theta \in \Theta^*$ and any $(a^1, a^2) \in A^1 \times A^2$ such that $(a^1, a^2) \in \varphi(\theta)$. Constrained monotonicity implies that there exists $m^2 \in M^2$ such that $g^2(m^2) = a^2$ and $proj_2\left\{L_{i,m^2_{-i}}(\theta, (a^1, a^2))\right\} = g^2\left(M^2_i, m^2_{-i}\right)$ for all $i \in I$. To observe that φ is constrained monotonic*, it suffices to show that $m^2 \in NE(\Gamma^2, R^2(\theta))$. To this end, fix any $i \in I$. Since $proj_2\left\{L_{i,m^2_{-i}}(\theta, (a^1, a^2))\right\} = g^2\left(M^2_i, m^2_{-i}\right)$ and since $R^2_i(\theta)$ is a sector-2 marginal ordering induced by $\theta \in \Theta^*$, it follows that $g^2\left(M^2_i, m^2_{-i}\right) \subseteq L_i(R^2(\theta), a^2)$. Since the choice of agent i is arbitrary, we obtain $m^2 \in NE(\Gamma^2, R^2(\theta))$, as we aimed to achieve. ■

Remark 1 Theorem 6 also holds in the case in which $\Theta^* = \Theta^{SEP}$.

Constrained monotonicity continues to be a necessary and almost sufficient condition for constrained implementation in the case in which the sector-1 designer does not care at all about sector 2. This situation can be modeled by defining the SCR $\varphi : \Theta \twoheadrightarrow A^1 \times A^2$ by $\varphi(\theta) = \varphi^1(\theta) \times g^2(M^2)$, for all $\theta \in \Theta$, where $\varphi^1 : \Theta \twoheadrightarrow A^1$ is a (nonempty) correspondence.[14] The reason is that the designer of sector 1 still needs to condition the outcome of the sector-1 mechanism to strategies played by agents in sector 2 because, for example, there are complementarities between sectors.

As already noted at the beginning of this subsection, constrained monotonicity is equivalent to sector-1 monotonicity provided that agents' preference orderings are fully separable, the SCR φ is decomposable, and Γ^2 implements the sector-2 SCR in Nash equilibrium or in one of its refinements.

Definition 12 *The sector-1 SCR $\varphi^1 : \Theta \twoheadrightarrow A^1$ is* sector-1 monotonic *provided that $\forall (a^1, a^2) \in A^1 \times A^2$ and $\forall \theta, \theta' \in \Theta$, if $a^1 \in \varphi^1(\theta)$ and*

$$L_i\left(R^1\left[\theta, a^2\right], a^1\right) \subseteq L_i\left(R^1\left[\theta', a^2\right], a^1\right) \ \forall i \in I,$$

then $a^1 \in \varphi^1(\theta')$.

We now pose the following question: are there cases in which sector-1 monotonicity is equivalent to or implies constrained monotonicity? The following examples summarize our answers.

Example 4 Let us consider the case in which $A^2 = \{a^2\}$ and $\varphi(\theta) = \varphi^1(\theta) \times g^2(M^2)$, for all $\theta \in \Theta$, where $\varphi^1 : \Theta \twoheadrightarrow A^1$ is a (nonempty) correspondence. In this setting, constrained monotonicity is equivalent to sector-1 monotonicity. Since $A^2 = \{a^2\}$, one can easily observe that constrained monotonicity implies sector-1 monotonicity. For the converse, take any $\theta \in \Theta$ and any $a^1 \in A^1$ such that $(a^1, a^2) \in \varphi(\theta)$. Part (i) of constrained monotonicity is satisfied, since, by definition of Γ^2, there exists $m^2 \in M^2$ such that $g^2(m^2) = a^2$ and $proj_2\left\{L_{i,m^2_{-i}}(\theta, (a^1, a^2))\right\} = \{a^2\}$ for each agent $i \in I$. Next, take any $\theta' \in \Theta$ so that $L_{i,m^2_{-i}}(\theta, (a^1, a^2)) \subseteq L_{i,m^2_{-i}}(\theta', (a^1, a^2))$ for each $i \in I$. Since $(a^1, a^2) \in \varphi(\theta)$, it follows that $a^1 \in \varphi^1(\theta)$. Moreover, given

[14] This result holds without restricting our attention to Θ^*.

that the set $L_{i,m_{-i}^2}\left(\bar{\theta},\left(a^1,a^2\right)\right)$ is equivalent to $L_i\left(R^1\left[\bar{\theta},a^2\right],a^1\right)$ for each $\bar{\theta}\in\Theta$, where $R^1\left[\bar{\theta},a^2\right]$ is the profile of marginal orderings induced by a^2 under $\bar{\theta}$, we obtain $L_i\left(R^1\left[\theta,a^2\right],a^1\right)\subseteq L_i\left(R^1\left[\theta',a^2\right],a^1\right)$ for each $i\in I$. Sector-1 monotonicity implies that $a^1\in\varphi^1\left(\theta'\right)$, and thus, $\left(a^1,a^2\right)\in\varphi\left(\theta'\right)$, by definition of φ.

Example 5 For the same reason shown in the above example, constrained monotonicity is equivalent to sector-1 monotonicity when Γ^2 is a constant mechanism, that is, when $g^2\left(m^2\right)=a^2\in A^2$ for all $m^2\in M^2$.

Example 6 Let $\Theta=\Theta^{SEP}$. Let Γ^2 be given. Let $\varphi^2:\mathcal{D}^2\twoheadrightarrow A^2$ denote a sector-2 SCR, where $\mathcal{D}^2$ is set of profiles of sector-2 marginal orderings induced by Θ^{SEP}. Assume that Γ^2 implements φ^2 in Nash equilibrium, that is, $\varphi^2\left(R^2\left(\theta\right)\right)=g^2\circ NE\left(\Gamma^2,R^2\left(\theta\right)\right)$ for all $\theta\in\Theta^{SEP}$. Define φ by $\varphi\left(\theta\right)=\varphi^1\left(\theta\right)\times g^2\circ NE\left(\Gamma^2,R^2\left(\theta\right)\right)$ for each $\theta\in\Theta^{SEP}$.

Let us show that constrained monotonicity is implied by sector-1 monotonicity. Suppose that φ^1 is sector-1 monotonic. Suppose that $\left(a^1,a^2\right)\in\varphi\left(\theta\right)$ for some $\theta\in\Theta^{SEP}$. Then, by definition of φ, there exists $m^2\in NE\left(\Gamma^2,R^2\left(\theta\right)\right)$ such that $g^2\left(m^2\right)=a^2$. Moreover, since $\Theta=\Theta^{SEP}$, by assumption, it follows that $proj_2\left\{L_{i,m_{-i}^2}\left(\theta,\left(a^1,a^2\right)\right)\right\}=g^2\left(M_i^2,m_{-i}^2\right)$ for each $i\in I$. Thus, φ satisfies part (i) of constrained monotonicity. To verify part (ii), take any $\theta'\in\Theta^{SEP}$ and suppose that $L_{i,m_{-i}^2}\left(\theta,\left(a^1,a^2\right)\right)\subseteq L_{i,m_{-i}^2}\left(\theta',\left(a^1,a^2\right)\right)$ for each $i\in I$. We show that $\left(a^1,a^2\right)\in\varphi\left(\theta'\right)$.

Let us first show that $m^2\in NE\left(\Gamma^2,R^2\left(\theta'\right)\right)$. Fix any agent i and any $\hat{m}_i^2\in M_i^2$. Since $m^2\in NE\left(\Gamma^2,R^2\left(\theta\right)\right)$, it follows that $a^2R_i\left(\theta\right)g^2\left(\hat{m}_i^2,m_{-i}^2\right)$. Since $\theta\in\Theta^{SEP}$, we obtain $\left(a^1,g^2\left(\hat{m}_i^2,m_{-i}^2\right)\right)\in L_{i,m_{-i}^2}\left(\theta,\left(a^1,a^2\right)\right)\subseteq L_{i,m_{-i}^2}\left(\theta',\left(a^1,a^2\right)\right)$. Then, by our supposition that $\theta'\in\Theta^{SEP}$, it follows that $a^2R_i^2\left(\theta'\right)g^2\left(\hat{m}_i^2,m_{-i}^2\right)$. Since the choice of $\hat{m}_i^2$, as well as that of agent i, is arbitrary, one can observe that $m^2\in NE\left(\Gamma^2,R^2\left(\theta'\right)\right)$, as we aimed to achieve.

Next, let us show that $a^1\in\varphi^1\left(\theta'\right)$. Note that $a^1\in\varphi^1\left(\theta\right)$, by definition of φ. Fix any $b^1\in A^1$ and any $i\in I$. Let us show that $a^1R_i^1\left(\theta\right)b^1\implies a^1R_i^1\left(\theta'\right)b^1$. Suppose that $a^1 R_i^1(\theta) b^1$. Then, since $\theta \in \Theta^{SEP}$, it follows that $(a^1, g^2(m^2)) R_i(\theta)(b^1, g^2(m^2))$. Since $L_{i,m_{-i}^2}(\theta, (a^1, a^2)) \subseteq L_{i,m_{-i}^2}(\theta', (a^1, a^2))$, it follows that $(a^1, g^2(m^2)) R_i(\theta')(b^1, g^2(m^2))$. Again, since $\theta \in \Theta^{SEP}$, we obtain $a^1 R_i^1(\theta') b^1$. Since the choice of b^1, as well as that of agent i, is arbitrary, we obtain $\forall i \in I$, $\forall b^1 \in A^1$, $a^1 R_i^1(\theta) b^1 \implies a^1 R_i^1(\theta') b^1$. Sector-1 monotonicity implies that $a^1 \in \varphi^1(\theta')$.

Finally, since we have established that $a^2 \in g^2 \circ NE(\Gamma^2, R^2(\theta'))$ and $a^1 \in \varphi^1(\theta')$, it follows that $(a^1, a^2) \in \varphi(\theta')$, by definition of φ.

It is noteworthy that sector-1 monotonicity implies decomposability of φ when $\Theta = \Theta^{SEP}$ and Γ^2 implements $\varphi^2 : \mathcal{D}^2 \twoheadrightarrow A^2$ in Nash equilibrium. The reason is that for each sector $s = 1, 2$, $a^s \in \varphi^s(\theta')$ if $a^s \in \varphi^s(\theta)$ and $R^s(\theta) = R^s(\theta')$ $\forall i \in I$.

7. Concluding remarks

In this study, we investigate the theory of implementation pioneered by Maskin (1999) under the assumption that some dimensions of the design problem are fixed and the designer works under this constraint. Specifically, the designer's constraint is represented by mechanisms "solving" implementation problems linked to the implementation problem at hand. Under the simplifying assumption that there are two sectors, issues, or objects, we assume that the dimension of sector 2 of the design problem is fixed. We denote this constraint by Γ^2. The implementation problem of the designer consists of designing a mechanism for sector 1, Γ^1, with the property that for any type of agents' preferences, the set of Nash equilibrium outcomes of $\Gamma^1 \times \Gamma^2$ coincides with the set of φ-optimal outcomes. If this design exercise can be accomplished, the SCR is said to be constrained implementable.

We identify a necessary condition for constrained implementation, named constrained monotonicity. This condition is a strengthened version of monotonicity. Furthermore, under an auxiliary condition, we show that constrained monotonicity is sufficient for constrained implementation. This result is obtained under the informational assumption that the designer can use the private information elicited from agents via Γ^2 to make a socially optimal decision for sector 1. Unlike the negative result of Hayashi and Lombardi (2017), our sufficiency result does not rule out any kind of complementarity between the two sectors.

This main result is obtained by imposing a very stringent assumption: there is complete information among agents. This assumption is not often met in reality. Moreover, the mechanism used in constructing the proof inherits all limitations of Nash implementation. Indeed, the devised mechanism has several technical problems, in addition to having strategy spaces that are difficult to interpret. For more on these limitations, see, in particular, Jackson (1992), Abreu and Matsushima (1992), and more recently Ollár and Penta (2017). The main result we obtain must be thought of as providing a theoretical benchmark; its applicability should not be exaggerated.

Appendix

Proof of Theorem 3

Let $\Gamma^2 = (M^2, g^2)$ be given. Take any constrained monotonic SCR $\varphi : \Theta \twoheadrightarrow A^1 \times A^2$ satisfying constrained no veto power. Let us define $\Gamma^1 = (M^1 \times M^2, g^1)$ as follows. For all $i \in I$, agent i's action space is defined by:

$$M_i^1 = \Theta \times A^1 \times A^2 \times \{0, 1, ..., n\}.$$

Let $m_i^1 = (\theta^i, a^{1i}, a^{2i}, k^i)$ denote a typical element of M_i^1. The outcome function g^1 is defined by the following four rules.

$\forall (m^1, m^2) \in M^1 \times M^2,$

Rule 1: If $m_j^1 = (\bar{\theta}, a^1, a^2, 0)$ for all $j \in I$, $(a^1, a^2) \in \varphi(\bar{\theta})$ and $m^2 = m_\varphi^2(\bar{\theta}, (a^1, a^2))$, then $(g^1 \times g^2)(m^1, m^2) = (a^1, g^2(m^2))$, where $m_\varphi^2(\bar{\theta}, (a^1, a^2))$ is the sector-2 message specified by part (i) of constrained monotonicity.

Rule 2: For all $i \in I$, if each agent $j \in I\backslash\{i\}$ plays $m_j^1 = (\bar{\theta}, a^1, a^2, 0)$, with $(a^1, a^2) \in \varphi\left(\bar{\theta}\right)$, and, moreover, for each $j \in I\backslash\{i\}$, m_j^2 coincides with the jth coordinate of $m_\varphi^2\left(\bar{\theta}, (a^1, a^2)\right)$, and agent i plays $m_i^1 = (\theta^i, a^{1i}, a^{2i}, k^i) \neq (\bar{\theta}, a^1, a^2, 0)$, m_i^2 does not coincide with the ith coordinate of $m_\varphi^2\left(\bar{\theta}, (a^1, a^2)\right)$, or both, then there are the following two cases.

1. If $(a^1, a^2)\, R_i\left(\bar{\theta}\right)(a^{1i}, g^2(m^2))$, then $(g^1 \times g^2)(m^1, m^2) = (a^{1i}, g^2(m^2))$.
2. Otherwise, let $(g^1 \times g^2)(m^1, m^2) = (b^1, g^2(m^2))$ such that $(a^1, a^2)\, R_i\left(\bar{\theta}\right)(b^1, g^2(m^2))$ for some $b^1 \in proj_1\left\{L_{i,g^2,m_{-i}^2}\left(\bar{\theta}, (a^1, a^2)\right)\right\}$, which is not empty by part (i) of constrained monotonicity.

Rule 3: Otherwise, a modulo game is played: divide the sum $\sum_{i\in I} k^i$ by the cardinality of I and identify the remainder, which can be 0, 1, $\cdots$, or $n-1$. The agent i^* with the same index of the remainder is declared the winner of the game and the alternative implemented is $(g^1 \times g^2)(m^1, m^2) = \left(a^{1i^*}, g^2(m^2)\right)$, with the convention that the winner is agent n if the remainder is 0.

Fix any $\theta \in \Theta$. We show that $\varphi(\theta) = (g^1 \times g^2)(NE(\Gamma^1 \times \Gamma^2, \theta))$. Let us first show that $\varphi(\theta) \subseteq (g^1 \times g^2)(NE(\Gamma^1 \times \Gamma^2, \theta))$. Fix any $(a^1, a^2) \in \varphi(\theta)$. Part (i) of constrained monotonicity implies that there is $m^2 = m_\varphi^2(\theta, (a^1, a^2))$ such that $g^2(m^2) = a^2$, and $proj_2\left\{L_{i,m_{-i}^2}(\theta, (a^1, a^2))\right\} = g^2\left(M_i^2, m_{-i}^2\right)$, for each $i \in I$. Then, let $m_i^1 = (\theta, a^1, a^2, 0)$, for each $i \in I$, and let agent $i \in I$ play m_i^2. Then, (m^1, m^2) falls into *Rule 1*, and thus, $(g^1 \times g^2)(m^1, m^2) = (a^1, a^2)$. Note that no agent can induce *Rule 3*. In addition, note that each agent can induce *Rule 2* by unilaterally deviating from (m^1, m^2). Since no agent can benefit by unilaterally deviating from (m^1, m^2)—since every agent $i \in I$ can obtain outcomes only in $L_{i,m_{-i}^2}(\theta, (a^1, a^2))$, it follows that $(m^1, m^2) \in NE(\Gamma^1 \times \Gamma^2, \theta)$, as we aimed to achieve.

For the converse, take any $(m^1, m^2) \in NE(\Gamma^1 \times \Gamma^2, \theta)$. We proceed according to the following cases.

Case 1: (m^1, m^2) falls into *Rule 1.*
Then, the outcome is $(g^1 \times g^2)(m^1, m^2) = (a^1, a^2)$. Take any agent $i \in I$. First, it can be easily checked that agent i can never induce *Rule 3.* Second, by part (i) of constrained monotonicity, take any

$$b^2 \in proj_2 \left\{ L_{i,m^2_{-i}} \left(\bar{\theta}, \left(a^1, a^2\right)\right) \right\} = g^2 \left(M^2_i, m^2_{-i}\right). \tag{3}$$

Then, there is $\hat{m}^2_i \in M^2_i$ such $g^2 \left(\hat{m}^2_i, m^2_{-i}\right) = b^2$. Moreover, take any $b^1 \in A^1$ such that $(a^1, a^2) R_i \left(\bar{\theta}\right) (b^1, b^2)$—note that there exists b^1 such that $(a^1, a^2) R_i \left(\bar{\theta}\right) (b^1, b^2)$ since (3) holds, by part (i) of constrained monotonicity. By changing (m^1_i, m^2_i) into $(\hat{m}^1_i, \hat{m}^2_i)$, agent i induces part (1) of *Rule 2* and obtains (b^1, b^2). Since the choice of (b^1, b^2), as well as that of agent i, is arbitrary, it follows that

$$L_{i,m^2_{-i}} \left(\bar{\theta}, \left(a^1, a^2\right)\right) \subseteq \left(g^1 \times g^2\right) \left(\left(M^1_i, m^1_{-i}\right), \left(M^2_i, m^2_{-i}\right)\right) \;\; \forall i \in I. \tag{4}$$

Since $(m^1, m^2) \in NE((\Gamma^1 \times \Gamma^2, \theta))$, it follows from (4) that

$$L_{i,m^2_{-i}} \left(\bar{\theta}, \left(a^1, a^2\right)\right) \subseteq L_{i,m^2_{-i}} \left(\theta, \left(a^1, a^2\right)\right) \;\; \forall i \in I.$$

Part (ii) of constrained monotonicity implies that $(g^1 \times g^2)(m^1, m^2) \in \varphi(\theta)$, as we aimed to achieve.

Case 2: (m^1, m^2) falls into *Rule 2.*
Fix any agent $j \in I \backslash \{i\}$. Take any $\hat{m}^2_j \in M^2_j$ and any $c^1 \in A^1$. Then, by changing $\left(m^1_j, m^2_j\right)$ into $\left(\hat{m}^1_j, \hat{m}^2_j\right)$, where $\hat{m}^1_j = \left(\bar{\theta}, c^1, g^2\left(\hat{m}^2_j, m^2_{-j}\right), k^j\right)$ and $k^j \neq 0$, agent j can induce *Rule 3.* To obtain $\left(c^1, g^2\left(\hat{m}^2_j, m^2_{-j}\right)\right)$, agent j has only to adjust k^j by which he becomes the winner of the modulo game. Since the choice of $\left(c^1, \hat{m}^2_j\right)$, as well as that of agent $j \in I \backslash \{i\}$, is arbitrary, we established that any agent $j \in I \backslash \{i\}$ can induce any outcome of $A^1 \times g^2\left(M^2_j, m^2_{-j}\right)$. Since $(m^1, m^2) \in NE(\Gamma^1 \times \Gamma^2, \theta)$, it

follows that $A^1 \times g^2\left(M_j^2, m_{-j}^2\right) \subseteq L_{j,g^2,m_{-j}^2}\left(\theta, \left(g^1 \times g^2\right)\left(m^1, m^2\right)\right)$, for all $j \in I \backslash \{i\}$. Constrained no veto power implies that $\left(g^1 \times g^2\right)\left(m^1, m^2\right) \in \varphi(\theta)$, as we aimed to achieve.

Case 3: (m^1, m^2) falls into *Rule 3*.

Fix any agent $i \in I$. Take any $\hat{m}_i^2 \in M_i^2$ and any $b^1 \in A^1$. Suppose that for all $j \in I \backslash \{i\}$, $\left(\theta^j, a^{1j}, a^{2j}, k^j\right) = \left(\bar{\theta}, a^1, a^2, k\right)$ for some $0 \leq k \leq n$. It is clear that agent i can induce *Rule 3* and obtain $\left(b^1, g^2\left(\hat{m}_i^2, m_{-i}^2\right)\right)$ when $k \neq 0$. Then, let $k = 0$. Since (m^1, m^2) falls into *Rule 3*, it follows that $(a^1, a^2) \notin \varphi\left(\bar{\theta}\right)$ or m_j^2 does not coincide with the jth coordinate of $m_{\varphi}^2\left(\bar{\theta}, \left(a^1, a^2\right)\right)$, for some agent $j \in I \backslash \{i\}$ - otherwise, (m^1, m^2) would fall either into *Rule 1* or into *Rule 2*. Since $|\Theta| \geq 2$, by assumption, take any $\tilde{\theta} \neq \bar{\theta}$. By changing (m_i^1, m_i^2) into $(\hat{m}_i^1, \hat{m}_i^2)$ such that $\hat{m}_i^1 = \left(\tilde{\theta}, b^1, g^2\left(\hat{m}_i^2, m_{-i}^2\right), k^i\right)$, agent i can induce *Rule 3*. To obtain $\left(b^1, g^2\left(\hat{m}_i^2, m_{-i}^2\right)\right)$, agent i has only to adjust k^i by which he becomes the winner of the modulo game. Thus, let us consider the case in which $\left(\theta^j, a^{1j}, a^{2j}, k^j\right) \neq \left(\theta^h, a^{1h}, a^{2h}, k^h\right)$ for some $j, h \in I \backslash \{i\}$, with $j \neq h$. In this case, note that agent i can always induce *Rule 3* by playing $k^i \neq 0$. In addition, note that agent i can freely determine the winner of the modulo game by playing an appropriate integer $k^i \neq 0$.[15] It follows that agent i can obtain the outcome $\left(b^1, g^2\left(\hat{m}_i^2, m_{-i}^2\right)\right)$. Since the choice $(b^1, \hat{m}_i^2)$, as well as the choice of agent i, is arbitrary, any agent $i \in I$ can induce any outcome in $A^1 \times g^2\left(M_i^2, m_{-i}^2\right)$. Since $(m^1, m^2) \in NE\left(\Gamma^1 \times \Gamma^2, \theta\right)$, it follows that $A^1 \times g^2\left(M_i^2, m_{-i}^2\right) \subseteq L_{i,g^2,m_{-i}^2}\left(\theta, \left(g^1 \times g^2\right)\left(m^1, m^2\right)\right)$, for all $i \in I$. Constrained no veto power implies that $\left(g^1 \times g^2\right)\left(m^1, m^2\right) \in \varphi(\theta)$, as we aimed to achieve. ■

[15] Note that if agent i needs to achieve the remainder of $\left(\sum_{j \in I \backslash \{i\}} k^j\right)/n$, he can obtain it by playing $k^i = n$.

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