Dynamic Reserves in Matching Markets*

Orhan Aygün[†] and Bertan Turhan[‡]

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Abstract

We study a school choice problem under affirmative action policies where authorities reserve a certain fraction of the slots at each school for specific student groups, and where students have preferences not only over the schools they are matched to but also the type of slots they receive. Such reservation policies might cause waste in instances of low demand from some student groups. To propose a solution to this issue, we construct a family of choice functions, *dynamic reserves choice functions,* for schools that respect within-group fairness and allow the transfer of otherwise vacant slots from low-demand groups to high-demand groups. We propose the cumulative offer mechanism (COM) as an allocation rule where each school uses a dynamic reserves choice function and show that it is *stable* with respect to schools' choice functions, is *strategy-proof*, and *respects improvements*. Furthermore, we show that transferring more of the otherwise vacant slots leads to strategy-proof Pareto improvement under the COM.

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†orhan.aygun@boun.edu.tr; Boğaziçi University, Department of Economics, Natuk Birkan Building, Bebek, Istanbul 34342, Turkey.

‡bertan@iastate.edu; Iowa State University, Department of Economics, Heady Hall, 518 Farm House Lane, Ames, IA 50011, USA.

1 Introduction

The theory of two-sided matching and its applications has been studied since the seminal work of *Gale and Shapley* (1962). Nevertheless, many real-life matching markets are subject to various constraints, such as affirmative action in school choice. Economists and policy makers are often faced with new challenges from such constraints. Admission policies in school choice systems often use reserves to grant applicants from certain backgrounds higher priority for some available slots. *Reservation in India* is such a process of setting aside a certain percentage of slots in government institutions for members of underrepresented communities, defined primarily by castes and tribes. We present engineering school admissions in India as an unprecedented matching problem with affirmative action in which students care about the category through which they are admitted.

In engineering school admissions in India, students from different backgrounds (namely, *scheduled castes* (SC), *scheduled tribes* (ST), *other backward classes* (OBC), and *general category* (GC))[1] are treated with different criteria. Schools reserve a certain fraction of their slots for students from SC, ST, and OBC categories. The remaining slots at each school, which are called *general category* (GC) slots, are open to competition. It is *optional* for SC, ST, and OBC students to declare their background information. Those who do declare their background information are considered for the reserved slots in their respective category, as well as for the GC slots. Students who do not belong to SC, ST, or OBC categories are considered only for GC slots. Students belonging to SC, ST, and OBC communities who do not reveal their background information are only considered for GC slots. *Aygün and Turhan* (2017) documented that students from SC, ST, and OBC categories have preferences not only for schools but also for the category through which they are admitted. Hence, students from these communities may prefer not to declare their caste and tribe information in the application process. Besides this strategic calculation burden on students, the current admission procedure[2] suffers from a crucial market failure: The assignment procedure fails to transfer some unfilled slots reserved for under-privileged castes and tribes to the use of remaining students. Hence, it is quite wasteful.

We address real-life applications as follows: There are schools and students to be matched. Each school initially reserves a certain number of its slots for different privilege groups (or student types). A given student may possibly match with a given school under more than one type. Each school has a pre-specified sequence[3] in which different sets of slots are considered, and where each set accepts students in a single privilege type. Different schools might have different orders. Since a student might have more than one privilege type, the set of students cannot be partitioned into

[1]Students who do not belong to SC, ST, and OBC categories are called general category (GC) applicants.

[2]Admission to the Indian Institute of Technologies (IITs) and its matching-theoretical shortcomings are explained in detail in *Aygün and Turhan* (2017).

[3]We will call this sequence a *precedence sequence,* which is different than the *precedence order* from *Kominers and Sönmez* (2016). Precedence order is a linear order over the set of student types. Precedence sequence, on the other hand, is more general in the sense that a given student type might appear multiple times. A technical definition will be given in the model section.

privilege groups. Each student has a preference over school-privilege type pairs. Students care not only about which school they are matched to but also about the privilege type under which they are admitted. Each school has a target distribution of its slots over privilege types, but they do not consider these target distributions as hard bounds[4]. If there is less demand from at least one privilege type, schools are given the opportunity to utilize vacant slots by transferring them over to other privilege types. Authorities might require a certain capacity transfer scheme so that each school has a complete plan where they state how they want to redistribute these slots. Thus, we take capacity transfer schemes exogenously given. The only mild condition imposed on the capacity transfer scheme is *monotonicity,*[5] which requires that (1) if more slots are left from one or more sets, the capacity of the sets considered later in the precedence sequence must be weakly higher, and (2) a school cannot decrease the total capacity in response to increased demand for some sets of slots.

We design choice functions for schools that allow them to transfer capacities from low-demand privilege types to high-demand privilege types. Each school respects an exogenously given precedence sequence between different sets of slots when it fills its slots. Each school has a strict priority ordering (possibly different than the other schools') over all students. For each school, priority orderings for different privilege types are straightforwardly derived from the school's priority ordering. There is an associated choice function, which we call a *"sub-choice function,"* for each set of slots. In Indian engineering school admissions, sub-choice functions are *q-responsive.* That is, a sub-choice function always selects the q-best students with respect to the priority ordering of the associated privilege type at that school, where q denotes the capacity.

The school starts filling its first set of slots according to its precedence sequence. Given the initial capacity of the first set of slots and a contract set, the sub-choice function associated with the first set selects contracts. The school then moves to the second set according to its precedence sequence. The (dynamic) capacity of the second set is a function of the number of unfilled slots in the first set. The exogenous capacity transfer function of the school specifies the capacity of the second set. The set of available contracts for the second set of slots is computed as follows: If a student has one of her contracts chosen by the first set, then all of her contracts are removed for the rest of the choice process. Given the set of remaining contracts and the capacity, the sub-choice function associated with the second set selects contracts. In general, the (dynamic) capacity of set k is a function of the number of vacant slots of the $k-1$ sets that precede it. The set of contracts available to the set of slots k is computed as follows: If a student has one of her contracts chosen by one of the $k-1$ sets of slots that precede the k^{th} set, then all of her contracts are removed. Given the set of remaining contracts for the set of slots k and its capacity, the sub-choice function associated with the set k selects contracts. The (overall) choice of a school is the union of sub-choices of its sets of slots.

[4] Hard bounds and soft bounds are analyzed in detail in *Hafalır et al.* (2013) and *Ehlers et al.* (2014).

[5] *Westkamp* (2013) introduces this monotonicity condition on capacity transfer schemes.

We propose a remedy for the Indian engineering school admissions problem through a matching with contracts model that has the ability to utilize vacant slots of certain types for other students. We have three design objectives: *stability, strategy-proofness* and *respect for improvements.* Stability ensures that (1) no student is matched with an unacceptable school-slot category pair, (2) schools' dynamic reserves choices are respected, and (3) no student desires a slot at which she has a justified claim under the priority and precedence structure. Strategy-proofness guarantees that students can never game the allocation mechanism via preference manipulation. In our framework, it also relieves students of the strategic manipulation burden, which involves whether or not students declare their background.[6] Respect for improvements[7] is an essential property in meritocratic systems. In allocation mechanisms that respect improvements, students have no incentive to lower their standings in schools' priority rankings.

We propose the cumulative offer mechanism (COM) as an allocation rule. We prove that the COM is stable with respect to schools' dynamic reserve choice functions (Theorem 1), is (weakly) group strategy-proof (Theorem 2), and respects improvements (Theorem 3). The main result of the paper (Theorem 4) states that when a single school's choice function becomes *"more flexible,"*[8] while those of the other schools remain unchanged, the outcome of the COM under the former (weakly) Pareto dominates the outcome under the latter. Theorem 4 is of particular importance because it describes a strategy-proof Pareto improvement. Finally, we investigate the relationship between families of dynamic reserves choice rules and Kominers and Sönmez's (2016) slot-specific priorities choice rules. We show that for every slot-specific priorities choice rule, there is an outcome equivalent dynamic reserves choice rule (Theorem 5). Moreover, we give an example of a dynamic reserves choice rule for which there is no outcome equivalent slot-specific priorities choice rule (Example 1).

Related Literature

The school choice problem was first introduced by the seminal paper of *Abdulkadiroğlu and Sönmez* (2003). The authors introduced a simple affirmative action policy with type-specific quotas. *Kojima* (2012) showed that the minority students who purported to be the beneficiaries might instead be made worse off under this type of affirmative action. To circumvent inefficiencies caused by majority quotas, *Hafalır et al.* (2013) offer minority reserves. *Westkamp* (2013) introduced a model of matching with complex constraints. His model permits priorities to vary across slots. In his model, students are considered to be indifferent between different slots of a given school.

[6]Strategy-proofness ensures that it is a weakly dominant strategy for each student to report their caste and tribe information.

[7]See *Kominers* (2019) for detailed discussion of respect for improvements in matching markets.

[8]We define *"more flexible"* criterion to compare two monotonic capacity transfer schemes given a precedence sequence. We say that a monotonic capacity transfer scheme $\widetilde{q}$ is more flexible than monotonic capacity transfer scheme q if $\widetilde{q}$ transfers at least as many otherwise vacant slots as q at every instance. There must also be an instance where $\widetilde{q}$ transfers strictly more otherwise vacant slots than q does.

However, in our framework, students have strict preferences for type-specific matches with schools. This crucial aspect differentiates our paper from *Westkamp* (2013). Moreover, our comparative statics result on transfer schemes does not have a counterpart in *Westkamp* (2013).

Kominers and Sönmez (2016) introduce another prominent family of choice functions—slot-specific priorities choice functions—to implement diversity objectives in many-to-one settings. We show that dynamic reserves choice rules nest slot-specific priorities choice rules. Moreover, we provide an example of a dynamic reserves choice rule that cannot be generated by a slot-specific priorities choice rule.

In a related work, *Biró et al.* (2010) analyze a college admission model with common and upper quotas in the context of Hungarian college admissions. They use choice functions for colleges that allow them to select multiple contracts of the same applicant. They show that a stable assignment exists. The completions of dynamic reserves choice functions, discussed in Appendix 7.2, satisfy the properties they impose. Hence, their result also implies the existence of a stable allocation in our framework. However, our main focus is different as we aim to show strategy-proof Pareto improvement by making capacity transfer function more flexible.

The matching problem with dynamic reserves choice functions is a special case of the matching with contracts model of *Fleiner* (2003)[9] and *Hatfield and Milgrom* (2005).[10] The analysis and results of *Hatfield and Kominers* (2019) are the technical backbone of our results regarding stable and strategy-proof mechanism design. We show that every dynamic reserves choice function has a completion that satisfies the irrelevance of rejected contracts condition of *Aygün and Sönmez* (2013), in conjunction with substitutability and the law of aggregate demand.

Hatfield et al. (2017) introduce a model of hospital choice in which each hospital has a set of divisions and *flexible allotment of capacities* to those divisions that vary as a function of the set of contracts available. These authors define choice functions that nest dynamic reserves choice functions while continuing to obtain stability and strategy-proofness for the COM. Our Theorems 3 and 4 do not have a counterpart in *Hatfield et al.* (2017).

Our work is also related with the research agenda on matching with constraints that is studied in a series of papers: *Kamada and Kojima* (2015), (2017), *Kojima et al.* (2018), and *Goto et al.* (2017). In these papers, constraints are imposed on subsets of institutions as a joint restriction, as opposed to at each individual institution. Our main results distinguish our work from these papers. We discuss the relationship between our stability notion and that of *Kamada and Kojima* (2017) in Section 3.

Another related paper is *Echenique and Yenmez* (2015). Dynamic reserves choice functions might seem similar to the family of choice functions the authors analyze: choice rules generated

[9]Fleiner's results cover these of *Hatfield and Milgrom* (2005) regarding stability. However, *Fleiner* (2003) does not analyze incentives.

[10]*Echenique* (2012) has shown that under the substitutes condition, which is thoroughly assumed in *Hatfield and Milgrom* (2005), the matching with contracts model can be embedded within the *Kelso and Crawford* (1982) labor market model. *Kelso and Crawford* (1982) built on the analysis of *Crawford and Knoer* (1981).

by reserves. However, dynamic reserves choice functions choose contracts whereas choice rules generated by reserves choose students.

Two recent papers, *Sönmez and Yenmez* (2019a,b), study affirmative action in India from a matching-theoretical perspective. The authors consider both vertical and horizontal reservations[11] while we consider only vertical reservations for simplicity. Even though they consider more general reserve structure than ours, the authors consider agents' preferences only over institutions and do not take agents' preferences over vertical categories they are admitted under into account. Moreover, they assume away capacity transfers between vertical categories. Therefore, their model does not contain our model and vice versa.

2 Model

There is a finite set of students $I = \{i_1, ..., i_n\}$, a finite set of schools $S = \{s_1, ..., s_m\}$, and a finite set of student privileges (types)[12] $T = \{t_1, ..., t_p\}$. We call $T_i \subseteq T$ the set of privileges that student i can claim and $\mathbf{T} = (T_i)_{i\in I}$ the profile of types that students can claim. We define $X_i = \{i\} \times S \times T_i$ as the set of all contracts associated with student $i \in I$. We let $X = \underset{i\in I}{\cup} X_i$ be the set of all contracts. Each contract $x \in X$ is between a student $\mathbf{i}(x)$ and a school $\mathbf{s}(x)$ and specifies a privilege $\mathbf{t}(x) \in T_{\mathbf{i}(x)}$. There may be many contracts for each student-school pair. We extend the notations $\mathbf{i}(\cdot)$, $\mathbf{s}(\cdot)$ and $\mathbf{t}(\cdot)$ to the set of contracts for any $Y \subseteq X$ by setting $\mathbf{i}(Y) \equiv \underset{y\in Y}{\cup} \{\mathbf{i}(y)\}$, $\mathbf{s}(Y) \equiv \underset{y\in Y}{\cup} \{\mathbf{s}(y)\}$ and $\mathbf{t}(Y) \equiv \underset{y\in Y}{\cup} \{\mathbf{t}(y)\}$. For $Y \subseteq X$, we denote $Y_i \equiv \{y \in Y \mid \mathbf{i}(y) = i\}$; analogously, we denote $Y_s \equiv \{y \in Y \mid \mathbf{s}(y) = s\}$ and $Y_t \equiv \{y \in Y \mid \mathbf{t}(y) = t\}$.

Each student $i \in I$ has a (linear) preference order P^i over contracts in $X_i = \{x \in X \mid \mathbf{i}(x) = i\}$ and an *outside option* $\emptyset$ which represents remaining unmatched. A contract $x \in X_i$ is *acceptable* for i (with respect to P^i) if $xP^i\emptyset$. We use the convention that $\emptyset P^i x$ if $x \in X \setminus X_i$. We say that the contracts $x \in X$ for which $\emptyset P^i x$ are *unacceptable* to i . The *at-least-as-well* relation R^i is obtained from P^i as follows: xR^ix' if and only if either xP^ix' or $x = x'$. Let $\mathcal{P}^i$ denote the set of all preferences over $X_i \cup \{\emptyset\}$. A preference profile of students is denoted by $P = (P^{i_1}, ..., P^{i_n}) \in \times_{i\in I}\mathcal{P}^i$. A preference profile of all students except student i_l is denoted by $P_{-i_l} = (P^{i_1}, ..., P^{i_{l-1}}, P^{i_{l+1}}, ..., P^{i_n}) \in \times_{i\neq i_l}\mathcal{P}^i$.

Students have *unit demand*, that is, they choose at most one contract from a set of contract offers. We assume that students always choose the best available contract, so that the choice $C^i(Y)$ of a student $i \in I$ from contract set $Y \subseteq X$ is the P^i-maximal element of Y (or the outside option if $\emptyset P^i y$ for all $y \in Y$).[13]

[11] Caste-based reservations for SC, ST, and OBC categories are called vertical reservations, also referred to as social reservations. Horizontal reservations, also referred to as special reservations, are intended for other disadvantaged groups of citizens, such as disabled persons, and women. Horizontal reservations are implemented within each vertical category. See *Sönmez and Yenmez* (2019a,b) for details.

[12] We use the terms *"type"* and *"privilege"* interchangeably.

[13] To simplify our notation, the individual contracts are treated as interchangeable with singleton contract sets.

For each school $s \in S$, $\overline{q}_s$ denotes the physical capacity of school $s \in S$. We call $\overline{q} = (\overline{q}_{s_1} ..., \overline{q}_{s_m})$ the vector of school capacities. Each school $s \in S$ has a priority order π^s, which is a linear order over $I \cup \{\emptyset\}$.[14] Let $\Pi = (\pi^{s_1}, ..., \pi^{s_m})$ denote the priority profile of schools. For each school $s \in S$, the priority ordering for students who can claim the privilege $t \in T$, denoted by π_t^s, is obtained from π^s as follows:

- for $i, j \in I$ such that $t \in T_i \setminus T_j$, $i\pi^s\emptyset$, and $j\pi^s\emptyset$, $i\pi_t^s\emptyset\pi_t^s j$,[15]
- for any other $i, j \in I$, $i\pi_t^s j$ if and only if $i\pi^s j$.

An *allocation* $Y \subseteq X$ is a set of contracts such that each student appears in *at most one* contract and no school appears in more contracts than its capacity allows. Let $\mathcal{X}$ denote the set of all allocations. Given a student i and an allocation Y, we refer to the pair $(\mathbf{s}(x), \mathbf{t}(x))$ such that $\mathbf{i}(x) = i$ as the *assignment* of student i under allocation Y. We extend student preferences over contracts to preferences over outcomes in the natural way. We say that an allocation $Y \subseteq X$ *Pareto dominates* allocation $Z \subseteq X$ if $Y_i R^i Z_i$ for all $i \in I$ and $Y_i P^i Z_i$ for at least one $i \in I$.

2.1 Dynamic Reserves Choice Functions

Each school $s \in S$ has multi-unit demand, and is endowed with a choice function $C^s(\cdot)$ that describes how s would choose from any offered set of contracts. Throughout the paper, we assume that for all $Y \subseteq X$ and for all $s \in S$, the choice function $C^s(\cdot)$:

1. only selects contracts to which s is a party, i.e., $C^s(Y) \subseteq Y_s$, and
2. selects at most one contract with any given student.

For any $Y \subseteq X$ and $s \in S$, we denote $R^s(Y) \equiv Y \setminus C^s(Y)$ as the set of contracts that s *rejects* from Y.

We now introduce a model of dynamic reserves choice functions in which each school $s \in S$ has λ_s *groups of slots*. School s fills its groups of slots according to a *precedence sequence,*[16] which is a surjective function $f^s : \{1, ..., \lambda_s\} \longrightarrow T$. The interpretation of f^s is that school s fills the first group of slots with $f^s(1)$-type students, the second group of slots with $f^s(2)$-type students, and so on. School $s \in S$ has a *target distribution* of its slots across different types $(\overline{q}_s^{t_1}, ..., \overline{q}_s^{t_p})$, which means that it has $\overline{q}_s^{t_1}$ slots to be reserved for privilege t_1, $\overline{q}_s^{t_2}$ slots to be reserved for privilege t_2,

[14] This priority order is often determined by performance on an admission exam, by a random lottery, or dictated by law. In engineering school admissions in India, each school ranks students according to test scores. Different schools might have different test score rankings because they use different weighted averages of math, physics, chemistry, and biology scores depending on the school. It is important to note that students whose test scores are under a certain threshold are deemed as unacceptable for each school.

[15] $\emptyset\pi_t^s j$ means student j is unacceptable for privilege t at school s.

[16] We take precedence sequences to be exogenously given. However, *Dur et al.* (2018) show that precedence sequences might have significant effects on distributional objectives in the context of Boston's school choice system.

and so on. To satisfy its target reserve structure, school s fills its slots according to the initially set capacities for each group of slots $(\overline{q}_s^1, \overline{q}_s^2, ..., \overline{q}_s^{\lambda_s})$ such that $\sum_{j \in (f^s)^{-1}(t)} \overline{q}_s^j = \overline{q}_s^t$ for all $t \in T$. If the target distribution cannot be achieved because too few students from one or more privileges apply, then school s use an exogenously given capacity transfer scheme that specifies how its capacity is to be redistributed. Technically, a capacity transfer scheme is defined as follows:

Definition 1. Given a precedence sequence f^s and a capacity of the first group of slots $\overline{q}_s^1$, a **capacity transfer scheme** of school s is a sequence of capacity functions $q_s = (\overline{q}_s^1, (q_s^k)_{k=2}^{\lambda_s})$, where $q_s^k : \mathbb{Z}_+^{k-1} \longrightarrow \mathbb{Z}_+$ such that $q_s^k(0, ..., 0) = \overline{q}_s^k$ for all $k \in \{2, ..., \lambda_s\}$.

We impose a mild condition, à la *Westkamp* (2013), on capacity transfer functions.

Definition 2. A capacity transfer scheme q_s is **monotonic** if, for all $j \in \{2, ..., \lambda_s\}$ and all pairs of sequences $(r_l, \widetilde{r}_l)$ such that $\widetilde{r}_l \geq r_l$ for all $l \leq j-1$,

- $q_s^j(\widetilde{r}_1, ..., \widetilde{r}_{j-1}) \geq q_s^j(r_1, ..., r_{j-1})$, and
- $\sum_{m=2}^{j} [q_s^m(\tilde{r}_1, ..., \widetilde{r}_{m-1}) - q_s^m(r_1, ..., r_{m-1})] \leq \sum_{m=1}^{j-1} [\widetilde{r}_m - r_m]$.

Monotonicity of capacity transfer schemes requires that (1) whenever weakly more slots are left unfilled in *every* groups of slots preceding the j^{th} group of slots, weakly more slots should be available for the j^{th} group, and (2) a school cannot decrease its total capacity in response to increased demand for some groups of slots.

Sub-choice functions

For each group of slots at school $s \in S$, there is an associated *sub-choice* function $c^s : 2^X \times \mathbb{Z}_+ \times T \longrightarrow 2^X$. Given a set of contracts $Y \subseteq X$, a nonnegative integer $\kappa \in \mathbb{Z}_+$, and a privilege $t \in T$, $c^s(Y, \kappa, t)$ denotes the set of chosen contracts that name privilege t up to the capacity κ from the set of contracts Y. We require sub-choice functions to be *q-responsive* given the ranking π_t^s.

Definition 3. [17]A sub-choice function $c^s(\cdot, \kappa, t)$ of a group of slots at school s for privilege type t is *q-responsive* if there exists a strict priority ordering π_t^s on the set of contracts naming privilege type t and a positive integer κ, such that for any $Y \subseteq (X_s \cap X_t)$,

$$c^s(Y, \kappa, t) = \bigcup_{i=1}^{\kappa} \{y_i^*\}$$

where y_i^* is defined as $y_1^* = \underset{\pi_t^s}{max} Y$ and, for $2 \leq i \leq \kappa$, $y_i^* = \underset{\pi_t^s}{max} Y \setminus \{y_1^*, ..., y_{i-1}^*\}$.

[17]We adapt this definition from *Chambers and Yenmez* (2017).

In other words, a sub-choice function is q-responsive[18] if there is a strict priority ordering over students who have privilege t for which the sub-choice function always selects the highest-ranked available students in privilege t up to the capacity.

Remark 1. Since our main real-life application is engineering school admissions in India, we shall assume that at each school $s \in S$, and for each group of slots reserved for privilege $t \in T$, the associated sub-choice function $c^s(\cdot, \cdot, t)$ is q-responsive and obtained from π^s_t.

Overall choice functions

The **overall choice function** of school s, $C^s(\cdot, f^s, q_s) : 2^X \longrightarrow 2^X$, runs its sub-choice functions in an orderly fashion given the precedence sequence f^s and capacity transfer scheme q_s. Given a set of contracts $Y \subseteq X$, $C^s(Y, f^s, q_s)$ denotes the set of chosen contracts from the set of contracts Y and is determined as follows:

- Given $\overline{q}^1_s$ and $Y = Y^0 \subseteq X$, let $Y_1 \equiv c^s_1(Y^0, \overline{q}^1_s, f^s(1))$ be the set of chosen contracts with privilege $f^s(1)$ from Y^0. Let $r_1 = \overline{q}^1_s- \mid Y_1 \mid$ be the number of vacant slots. Define $\widetilde{Y}_1 \equiv \{y \in Y^0 \mid \mathbf{i}(y) \in \mathbf{i}(Y_1)\}$ as the set of all contracts of students whose contracts are chosen by sub-choice function $c^s_1(\cdot, \overline{q}^1_s, f^s(1))$. If a contract of a student is chosen, then all of the contracts naming that student shall be removed from the set of available contracts for the rest of the procedure. The set of remaining contracts is then $Y^1 = Y^0 \setminus \widetilde{Y}_1$.

- In general, let $Y_k = c^s_k(Y^{k-1}, q^k_s, f^s(k))$ be the set of chosen contracts with privilege $f^s(k)$ from the set of available contracts Y^{k-1} , where $q^k_s = q^k_s(r_1, ..., r_{k-1})$ is the dynamic capacity of group of slots k as a function of the vector of the number of unfilled slots $(r_1, ..., r_{k-1})$. Let $r_k = q^k_s- \mid Y_k \mid$ be the number of vacant slots. Define $\widetilde{Y}_k = \{y \in Y^{k-1} \mid \mathbf{i}(y) \in \mathbf{i}(Y_k)\}$. The set of remaining contracts is then $Y^k = Y^{k-1} \setminus \widetilde{Y}_k$.

- Given $Y = Y^0 \subseteq X$ and the capacity of the first group of slots $\overline{q}^1_s$, we define the *overall choice function* of school s as $C^s(Y, f^s, q_s) = c^s_1(Y^0, \overline{q}^1_s, f^s(1)) \cup (\bigcup_{k=2}^{\lambda_s} c^s_k(Y^{k-1}, q^k_s(r_1, ..., r_{k-1}), f^s(k)))$.

The primitives of the overall choices for each school $s \in S$ are the precedence sequence f^s, the capacity transfer scheme q_s, and the priority order π^s. Since an overall choice is computed by using these primitives, it is not one of the primitives in our model. The list $(I, S, \mathbf{T}, X, P, \Pi, (f^s, q_s, \pi^s)_{s \in S})$ denotes a problem.

[18] These types of sub-choice functions are often used in real-life applications. For example, in the cadet branch matching processes in the USMA and ROTC, each sub-choice function is induced from a strict ranking of students according to test scores. See *Sönmez and Switzer* (2013) and *Sönmez* (2013) for further details.

3 Stability Concept

Stability has emerged as the key to a successful matching market design. We follow the *Gale and Shapley* (1962) tradition in focusing on outcomes that are stable. In the matching with contracts framework, *Hatfield and Milgrom* (2005) define stability as follows: An outcome $Y \subseteq X$ is **stable** if

1. $Y_i R^i \emptyset$ for all $i \in I$,

2. $C^s(Y) = Y_s$ for all $s \in S$, and

3. there does not exist a school $s \in S$ and a **blocking set** $Z \neq C^s(Y)$ such that $Z_s \subseteq C^s(Y \cup Z)$ and $Z_i = C^i(Y \cup Z)$ for all i$\in \mathbf{i}(Z)$.

If the first requirement (*individual rationality for students*) fails, then there is a student who prefers to reject a contract that involves her (or, equivalently, there is a student who is given an unacceptable contract). In our context, the second condition (*individual rationality for schools*) requires that the schools' choice functions are respected. If the third condition (*unblockedness*) fails, then there is an alternative set of contracts that a school and students associated with a contract in that set strictly prefers.

Remark 2. Our stability notion is related to the *weak stability* notion of *Kamada and Kojima* (2017). The authors define the *feasibility constraint* as a map $\phi : \mathbb{Z}_+^{|H|} \longrightarrow \{0, 1\}$, such that $\phi(w) \geq \phi(w^{'})$ whenever $w \leq w^{'}$. Their interpretation is that each coordinate in w corresponds to a hospital and the number in that coordinate represents the number of doctors matched to that hospital. $\phi(w) = 1$ means that w is feasible and $\phi(w) = 0$ means it is not. They say that matching μ is *feasible* if and only if $\phi(w(\mu)) = 1$, where $w(\mu) := (\mid \mu_h \mid)_{h \in H}$ is a vector of nonnegative integers indexed by hospitals whose coordinates corresponding to h are $\mid \mu_h \mid$. Capacity transfer functions in our setting can be represented by *the* feasibility constraint map from their paper. Condition 2 in our stability definition takes into account not only dynamic capacities of groups of seats in each school but also their precedence sequences. It is a feasibility condition. *Westkamp* (2013) defines a similar condition in his *"procedural stability"* definition in a simpler matching model without contracts.

4 The Cumulative Offer Mechanism and its Properties under Dynamic Reserves Choice Functions

A *direct mechanism* is a mechanism where the strategy space is the set of preferences $\mathcal{P}$ for each student $i \in I$, i.e., a function $\psi : \mathcal{P}^n \longrightarrow \mathcal{X}$ that selects an allocation for each preference profile. We propose the COM as our allocation function. Given the student preferences and schools' overall choice functions, the outcome of the COM is computed by the *cumulative offer algorithm*. This is the generalization of the agent-proposing deferred acceptance algorithm of *Gale and Shapley* (1962). We now introduce the cumulative offer process (COP)[19] for matching with contracts. Here, we provide an intuitive description of this algorithm; we give a more technical description in Appendix 7.1.

Definition 4. In the COP, students propose contracts to schools in a sequence of steps $l = 1, 2 \ldots$:

Step 1 : Some student $i^1 \in I$ proposes his most-preferred contract, $x^1 \in X_{i^1}$. School $s(x^1)$ holds x^1 if $x^1 \in C^{s\left(x^1\right)}(\{x^1\})$, and rejects x^1 otherwise. Set $A^2_{s(x^1)} = \{x^1\}$, and set $A^2_{s'} = \emptyset$ for each $s' \neq s(x^1)$; these are the sets of contracts available to schools at the beginning of Step 2.

Step 2 : Some student $i^2 \in I$, for whom no school currently holds a contract, proposes his most-preferred contact that has not yet been rejected, $x^2 \in X_{i^2}$. School $s(x^2)$ holds the contract in $C^{s\left(x^2\right)}\left(A^2_{s(x^2)} \cup \{x^2\}\right)$ and rejects all other contracts in $A^2_{s(x^2)} \cup \{x^2\}$; schools $s' \neq s(x^2)$ continue to hold all contracts they held at the end of Step 1. Set $A^3_{s(x^2)} = A^2_{s(x^2)} \cup \{x^2\}$, and set $A^3_{s'} = A^2_{s'}$ for each $s' \neq s(x^2)$.

Step l : Some student $i^l \in I$, for whom no school currently holds a contract, proposes his most-preferred contact that has not yet been rejected, $x^l \in X_{i^l}$. School $s\left(x^l\right)$ holds the contract in $C^{s\left(x^l\right)}\left(A^l_{s\left(x^l\right)} \cup \left\{x^l\right\}\right)$ and rejects all other contracts in $A^l_{s\left(x^l\right)} \cup \left\{x^l\right\}$; schools $s' \neq s\left(x^l\right)$ continue to hold all contracts they held at the end of Step $l-1$. Set $A^{l+1}_{s\left(x^l\right)} = A^l_{s\left(x^l\right)} \cup \left\{x^l\right\}$, and set $A^{l+1}_{s'} = A^l_{s'}$ for each $s' \neq s\left(x^l\right)$.

If at any time no student is able to propose a new contract—that is, if all students for whom no contracts are on hold have proposed all contract they find acceptable—then the algorithm terminates. The outcome of the COP is the set of contracts held by schools at the end of the last step before termination.

In the COP, students propose contracts sequentially. Schools accumulate offers, choosing at each step (according to their choice functions) a set of contracts to hold from the set of all previous offers. The process terminates when no student wishes to propose a contract.

Given a preference profile of students $P = (P_i)_{i \in I}$ and a profile of choice functions for schools $C = (C^s)_{s \in S}$, let $\Phi(P, C)$ denote the outcome of the COM. Let $\Phi_i(P, C)$ denote the assignment of student $i \in I$ and $\Phi_s(P, C)$ denote the assignment of school $s \in S$.

Remark 3. We do not explicitly specify the order in which students make proposals. *Hirata and Kasuya* (2014) show that in the matching with contracts model, the outcome of the COP is *order-independent* if the overall choice function of every school satisfies the bilateral substitutability (BLS) and the irrelevance of rejected contracts (IRC) conditions. Dynamic reserves choice functions satisfy BLS and IRC. Hence, the order-independence of the COP holds.

[19]See *Hatfield and Milgrom* (2005) for more details.

A mechanism φ is *stable* if for every preference profile $P \in \mathcal{P}^{|I|}$ the outcome $\varphi(P)$ is stable with respect to the schools' overall choice functions. Since the COP gives a stable outcome for every input if each school's capacity transfer scheme is monotonic, the COM is a stable mechanism.

Theorem 1. *The cumulative offer mechanism is stable with respect to dynamic reserves choice functions.*

Proof. See Appendix 7.3.

To analyze the incentive properties of the COM when schools use dynamic reserves choice functions, we first define standard strategy-proofness and (weak) group strategy-proofness in relation to a direct mechanism.

Definition 5. A direct mechanism φ is said to be **strategy-proof** if there does not exist a preference profile P, a student $i \in I$, and preferences P_i' of student i such that

$$\varphi_i(P_i', P_{-i})\, P_i \varphi_i(P).$$

That is, no matter which student we consider, no matter what her true preferences P_i are, no matter what other preferences P_{-i} other students report (true or not), and no matter which potential "misrepresentation" P_i' student i considers, a truthful preference revelation is in her best interest. Hence, students can never benefit from gaming the mechanism φ.

Definition 6. A direct mechanism φ is said to be **weakly group strategy-proof** if there is no preference profile P, a subset of students $I' \subseteq I$, and a preference profile $(P_i)_{i\in I'}$ of students in I' such that

$$\varphi_i\left((P_i')_{i\in I'}, (P_j)_{j\in I\setminus I'}\right) P_i \varphi_i(P)$$

for all $i \in I'$.

That is, no subset of students can jointly misreport their preferences to receive a strictly preferred outcome for every member of the coalition.

Hatfield and Kominers (2019) show that if schools' choice functions have substitutable completions so that these completions satisfy the LAD, then the COP becomes weakly group strategy-proof.

Theorem 2. *Suppose that each school uses a dynamic reserves choice function. Then, the cumulative offer mechanism is weakly group strategy-proof.*

Proof. See Appendix 7.3.

Respect for Unambiguous Improvements

We say that priority profile $\overline{\Pi}$ is an *unambiguous improvement over priority profile* Π for student $i \in I$ if, for all schools $s \in S$, the following conditions hold:

1. For all $x \in X_i$ and $y \in \left(X_{I\setminus\{i\}} \cup \{\emptyset\}\right)$, if $x\pi^s y$ then $x\overline{\pi}^s y$.

2. For all $y, z \in X_{I\setminus\{i\}}$, $y\pi^s z$ if and only if $y\overline{\pi}^s z$.

That is, $\overline{\Pi}$ is an unambiguous improvement over priority profile Π for student i if $\overline{\Pi}$ is obtained from Π by increasing the priority of some of i's contracts while leaving the relative priority of other students' contracts unchanged.

Definition 7. A mechanism φ **respects unambiguous improvements** for $i \in I$ if for any preference profile $P \in \times_{i\in I}\mathcal{P}^i$

$$\varphi_i(P;\overline{\Pi})R^i\varphi_i(P;\Pi)$$

whenever $\overline{\Pi}$ is an unambiguous improvement over Π for i. We say that φ respects unambiguous improvement*s* if it respects unambiguous improvements for each student $i \in I$.

Respect for improvements is essential in settings like ours where it implies that students never want to intentionally decrease their test scores and, in turn, their rankings. Similarly, it is also important in cadet-branch matching where cadets can influence their priority rankings directly. *Sönmez* (2013) argues that cadets take perverse steps to lower their priorities because the mechanism used by the Reserve Officer Training Corps (ROTC) to match its cadets to branches fails the respecting improvements property.

Theorem 3. *The cumulative offer mechanism with respect to dynamic reserves choice functions respects unambiguous improvements.*

Proof. See Appendix 7.3.

5 Comparative Statics on Monotonic Capacity Transfer Schemes

In this section, we first define a comparison criteria between two monotone capacity transfer schemes. Consider a school $s \in S$ with a given precedence sequence f^s and target distribution $\overline{q}_s = (\overline{q}_s^1, ..., \overline{q}_s^{\lambda_s})$. Let q_s and $\widetilde{q}_s$ be two monotone capacity transfer schemes: given a vector of unused slots from group of slots 1 to $j-1$, $(r_1, ..., r_{j-1}) \in \mathbb{Z}_+^{j-1}$, the dynamic capacity of the j^{th} group under capacity transfer schemes q_s and $\widetilde{q}_s$ are $q_s^j = q_s^j(r_1, ..., r_{j-1})$ and $\widetilde{q}_s^j = \widetilde{q}_s^j(r_1, ..., r_{j-1})$, respectively, for all $j \geq 2$ and, $q_s^1 = \widetilde{q}_s^1 = \overline{q}_s^1$.

Let q_s and $\widetilde{q}_s$ be two monotone capacity transfer schemes that are compatible with the precedence sequence f^s and target capacity vector $\overline{q}_s$ of school $s \in S$. We say that the monotone capacity transfer scheme $\widetilde{q}^s$ is **more flexible** than the monotone capacity transfer scheme q^s if

1. there exists $l \in \{2, ..., \lambda_s\}$ and $(\hat{r}_1, ..., \hat{r}_{l-1}) \in \mathbb{Z}_+^{l-1}$ such that $\widetilde{q}_s^l(\hat{r}_1, ..., \hat{r}_{l-1}) > q_s^l(\hat{r}_1, ..., \hat{r}_{l-1})$, and

2. for all $j \in \{2, ..., \lambda_s\}$ and $(r_1, ..., r_{j-1}) \in \mathbb{Z}_+^{j-1}$, if $j \neq l$ or $(r_1, ..., r_{j-1}) \neq (\hat{r}, ..., \hat{r}_{l-1})$, then $\widetilde{q}_s^j(r_1, ..., r_{j-1}) \geq q_s^j(r_1, ..., r_{j-1})$.

The definition states that one monotonic capacity transfer scheme is more flexible than another if it transfers at least as many vacant slots as the other at every instance (i.e., the vectors of the number of unused slots). There must also be an instance where the first one transfers strictly more vacant slots than the second one to the next group of slots according to the precedence sequence. Also, both of the monotonic capacity transfer schemes take the capacity of the first group of slots with respect to the precedence sequence equal to its target capacity. Holding all else constant, when the capacity transfer scheme becomes more flexible, it defines a particular choice function expansion.[20]

Expanding the overall choice function of a single school leads to Pareto improvement for students under the COM.[21]

Theorem 4. *Let $C = (C^{s_1}, ..., C^{s_m})$ be the profile of schools' overall choice functions. Fix a school $s \in S$. Suppose that $\widetilde{C}^s$ takes a capacity transfer scheme that is more flexible than that of C^s, holding all else constant. Then, the outcome of the cumulative offer mechanism with respect to $(\widetilde{C}^s, C_{-s})$ weakly Pareto dominates the outcome of the cumulative offer mechanism with respect to C.*

Proof. See Appendix 7.3.

Theorem 4 is of particular importance because it indicates that increasing the transferability of capacity from low-demand to high-demand groups leads to strategy-proof Pareto improvement with the cumulative offer algorithm. This result provides a normative foundation for recommending a more flexible interpretation of type-specific quotas. This result establishes that to maximize students' welfare, schools' choice functions should be expanded as much as possible.

It is important to note that when more than one school's capacity transfer scheme become more flexible, a simple iteration of Theorem 4, one school at a time, ensures (weak) Pareto improvement. Therefore, a more flexible capacity transfer profile of schools implies that the COM with the new capacity transfer scheme (weakly) Pareto improves the original transfer scheme.

[20] The type of choice function expansion here is different than the one *Chambers and Yenmez* (2017) define. Their notion of expansion is in the sense of set inclusion while ours is not. They say that a choice function $C^{'}$ is an expansion of another choice function C if for every offer set Y, $C(Y) \subseteq C^{'}(Y)$. According to the expansion via a more flexible capacity transfer scheme, when a choice function C expands to $C^{'}$ it is possible to have $C(Y) \nsubseteq C^{'}(Y)$ for some Y.

[21] This result does not contradict the findings of *Alva and Manjunath* (2019), because increasing flexibility of the capacity transfers changes the choice functions, and therefore the set of contracts that are feasible in their context. Theorem 4 achieves the improvement by considering a dominating mechanism that is infeasible under the original transfer scheme.

6 Relationship Between Slot-specific Priorities and Dynamic Reserves Choice Rules

In this section, we investigate the relationship between the families of slot-specific priorities choice rules and dynamic reserves choice rules. To do so, we first describe slot-specific priorities choice rules.

Each school $s \in S$ has a set of slots $\mathcal{B}_s$. Each slot can be assigned at most one contract in X_s. Slots $b \in \mathcal{B}_s$ have linear priority orders π^b over contracts in X_s. Each slot b ranks a null contract $\emptyset_b$ that represents remaining unassigned. Schools $s \in S$ may be assigned as many as $| \mathcal{B}_s |$ contracts from an offer set $Y \subseteq X$—one for each slot in $\mathcal{B}_s$— but may hold no more than one contract with a given student. The slots in $\mathcal{B}_s$ are ordered according to a linear order of precedence $\triangleright^s$. We denote $\mathcal{B}_s \equiv \{b_1, ..., b_{\overline{q}_s}\}$ with $| \mathcal{B}_s |= \overline{q}_s$. The interpretation of $\triangleright^s$ is that if $b_l \triangleright^s b_{l+1}$, then—whenever possible—school s fills slot b_l before filling slot b_{l+1}. Formally, the choice $C^s(Y)$ of a school $s \in S$ from contract set $Y \subseteq X$ is defined as follows:

- First, slot b_1 is assigned the contract y_1 that is π^{b_1}-maximal among contracts in Y.
- Then, slot b_2 is assigned the contract y_2 that is π^{b_2}-maximal among contracts in the set $Y \setminus Y_{\mathbf{i}(y_1)}$ of contracts in Y with agents other than $\mathbf{i}(y_1)$.
- This process continues in sequence, with each slot b_l being assigned to the contract y_l that is π^{b_l}-maximal among contracts in the set $Y \setminus Y_{\mathbf{i}(\{y_1,...,y_{l-1}\})}$.

If no contract is assigned to a slot $b_l \in \mathcal{B}_s$ in the computation of $C^s(Y)$, then b_l is assigned the null contract $\emptyset_{b_l}$.

We first give an example of a dynamic reserves choice rule that cannot be generated by a slot-specific priorities choice rule.

Example 1. Consider $I = \{i, j, k, l\}$, $S = \{s\}$ with $q_s = 2$, and $\Theta = \{t_1, t_2, t_3\}$. Student i only has type t_1 and a single contract x_1. Student j only has type t_2 and a single contract y_2. Student k has types t_2 and t_3, and two contracts related to these types z_2 and z_3, respectively. Finally, student l has types t_1 and t_3, and two contracts related to these types w_1 and w_3, respectively. The set of contracts for this problem is $X = \{x_1, y_2, z_2, z_3, w_1, w_3\}$. Students are ordered with respect to their exam scores from highest to lowest as follows: $i - j - k - l$.

The school reserves the first seat for type t_1, and the second seat for type t_2. If either the first seat or the second seat cannot be filled with the students they are reserved for, they are filled with a type t_3 student(s). The precedence order is such that the first seat is filled first with a type t_1 student if possible, and then the second seat is filled with a type t_2 student, if possible. If any of these seats cannot be filled with the intended student types, all of the vacant seats are filled with type t_3 students at the very end, if possible.

We can represent the distributional objective described above by capacity-transfers as follows: Initially $\overline{q}_{t_1} = \overline{q}_{t_2} = 1$ and $\overline{q}_{t_3} = 0$. The dynamic capacity of the third seat is given by $q_{t_3} = r_1 + r_2$, where $r_1, r_2 \in \{0, 1\}$. Some of the choice situations under the capacity-transfer described above are given below:

Y	$C(Y)$
$\{x_1, y_2, z_2, z_3, w_1, w_3\}$	$\{x_1, y_2\}$
$\{y_2, z_2, z_3\}$	$\{y_2, z_3\}$
$\{x_1, z_2, z_3\}$	$\{x_1, z_2\}$
$\{y_2, w_1, w_3\}$	$\{y_2, w_1\}$
$\{x_1, w_1, w_3\}$	$\{x_1, w_3\}$
$\{z_2, z_3\}$	$\{z_2\}$
$\{w_1, w_3\}$	$\{w_1\}$

In order to implement the choices above with slot-specific priorities, we need to find a strict ranking of the contracts in X for both of the slots. Note that $\{x_1, y_2\}$ is chosen from $\{x_1, y_2, z_2, z_3, w_1, w_2\}$. Then, x_1 must be chosen for one of the slots and y_2 must be chosen for the other. There are two cases to consider.

Case 1: x_1 is chosen from slot 1 and y_2 is chosen from slot 2. Then, x_1 is the highest priority contract in slot 1. We have $C(\{x_1, z_2, z_3\}) = \{x_1, z_2\}$. Then, z_2 must have higher priority than z_3 in the strict priority ranking of slot 2 because x_1 will be chosen from the first slot. Notice that both z_2 and z_3 must have lower priority than y_2 in the strict ranking of slot 2. Also, since $C(\{y_2, z_2, z_3\}) = \{y_2, z_3\}$, then it must be the case that z_3 has higher priority than z_2 in the strict priority of the first slot. Notice that z_3 cannot be chosen from the second slot as z_2 has higher priority. However, $C(\{z_2, z_3\}) = \{z_2\}$. This is a contradiction.

Case 2: y_2 is chosen from slot 1 and x_1 is chosen from slot 2. Then, y_2 has the highest priority in slot 1. We have $C(\{y_2, w_1, w_3\}) = \{y_2, w_1\}$. Therefore, in the ranking of slot 2, w_1 must have higher priority than w_3. Also, since $C(\{x_1, w_1, w_3\}) = \{x_1, w_3\}$, it follows that in the ranking of slot 1 w_3 must have higher priority than w_1. This is because w_3 cannot be chosen from slot 2 as it has a lower priority than w_1 there. However, $C(\{w_1, w_3\}) = \{w_1\}$. This is a contradiction.

Hence, we cannot find a strict rankings of the contracts in X for these two slots that generate the dynamic reserves choice rule defined above.

Our last result states that the family of dynamic reserves choice rules nests the family of slot-specific priorities choice rules.

Theorem 5. *Every slot-specific priorities choice rule can be generated by a dynamic reserves choice rule.*

Proof. See Appendix 7.3.

7 Conclusion

This paper studies a school choice problem with distributional objectives where students care about both the school they are matched with as well as the category through which they are admitted. Each school can be thought of as union of different groups of slots, where each group is associated with exactly one category. Schools have target distributions over their groups of slots in the form of reserves. If these reserves are considered to be hard bounds, then some slots will remain empty in instances where demand for particular categories is less than their target capacities. To overcome this problem and to increase efficiency, we design a family of dynamic reserves choice functions. We do so by allowing monotonic capacity transfers across groups of slots when one or more of the groups is not able to fill to its target capacity. The capacity transfer scheme is exogenously given for each school and governs the dynamic capacities of groups, each of which has a q-responsive sub-choice function. The overall choice function of a school can be thought of as the union of choices with these sub-choice functions of its groups.

We offer the COM with respect to dynamic reserves choice functions as an allocation rule. We show that the COM is stable and strategy-proof in our framework. Moreover, the COM respects improvements. We introduce a comparison criteria between two monotonic capacity transfer schemes. If a monotone capacity transfer scheme transfers at least as many vacancies in every contingency compared to another monotone capacity transfer scheme, we say that the first is more flexible than the second. We show that when capacity transfer scheme of a school becomes more flexible, while other school choice functions remain unchanged, the outcome of the COM under the modified profile of choice functions Pareto dominates the outcome of the COM under the original profile. This result is the main message of our paper, as it describes a strategy-proof Pareto improvement by making capacity transfers more flexible.

8 APPENDICES

8.1 Formal Description of the Cumulative Offer Process

Cumulative Offer Process (COP): Consider the outcome the COM as denoted by $\Phi^{\Gamma}(P, C)$. For any preference profile P of students, profile of choice functions of schools C, and an ordering Γ of the elements of X, the outcome is determined by the *COP with respect to* Γ*,* P and C as follows:

Step 0: Initialize the set of contracts *available* to the schools as $A^0 = \emptyset$.

Step $t \geq 1$: Consider the set

$$U^t \equiv \left\{x \in X \setminus A^{t-1} : \mathbf{i}(x) \notin \mathbf{i}\left(C^S(A^{t-1})\right) \textit{ and } \nexists z \in \left(X_{\mathbf{i}(x)} \setminus A^{t-1}\right) \cup \{\emptyset\} \textit{ such that } zP^{\mathbf{i}(x)}x\right\}.$$

If U^t is empty, then the algorithm terminates and the outcome is given by $C^S(A^{t-1})$.[22] Otherwise, letting y^t be the highest-ranked element of U^t according to Γ, we say that y^t is *proposed* and set $A^t = A^{t-1} \cup \{y^t\}$ and proceed to step $t+1$.

A COP begins with no contracts available to the schools (i.e., $A^0 = \emptyset$). Then, at each step t, we construct U^t, the set of contracts that (1) have not yet been proposed, (2) are not associated to students with contracts chosen by schools from the currently available set of contracts, and (3) are both acceptable and the most-preferred by their associated students among all contracts not yet proposed. If U^t is empty, then every student i either has some associated contract chosen by some school, i.e., $i \in \mathbf{i}\left(C^S(A^{t-1})\right)$, or has no acceptable contracts left to propose, and so the COP ends. Otherwise, the contract in U^t that is highest-ranked according to Γ is proposed by its associated student, and the process proceeds to the next step. Note that at some step this process must end as the number of contracts is finite.

Letting T denote the last step of the COP, we call A^T the set of contracts observed in the COP with respect to Γ, P, and C.

8.2 Substitutable Completion of Dynamic Reserves Choice Functions

Definition 8. A choice function $C^s(\cdot)$ satisfies the **irrelevance of rejected contracts** (IRC) condition if for all $Y \subset X$, for all $z \in X \setminus Y$, and $z \notin C^s(Y \cup \{z\}) \Longrightarrow C^s(Y) = C^s(Y \cup \{z\})$.

Hatfield and Milgrom (2005) introduce the substitutability condition, which generalizes the earlier gross substitutes condition of *Kelso and Crawford* (1982).

Definition 9. A choice function $C^s(\cdot)$ satisfies **substitutability** if for all z, $z' \in X$, and $Y \subseteq X$, $z \notin C^s(Y \cup \{z\}) \Longrightarrow z \notin C^s(Y \cup \{z, z'\})$.

Definition 10. A choice function $C^s(\cdot)$ satisfies the **law of aggregate demand** (LAD) if $Y \subseteq Y' \Longrightarrow \mid C^s(Y) \mid \leq \mid C^s(Y') \mid$.

The following definitions are from *Hatfield and Kominers* (2019). A *completion* of a many-to-one choice function $C^s(\cdot)$ of school $s \in S$ is a choice function $\overline{C}^s(\cdot)$, such that for all $Y \subseteq X$, either $\overline{C}^s(Y) = C^s(Y)$ or there exists a distinct $z, z' \in \overline{C}^s(Y)$ such that $i(z) = i(z')$. If a choice function $C^s(\cdot)$ has a completion that satisfies the substitutability and *IRC* condition, then we say that $C^s(\cdot)$ is *substitutably completable.* If every choice function in a profile $C = (C^s(\cdot))_{s \in S}$ is substitutably completable, then we say that C is *substitutably completable.*

[22] We denote by $C^S(Y) \equiv \cup_{s \in S} C^s(Y)$ the set of contracts chosen by the set of schools from a set of contracts $Y \subseteq X$.

Let $C^s(\cdot, f^s, q_s)$ be a dynamic reserve choice function given the precedence sequence f^s and the capacity transfer scheme q_s. We define a related choice function $\overline{C}^s(\cdot, f^s, q_s)$. Given a set of contracts $Y \subseteq X$, $\overline{C}^s(Y, f^s, q_s)$ denotes the set of chosen contracts from set Y and is determined as follows:

- Given $\overline{q}_s^1$ and $Y = Y^0 \subseteq X$, let $Y_1 \equiv c_1^s(Y^0, \overline{q}_s^1, f^s(1))$ be the set of chosen contracts with privilege $f^s(1)$ from Y^0. Let $r_1 = \overline{q}_s^1 - \mid Y_1 \mid$ be the number of vacant slots. The set of remaining contracts is then $Y^1 = Y^0 \setminus Y_1$.

- In general, let $Y_k = c_k^s(Y^{k-1}, q_s^k, f^s(k))$ be the set of chosen contracts with privilege $f^s(k)$ from the set of available contracts Y^{k-1} , where $q_s^k = q_s^k(r_1, ..., r_{k-1})$ is the dynamic capacity of group of slots k as a function of the vector of the number of unfilled slots $(r_1, ..., r_{k-1})$. Let $r_k = q_s^k - \mid Y_k \mid$ be the number of vacant slots. The set of remaining contracts is then $Y^k = Y^{k-1} \setminus Y_{k-1}$.

- Given $Y = Y^0 \subseteq X$ and the capacity of the first group of slots $\overline{q}_s^1$, we define $\overline{C}^s(Y, f^s, q_s) = c_1^s(Y^0, \overline{q}_s^1, f^s(1)) \cup (\bigcup_{k=2}^{\lambda_s} c_k^s(Y^{k-1}, q_s^k(r_1, ..., r_{k-1}), f^s(k)))$.

The difference between $C^s(\cdot)$ and $\overline{C}^s(\cdot)$ is as follows: In the computation of $C^s(\cdot)$, if a contract of a student is chosen by some group of slots then his/her other contracts are removed for the rest of the choice procedure. However, in the computation of $\overline{C}^s(\cdot)$ this is not the case. According to the choice procedure $\overline{C}^s(\cdot)$, if a contract of a student is chosen, say, by group of slots k, then his/her other contracts will still be available for the following groups of slots.

The following proposition shows that $\overline{C}^s(\cdot)$ defined above is the completion of the dynamic reserves choice function $C^s(\cdot)$.

Proposition 1. *$\overline{C}^s(\cdot)$ is a completion of $C^s(\cdot)$.*

Proof. Let f^s and q_s be the precedence sequence and capacity transfer scheme of school $s \in S$, respectively. Take an offer set $Y = Y^0 \subseteq X$ and assume there is no pair of contracts $z, z' \in Y^0$ such that $i(z) = i(z')$ and $z, z' \in \overline{C}^s(Y, f^s, q_s)$. We want to show that

$$\overline{C}^s(Y, f^s, q_s) = C^s(Y, f^s, q_s).$$

Let Y_j be the set of contracts chosen by group of slots j and let Y^j be the set of contracts that remains in the choice procedure after group j selects according to dynamic reserve choice function $C(\cdot)$. Similarly, let $\overline{Y}_j$ be the set of contracts chosen by group of slots j and let $\overline{Y}^j$ be the set of contracts that remains in the choice procedure after group j selects according to the completion $\overline{C}(\cdot)$. Notice that $Y^0 = \overline{Y}^0$. Let r_j and $\overline{r}_j$ be the number of vacant slots in group of slots j in the choice procedures $C^s(Y, f^s, q_s)$ and $\overline{C}^s(Y, f^s, q_s)$, respectively. Also, let

$q_s^j(r_1, ..., r_{j-1})$ and $\overline{q}_s^j(\overline{r}_1, ..., \overline{r}_{j-1})$ denote the dynamic capacities of group of slots j under choice procedures $C^s(Y, f^s, q_s)$ and $\overline{C}^s(Y, f^s, q_s)$, respectively.

Given $\bar{q}_s^1$ and $Y^0 = \overline{Y}^0$, we have $\overline{Y}_1 = c_1^s(Y^0, \bar{q}_s^1, f^s(1)) = Y_1$ by the construction of $\overline{C}^s$. Moreover, $\overline{r}_1 = r_1$ and $\overline{q}_s^2(\overline{r}_1) = q_s^2(r_1)$.

Suppose that for all $j \in \{2, ..., k-1\}$ we have $Y_j = \overline{Y}_j$. We need to show that it holds for group of slots k, i.e., $Y_k = \overline{Y}_k$. Since the chosen set is the same in every group from 1 to $k-1$ under $C(\cdot)$ and $\overline{C}(\cdot)$, the number of remaining slots in each group is the same as well. Then, the dynamic capacity of the group of slots k are the same under choice procedures $C^s(Y, f^s, q_s)$ and $\overline{C}^s(Y, f^s, q_s)$, i.e., $q_s^k(r_1, ..., r_{k-1}) = \overline{q}_s^k(\overline{r}_1, ..., \overline{r}_{k-1})$. Since there are no two contracts of an agent chosen by $\overline{C}^s(Y, f^s, q_s)$, one can deduce that all of the remaining contracts of agents, whose contracts were chosen by previous sub-choice functions, are rejected by $c_k^s(\overline{Y}^{k-1}, \overline{q}_s^k(\overline{r}_1, ..., \overline{r}_{k-1}), f^s(k))$. Therefore, the IRC of the sub-choice function implies that

$$c_k^s(\overline{Y}^{k-1}, \overline{q}_s^k(\overline{r}_1, ..., \overline{r}_{k-1}), f^s(k)) = c_k^s(Y^{k-1}, q_s^k(r_1, ..., r_{k-1}), f^s(k)).$$

Hence, we have $\overline{Y}_k = Y_k$, $\overline{r}_k = r_k$, and $\overline{q}_s^{k+1}(\overline{r}_1, ..., \overline{r}_k) = q_s^{k+1}(r_1, ..., r_k)$.

Since in each group of slots the same sets of contracts are chosen by the dynamic reserve choice function and its completion, the result follows.

Proposition 2. *$\overline{C}^s(\cdot)$ satisfies the IRC.*

Proof. For any $Y \subseteq X$ such that $Y \neq \overline{C}^s(Y, f^s, q_s)$, let x be one of the rejected contracts, i.e., $x \in Y \setminus \overline{C}^s(Y, f^s, q_s)$. To show that the IRC is satisfied, we need to prove that

$$\overline{C}^s(Y, f^s, q_s) = \overline{C}^s(Y \setminus \{x\}, f^s, q_s).$$

Let $\tilde{Y} = Y \setminus \{x\}$. Let $(\overline{Y}_j, \bar{r}_j, \overline{Y}^j)$ be the sequence of the set of chosen contracts, the number of vacant slots, and the remaining set of contracts for group $j = 1, ..., \lambda_s$ from Y under $\overline{C}(\cdot)$. Similarly, let $(\tilde{Y}_j, \tilde{r}_j, \tilde{Y}^j)$ be the sequence of the set of chosen contracts, the number of vacant slots, and the remaining set of contracts for group $j = 1, ..., \lambda_s$ from $\tilde{Y}$ under $\overline{C}(\cdot)$.

For the first group of slots, since the sub-choice functions satisfy the IRC, we have $\overline{Y}_1 = \tilde{Y}_1$. Moreover, $\bar{r}_1 = \tilde{r}_1$ and $\overline{Y}^1 \setminus \{x\} = \tilde{Y}^1$. By induction, for each $j = 2, ..., k-1$, assume that

$$\overline{Y}_j = \tilde{Y}_j, \ \bar{r}_j = \tilde{r}_j, \ \textit{and} \ \overline{Y}^j \setminus \{x\} = \tilde{Y}^j.$$

We need to show that the above equalities hold for $j = k$. Since , $x \notin \overline{C}^s(Y, f^s, q_s)$ and the sub-choice functions satisfy the IRC condition we have

$$c_k^s(\overline{Y}^{k-1}, q_s^k(\overline{r}_1, ..., \overline{r}_{k-1}), f^s(k)) = c_k^s(\tilde{Y}^{k-1}, q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}), f^s(k)).$$

The same set of contracts is chosen for group k in the choice processes beginning with Y and $Y \cup \{x\}$, respectively. By our inductive assumption that $\bar{r}_j = \tilde{r}_j$ for each $j = 2, ..., k-1$, the dynamic capacity of group k is the same under both choice processes. The number of remaining slots is the same as well, i.e., $\bar{r}_k = \tilde{r}_k$. Finally, we know that x is chosen from the set $\tilde{Y}^{k-1} \cup \{x\}$, then we have

$$\overline{Y}^k = \tilde{Y}^k \cup \{x\}.$$

Since for all $j \in \{1, ..., \lambda_s\}$, $\overline{Y}_j = \tilde{Y}_j$, we have $\overline{C}^s(Y, f^s, q_s) = \overline{C}^s(\tilde{Y}, f^s, q_s)$. Hence, $\overline{C}^s(\cdot, f^s, q_s)$ satisfies the IRC.

Proposition 3. *$\overline{C}^s(\cdot)$ satisfies the substitutability.*

Proof. Consider an offer set $Y \subseteq X$ such that $Y \neq \overline{C}^s(Y, f^s, q_s)$. Let x be one of the rejected contracts, i.e., $x \in Y \setminus \overline{C}^s(Y, f^s, q_s)$, and let z be an arbitrary contract in $X \setminus Y$. To show substitutability, we need to show that

$$x \notin \overline{C}^s(Y \cup \{z\}, f^s, q_s).$$

Consider $\tilde{Y} = Y \cup \{z\}$. Let (Y_j, r_j, Y^j) be the sequence of the set chosen contracts, the number of vacant slots, and the set of remaining contracts for group of slots $j = 1, ..., \lambda_s$ from Y under $\overline{C}(\cdot)$. Similarly, let $(\tilde{Y}_j, \tilde{r}_j, \tilde{Y}^j)$ be the sequence of the set chosen contracts, the number of vacant slots, and the set of remaining contracts for group of slots $j = 1, ..., \lambda_s$ from $\tilde{Y}$ under $\overline{C}(\cdot)$. There are two cases to consider:

Case 1 $z \in \tilde{Y} \setminus \overline{C}^s(\tilde{Y}, f^s, q_s)$.

In this case, the IRC of $\overline{C}^s$ implies $\overline{C}^s(\tilde{Y}, f^s, q_s) = \overline{C}^s(Y, f^s, q_s)$. Therefore, $x \notin \overline{C}^s(\tilde{Y}, f^s, q_s)$.

Case 2 $z \in \overline{C}^s(\tilde{Y}, f^s, q_s)$.

Let j be the group of slots such that $z \in \tilde{Y}_j$. By the IRC of sub-choice functions, $x \notin \tilde{Y}_j = Y_j$, for all $j' = 1, ..., j-1$. Moreover, $\tilde{Y}^{j'-1} = Y^{j'-1} \cup \{z\}$ and $\tilde{r}_{j'} = r_{j'}$, for all $j' = 1, ..., j-1$.

First note that the dynamic capacity of group j is the same under choice procedures beginning with $Y = Y^0$ and $Y \cup \{z\} = \tilde{Y}^0$, respectively. This is because the number of unused slots from groups 1 to $j-1$ are the same under the two choice procedures. We know that z is chosen exactly at group j in the process beginning with $\tilde{Y}^0$. There are two cases here:

(a) The dynamic capacity of group j is exhausted in the process beginning with Y^0. In this case, by choosing z from $\tilde{Y}^0$ another contract, we say that say $y \in \tilde{Y}^0$ is rejected even though y was chosen at group j in the process beginning with Y^0.

(b) The dynamic capacity of group j is **not** exhausted in the choice process beginning with Y^0. In this case, z is chosen at group j in the process beginning with $\tilde{Y}^0$ without rejecting any contract that was chosen in the process beginning with Y^0 at group j.

In the case of (a),

$$| c_j^s(Y^{j-1}, q_s^j(r_1, ..., r_{j-1}), f^s(j)) |= q_s^j(r_1, ..., r_{j-1})$$

and

$$z \in c_j^s(\tilde{Y}^{j-1}, q_s^j(r_1, ..., r_{j-1}), f^s(j))$$

implies that there exists a contract y such that

$$y \in c_j^s(Y^{j-1}, q_s^j(r_1, ..., r_{j-1}), f^s(j)) \setminus c_j^s(\tilde{Y}^{j-1}, q_s^j(\tilde{r}_1, ..., \tilde{r}_{j-1}), f^s(j)).$$

This implies that $\tilde{Y}^j = Y^j \cup \{y\}$. Since the capacity of group j is exhausted under both choice processes, the number of vacant slots for group j will be 0 in both choice processes. Thus, the capacity will be the same for group $j+1$ under both.

Notice that

$$x \notin Y_j \implies x \notin \tilde{Y}_j$$

because

$$c_j^s(Y^{j-1}, q_s^j(r_1, ..., r_{j-1}), f^s(j)) \cup \{z\} \setminus \{y\} = c_j^s(\tilde{Y}^{j-1}, q_s^j(\tilde{r}_1, ..., \tilde{r}_{j-1}), f^s(j)).$$

In case (b), we have

$$| c_j^s(Y^{j-1}, q_s^j(r_1, ..., r_{j-1}), f^s(j)) |< q_s^j(r_1, ..., r_{j-1}).$$

Hence, $r_j > 0$. Then, since the sub-choice functions are responsive, we have

$$c_j^s(\tilde{Y}^{j-1}, q_s^j(\tilde{r}_1, ..., \tilde{r}_{j-1}), f^s(j)) = \{z\} \cup c_j^s(Y^{j-1}, q_s^j(r_1, ..., r_{j-1}), f^s(j)).$$

Therefore,

$$x \notin Y_j \implies x \notin \tilde{Y}_j.$$

We also have $r_j = \tilde{r}_j + 1$. Moreover, the set of remaining contracts under both choice processes will be the same, i.e., $\tilde{Y}^j = Y^j$. The facts $r_{j'} = \tilde{r}_{j'}$ for all $j' = 1, ..., j-1$ and $r_j = \tilde{r}_j + 1$ implies—by the monotonicity of capacity transfer schemes—that either

$$q_s^{j+1}(r_1, ..., r_j) = q_s^{j+1}(\tilde{r}_1, ..., \tilde{r}_j)$$

or

$$q_s^{j+1}(r_1,...,r_j) = 1 + q_s^{j+1}(\tilde{r}_1,...,\tilde{r}_j)$$

hold.

Suppose now that for all $\gamma = j,...,k-1$ we have that either

$$\left[\tilde{Y}^\gamma = Y^\gamma \cup \{\widetilde{y}\} \text{ for some } \widetilde{y} \text{ and } q_s^{\gamma+1}(\tilde{r}_1,...,\tilde{r}_\gamma) = q_s^{\gamma+1}(r_1,...,r_\gamma)\right]$$

or

$$\left[\tilde{Y}^\gamma = Y^\gamma \text{ and } q_s^{\gamma+1}(\tilde{r}_1,...,\tilde{r}_\gamma) \leq q_s^{\gamma+1}(r_1,...,r_\gamma) \leq 1 + q_s^{\gamma+1}(\tilde{r}_1,...,\tilde{r}_\gamma)\right].$$

We have already shown that it holds for $\gamma = j$ and we will now show that it also holds for $\gamma = k$.

We will first analyze the former case. By inductive assumption, we have $\tilde{Y}^{k-1} = Y^{k-1} \cup \{\widetilde{y}\}$ for some contract $\widetilde{y}$. If $\widetilde{y}$ is not chosen from the set $\widetilde{Y}^{k-1}$ then exactly the same set of contracts will be chosen from Y^{k-1} and $\tilde{Y}^{k-1}$ since the capacities of group k are the same under both choice processes and the sub-choice function satisfies the IRC condition. Then, we will have $\tilde{Y}^k = Y^k \cup \{\widetilde{y}\}$. Moreover, since the number of vacant slots at group k will be the same under both processes, we will have $q_s^{k+1}(r_1,...,r_j) = q_s^{k+1}(\tilde{r}_1,...,\tilde{r}_j)$. If $\widetilde{y}$ is chosen from the set $\tilde{Y}^{k-1}$, we have two sub-cases, depending on if the dynamic capacity of group k is exhausted under the choice process beginning with Y^0. If it is not exhausted, then we will have

$$c_k^s(\tilde{Y}^{k-1}, q_s^k(\tilde{r}_1,...,\tilde{r}_{k-1}), f^s(k)) = \{\widetilde{y}\} \cup c_k^s(Y^{k-1}, q_s^k(r_1,...,r_{k-1}), f^s(k)),$$

which implies that $\tilde{Y}^k = Y^k$. Moreover, we will have $r_k = \tilde{r}_k + 1$. The monotonicity of capacity transfer scheme implies that

$$q_s^{k+1}(\tilde{r}_1,...,\tilde{r}_k) \leq q_s^{k+1}(r_1,...,r_k) \leq 1 + q_s^{k+1}(\tilde{r}_1,...,\tilde{r}_k).$$

The first inequality follows from the fact that $\tilde{r}_i \leq r_i$ for all $i = 1,...,k$. The second inequality follows from the second condition of the monotonicity of the capacity transfer schemes.

On the other hand, if the dynamic capacity of group k is exhausted in the choice procedure beginning with Y^0, then choosing $\widetilde{y}$ from the set $\tilde{Y}^{k-1}$ implies that there exists a contract $\overline{y}$ that is chosen from Y^{k-1} but rejected from $\tilde{Y}^{k-1}$. Then, we will have $\tilde{Y}^k = Y^k \cup \{\overline{y}\}$ since the sub-choice function is q-responsive and group k's capacities are the same under both choice processes. In this case, we will have $r_k = \tilde{r}_k = 0$. Since $\tilde{r}_i \leq r_i$ for all $i = 1,...,k$, we will have $q_s^{k+1}(\tilde{r}_1,...,\tilde{r}_k) \leq q_s^{k+1}(r_1,...,r_k)$ from the first condition of the monotonicity of the capacity transfer scheme. Since $q_s^k(\tilde{r}_1,...,\tilde{r}_{k-1}) = q_s^k(r_1,...,r_{k-1})$ and $\tilde{r}_k = r_k$, we will have $q_s^{k+1}(\tilde{r}_1,...,\tilde{r}_k) \geq q_s^{k+1}(r_1,...,r_k)$ by the second condition of the monotonicity of capacity transfer schemes.[23]

[23]In the second condition of the monotonicity of the capacity transfer schemes, if the number of vacant slots is

We will now analyze the latter case in which we have $\tilde{Y}^{k-1} = Y^{k-1}$ and either $q_s^k(r_1, ..., r_{k-1}) = q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1})$ or $q_s^k(r_1, ..., r_{k-1}) = 1 + q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1})$.

If $q_s^k(r_1, ..., r_{k-1}) = q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1})$, then given that $\tilde{Y}^{k-1} = Y^{k-1}$, we will have $\tilde{Y}^k = Y^k$. This also implies $r_k = \tilde{r}_k$. Moreover, we obtain $q_s^{k+1}(r_1, ..., r_k) = q_s^{k+1}(\tilde{r}_1, ..., \tilde{r}_k)$ by the monotonicity of capacity transfer scheme. Note that $\tilde{r}_i \leq r_i$ for all $i = 1, ..., k$ implies $q_s^{k+1}(r_1, ..., r_k) \geq q_s^{k+1}(\tilde{r}_1, ..., \tilde{r}_k)$ by the first condition of the monotonicity of capacity transfers. The second condition of the monotonicity of capacity transfers implies $q_s^{k+1}(r_1, ..., r_k) \leq q_s^{k+1}(\tilde{r}_1, ..., \tilde{r}_k)$.

If $q_s^k(r_1, ..., r_{k-1}) = 1 + q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1})$, then given $\tilde{Y}^{k-1} = Y^{k-1}$, we have two sub-cases here.

Sub-case 1. If

$$c_k^s(\tilde{Y}^{k-1}, q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}), f^s(k)) = c_k^s(Y^{k-1}, q_s^k(r_1, ..., r_{k-1}), f^s(k)),$$

then we will have $\tilde{Y}^k = Y^k$. Also, the monotonicity of capacity transfer scheme implies that

$$q_s^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) \leq q_s^{k+1}(r_1, ..., r_k) \leq 1 + q_s^{k+1}(\tilde{r}_1, ..., \tilde{r}_k).$$

Sub-case 2. If

$$c_k^s(\tilde{Y}^{k-1}, q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}), f^s(k)) \cup \{y^*\} = c_k^s(Y^{k-1}, q_s^k(r_1, ..., r_{k-1}), f^s(k))$$

for some y^*, then we will have $\tilde{Y}^k = Y^k \cup \{y^*\}$. Moreover, the monotonicity of capacity transfer schemes in this case implies that

$$q_s^{k+1}(r_1, ..., r_k) = q_s^{k+1}(\tilde{r}_1, ..., \tilde{r}_k).$$

This is because given $\tilde{r}_i \leq r_i$ for all $i = 1, ..., k$ the first condition of the monotonicity of the capacity transfers implies that $q_s^{k+1}(r_1, ..., r_k) \geq q_s^{k+1}(\tilde{r}_1, ..., \tilde{r}_k)$. On the other hand, the second condition of the monotonicity of the capacity transfers implies that $q_s^{k+1}(r_1, ..., r_k) \leq q_s^{k+1}(\tilde{r}_1, ..., \tilde{r}_k)$.

Since $x \notin Y_k$, we will have $x \notin \tilde{Y}_k$ for all $k = 1, ..., \lambda_s$. Thus, we can conclude that $x \notin \overline{C}^s(Y \cup \{z\}, f^s, q_s)$, which tells us that the completion $\overline{C}^s$ satisfies the substitutability condition.

Proposition 4. *$\overline{C}^s(\cdot)$ satisfies the LAD.*

written as the dynamic capacity of the group minus the number of chosen contracts then we will have the following: the dynamic capacity of the group $k+1$ in the choice process beginning with Y minus the dynamic capacity of the group $k+1$ in the choice process beginning with $Y \cup \{z\} = \tilde{Y}^0$ must be less than or equal to the summation of the difference of the number of chosen contracts from group 1 to group k, which is 0 in this specific case.

Proof. Consider two sets of contracts Y and $\tilde{Y}$ such that $Y \subseteq \tilde{Y} \subseteq X$. Let f^s and q_s be the precedence sequence and the capacity transfer scheme of school $s \in S$. We want to show that

$$\mid \overline{C}^s(Y, f_s, q_s) \mid \leq \mid \overline{C}^s(\tilde{Y}, f^s, q_s) \mid .$$

Let (Y_j, r_j, Y^j) be the sequences of sets of chosen contracts, numbers of vacant slots and sets of remaining contracts for groups $j = 1, ..., \lambda_s$ under choice processes beginning with $Y = Y^0$. Similarly, let $(\tilde{Y}_j, \tilde{r}_j, \tilde{Y}^j)$ be the sequences of sets of chosen contracts, numbers of vacant slots and sets of remaining contracts for groups $j = 1, ..., \lambda_s$ under choice processes beginning with $\tilde{Y}^0 = \tilde{Y}$.

For the first group with capacity $\overline{q}_s^1$, since the sub-choice function is q-responsive (and thus implies the LAD), we have

$$\mid Y_1 \mid = \mid c_1^s(Y^0, \overline{q}_s^1, f^s(1)) \mid \leq \mid c_1^s(\tilde{Y}^0, \overline{q}_s^1, f^s(1)) \mid = \mid \tilde{Y}_1 \mid .$$

Then, it implies that $r_1 = \overline{q}_s^1 - \mid Y_1 \mid \geq \tilde{r}_1 = \overline{q}_s^1 - \mid \tilde{Y}_1 \mid$. Moreover, we have $Y^1 \subseteq \tilde{Y}^1$. To see this, consider a $y \in Y^1$. It means that $y \notin Y_1$. If y is not chosen from a smaller set Y^0, then it cannot be chosen from a larger set $\tilde{Y}^0$ because sub-choice function is q-responsive (hence, substitutable).

Suppose that $\tilde{r}_j \leq r_j$ and $Y^j \subseteq \tilde{Y}^j$ hold for all $j = 1, ..., k-1$. We need to show that both of them hold for group k.

Given that $\tilde{r}_j \leq r_j$ for all $j = 1, ..., k-1$, the first condition of the monotonicity implies that $q_s^k(r_1, ..., r_{k-1}) \geq q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1})$. The second condition of the monotonicity puts an upper bound for the difference between $q_s^k(r_1, ..., r_{k-1})$ and $q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1})$. For group k

$$\mid Y_k \mid - \mid \tilde{Y}_k \mid \leq \mid Y_k \mid - \mid c_k^s(Y^{k-1}, q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}, f^s(k)) \mid$$

because

$$\mid \tilde{Y}_k \mid = \mid c_k^s(\tilde{Y}^{k-1}, q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}), f^s(k)) \mid \geq \mid c_k^s(Y^{k-1}, q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}), f^s(k)) \mid$$

by the q-responsiveness of the sub-choice function. We then have

$$\mid Y_k \mid - \mid c_k^s(Y^{k-1}, q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}, f^s(k)) \mid \leq q_s^k(r_1, ..., r_{k-1}) - q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}).$$

This follows from q-responsiveness because $\mid Y_k \mid - \mid c_k^s(Y^{k-1}, q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}, f^s(k)) \mid$ is the difference between the number of chosen contracts when the capacity is (weakly) increased from $q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1})$ to $q_s^k(r_1, ..., r_{k-1})$. Hence, the difference $\mid Y_k \mid - \mid c_k^s(Y^{k-1}, q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}, f^s(k)) \mid$ cannot exceed the increase in the capacity which is $q_s^k(r_1, ..., r_{k-1}) - q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1})$. Therefore, now we have

$$\mid Y_k \mid - \mid \tilde{Y}_k \mid \leq q_s^k(r_1, ..., r_{k-1}) - q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}).$$

Rearranging gives us

$$q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}) - \mid \tilde{Y}_k \mid \leq q_s^k(r_1, ..., r_{k-1}) - \mid Y_k \mid,$$

which is $\tilde{r}_k \leq r_k$.

Given that $Y^{k-1} \subseteq \tilde{Y}^{k-1}$ and $q_s^k(r_1, ..., r_{k-1}) \geq q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1})$, we will have $Y^k \subseteq \tilde{Y}^k$. For an explanation, consider a contract $x \in Y^k$. That means that $x \in Y^{k-1}$ but x is not chosen from Y^{k-1} when the capacity is $q_s^k(r_1, ..., r_{k-1})$, i.e., $x \notin c_k^s(Y^{k-1}, q_s^k(r_1, ..., r_{k-1}), f^s(x))$. When the capacity is reduced to $q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1})$ and the set Y^{k-1} is expanded to $\tilde{Y}^{k-1}$, x cannot be chosen because the sub-choice function is q-responsive. Hence, it must be the case that $x \in \tilde{Y}^k$.

Now let $\eta_j = r_j - \tilde{r}_j$. As we just proved above, $\eta_j \geq 0$ for all $j = 1, ..., \lambda_s$. Plugging $r_j = q_s^j(r_1, ..., r_{j-1}) - \mid Y_j \mid$ and $\tilde{r}_j = q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}) - \mid \tilde{Y}_j \mid$ in $\eta_j = r_j - \tilde{r}_j$ gives us

$$\mid \tilde{Y}_j \mid = q_s^j(r_1, ..., r_{j-1}) - q_s^j(\tilde{r}_1, ..., \tilde{r}_{j-1}) + \mid Y^j \mid + \eta_j.$$

Summing both the right and left hand sides for $j = 1, ..., \lambda_s$ yields

$$\sum_{j=1}^{\lambda_s} \mid \tilde{Y}_j \mid = \sum_{j=1}^{\lambda_s} \mid Y_j \mid + \sum_{j=2}^{\lambda_s} \left[q_s^j(r_1, ..., r_{j-1}) - q_s^j(\tilde{r}_1, ..., \tilde{r}_{j-1}) \right] + \sum_{j=1}^{\lambda_s} \eta_j.$$

Since each $\eta_j \geq 0$, we have

$$\sum_{j=1}^{\lambda_s} \mid \tilde{Y}_j \mid \geq \sum_{j=1}^{\lambda_s} \mid Y_j \mid + \sum_{j=2}^{\lambda_s} \left[q_s^j(r_1, ..., r_{j-1}) - q_s^j(\tilde{r}_1, ..., \tilde{r}_{j-1}) \right].$$

Also, we know that $q_s^j(r_1, ..., r_{j-1}) \geq q_s^j(\tilde{r}_1, ..., \tilde{r}_{j-1})$ for all $j = 2, ..., \lambda_s$ by the first condition of the monotonicity of the capacity transfer scheme as, $r_i \geq \tilde{r}_i$ for all $i = 1, ..., j-1$ (Notice that for $j = 1$, the capacity is fixed to $\overline{q}_s^1$ under both processes.) Therefore, we have

$$\sum_{j=1}^{\lambda_s} \mid \tilde{Y}_j \mid \geq \sum_{j=1}^{\lambda_s} \mid Y_j \mid,$$

which means $\mid \overline{C}^s(Y, f^s, q_s) \mid \leq \mid \overline{C}^s(\tilde{Y}, f^s, q_s) \mid$.

8.3 Proofs of Theorems

Proof of Theorem 1

In Proposition 1 we showed that each dynamic reserve choice function has a completion. Propositions 2 and 3 show that the completion satisfies the IRC and substitutability conditions, respectively. Then, by Theorem 2 of *Hatfield and Kominers* (2019), there exists a stable outcome with respect to the profile of schools' choice functions.

Proof of Theorem 2

In Proposition 4 we showed that the substitutable completion satisfies the LAD. Then, by the Theorem 3 of *Hatfield and Kominers* (2019), the COM is (weakly) group strategy-proof for students.

Proof of Theorem 3

Assume, toward a contradiction, that the COM does not respect unambiguous improvements. Then, there exists a student $i \in I$, a preference profile of students $P \in \times_{i \in I} \mathcal{P}^i$, and priority profiles $\overline{\Pi}$ and Π such that $\overline{\Pi}$ is an unambiguous improvement over Π for student i and

$$\varphi_i(P;\Pi) P^i \varphi_i(P;\overline{\Pi}).$$

Let $\varphi_i(P;\Pi) = x$ and $\varphi_i(P;\overline{\Pi}) = \overline{x}$. Consider a preference $\widetilde{P}^i$ of student i according to which the only acceptable contract is x, i.e., $\widetilde{P}^i : x - \emptyset_i$. Let $\widetilde{P} = (\widetilde{P}^i, P_{-i})$. We will first prove the following claim:

Claim: $\varphi_i(\widetilde{P};\Pi) = x \implies \varphi_i(\widetilde{P};\overline{\Pi}) = x$.

Proof of the Claim: Consider the outcome of the COM under priority profile Π given the preference profile of students $\widetilde{P}$. Recall that the order in which students make offers has no impact on the outcome of the COP. We can thus completely ignore student i and run the COP until it stops. Let Y be the resulting set of contracts. At this point, student i makes an offer for his only contract x. This might create a chain of rejections, but it does not reach student i. So, his contract x is chosen by $\mathbf{s}(x)$ by, say, the group k with respect to the precedence sequence $f^{\mathbf{s}(x)}$ of school $\mathbf{s}(x)$. Now consider the COP under priority profile $\overline{\Pi}$. Again, we completely ignore student i and run the COP until it stops. The same outcome Y is obtained, because the only difference between the two COPs is student i's position in the priority rankings. At this point, student i makes an offer for his only contract x. If x is chosen by the same group k, then the same rejection chain (if there was one in the COP under the priority profile Π) will occur and it does not reach student i; otherwise, we would have a contradiction with the case under priority profile Π. The only other possibility is the following: since student i's ranking is now (weakly) better under $\overline{\pi}^{\mathbf{s}(x)}$ compared to $\pi^{\mathbf{s}(x)}$, his contract x might be chosen by group $l < k$. Then, it must be the case that $r_l = 0$ in the COP under both priority profiles Π and $\overline{\Pi}$. Therefore, by selecting x, the group l must reject some other contract. Let us call this contract y. If no contract of student $\mathbf{i}(y) = j$ is chosen between groups l and k, then, by the q-responsiveness of sub-choice functions, the groups' chosen sets between l and k under both priority profiles are the same. Hence, the number of remaining slots would be the same. In this case, y is chosen in the group k. Thus, if a rejection chain starts, it will not reach student i; otherwise, we could have a contradiction due to the fact that x was chosen at the end of the COP under priority profile Π. A different contract of student j cannot be chosen between groups l and k; otherwise, the observable substitutability[24] of dynamic reserves choice function of school $\mathbf{s}(x)$ would be violated. Therefore, if any contract of student j is chosen by these groups between l and k, it must be y. If y is chosen by a group that precedes k, then it must replace a contract—we call it z. By the same reasoning, no other contract of student $\mathbf{i}(z)$ can be chosen before group k; otherwise, we would violate the observable substitutability of the dynamic reserve choice function of school $\mathbf{s}(x)$. Proceeding in this fashion leads the same contract in group k to be rejected and initiates the same rejection chain that occurs under priority profile Π. Since the same rejection chain does not reach student i under priority profile Π, it will not reach student i under priority profile $\overline{\Pi}$, which ends our proof for the claim.

Since $\varphi_i(P;\Pi) = x$ and $\varphi_i(P;\overline{\Pi}) = \overline{x}$ such that $xP^i\overline{x}$, if student i misreports and submits $\widetilde{P}^i$ under priority profile $\overline{\Pi}$, then she can successfully manipulate the COM. This is a contradiction because we have already established that the COM is strategy-proof.

Proof of Theorem 4

Consider school $s \in S$ with a precedence sequence f^s and a target capacity vector $(\overline{q}_s^1, ..., \overline{q}_s^{\lambda_s})$. Let $\widetilde{q}_s$ and q_s be two capacity transfer schemes that are compatible with the precedence sequence f^s and the target capacity vector $(\overline{q}_s^1, ..., \overline{q}_s^{\lambda_s})$. Suppose that the following two conditions hold:

- there exists $l \in \{2, ..., \lambda_s\}$ and $(\hat{r}_1, ..., \hat{r}_{l-1}) \in \mathbb{Z}_+^{l-1}$, such that $\widetilde{q}_s^l(\hat{r}_1, ..., \hat{r}_{l-1}) = 1 + q_s^l(\hat{r}_1, ..., \hat{r}_{l-1})$, and

- for all $j \in \{2, ..., \lambda_s\}$ and $(r_1, ..., r_{j-1}) \in \mathbb{Z}_+^{j-1}$, if $j \neq l$ or $(r_1, ..., r_{j-1}) \neq (\hat{r}_1, ..., \hat{r}_{l-1})$, then $\widetilde{q}_s^j(r_1, ..., r_{j-1}) = q_s^j(r_1, ..., r_{j-1})$.

Let $\widetilde{C}^s$ and C^s be dynamic reserves choice functions $\widetilde{C}^s(\cdot, f^s, \widetilde{q}_s)$ and $C^s(\cdot, f^s, q_s)$, respectively. Let $\widetilde{C} = \left(\widetilde{C}^s, C_{-s}\right)$ and $C = (C^s, C_{-s})$. Let the outcomes of the cumulative offer algorithm at $\left(P, \widetilde{C}\right)$ and (P, C) be $\widetilde{Z}$ and Z, respectively. If $\widetilde{Z} = Z$, then there is nothing to prove because it means the capacity flexibility of school s does not bite.

Suppose that $\widetilde{Z} \neq Z$. That is, the capacity flexibility of school s bites, which means that there is a student who was rejected under C^s who is no longer rejected under $\widetilde{C}^s$. We now define an *improvement chains* algorithm that starts with outcome Z. Since the capacity flexibility bites, the vector $(\hat{r}_1, ..., \hat{r}_{l-1})$ must occur in the choice procedure of school s.

[24] Dynamic reserves choice functions satisfy *observable substitutability* condition of *Hatfield et al.* (2019). We refer readers to *Hatfield et al.* (2019) for the definitions of observable offer processes and observable substitutability. Since dynamic reserves choice functions have substitutable completion that satisfies the size monotonicity, it satisfies observable substitutability.

Step 1: Consider students who prefer $(s, f^s(l))$ to their assignments under Z, i.e.,

$$\widetilde{I}_1^{(s,f^s(l))} = \{i \in I \mid (s, f^s(l))P^i Z_i\}.$$

We choose π^s-maximal student in $\widetilde{I}_1^{(s,f^s(l))}$ (if any), call her $\widetilde{i}_1$, and assign her $\widetilde{x}_1 = (\widetilde{i}_1, s, f^s(l))$. Update the outcome to $\widetilde{Z}_1 = Z \cup \{\widetilde{x}_1\} \setminus z_1$ where z_1 is the contract student $\widetilde{i}_1$ receives under Z.

If $(\mathbf{s}(z_1), \mathbf{t}(z_1)) = \emptyset$, then the improvement process ends and we have $\widetilde{Z} = \widetilde{Z}_1 = Z \cup \{\widetilde{x}_1\}$. Otherwise, we move to Step 2 because by assigning $\widetilde{i}_1$ to $(s, f^s(l))$ we create a vacancy in school $\mathbf{s}(z_1)$ within the privilege $\mathbf{t}(z_1)$.

If $\widetilde{I}_1^{(s,f^s(l))} = \emptyset$, then the number of vacant slots at the *last* group accepting students in type $f^s(l)$ will increase by one. When the capacity transfer scheme of school s does not transfer this extra vacancy to any other group following the last group in type $f^s(l)$ in the computation of $C^s(Z_s, f^s, \widetilde{q}_s)$, the improvement chain process ends and we have $\widetilde{Z} = Z$. If the extra slot is transferred to the group l' that follows the last group in type $f^s(l)$ in the computation of $C^s(Z_s, f^s, \widetilde{q}_s)$, then we consider students who prefer $(s, f^s(l'))$ over their assignments under Z, i.e.,

$$I_1^{(s,f^s(l'))} = \{i \in I \mid (s, f^s(l'))P^i Z_i\}.$$

We choose π^s-maximal student in $I_1^{(s,f^s(l'))}$ (if there is any), call her $\widetilde{i}_1$, and assign her $\widetilde{x}_1 = (\widetilde{i}_1, s, f^s(l'))$. Update the outcome to $\widetilde{Z}_1 = Z \cup \{\widetilde{x}_1\} \setminus z_1$ where z_1 is the contract $\widetilde{i}_1$ receives under Z.

If $(\mathbf{s}(z_1), \mathbf{t}(z_1)) = \emptyset$, then the improvement process ends and we have $\widetilde{Z} = \widetilde{Z}_1 = Z \cup \{\widetilde{x}_1\}$. Otherwise, we move to Step 2. Because assigning $\widetilde{i}_1$ to $(s, f^s(l'))$ creates a vacancy in school $\mathbf{s}(z_1)$ within the privilege $\mathbf{t}(z_1)$.

If $\widetilde{I}_1^{(s,f^s(l'))} = \emptyset$, then the number of vacant slots at the last group that accepts students in type $f^s(l')$ will increase by one. If the capacity transfer scheme of school s does not transfer this extra vacancy to any other group following the last group that accepts students of type $f^s(l')$ in the computation of $C^s(Z_s, f^s, \widetilde{q}_s)$, then the improvement chain process ends and we have $\widetilde{Z} = Z$. If the extra slot is transferred to the group l'' that follows the last group that accepts students in type $f^s(l')$ in the computation of $C^s(Z_s, f^s, \widetilde{q}_s)$, then we consider students who prefer $(s, f^s(l''))$ over their assignments under Z, and so on.

Since school s has finitely many groups, Step 1 ends in finitely many iterations. If no extra student is assigned to school s by the end of Step 1, then the improvement chains algorithm ends and we have $\widetilde{Z} = Z$. If an extra student is assigned to school s by the end of Step 1, then we move on to Step 2.

Step t>1: Consider students who prefer $(\mathbf{s}(z_{t-1}), \mathbf{t}(z_{t-1}))$ to their assignments under $\widetilde{Z}_{t-1}$, i.e.,

$$\widetilde{I}_t^{(\mathbf{s}(z_{t-1}),\mathbf{t}(z_{t-1}))} = \{i \in I \mid (\mathbf{s}(z_{t-1}), \mathbf{t}(z_{t-1}))P^i(\widetilde{Z}_{t-1})_i\}.$$

We choose $\pi^{\mathbf{s}(z_{t-1})}$-maximal student in $\widetilde{I}_t^{(\mathbf{s}(z_{t-1}),\mathbf{t}(z_{t-1}))}$, call her $\widetilde{i}_t$, and assign her $\widetilde{x}_t = (\widetilde{i}_t, \mathbf{s}(z_{t-1}), \mathbf{t}(z_{t-1}))$. Update the outcome to $\widetilde{Z}_t = \widetilde{Z}_{t-1} \cup \{\widetilde{x}_t\} \setminus z_t$ where z_t is the contract student $\widetilde{i}_t$ receives under $\widetilde{Z}_{t-1}$.

If $(\mathbf{s}(z_{t-1}), \mathbf{t}(z_{t-1})) = \emptyset$, then the improvement algorithm ends and we have $\widetilde{Z} = \widetilde{Z}_t = \widetilde{Z}_{t-1} \cup \{\widetilde{x}_t\}$. Otherwise, we move to Step $t+1$. Because assigning $\widetilde{i}_t$ to $(\mathbf{s}(z_{t-1}), \mathbf{t}(z_{t-1}))$ creates a vacancy in school $\mathbf{s}(z_t)$ within type $\mathbf{t}(z_t)$.

If $\widetilde{I}_t^{(\mathbf{s}(z_{t-1}),\mathbf{t}(z_{t-1}))} = \emptyset$, then the number of vacant slots at the last group that accepts students in type $f^{\mathbf{s}(z_{t-1})}$ will increase by one. If the capacity transfer scheme of school $\mathbf{s}(z_{t-1})$ does not transfer this extra capacity to any other group following the last group that accepts students in type $\mathbf{t}(z_{t-1})$ in the computation of $C^{\mathbf{s}(z_{t-1})}((\widetilde{Z}_{t-1})_{\mathbf{s}(z_{t-1})}, f^{\mathbf{s}(z_{t-1})}, q_{\mathbf{s}(z_{t-1})})$, then the improvement chains process ends and we have $\widetilde{Z} = \widetilde{Z}_{t-1}$. If the extra slot is transferred to the group of slot m that follows the last group that accepts students in type $\mathbf{t}(z_{t-1})$ in the computation of $C^{\mathbf{s}(z_{t-1})}((\widetilde{Z}_{t-1})_{\mathbf{s}(z_{t-1})}, f^{\mathbf{s}(z_{t-1})}, q_{\mathbf{s}(z_{t-1})})$, then we consider students who prefer $(\mathbf{s}(z_{t-1}), f^{\mathbf{s}(z_{t-1})}(m))$ over their assignments under $\widetilde{Z}_{t-1}$, i.e.,

$$\widetilde{I}_t^{(\mathbf{s}(z_{t-1}),f^{\mathbf{s}(\mathbf{z_{t-1}})}(m))} = \{i \in I \mid (\mathbf{s}(z_{t-1}), f^{\mathbf{s}(z_{t-1})}(m)) P^i (\widetilde{Z}_{t-1})_i\}.$$

We choose $\pi^{\mathbf{s}(z_{t-1})}$-maximal student in $\widetilde{I}_t^{(\mathbf{s}(z_{t-1}),f^{\mathbf{s}(z_{t-1})}(m))}$, call her $\widetilde{i}_t$, and assign her $\widetilde{x}_t = (\widetilde{i}_t, \mathbf{s}(z_{t-1}), f^{\mathbf{s}(z_{t-1})}(m))$. Update the outcome to $\widetilde{Z}_t = \widetilde{Z}_{t-1} \cup \{\widetilde{x}_t\} \setminus z_t$ where z_t is the contract student $\widetilde{i}_t$ receives under $\widetilde{Z}_{t-1}$.

If $(\mathbf{s}(z_{t-1}), f^{\mathbf{s}(z_{t-1})}(m)) = \emptyset$, then the improvement algorithm ends and we have $\widetilde{Z} = \widetilde{Z}_t = \widetilde{Z}_{t-1} \cup \{\widetilde{x}_t\}$. Otherwise, we move to Step $t+1$. Because assigning $\widetilde{i}_t$ to $(\mathbf{s}(z_{t-1}), f^{\mathbf{s}(z_{t-1})}(m))$ creates a vacancy in school $\mathbf{s}(z_t)$ within type $\mathbf{t}(z_t)$.

If $\widetilde{I}_t^{(\mathbf{s}(z_{t-1}),\mathbf{t}(z_{t-1}))} = \emptyset$, then the number of vacant slots at the last group that accepts students in type $f^{\mathbf{s}(z_{t-1})}$ will increase by one. If the capacity transfer scheme of school $\mathbf{s}(z_{t-1})$ does not transfer this extra capacity to any other group following the last group that accepts students in type $f^{\mathbf{s}(z_{t-1})}(m)$ in the computation of $C^{\mathbf{s}(z_{t-1})}((\widetilde{Z}_{t-1})_{\mathbf{s}(z_{t-1})}, f^{\mathbf{s}(z_{t-1})}, q_{\mathbf{s}(z_{t-1})})$, then the improvement chains process ends and we have $\widetilde{Z} = \widetilde{Z}_{t-1}$. If the extra slot is transferred to the group of slot m' that follows the last group that accepts students in type $f^{\mathbf{s}(z_{t-1})}(m)$ in the computation of $C^{\mathbf{s}(z_{t-1})}((\widetilde{Z}_{t-1})_{\mathbf{s}(z_{t-1})}, f^{\mathbf{s}(z_{t-1})}, q_{\mathbf{s}(z_{t-1})})$, then we consider students who prefer $(\mathbf{s}(z_{t-1}), f^{\mathbf{s}(z_{t-1})}(m'))$ over their assignments under $\widetilde{Z}_{t-1}$, and so on.

Since school $\mathbf{s}(z_{t-1})$ has finitely many groups , Step t ends in finitely many iterations. If no extra student is assigned to school $\mathbf{s}(z_{t-1})$ by the end of Step t, then the improvement chains algorithm ends and we have $\widetilde{Z} = \widetilde{Z}_{t-1}$. If an extra student is assigned to school $\mathbf{s}(z_{t-1})$ by the end of Step t, then we move on to Step $t+1$.

This process ends in finitely many iterations because there are finitely many contracts and when we move to the next step it means a student is made strictly better off. Also, notice that no student is worse off during the execution of the improvement chains algorithm. The improvement algorithm, by construction, starts with the outcome $\Phi(P, C)$ and ends at $\Phi(P, \widetilde{C})$. Hence, we have $\Phi_i(P, \widetilde{C}) R^i \Phi_i(P, C)$ for all $i \in I$.

We define the sequence of capacity transfer schemes and dynamic reserve choice functions for school $s \in S$: $((q_s)^1, (q_s)^2, ...)$ and $(C^s(Y, f^s, (q_s)^1), C^s(Y, f^s, (q_s)^2), ...)$. Let the sequence $\Phi(P, C^1)$, $\Phi(P, C^2)$,... denote the outcomes of the COPs at profiles $(P, (C^s(\cdot, f^s, (q_s)^1), C_{-s}))$ and $(P, (C^s(\cdot, f^s, (q_s)^2), C_{-s}))$,..., respectively. Hence, by construction, we have $\Phi_i(P, C^{a+1}) R^i \Phi_i(P, C^a)$ for all $i \in I$ and $a \geq 1$. By the transitivity of weak preferences, we have $\Phi_i(P, \widetilde{C}) R^i \Phi(P, C)$ for all $i \in I$.

Proof of Theorem 5

Our proof is constructive. We first define an associated type space. Let X be the set of all contracts. We define a distinct "type" for each contract in X. Let $g : X \rightarrow \mathcal{T} = \{\tau_1, ..., \tau_{|X|}\}$ be a bijective function. The interpretation of the g function is that the artificial type of a contract $x \in X$ is $g(x) \in \{\tau_1, ..., \tau_{|X|}\}$. Therefore, each contract in X is associated with a distinct (artificial) type.

Consider a slot $b_l \in \mathcal{B}_s$ with priority order π^{b_l}. Let $\mid \pi^{b_l} \mid$ denote the number of contracts that the slot b_l finds acceptable, i.e., ranks higher than the null contract which corresponds to remaining unassigned. Let x_l^1, x_l^2,...,$x_l^{|\pi^{b_l}|}$ be the acceptable contracts for slot b_l such that

$$x_l^1 \pi^{b_l} x_l^2 \pi^{b_l} \cdots \pi^{b_l} x_l^{|\pi^{b_l}|}.$$

For the slot b_l in school s in the true market, we create a *sequence of slots*—$\mid \pi^{b_l} \mid$ many slots— in the associated market, i.e., $\{b_l^1, ... b_l^{|\pi^{b_l}|}\}$. The initial capacity of b_l^1 is 1, i.e., $\overline{q}_{b_l^1} = 1$, and the initial capacities of $b_l^2, b_l^3, ..., b_l^{|\pi^{b_l}|}$ are 0, i.e., $\overline{q}_{b_l^k} = 0$ for all $k = 2, ..., \mid \pi^{b_l} \mid$. Define $r_{b_l^k}$ such that $r_{b_l^k} = 0$ if slot b_l^k is filled and $r_{b_l^k} = 1$ if slot b_l^k remains vacant. The dynamic capacity of the slot b_l^k, for all $k = 2, ..., \mid \pi^{b_l} \mid$, is defined as $q_{b_l^k}(r_{b_l^1}, ..., r_{b_l^{k-1}}) = r_{b_l^{k-1}}$. That is, if the slot b_l^{k-1} remains vacant, then the capacity of the slot b_l^k becomes 1. Note that if a slot b_l^{k-1} is filled, then the dynamic capacity of slots that come after b_l^{k-1} become 0.

Each slot b_l^k is associated with a sub-choice rule $c^s_{b_l^k}(\cdot, q_{b_l^k}, \cdot)$ that is defined as follows: The sub-choice rule $c^s_{b_l^k}(\cdot, q_{b_l^k}, \cdot)$ can only considers contracts with artificial type $g^{-1}(x_l^k)$, therefore only the contract x_l^k. Given a set of contracts $Y \subseteq X$,

$$c^s_{b_l^k}(Y, q_{b_l^k}, g^{-1}(x_l^k)) = \begin{cases} \{x_l^k\} & \text{if } x_l^k \in Y \text{ and } q_{b_l^k} = 1, \\ \emptyset & \text{otherwise} \end{cases}.$$

Note that $c^s_{b_l^k}$ is a q-responsive choice function. We now describe a dynamic reserves choice rule $\widetilde{C}^s(\cdot)$ that is outcome equivalent to the slot-specific choice rule $C^s(\cdot)$. Let $Y \subseteq X$ be a set of contracts.

Step 1 Consider slots $\{b_1^1, b_1^2, ..., b_1^{|\pi^{b_1}|}\}$ in this step.

Step 1.1 Apply the sub-choice function $c^s_{b^1_1}$. If a contract is chosen, then end Step 1, and move to Step 2 due to the capacity transfer rule described above. Otherwise, move to Step 1.2.

Step 1.2 Apply the sub-choice function $c^s_{b^2_1}$. If a contract is chosen, then end Step 1, and move to Step 2 due to the capacity transfer rule described above. Otherwise, move to Step 1.3.

This process continues in sequence. If a contract chosen in Step 1, then all of the contracts associated with the student whose contract is chosen is removed for the rest of the procedure. Let y^1 be the chosen contract in this step. Then, the set of remaining contracts is $Y \setminus Y_{\mathbf{i}(y^1)}$.

Step n ≥ 2 Consider slots $\{b^1_n, b^2_n, ..., b^{|\pi^{b_n}|}_n\}$ in this step.

Step n.1 Apply the sub-choice function $c^s_{b^1_n}$. If a contract is chosen, then end Step n, and move to Step $(n+1)$ due to the capacity transfer rule described above. Otherwise, move to Step n.2.

Step n.2 Apply the sub-choice function $c^s_{b^2_n}$. If a contract is chosen, then end Step n, and move to Step $(n+1)$ due to the capacity transfer rule described above. Otherwise, move to Step n.3.

This process continues in sequence. If a contract chosen in Step n, then all of the contracts associated with the student whose contract is chosen is removed for the rest of the procedure. Let y^n be the chosen contract in this step. Then, the set of remaining contracts is $Y \setminus Y_{\mathbf{i}(y^1,...,y^n)}$.

By construction, for any given set of contracts $Y \subseteq X$, for each slot b_l in the process of the slot-specific priorities choice function $C^s(\cdot)$ and Step l of the dynamic reserves choice function $\widetilde{C}^s(\cdot)$ the set of available contracts, and hence, the chosen contract are the same. Therefore, these two choice functions select the same set of contracts, i.e., $C^s(Y) = \widetilde{C}^s(Y)$. This ends our proof.

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