CESifo WORKING PAPERS

8993 2021

April 2021

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Ka-Kit Iong, Andreas Irmen

CESifo

Impressum:

CESifo Working Papers
ISSN 2364-1428 (electronic version)
Publisher and distributor: Munich Society for the Promotion of Economic Research - CESifo GmbH
The international platform of Ludwigs-Maximilians University's Center for Economic Studies and the ifo Institute
Poschingerstr. 5, 81679 Munich, Germany
Telephone +49 (0)89 2180-2740, Telefax +49 (0)89 2180-17845, email office@cesifo.de
Editor: Clemens Fuest
https://www.cesifo.org/en/wp
An electronic version of the paper may be downloaded
· from the SSRN website: www.SSRN.com
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CESifo *Working Paper No. 8993*

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Abstract

Declining hours of work per worker in conjunction with a growing work force may give rise to fluctuations between growth regimes. This is shown in an overlapping generations model with two-period lived individuals endowed with Boppart-Krusell preferences (Boppart and Krusell (2020)). On the supply side, economic growth is due to the expansion of consumption-good varieties through endogenous research. A sufficiently negative equilibrium elasticity of the individual supply of hours worked to an expansion in the set of consumption-good varieties destabilizes the steady state so that equilibrium trajectories may fluctuate between two growth regimes, one with and the other without an active research sector. Fluctuations affect intergenerational welfare, the evolution of GDP, and the functional income distribution. A stabilization policy can shift the economy onto its steady-state path. Fluctuations arise for empirically reasonable parameter constellations. The economics of fluctuations between growth regimes is linked to the intergenerational trade of shares and their pricing in the asset market.

JEL-Codes: J220, O330, O410.

Keywords: endogenous fluctuations, growth regimes, endogenous technological change, endogenous labor supply, OLG-model.

Ka-Kit Iong
Department of Economics and Management
University of Luxembourg
Campus Kirchberg (Bloc G)
6, Rue Richard Coudenhove-Kalergi
Luxembourg - 1359 Luxembourg
kakitiong@gmail.com

*Andreas Irmen**
Department of Economics and Management
University of Luxembourg
Campus Kirchberg (Bloc G)
6, Rue Richard Coudenhove-Kalergi
Luxembourg - 1359 Luxembourg
airmen@uni.lu

*corresponding author

This version: March 29, 2021
This paper is a revised and extended version of our discussion paper Iong and Irmen (2020). We would like to thank two anonymous referees, an associate editor, as well as Karl-Josef Koch, Christos Koulovatianos, Anastasia Litina, and Benteng Zou for valuable comments.

1 Introduction

At least since 1870 hours worked per worker have substantially declined in many of today's industrialized countries. Estimates of Huberman (2004) and Huberman and Minns (2007) suggest that a male full-time production worker in the U.K. had a weekly workload of 56.9 hours in the year 1870. In 2000 this number comes down to 42 hours of work, an absolute decline of roughly 35%. According to these authors a similar tendency can be found in Australia, Belgium, Canada, Denmark, France, Germany, Ireland, Italy, the Netherlands, Spain, Sweden, Switzerland, and the US. At the same time the total workforce of these countries has been increasing.

Boppart and Krusell (2020) and Irmen (2018) interpret these stylized facts as properties of country-specific balanced growth paths driven by exogenous technical change. In contrast to this view, the present paper shows that declining hours of work per worker may cause fluctuations between growth regimes when technical change is endogenous. Over time the evolution is unbalanced, yet, consistent with the data income and wages increase and individual hours of work decline.

We obtain these features in an overlapping generations model with exogenous population growth where the individual supply of hours worked and technological change are both endogenous. The household side has two-period lived individuals endowed with a periodic log-log utility function that belongs to the Boppart-Krusell class (Boppart and Krusell (2020)). Individuals work when young and retire when old. Therefore, the Marshallian wage elasticity of their supply of hours worked is negative and constant (Irmen (2018)). In the spirit of Grossman and Helpman (1991) and Jones (1995), the production side has endogenous technological change through an expansion of the set of consumption-good varieties. New varieties are invented by research firms. There are two growth regimes. One regime has an active research sector, in the other the research sector is inactive. The global dynamics exhibits fluctuations between these two growth regimes.

The occurrence of two growth regimes is closely linked to the mechanics of the market for assets and its repercussions on the evolution of the economy over time. In this market, ownership shares of the firms that produce a variety of the consumption good are traded. The demand side has the current young with their need to save for old age. In the regime without an operating research sector, the supply of shares stems solely from the current old. They own the shares of all firms and sell them to finance their consumption demand. In the regime with an active research sector, there is an additional source of share supply. New firms emit ownership shares to finance the purchase of a blueprint that grants them the right to market a new consumption-good variety invented by a research firm.

In the regime without an active research sector the amount of existing consumption-good varieties is large relative to the cohort size of the current young. Accordingly, the supply of shares is large relative to its demand and the equilibrium share price will be low. Therefore, the financial resources that new firms may raise to pay for a blueprint are too small for a research firm to break even. As a result, the research sector remains inactive.

In the regime with an active research sector the amount of existing consumption-good varieties is small relative to the cohort size of the current young. Therefore, in spite of the additional supply of shares through the primary offering of new firms, the equilibrium price of shares is sufficiently high so that research firms enter the market and break even.

The supply of hours worked plays a key role for the fluctuations between the two regimes. The transition into the regime with an active research sector is driven by the extensive margin of the supply of hours worked. Without an active research sector wages, the individual supply of hours worked, individual savings, and the amount of traded shares remains constant. However, through population growth the cohort size of the young increases over time. This boosts the total demand for shares. Therefore, the equilibrium share price increases over time and, eventually, reaches a level at which an additional share supply of new firms can be absorbed and research firms break even.

The transition into the regime without an active research sector is driven by the effect of newly invented consumption-good varieties on the intensive margin of the supply of hours worked and its repercussions for the asset market. On its demand side, new varieties increase the wage which leads to a reduction in the individual supply of hours worked. Therefore, individual incomes and savings increase by less than wages. This attenuates the increase in the total demand for shares. On the supply side, new varieties affect the cost of research firms through higher wages and an intertemporal knowledge spillover. Since the spillover increases the productivity of research labor the break-even price of a blueprint increases by less than wages. Then, what matters is the relative strength of the effects of new varieties on the reduction in the supply of hours worked and on the cost reduction through the knowledge spillover. If the former channel dominates the latter, then, in spite of the cost reduction through the knowledge spillover, the demand for shares becomes so weak that the asset market equilibrium cannot support a share price at which research firms can break even.

To further highlight the role of an endogenous supply of hours worked for the occurrence of fluctuations between growth regimes, observe that the economy has a unique steady state in the regime with an active research sector. This steady state is globally stable with monotone convergence if the wage elasticity of the individual supply of hours worked is zero. If this elasticity becomes negative, then the stability properties of the steady state and the occurrence of fluctuations between growth regimes hinge in the *equilibrium elasticity of the individual supply of hours worked to an expansion in the set of consumption-good varieties*. If the latter is slightly negative, then the steady state remains stable, possibly with oscillatory convergence. If it becomes sufficiently negative then the steady state is unstable and the global dynamics give rise to fluctuations between the two growth regimes.

A closer look at the economic conditions that determine the local stability properties of the steady state reveal a tension between two channels through which an active research sector today affects the economy tomorrow. The first channel operates through the expansion of the set of consumption good varieties. Since the equilibrium elasticity of the individual supply of hours worked to an expansion in the set of consumption-good varieties is negative the individual supply of hours worked, aggregate wage income, and the demand for shares decline. Hence, for a given share price fewer primary offers can be placed and, therefore, fewer blueprints will be invented. The second channel is due to positive intertemporal knowledge spillovers that render each hour worked in the research sector more productive. This reduces the costs of creating a blueprint. For a given demand for shares, more primary offers can be placed and, therefore, more blueprints will be invented.

If the wage elasticity of the individual supply of hours worked is slightly negative (or zero), then the second channel dominates the first and the intertemporal knowledge spillover that operates under diminishing returns is the source of stability. The growth rate of newly invented blueprints tomorrow increases in the respective growth rate today in a way that assures a monotone convergence to its steady-state level. However, if the first channel dominates then the growth rate of newly invented blueprints tomorrow decreases in the respective growth rate today. This gives either rise to oscillatory convergence, or, if the equilibrium elasticity of the individual supply of hours worked to an expansion in the set of consumption-good varieties is sufficiently negative, to instability. In the latter case an economy starting out in the vicinity of the steady state will eventually fluctuate between the two growth regimes.

We establish that these fluctuations enter an absorbing interval in finite time. Moreover, we show that the average geometric growth factor of the number of consumption-good varieties associated with any equilibrium paths converges to the growth factor of the steady state as the number of periods becomes unbounded.

In our discussion section, we establish additional properties of equilibrium paths and inquire further into the economic consequences of fluctuations between growth regimes. First, we study the evolution of intergenerational welfare of two overlapping cohorts along different equilibrium paths. The welfare comparison hinges on the evolution of wages and interest factors faced by the two cohorts. We identify two, possibly opposing channels that determine whether the welfare of the earlier or the one of the later cohort is higher. As a consequence, the relative welfare of overlapping generations hinges on whether fluctuations between growth regimes occur, and, if they occur, on their direction, and on whether they occur during the lifetime of the earlier or the later cohort.

Second, we analyze the evolution of GDP in absolute and per-capita terms as well as the evolution of the functional income distribution along different equilibrium paths. GDP is defined as the sum of the value added in the production and in the research sector. Since the production sector operates under monopolistic competition its value added per hour worked is greater than in the perfectly competitive research sector. Therefore, an increase in the fraction of the workforce employed in the research sector reduces GDP.

We find that GDP steadily increases over time, though at different rates. Since the growth rate of GDP reflects exogenous population growth, the growth factor of wages and the evolution of the fraction of the workforce employed in the research sector, GDP in absolute terms may increase over time while per-capita GDP declines. For instance, a switch from the regime without an active research sector into the regime with an active research sector reduces per-capita GDP since it requires some workers to join the research sector where the value added per hour worked is lower.

The economy features two types of income, dividend income and wage income. We study the evolution of the functional income distribution through the lens of the labor share. We show that the latter increases in the fraction of the workforce allocated to the research sector. As labor earns the same wage in the production and the research sector such a reallocation leaves the economy's aggregate wage bill unaffected. However, for the reason set out above, it reduces GDP, i. e., the sum of total incomes earned, and the labor share increases. As a consequence, a switch into the regime with an active research sector will increase the labor share whereas a switch in the opposite direction reduces it.

Third, we consider the possibility of a stabilization policy that upon its implementation avoids an evolution involving fluctuations. Moreover, we require this policy to be consistent with a balanced budget. We show that such a policy exists. Depending on the current state of the economy it involves either a tax on wage income in conjunction with a subsidy to capital income or vice versa. The purpose of this policy is to move the economy instantaneously into its steady state that is unstable and, therefore, can not be attained.

If the research sector of the economy is too small relative to its steady-state size then the policy involves a subsidy to wages. This pushes the demand for ownership shares up so that the appropriate amount of research firms becomes active. To balance the budget, this policy involves a tax on asset income. Hence, the policy involves a transfer from the current old to the current young. If the research sector is too big relative to its steady state level, then the policy prescribes a tax on wages to weaken the demand for ownership shares and to reduce the amount of active research firms. Then, the policy will involve a transfer from the current young to the current old.

Finally, we provide some evidence on how the possibility of fluctuations between growth regimes may be linked to empirical evidence. In particular, we show that the condition for the instability of the steady state is satisfied for empirically reasonable parameter constellations.

This paper contributes to at least two strands of the literature. First, it contributes to the growth literature ignited by Boppart and Krusell (2020) that incorporates the secular decline in hours worked per worker into the neoclassical growth model of Ramsey (1928), Cass (1965), and Koopmans (1965). Irmen (2018) applies it to an overlapping generations setting. In both papers technical change is exogenous. Irmen (2020) studies the economic consequences of automation in a model with endogenous technical change where households are endowed with Boppart-Krusell preferences. These contributions have in common that the steady state is either a globally stable or a saddle-path stable balanced growth path. In contrast, the present paper establishes, that Boppart-Krusell preferences may be the source of fluctuations between growth regimes. Indeed, if the individual supply of hours worked does not respond to movements in the real wage, then the steady state of our model is globally stable. It is the negative wage elasticity of the individual supply of hours worked that opens the possibility of fluctuations between the regime with and the one without an active research sector.

Second, our analysis contributes to the literature on endogenous fluctuations and growth regimes. Unlike the present paper, this literature maintains the assumption of an exogenous labor supply. Hence, the mechanics behind the fluctuations that we identify are new to this literature.

A paper closely related to our study is Matsuyama (1999) who studies a variant of the lab-equipment model of Rivera-Batiz and Romer (1991). Matsuyama argues that the growth process may involve fluctuations between growth regimes including cycles where an economy moves back and forth between a regime with capital accumulation alone and a regime that has capital accumulation and innovation. Moreover, since the accumulation of capital is subject to diminishing returns, the economy may be trapped in a regime without long-run growth.[1]

Our rationale for fluctuations between growth regimes substantially differs from Matsuyama's in at least two ways. First, there is no capital accumulation in our setup. Rather, the economy "accumulates" workers through population growth. As this accumulation is not subject to diminishing returns the economy either converges to a steady state with growth of per-capita variables or keeps on fluctuating between the regime with and without an active research sector.

Second, we maintain the assumption that new blueprints are sold in conjunction with a perpetual patent. Hence, a key mechanism for the emergence of cycles in Matsuyama (1999), namely, the temporary monopoly power of new products, is mute in our model. In contrast, we emphasize in line with the empirical evidence that individuals reduce their supply of hours worked in response to higher wages. This is the key driver of instability of the unique steady state.

The remainder of this paper is organized as follows. Section 2 presents the model. Section 3 defines the intertemporal general equilibrium, proves its existence, derives the dynamical system, and determines the unique steady state. Section 4 deals with the transitional dynamics. Here, we derive the main results of our analysis on the possibility of fluctuations between the two growth regimes. Section 5 discusses relevant economic implications of fluctuating trajectories. The focus of Section 5.1 is on the evolution of intergenerational welfare along different equilibrium paths. Section 5.2 studies the implications of fluctuations for the evolution of GDP in absolute and per-capita terms as well as for the functional income distribution. In Section 5.3, we devise a stabilization policy that eliminates fluctuations. Finally, Section 5.4 shows that the condition under which fluctuations arise is satisfied for empirically plausible parameter constellations. Section 6 concludes. All proofs are contained in Section 7, Appendix A. The (online)

[1] Other studies emphasizing the presence of physical capital and research as a source of growth cycles include Matsuyama (2001) with cyclical deterministic growth and Bental and Peled (1996), Wälde (2002), and Wälde (2005) with cyclical stochastic growth. The growth process in the presence of a regime with capital accumulation alone and a regime that has capital accumulation and research can also be construed as globally stable as in Irmen (2005).

Appendix B contains additional results.

2 The Model

The economy has a household sector with overlapping two-period lived individuals, a production sector where monopolistically competitive firms manufacture differentiated varieties of a consumption good, and a research sector where new varieties of the consumption good are invented. Time is discrete and extends from one to infinity, i. e., $t = 1, 2, ..., \infty$.

In all periods, there are markets for the following objects of exchange. First, there are markets for a continuum of differentiated varieties of the consumption good. These varieties are supplied by the production sector and demanded by the household sector. Second, there is a labor market where the current young supply hours of work that the firms of the production and the research sector demand. Third, there is a market for the blueprints of newly invented varieties of the consumption good. These blueprints are supplied by the inventing research firms and demanded by new firms that enter the production sector. Finally, there is an asset market where bonds (in zero net supply) and ownership shares in the firms of the production sector are traded. At the beginning of each period, all assets are owned by the current old. To finance their consumption, the old sell their ownership shares to the young. In addition, there are primary stock offerings by the new firms of the production sector. They need to finance their purchase of a blueprint for a newly invented variety of the consumption good. The demand for both types of shares corresponds to the savings of the current young.

The Household Sector The population at t consists of L_t young and L_{t-1} old individuals. Except for their age, individuals are identical. The number of young individuals between two adjacent periods grows at the exogenous rate $g_L > 0$. For short, we shall refer to g_L as the growth rate of the labor force.

When young, individuals supply labor, earn wage income, save, and enjoy leisure as well as the consumption goods. When old, they retire, sell their wealth, and consume the receipts.

For cohort t, denote consumption when young and old by c_t^y and c_{t+1}^o, savings by s_t, and leisure time enjoyed when young by l_t. We normalize the maximum per-period time endowment supplied to the labor market to unity. Then, $1 - l_t = h_t$, where $h_t \in [0, 1]$ is the individual supply of hours worked when young. Individuals of all cohorts assess bundles (c_t^y, l_t, c_{t+1}^o) according to the lifetime utility function[2]

$$U(c_t^y, l_t, c_{t+1}^o) = \ln c_t^y + \ln \left(1 - \phi(1 - l_t)(c_t^y)^{\frac{\nu}{1-\nu}}\right) + \beta \ln c_{t+1}^o; \tag{2.1}$$

here, the parameter $\phi > 0$ determines the strength of the disutility of labor, and $\beta \in (0, 1)$ is the discount factor. As shown in Irmen (2018), $\nu \in (0, 1)$ implies that consumption when young

[2] The two periodic utility functions featured in $U(c_t^y, l_t, c_{t+1}^o)$ are of the generalized log-log type proposed for applied use in Boppart and Krusell (2020), Section V. As retirement in old age means $l_{t+1} = 1$ utility when old boils down to $\ln c_{t+1}^o$.

and leisure are complements in the cardinal sense of $\partial^2 U/\partial c^y \partial l > 0$.

Consumption when young and old, c_t^y and c_{t+1}^o, represent bundles of differentiated consumption goods, i. e.,

$$c_t^y = A_t^{\sigma-\frac{1}{\alpha}} \left[\int_0^{A_t} (x_t^y(j))^\alpha dj \right]^{\frac{1}{\alpha}} \quad \text{and} \quad c_{t+1}^o = A_{t+1}^{\sigma-\frac{1}{\alpha}} \left[\int_0^{A_{t+1}} (x_{t+1}^o(j))^\alpha dj \right]^{\frac{1}{\alpha}}, \tag{2.2}$$

where $\sigma > 1$ and $\alpha \in (0,1)$. Here, $x_t^y(j)$ and $x_{t+1}^o(j)$ denote the respective quantity of consumption good j consumed when young and old. The "number" of available consumption goods at any time t is given by A_t. As $\sigma > 1$ there is a "taste for variety". The parameter $\alpha \in (0,1)$ determines the elasticity of substitution between any pair of existing consumption good varieties. As α increases consumption goods become better substitutes.

The optimal behavior of an individual of cohort t results from a two-stage budgeting procedure. First, it allocates its spending across the differentiated consumption goods available when young and old age. Second, it determines its labor supply when young as well as the consumption profile over her life cycle. The first stage delivers the following conditional demands for each differentiated consumption good

$$x_t^y(j) = \frac{p_t(j)^{-\frac{1}{1-\alpha}} c_t^y}{A_t^{\sigma-\frac{1}{\alpha}} \left[\int_0^{A_t} p_t(j')^{-\frac{\alpha}{1-\alpha}} dj' \right]^{\frac{1}{\alpha}}} \quad \text{and} \quad x_{t+1}^o(j) = \frac{p_{t+1}(j)^{-\frac{1}{1-\alpha}} c_{t+1}^o}{A_{t+1}^{\sigma-\frac{1}{\alpha}} \left[\int_0^{A_{t+1}} p_{t+1}(j')^{-\frac{\alpha}{1-\alpha}} dj' \right]^{\frac{1}{\alpha}}} \tag{2.3}$$

as well as the two expenditure functions pertaining to period t, $P_t c_t^y$ for all young and $P_t c_t^o$ for all old individuals. Here,

$$P_t = A_t^{\frac{1}{\alpha}-\sigma} \left[\int_0^{A_t} p_t(j')^{-\frac{\alpha}{1-\alpha}} dj' \right]^{-\frac{1-\alpha}{\alpha}} \tag{2.4}$$

is the *ideal price index*.

In the second stage, each member of cohort t chooses a plan $(c_t^y, h_t, c_{t+1}^o, s_t)$ that maximizes her lifetime utility subject to two periodic budget constraints. We follow Irmen (2018) and denote by $\mathscr{P}$ the set of *permissible* bundles (c_t^y, h_t, c_{t+1}^o) out of which the individual chooses. Elements of this set satisfy the condition

$$1 - 2\nu - (1-\nu)\phi h_t (c_t^y)^{\frac{\nu}{1-\nu}} > 0, \tag{2.5}$$

which assures that U is strictly concave.

Let $w_t > 0$ denote the wage per hour worked and $R_{t+1} > 0$ the (perfect foresight) interest factor paid per unit saved.[3] Then, cohort t solves

$$\max_{(c_t^y, h_t, c_{t+1}^o, s_t) \in \mathscr{P} \times \mathbb{R}} \quad \ln c_t^y + \ln\left(1 - \phi h_t (c_t^y)^{\frac{\nu}{1-\nu}}\right) + \beta \ln c_{t+1}^o$$

$$\text{s. t.} \quad P_t c_t^y + P_t s_t \leq w_t h_t \quad \text{and} \quad P_{t+1} c_{t+1}^o \leq R_{t+1} P_t s_t. \tag{2.6}$$

Before we state and discuss the solution to this maximization problem it proves useful to define

$$w_c \equiv \left(\frac{(1+\beta)(1-\nu)}{\left(\phi\left(1+(1+\beta)(1-\nu)\right)\right)^{1-\nu}\left(1-\nu(1+\beta)\right)^{\nu}} \right)^{\frac{1}{\nu}}, \tag{2.7}$$

and to make the following assumption:

Assumption 1 *It holds that*

$$0 < \nu < \bar{\nu}(\beta) \equiv \frac{3 + \beta - \sqrt{5 + \beta(2+\beta)}}{2(1+\beta)},$$

and, for all t,

$$\frac{w_t}{P_t} \geq w_c.$$

Assumption 1 serves two purposes. The first inequality assures that the optimal plan derived in the following proposition satisfies condition (2.5). The second inequality implies that the individual chooses a positive demand for leisure.

Proposition 2.1 (Optimal Plan of Cohort t)

Suppose Assumption 1 holds. Then, the optimal plan of cohort t involves the conditional demands (2.3) and

$$h_t = w_c^{\nu} \left(\frac{w_t}{P_t}\right)^{-\nu}, \qquad P_t s_t = \frac{\beta}{(1+\beta)(1-\nu)} w_c^{\nu} \left(\frac{w_t}{P_t}\right)^{1-\nu},$$

$$P_t c_t^y = \frac{1-\nu(1+\beta)}{(1+\beta)(1-\nu)} w_c^{\nu} \left(\frac{w_t}{P_t}\right)^{1-\nu}, \qquad P_{t+1} c_{t+1}^o = \frac{\beta}{(1+\beta)(1-\nu)} R_{t+1} w_c^{\nu} \left(\frac{w_t}{P_t}\right)^{1-\nu}.$$

Hence, the optimal plan of cohort t hinges critically on the ratio w_t/P_t which has an interpretation as the real wage in units of contemporary consumption. Assumption 1 assures a positive demand for leisure since w_t/P_t is sufficiently high. Moreover, $(-\nu)$ is the wage elasticity of the individual labor supply. Accordingly, the individual supply of hours worked declines in the wage. Finally, Proposition 2.1 implies that consumption when young and leisure are also

[3] For simplicity our notation does not distinguish between a wage rate paid in the production and a wage rate paid in the research sector. As individuals are identical they may supply homogeneous labor to either sector. Hence, in any constellation that has both sectors operating there can only be one wage. Moreover, observe that the individual supply of hours worked is assumed to be perfectly divisible across occupations.

"demand complements," i. e, in response to a higher P_t or a lower w_t, both, c_t^y and l_t fall.

The Production Sector At all t there are A_t monopolistically competitive firms. Each firm possesses the blueprint and a perpetual patent for the exclusive production of one variety $j \in [0, A_t]$ that it acquired in the past. All firms produce their variety with the same linear production function,

$$x_t(j) = h_{x,t}(j), \tag{2.8}$$

where $h_x(j)$ is the amount of working hours hired by firm j to produce $x_t(j)$ units of consumption good j. Each firm's profit is

$$\pi_t(j) = p_t(j)x_t(j) - w_t h_{x,t}(j), \tag{2.9}$$

where $x_t(j)$ and $h_{x,t}(j)$ are linked via (2.8), and, in light of (2.3),

$$x_t(j) = L_{t-1}x_t^o(j) + L_t x_t^y(j). \tag{2.10}$$

Then, profit maximization delivers the price set by all firms as

$$p_t = p_t(j) = \frac{w_t}{\alpha}. \tag{2.11}$$

The less substitutable the differentiated consumption goods are, the higher their price. Since all firms charge the same price, p_t, they all supply the same quantity, i. e., $x_t^y = x_t^y(j)$, $x_t^o = x_t^o(j)$, and $x_t = x_t(j)$. Then, (2.8), (2.9) and (2.11) imply that the profit of each firm at t is

$$\pi_t = \pi_t(j) = (1-\alpha)p_t x_t. \tag{2.12}$$

These profits are paid as dividends to old individuals who are the shareholders of the firms in the production sector. Finally, since $h_{x,t} = h_{x,t}(j)$ the total amount of hours worked demanded by the production sector, $H_{x,t}$, equals

$$H_{x,t} = A_t h_{x,t} = A_t x_t. \tag{2.13}$$

The Research Sector At all t there are many small competitive research firms. They may either enter the market or remain inactive. If they enter the market then they hire labor, invent new varieties of the consumption good, and sell the respective blueprints to a newly created firm of the production sector. All research firms have access to the same technology for the invention of new varieties. Therefore, the analysis of the research sector can be done through the lens of a competitive representative firm in conjunction with a free-entry condition.

Following Jones (1995), the representative firm has access to a technology for the creation of new consumption-good varieties given by

$$\Delta A_t = \frac{H_{A,t}}{a} A_t^{\psi}, \quad 0 < \psi < 1. \tag{2.14}$$

Here, $\Delta A_t \equiv A_{t+1} - A_t$ denotes the additional varieties invented in period t, $H_{A,t} \geq 0$ is the total amount of working hours demanded in the research sector, and $a > 0$ determines the productivity of labor in research. Since $\psi > 0$ the productivity of hours worked in research increases in the number of varieties invented in the past, A_t. The assumption $\psi < 1$ constrains the extent of intertemporal knowledge spillovers.

Let v_t denote the revenue generated from selling a blueprint of a newly created variety. Then, the profit associated with an invention is

$$v_t - w_t \frac{a}{A_t^{\psi}}, \tag{2.15}$$

where a/A_t^{ψ} is the amount of hours worked in the research sector required to invent one new variety.

Since the total amount of hours worked demanded by research firms must be finite the profit of (2.15) cannot be strictly positive in equilibrium. This gives rise to the following equilibrium free-entry condition:

$$v_t \leq w_t \frac{a}{A_t^{\psi}}, \quad \text{with "=", if } \Delta A_t > 0. \tag{2.16}$$

Hence, if at t the revenue obtained from selling a blueprint of a new variety is too low, then the research sector will not be active and $\Delta A_t = 0$. However, if the research sector is active at t, then condition (2.16) must hold as equality since in equilibrium entering research firms must be just as well-off as non-entering ones. This distinction plays a key role in our analysis of the economy's transitional dynamics below.

3 Intertemporal General Equilibrium

This section states and interprets the definition of the intertemporal general equilibrium, develops the dynamical system, and establishes the existence of a unique steady state.

Definition For all $j \in [0, A_t]$, a price system is a sequence $\{w_t, R_t, P_t, p_t(j), \pi_t(j), v_t\}_{t=1}^{\infty}$, an allocation is a sequence $\{c_t^y, l_t, c_t^o, s_t, x_t^y(j), x_t^o(j), x_t(j), H_{x,t}, H_{A,t}, A_t\}_{t=1}^{\infty}$. The latter comprises a plan $\{c_t^y, l_t, c_t^o, s_t, x_t^y(j), x_t^o(j)\}_{t=1}^{\infty}$ for all cohorts, consumption of the old at $t = 1$, c_t^o, and strategies for the production and the research sector $\{x_t(j), H_{x,t}, H_{A,t}, A_t\}_{t=1}^{\infty}$.

For an exogenous evolution of the labor force, $L_t = L_1(1 + g_L)^{t-1}$, $g_L > 0$, with $L_1 > 0$ and a given initial stock of varieties of the consumption good, $A_1 > 0$, an intertemporal general equilibrium with perfect foresight corresponds to a price system and an allocation that satisfy for all $t = 1, 2, \cdots, \infty$ Proposition 2.1, (2.11), (2.14), (2.16), and the respective market clearing conditions for the consumption good and for hours of work, $P_t c_t^y L_t + P_t c_t^o L_{t-1} = p_t A_t x_t$ and $H_{x,t} + H_{A,t} = h_t L_t$. Moreover, the asset market clears and values all shares according to fundamentals.

In the asset market at t, the ownership shares of the A_t existing varieties and of the ΔA_t new varieties are traded. Since these shares are perfect substitutes as stores of value, they must have the same price denoted by v_t. Moreover, a market valuation of shares according to fundamentals requires

$$v_t = \frac{\pi_{t+1}}{R_{t+1}} + \frac{\pi_{t+2}}{R_{t+1}R_{t+2}} + \cdots, \tag{3.1}$$

where π_{t+i}, $i = 1, 2, ...$, are the (perfect foresight) dividends paid to the shareholders in the future. The latter implies the following no-arbitrage condition for bonds and stocks

$$\frac{v_{t+1}}{v_t} + \frac{\pi_{t+1}}{v_t} = R_{t+1}. \tag{3.2}$$

The equilibrium conditions imply for all t that the market for ownership shares satisfies

$$s_t L_t = v_t \left(A_t + \Delta A_t\right). \tag{3.3}$$

In other words, the savings of the young is equal to the period-t market capitalization of *all* consumption-good varieties available in $t+1$, i. e., those already produced at t and those invented at t. Condition (3.3) reflects two economic transactions. First, the old at t sell their A_t shares to the current young at a price v_t. Second, the firms of the production sector that purchase by auction at t one of the ΔA_t new blueprints at a price v_t need to finance this purchase. This is done through the issue of new shares with an aggregate value of $v_t \Delta A_t$. Clearly, if (2.16) holds as inequality then $\Delta A_t = 0$ and $A_{t+1} = A_t$, if it holds as equality then $A_{t+1} = A_t + \Delta A_t$.

Henceforth, we choose the consumption aggregate when young as the numéraire, i. e., $P_t = 1$ for all t. In view of (2.4) this choice implies for any symmetric configuration that

$$p_t = A_t^{\sigma-1}. \tag{3.4}$$

Due to the "taste for variety," i. e., $\sigma > 1$, the real price of each consumption good variety increases in the number of differentiated consumption goods. Moreover, with the latter in the mark-up formula (2.11) the equilibrium real wage is

$$w_t = \alpha A_t^{\sigma-1}. \tag{3.5}$$

Accordingly, from Proposition 2.1 a higher A_t reduces the supply of hours worked and increases savings. Henceforth, we refer to $-\nu(\sigma-1)$ as the *equilibrium elasticity of the individual supply of hours worked to changes in A_t*. It indicates how young individuals convert a 1 percentage point gain in the number of varieties and the associated productivity gain into more leisure as opposed to more consumption.

Dynamical System Throughout our focus is on equilibria with a positive demand for leisure. From Proposition 2.1 this requires $w_t \geq w_c$. Since A_t cannot decline and the equilibrium real wage is given by (3.5) this inequality holds for all t if the following assumption is satisfied.

Assumption 2 *It holds that*

$$A_1 \geq \left(\frac{w_c}{\alpha}\right)^{\frac{1}{\sigma-1}}.$$

The transitional dynamics of the intertemporal general equilibrium may be studied through the evolution of the transformed variable

$$z_t \equiv \frac{L_t}{A_t^{\eta}}, \tag{3.6}$$

where $\eta \equiv 1 - \psi + \nu(\sigma - 1) > 0$, and

$$z_c \equiv \left(\frac{\alpha}{w_c}\right)^{\nu} \frac{a(1+\beta)(1-\nu)}{\beta} \tag{3.7}$$

denotes a critical value of z_t. To interpret the variable z_t, observe that in equilibrium the total amount of hours worked, $H_t \equiv h_t L_t$, can be expressed as $H_t = (w_c/\alpha)^{\nu} A_t^{-\nu(\sigma-1)} L_t$ where use is made of Proposition 2.1 and (3.5). Let $\gamma_{A,t} \equiv H_{A,t}/H_t$ denote the fraction of total hours worked in research. Then, given $\gamma_{A,t}$ equation (2.14) implies that the productivity of existing consumption-good varieties in the invention of new ones, $\Delta A_t/A_t$, is proportionate to $A_t^{-\nu(\sigma-1)} L_t A_t^{\psi-1} = z_t$. Accordingly, keeping $\gamma_{A,t}$ constant, η is the elasticity of $\Delta A_t/A_t$ with respect to changes in A_t.

The following proposition shows that, depending on how z_t relates to z_c, the economy is in one of two distinct regimes. If $z_t \leq z_c$ then the economy is said to be in Regime 0, and the research sector is inactive. If $z_t \geq z_c$ then the research sector is active, and the economy is said to be in Regime 1.

Proposition 3.1 (Existence, Uniqueness, and Dynamical System)

Suppose Assumption 2 holds. Then, a unique intertemporal general equilibrium exists. Moreover, the transitional dynamics of this equilibrium is given by a unique equilibrium sequence, $\{z_t\}_{t=1}^{\infty}$, generated by the piecewise defined difference equation $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ where

$$z_{t+1} = \Phi(z_t) \equiv \begin{cases} (1+g_L) z_t & \text{if} \quad z_t \leq z_c, \\ \\ (1+g_L) z_c \left(\frac{z_t}{z_c}\right)^{\psi - \nu(\sigma-1)} & \text{if} \quad z_t \geq z_c. \end{cases} \tag{3.8}$$

Proposition 3.1 establishes the existence and the uniqueness of an equilibrium for any permissible initial value $z_1 > 0$. Moreover, it shows that the transitional dynamics may involve a passage through two regimes. In Regime 0, z_t is small, i. e., cohort t is small and/or the stock of existing varieties is large. Under these circumstances inequality (2.16) is strict in equilibrium, and the research sector remains inactive. In Regime 1, z_t is large, (2.16) holds as equality, and the research sector is active.[4]

Why are there two regimes? The explanation starts with the observation that the free-entry condition (2.16) in conjunction with (3.5) determines the equilibrium value of a blueprint of a newly invented variety if $\Delta A_t > 0$ as

$$v_t = a\alpha A_t^{\sigma-1-\psi}. \tag{3.9}$$

Hence, if the research sector is active then the equilibrium value of a new variety equals the total labor cost associated with its invention. The exponent $\sigma - 1 - \psi$ reflects the two channels through which A_t affects these costs. First, from (3.5) a higher A_t increases the wage per hour worked, i. e., to break even v_t must increase. Second, a higher A_t increases the productivity of labor in research. Hence, research firms break even at a lower v_t.

The question is then whether the current young are ready to buy some $\Delta A_t > 0$ primary offerings of the newly invented varieties in addition to the shares of existing varieties at the break-even price of equation (3.9). The equilibrium condition of the market for ownership shares stated in (3.3) has the answer. Here, given A_t, the demand increases in L_t. This supports a higher equilibrium price of shares. Indeed, with Proposition 2.1 one readily verifies that the current young are ready to buy some $\Delta A_t > 0$ newly emitted shares at a price equal to v_t of (3.9) if $L_t > z_c A_t^{\eta}$ or $z_t > z_c$. Hence, in Regime 1, cohort t is sufficiently large and/or the stock of existing varieties is sufficiently small.

In Regime 0 the equilibrium value of ownership shares obtains directly from the equilibrium condition of the market for ownership shares. In other words, v_t is the equilibrium value of shares that solves (3.3) for $\Delta A_t = 0$. The comparison between v_t and $a\alpha A_t^{\sigma-1-\psi}$, the break-even price of a new blueprint as stated in (3.9), reveals that (see equation (7.13) in the Proof of Proposition 3.1) $z_t < z_c \Rightarrow v_t < a\alpha A_t^{\sigma-1-\psi}$. Hence, in equilibrium inequality (2.16) is strict. The intuition is the following. If $v_t < a\alpha A_t^{\sigma-1-\psi}$ then the shares of existing varieties are cheaper than those of newly invented ones. Since both types of shares generate the same stream of returns investors will only buy the shares of existing varieties. Accordingly, there is no demand for newly emitted shares. In equilibrium, potential research firms anticipate this and, therefore, will not become active.

Steady State Define a steady-state equilibrium as a path along which all variables except leisure grow at constant rates. Let $g_{m,t} \equiv \Delta m_t/m_t$, $\Delta m_t \equiv m_{t+1} - m_t$, denote the growth rate of an arbitrary variable m_t, and m and g_m their respective steady-state values.

From (3.8) one readily verifies that the steady state has $z_t = z$ for all t, is unique, and given by

$$z \equiv z_c(1+g_L)^{\frac{1}{\eta}} > z_c. \tag{3.10}$$

[4] Observe that equations (3.6) - (3.8) include the case $\nu = 0$ for which the individual supply of hours worked is time-invariant and equal to $h = (1+\beta)/[\phi(2+\beta)]$. The parameter restriction $\phi \geq (1+\beta)/(2+\beta)$ ensuring that $h \leq 1$ will then replace Assumption 2. Details for this case are available from the authors upon request.

Hence, z is in Regime 1.[5] In steady state the endogenous growth rate of A_t^η adjusts to the exogenous growth rate of L_t. This reflects two steady-state requirements. First, from (2.14) the invention of new consumption-good varieties requires $H_{A,t}$ to grow at rate $(1+g_A)^{1-\psi}-1$. Hence, the total demand for of hours worked must grow at this rate to keep the fraction of the workforce employed in both sectors constant. Second, with Proposition 2.1 and (3.5) the total supply of hours worked grows at rate $(1+g_L)(1+g_A)^{-\nu(\sigma-1)}-1$. Hence, z_t is constant if $(1+g_A)^{1-\psi}=(1+g_L)(1+g_A)^{-\nu(\sigma-1)}$ or

$$1+g_A=(1+g_L)^{\frac{1}{\eta}} \tag{3.11}$$

which is the steady-state growth factor of A.[6] Interestingly, the steady-state growth rate of A is not only determined by population growth and technology parameters as in Jones (1995). It also depends on the preference parameters ν and σ.

4 Transitional Dynamics

This section develops the main result of the paper on the transitional dynamics of the dynamical system of Proposition 3.1. Using z of (3.10) allows to write the equilibrium difference equation $\Phi(z_t)$ for $z_t>z_c$ as

$$z_{t+1}=z^\eta z_t^{\psi-\nu(\sigma-1)}. \tag{4.1}$$

Then, the transitional dynamics follow from the stability properties of the steady state.[7]

Proposition 4.1 (Transitional Dynamics and the Stability Properties of the Steady State)

Consider the dynamical system of Proposition 3.1. Its steady state, z, is locally stable if and only if

$$\left|\psi-\nu(\sigma-1)\right|<1.$$

Moreover, for any initial value $z_1\in\mathbb{R}_{++}$, $z_1\neq z$, the evolution of z_t for $t>1$ satisfies the following:

[5] A simple argument shows why the steady state has to be in Regime 1. Suppose the economy starts in Regime 0. Then, the initial values are such that $z_1<z_c$. The research sector is not competitive and remains inactive since $v_1<a\alpha A_1^{\sigma-1-\psi}$. As long as the economy remains in Regime 0, the real wage, the individual supply of hours worked, and individual savings remain constant. However, the cohort size and, therefore, the demand for shares and the equilibrium share price grow exponentially at rate g_L. Then, there is a period $\tau_c\geq 2$ such that $v_1(1+g_L)^{\tau_c-1}\geq a\alpha A_1^{\sigma-1-\psi}$, or, equivalently, $z_1(1+g_L)^{\tau_c-1}\geq z_c$. The equilibrium at τ_c involves $v_{\tau_c}=a\alpha A_1^{\sigma-1-\psi}$ and $\Delta A_{\tau_c}\geq 0$. In other words, ongoing population growth implies in finite time that the demand for ownership shares becomes sufficiently large and the young are willing to buy $A_1+\Delta A_{\tau_c}$ shares at a price v_{τ_c}.

[6] See Section 8.1 of Appendix B for a detailed analysis of the structural properties of the steady state.

[7] See Proposition 5.1 and 5.2 in Iong and Irmen (2020) for a discussion of the transitional dynamics under the non-generic parameter constellations $-\nu(\sigma-1)=-\psi$ and $-\nu(\sigma-1)=-(1+\psi)$.

1. *if* $-\nu(\sigma-1) \in (-\psi, 0)$ *then* $\lim_{t\to\infty} z_t = z$ *with monotone convergence;*
2. *if* $-\nu(\sigma-1) \in (-(1+\psi), -\psi)$ *then* $\lim_{t\to\infty} z_t = z$ *with oscillating convergence in Regime* 1*. Moreover,*

   (a) *if* $z_1 < z_c$ *then the economy eventually settles down in Regime* 1*, then oscillates about and eventually converges to the steady state,*

   (b) *if* $z_c(1+g_L)^{-\frac{1}{\psi-\nu(\sigma-1)}} \geq z_1 \geq z_c$ *then the economy evolves in Regime* 1 *with oscillating convergence to the steady state,*

   (c) *if* $z_1 > z_c(1+g_L)^{-\frac{1}{\psi-\nu(\sigma-1)}}$*, then the economy immediately transits to Regime* 0*, and after a finite amount of periods moves back to Regime* 1*, where it oscillates about and eventually converges to the steady state* z*;*

3. *if* $-\nu(\sigma-1) < -(1+\psi)$ *then the economy enters an absorbing interval* $[\Phi^2(z_c), \Phi(z_c)]$ *in finite time. Inside this region the economy fluctuates between Regime* 0 *and* 1*.*

Hence, the local stability properties of the steady state hinge on how the intertemporal knowledge spillover represented by $\psi \in (0,1)$ relates to $-\nu(\sigma-1)$, the equilibrium elasticity of the individual supply of hours worked to changes in A_t. The latter vanishes for $\nu = 0$. In this case, the individual supply of hours worked does not respond to changes in the real wage, and the steady state is unequivocally locally stable with monotone convergence. Hence, any deviation from this transitional behavior results since $\nu \in (0,1)$. Figures 4.1 and 4.2 provide an illustration for the local and the global dynamics of Case 1 and Case 2(a).

To develop an intuition for the local dynamics about the steady state consider the equilibrium condition of the market for ownership shares (3.3) at t and $t+1$ for an economy in Regime 1

$$\frac{s_{t+1}L_{t+1}}{s_t L_t} = \frac{v_{t+1}\left(A_{t+1} + \Delta A_{t+1}\right)}{v_t\left(A_t + \Delta A_t\right)}. \tag{4.2}$$

From Proposition 2.1 and (3.5) we have $s_{t+1}/s_t = (h_{t+1}/h_t)(w_{t+1}/w_t) = (1+g_{A,t})^{(1-\nu)(\sigma-1)}$. Hence, the growth factor of individual savings reflects an effect of an increasing amount of varieties on wages and an effect on the supply of hours worked. Similarly, with (2.16) and (3.9) we obtain $v_{t+1}/v_t = (w_{t+1}/w_t)(A_{t+1}/A_t)^{-\psi} = (1+g_{A,t})^{\sigma-1-\psi}$. Hence, the growth factor of the share price reflects an effect of an increasing amount of varieties on wages and on the productivity of labor in research through the knowledge spillover. As the effects through wages on s_{t+1}/s_t and v_{t+1}/v_t on the left and the right-hand side of (4.2) cancel, we write this equation as[8]

$$(1+g_{A,t})^{\psi-\nu(\sigma-1)}(1+g_L) = 1 + g_{A,t+1}. \tag{4.3}$$

[8] Equation (4.3) is equivalent to the second difference equation in the dynamical system (3.8).

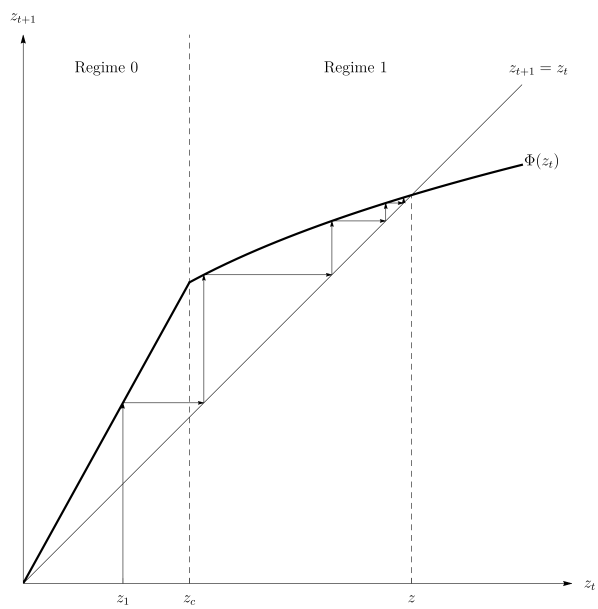


Figure 4.1: Case 1: Local and Global Dynamics of a Stable Steady-State Equilibrium for $-\nu(\sigma - 1) \in (-\psi, 0)$.

The steady state of this difference equation is stated in (3.11). Its local stability properties hinge on the exponent $\psi - \nu(\sigma - 1)$ which captures the respective effect of $g_{A,t}$ on the demand and the supply side of the market for ownership shares. On the demand side, $-\nu(\sigma - 1) < 0$ means that a higher $g_{A,t}$ reduces the individual supply of hours worked, the individual wage income and savings at $t + 1$. Accordingly, for a given share price at $t + 1$ fewer primary offerings can be placed and ΔA_{t+1} falls. On the supply side, $\psi > 0$ means that a higher $g_{A,t}$ reduces the costs of creating a blueprint. Accordingly, for a given demand for shares, more primary offers can be placed and ΔA_{t+1} increases.

If $\psi > \nu(\sigma - 1)$ then the effect a higher $g_{A,t}$ on the supply side dominates and supports a higher $g_{A,t+1}$. As $\psi < 1$ the strength of the supply side effect is limited. Accordingly, the steady state is locally stable with monotone convergence. If $\psi < \nu(\sigma - 1)$ then the effect a higher $g_{A,t}$ on the demand side dominates and the relationship between $g_{A,t}$ and $g_{A,t+1}$ becomes negative. As long as this dominance is not too pronounced, i. e., $\nu(\sigma - 1) < 1 + \psi$, the steady state remains locally stable with oscillatory convergence. However, for $\nu(\sigma - 1) > 1 + \psi$, it becomes unstable.

The qualitative properties of the global dynamics are driven by the extensive and the intensive margin of the supply of hours worked. On the one hand, a growing labor force enlarges the demand for ownership shares. This force pushes the economy into Regime 1 in finite time irrespective of its initial position in Regime 0. On the other hand, the decline in the individual supply of hours worked to a higher wage determines the stability of the steady state. The value of $-\nu(\sigma - 1)$ decides whether the economy remains eventually inside Regime 1 or keeps on fluctuating between both regimes.

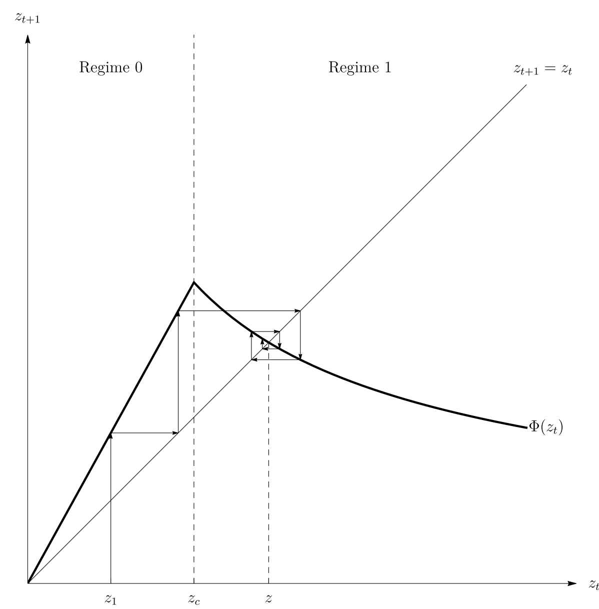


Figure 4.2: Case 2(a): Local and Global Dynamics of a Stable Steady-State Equilibrium for $-\nu(\sigma - 1) \in (-(1+\psi), -\psi)$.

Case 1 and 2 exhibit global convergence. The comparison between Case 2(b) and 2(c) reveals why the economy may fluctuate between growth regimes. In both cases, the economy starts in Regime 1, however, with different initial values. The equilibrium in the market for ownership shares, $z_1/z_c = A_2/A_1$, reveals that A_2 is smaller in Case 2(b) than in Case 2(c). Therefore, Case 2(b) has the economy remaining inside Regime 1 with oscillatory convergence towards the steady state. In contrast, A_2 will be large in Case 2(c). As $-\nu(\sigma - 1) < -\psi$ the induced reduction in the individual supply of hours worked and savings will be more pronounced than the increase in the productivity of research labor. As a consequence, inequality (2.16) will be strict in period 2. Hence, a switch from Regime 1 into Regime 0 is possible for $-(1+\psi) < -\nu(\sigma - 1) < -\psi$ if the initial value z_1 is sufficiently high.[9]

If $-\nu(\sigma - 1) < -(1+\psi)$ then the economy eventually enters an absorbing interval given by $\left[\Phi^2(z_c), \Phi(z_c)\right]$ that ranges over Regime 0 and 1 (see Figure 4.3 for an illustration). The economy will then fluctuate between both regimes. These fluctuations may be chaotic or involve unstable basic cycles of any finite periodicity.[10] Figure 4.4 shows four possible trajectories.

[9] Formally, one readily verifies that $v_2 = s_2 L_2 / A_2 < a\alpha A_2^{\sigma-1-\psi}$ is equivalent to $z_2 < z_c$. In turn, from the dynamical system of (3.8), the latter holds if $z_1 > z_c (1+g_L)^{-\frac{1}{\psi - \nu(\sigma-1)}}$ as stated in Proposition 4.1.

[10] A rigorous and comprehensive mathematical analysis of the fluctuations between the two growth regimes is beyond the scope of this paper. In Iong and Irmen (2020) we prove the possibility of chaotic fluctuations with the construction of a 3-cycle in conjunction with "period three implies chaos" (Li and Yorke (1975)). In the latter paper, we also characterize unstable basic cycles of any finite periodicity. More advanced methods to establish these and to elicit further mathematical properties of possible evolutions are described, e. g., in Sushko, Avrutin, and Gardini (2015). We leave this for future research. In any case, the underlying economic mechanisms that explain fluctuations between the two growth regimes as well as an evolution within a given growth regime are comprehensively described in the

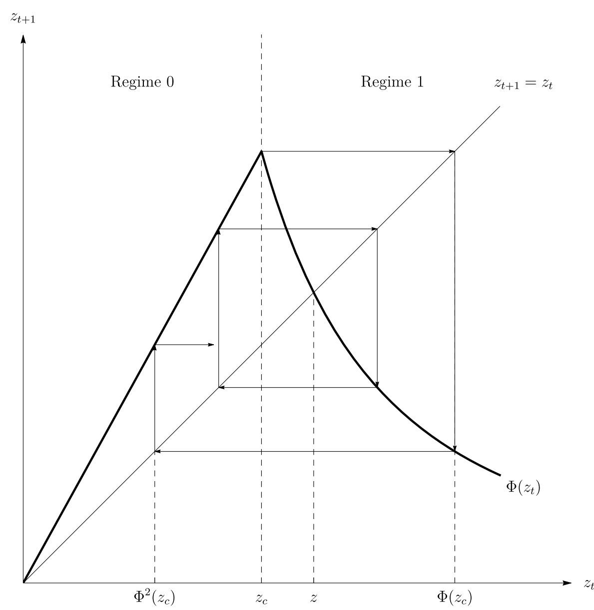


Figure 4.3: The Absorbing Interval and a 2 - Cycle Arising if $-\nu(\sigma - 1) < -(1 + \psi)$.

Nevertheless, the following corollary shows that irrespective of the type of fluctuation the average growth factor of A_t converges to its steady-state growth rate given in (3.11).

Corollary 4.1 (Convergence of the Average Growth Factor)

Consider the dynamical system of Proposition 3.1. It holds that

$$\lim_{T\to\infty}\left[\prod_{t=1}^{T}(1+g_{A,t})\right]^{\frac{1}{T}} = (1+g_L)^{\frac{1}{\eta}} = 1 + g_A.$$

Since the proof of Corollary 4.1 highlights the underlying intuition we develop it here. In fact, from the definition of z_t given in (3.6) one readily verifies that

$$\frac{A_{t+1}}{A_t} = (1+g_L)^{\frac{1}{\eta}}\left(\frac{z_{t+1}}{z_t}\right)^{-\frac{1}{\eta}}.$$

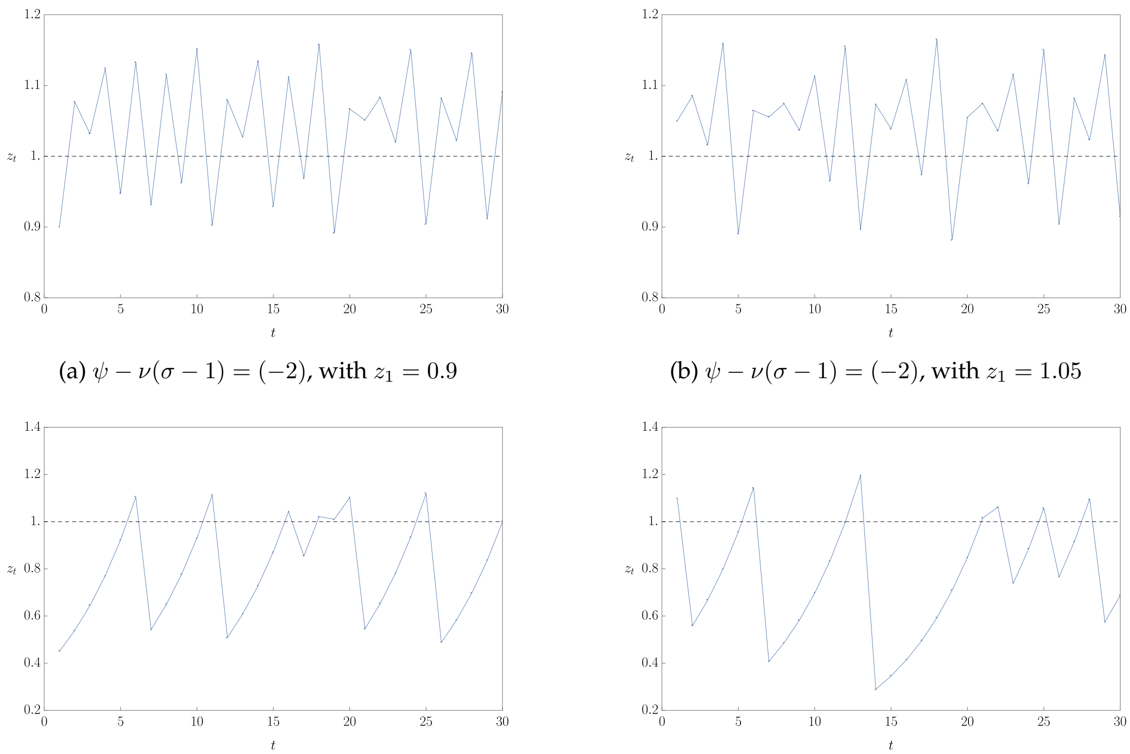
Then, for any finite $T > 1$ the average geometric growth factor of A_t can be expressed as

$$\left[\prod_{t=1}^{T}(1+g_{A,t})\right]^{\frac{1}{T}} = (1+g_L)^{\frac{1}{\eta}}\left(\frac{z_{T+1}}{z_1}\right)^{-\frac{1}{\eta T}}.$$

Hence, the average growth factor of A_t exceeds its steady state level whenever $z_{T+1} < z_1$ and vice versa.[11] Then, the corollary follows since z_t enters the absorbing interval $[\Phi^2(z_c), \Phi(z_c)] \subset$

discussion ensuing Proposition 4.1.

[11] Observe that for all trajectories satisfying $z_{T+1} = z_1$ the average growth factor after T periods coincides



(a) $\psi - \nu(\sigma - 1) = (-2)$, with $z_1 = 0.9$

(b) $\psi - \nu(\sigma - 1) = (-2)$, with $z_1 = 1.05$

(c) $\psi - \nu(\sigma - 1) = (-8)$, with $z_1 = 0.45$

(d) $\psi - \nu(\sigma - 1) = (-8)$, with $z_1 = 1.1$

Figure 4.4: Four Possible Trajectories if $-\nu(\sigma - 1) < -(1 + \psi)$. Throughout, we set $1 + g_L = 1.006^{30}$ and choose parameters such that $z_c = 1$.

$\mathbb{R}_{++}$ in finite time. Accordingly, $\lim_{T\to\infty}(z_{T+1}/z_1)^{-\frac{1}{\eta T}} = 1$.

5 Discussion

5.1 The Evolution of Intergenerational Welfare

How does the welfare of two overlapping cohorts evolve along different equilibrium paths? To address this question let V_t denote the indirect lifetime utility of cohort t. Then, with Proposition 2.1, the indirect lifetime utility function of cohort t is

$$V_t \equiv V(w_t, R_{t+1}) = (1 + \beta)(1 - \nu) \ln w_t + \beta \ln R_{t+1} + \omega, \tag{5.1}$$

where ω is a constant reflecting parameters that remain unchanged across cohorts. Hence, with ΔV_t denoting the difference between the indirect lifetime utility of cohort t and $t - 1$ we have

$$\Delta V_t \equiv V_t - V_{t-1} = (1 + \beta)(1 - \nu) \ln \left(\frac{w_t}{w_{t-1}} \right) + \beta \ln \left(\frac{R_{t+1}}{R_t} \right) \tag{5.2}$$

and the following holds.

with the steady state growth factor. Therefore, any period-T cycle satisfies this property.

Proposition 5.1 (Evolution of Intergenerational Welfare)

Consider the dynamical system of Proposition 3.1 and suppose that $-\nu(\sigma-1) < -(1+\psi)$.

1. *If* $0 < z_{t-1} \leq z_c(1+g_L)^{-2}$ *then* $\Delta V_t = 0$,
2. *if* $z_c(1+g_L)^{-2} < z_{t-1} \leq z_c(1+g_L)^{-1}$ *then* $\Delta V_t < 0$,
3. *if* $z_c(1+g_L)^{-1} < z_{t-1} < z_c(1+g_L)^{\frac{\psi-\nu(\sigma-1)}{\eta}}$ *then* $\Delta V_t \gtreqqless 0$,
4. *if* $z_c(1+g_L)^{\frac{\psi-\nu(\sigma-1)}{\eta}} \leq z_{t-1}$ *then* $\Delta V_t > 0$.

Proposition 5.1 compares the welfare of cohort $t-1$ and t for all permissible starting values, z_{t-1}, ordered from small to large, and all possible evolutions under the parameter constellation for which the steady state is unstable.[12] The key finding is that the sign of ΔV_t is not unequivocal. It depends on the qualitative and the quantitative properties of the sequence $\{z_{t-1}, z_t, z_{t+1}\}$. The intuition is the following.

In Case 1, z_{t-1} is so small that cohort $t-1$ and cohort t spend their entire lives in Regime 0. Then, $A_{t-1} = A_t$, $w_{t-1} = w_t$, $h_{t-1} = h_t$, $c^y_{t-1} = c^y_t$, $R_t = R_{t+1}$, and $c^o_t = c^o_{t+1}$, hence, $\Delta V_t = 0$.

In Case 2, cohort $t-1$ spends its entire life in Regime 0 whereas cohort t has its old age in Regime 1. Then, $A_{t-1} = A_t$, $w_{t-1} = w_t$, $h_{t-1} = h_t$, and $c^y_{t-1} = c^y_t$. However, $\Delta A_{t+1} > 0$. This affects R_{t+1} in two ways. First, as some labor shifts into the research sector, the scale of production, x_{t+1}, profits, π_{t+1}, and the dividend yield, π_{t+1}/v_t, is smaller than in Case 1. Second, in the asset market there will be additional primary share offerings at $t+1$. This increase in the supply of shares puts pressure on v_{t+1} so that v_{t+1}/v_t is smaller than in Case 1. As a consequence, the no-arbitrage condition (3.2) for t and $t+1$ delivers $R_{t+1} < R_t$. Hence, $c^o_{t+1} < c^o_t$ and $\Delta V_t < 0$.

In Case 3, cohort $t-1$ spends its youth in Regime 0 and its old age in Regime 1 whereas cohort t spends its entire life in Regime 1. In addition, the evolution of the state variable is monotonous, i. e., $z_{t-1} < z_c < z_t < z_{t+1}$. Then, $A_{t-1} = A_t$, $w_{t-1} = w_t$, $h_{t-1} = h_t$, and $c^y_{t-1} = c^y_t$. However, $\Delta A_t > 0$ and $\Delta A_{t+1} > 0$ so that the induced changes in the growth factor of the share price and in the dividend yield may either deliver $R_t \geq R_{t+1}$ or $R_t \leq R_{t+1}$. The latter implies $\Delta V_t \leq 0$ or $\Delta V_t \geq 0$, respectively.

In Case 4, z_{t-1} is large enough so that cohort $t-1$ spends at least one period of its life in Regime 1. Then, all possible sequences of the state variable, $\{z_{t-1}, z_t, z_{t+1}\}$, deliver $\Delta V_t > 0$. As becomes clear from the proof of Proposition 5.1, if cohort $t-1$ lives its youth in Regime 0 the sign of ΔV_t is due to $R_{t+1} > R_t$.[13] If cohort $t-1$ lives its youth in Regime 1 then cohort t

[12] Mutatis mutandis, the qualitative results stated in Proposition 5.1 carry over to parameter constellations that satisfy $0 > -\nu(\sigma-1) \geq -(1+\psi)$. We provide a comprehensive analysis of this case in Section 8.2.

benefits from the research done in period $t-1$ as $A_{t-1} < A_t$ means that $w_{t-1} < w_t$, $h_{t-1} > h_t$, and $c_{t-1}^y < c_t^y$. This channel determines the sign of ΔV_t irrespective of whether $R_{t+1} > R_t$ or $R_{t+1} < R_t$.

What is the role of fluctuations for the welfare comparison of two overlapping cohorts? Proposition 5.1 suggests that the answer hinges on the direction of the regime switch as well as on whether the regime switch occurs during the lifetime of cohort $t-1$ or of cohort t. A regime switch from Regime 0 into Regime 1 during the lifetime of cohort t decreases its relative welfare (Case 2). If this regime switch occurs during the lifetime of cohort $t-1$ then the welfare comparison is not unequivocal (Case 3 and 4). The regime switch from Regime 1 into Regime 0 is covered by Case 4. Here, the welfare comparison is unequivocal. Irrespective of whether cohort $t-1$ or cohort t experiences the regime switch during their respective lifetime the welfare of cohort t will be higher.

5.2 The Evolution of GDP and the Functional Income Distribution

How does GDP in absolute and per-capita terms as well as the functional income distribution evolve for different equilibrium paths? To address this question, define GDP_t as the economy's total value added at t. The latter is equal to the sum of the value added in the production and the research sector, i. e., $GDP_t = A_t p_t x_t + v_t \Delta A_t$. Using $p_t = w_t/\alpha$, $x_t = H_{x,t}/A_t$, $v_t \Delta A_t = w_t H_{A,t}$, and $\gamma_{A,t}$, GDP_t may be expressed as

$$GDP_t = \frac{w_t H_t}{\alpha} \left(1 - \gamma_{A,t}(1-\alpha)\right), \tag{5.3}$$

where $\gamma_{A,t} = 0$ holds in Regime 0 and $0 < \gamma_{A,t} < 1$ in Regime 1. A higher fraction of hours worked in research reduces GDP_t. Due to monopolistic competition the value added per hour worked in the production sector is equal to w_t/α and higher than w_t, the value added of an hour worked in the competitive research sector. Let gdp_t denote per-capita GDP at t so that

$$gdp_t \equiv \frac{1+g_L}{2+g_L} \times \frac{GDP_t}{L_t}.$$

To describe the evolution of GDP and gdp over two consecutive periods of different equilibrium paths, four cases must be distinguished. The first and simplest is the one with the economy staying in Regime 0 for the two periods, t and $t+1$. In these periods, the economy differs only with respect to its population size, i. e., $H_{t+1} = h_{t+1} L_{t+1}$, $H_t = h_t L_t$, $h_{t+1} = h_t$, and $L_{t+1} = (1+g_L) L_t$. Hence, $GDP_{t+1} = (1+g_L) GDP_t$ and $gdp_{t+1} = gdp_t$.

The second case has the economy in Regime 1 for two consecutive periods. Then,

$$GDP_t = A_t^{\sigma-1} H_t \left(1 - \gamma_{A,t}(1-\alpha)\right) \quad \text{and} \quad GDP_{t+1} = A_{t+1}^{\sigma-1} H_{t+1} \left(1 - \gamma_{A,t+1}(1-\alpha)\right),$$

[13] As in Case 3, cohort $t-1$ may spend its youth in Regime 0. However, in Case 4 the subsequent evolution of the state variable must not be monotonous. Moreover, observe that Case 4 includes the steady state. Here, $\Delta V_t > 0$ as $w_t > w_{t-1}$ and $R_t = R_{t+1}$.

with $H_t = w_c^{\nu} A_t^{-\nu(\sigma-1)} L_t$, $H_{t+1} = w_c^{\nu} A_{t+1}^{-\nu(\sigma-1)} L_{t+1}$, $0 < \gamma_{A,t} < 1$, and $0 < \gamma_{A,t+1} < 1$. Accordingly, we have

$$\frac{GDP_{t+1}}{GDP_t} = (1+g_{A,t})^{(1-\nu)(\sigma-1)}(1+g_L)\left(\frac{1-\gamma_{A,t+1}(1-\alpha)}{1-\gamma_{A,t}(1-\alpha)}\right) > 1, \tag{5.4}$$

where the sign follows from the proof of Proposition 5.2 below. Intuitively, $GDP_{t+1} > GDP_t$ is the result of population growth and higher wages at $t+1$. These forces may be weakened but cannot be dominated by $\gamma_{A,t+1} > \gamma_{A,t}$, i. e., when a larger fraction of the workforce moves into the sector with the lower value added.

However, wage growth is not sufficient for per-capita GDP to increase, i. e.,

$$\frac{gdp_{t+1}}{gdp_t} = (1+g_{A,t})^{(1-\nu)(\sigma-1)}\left(\frac{1-\gamma_{A,t+1}(1-\alpha)}{1-\gamma_{A,t}(1-\alpha)}\right) \gtreqless 1. \tag{5.5}$$

Clearly, $gdp_{t+1} > gdp_t$ will obtain if $\gamma_{A,t} > \gamma_{A,t+1}$. For this constellation the research sector at t is large so that the wage growth between t and $t+1$ is strong. Moreover, the growth of gdp benefits from the allocation of a larger fraction of hours worked to the production sector at $t+1$.[14]

Third, consider the case of an economy starting in Regime 0 at t and switching into Regime 1 at $t+1$. Then, $A_{t+1} = A_t$ and $w_{t+1} = w_t$ so that

$$\frac{GDP_{t+1}}{GDP_t} = (1+g_L)\left(1-\gamma_{A,t}(1-\alpha)\right) > 1,$$

where again the sign follows from the proof of Proposition 5.2 below. Hence, the dampening effect of moving hours of work into the research sector cannot outweigh population growth. However, it implies that per-capita GDP unequivocally falls, i. e., $gdp_{t+1} < gdp_t$.

Finally, consider the case where the economy is in Regime 1 at t and switches to Regime 0 in period $t+1$. Then, $GDP_t = A_t^{\sigma-1} H_t\left(1-\gamma_{A,t}(1-\alpha)\right)$ whereas $GDP_{t+1} = A_{t+1}^{\sigma-1} H_{t+1}$. Since $H_t = w_c^{\nu} A_t^{-\nu(\sigma-1)} L_t$ and $H_{t+1} = w_c^{\nu} A_{t+1}^{-\nu(\sigma-1)} L_{t+1}$ we have

$$\frac{GDP_{t+1}}{GDP_t} = \frac{(1+g_{A,t})^{(\sigma-1)(1-\nu)}(1+g_L)}{1-\gamma_{A,t}(1-\alpha)} > 1.$$

The latter reveals that GDP growth exceeds population growth for two reasons. First, research at t delivers wage growth between t and $t+1$. Second, as the research sector closes at $t+1$ the whole workforce will be employed in the production sector where the value added per hour worked is higher. As a consequence, per-capita GDP will also grow between t and $t+1$, i. e., $gdp_{t+1} > gdp_t$.

[14] The proof of Proposition 5.2 below reveals that an economy starting to the right of its steady state will exhibit $gdp_{t+1} > gdp_t$.

The following proposition summarizes the findings of the above discussion.

Proposition 5.2 (Evolution of GDP and gdp)

Consider the evolution of GDP and gdp as defined above over two consecutive periods, t and $t+1$. Then, irrespective of the economy's evolution $GDP_{t+1} > GDP_t$. Moreover,

1. *if the economy starts in Regime 0 and*
   - *stays there, then $gdp_{t+1} = gdp_t$;*
   - *switches into Regime 1, then $gdp_{t+1} < gdp_t$.*

2. *if the economy starts in Regime 1 and*
   - *stays there, then $gdp_{t+1} \gtreqless gdp_t$, and $z_t > z$ implies $gdp_{t+1} > gdp_t$;*
   - *switches into Regime 0, then $gdp_{t+1} > gdp_t$.*

The functional income distribution at t has to account for the labor income earned in the production and the research sector as well as for the dividend income that accrues to the owners of the A_t producing firms. One readily verifies that the value added in the production sector can be expressed as $A_t p_t x_t = A_t \pi_t + w_t H_{x,t}$. Hence, GDP_t is also equal to total incomes earned in the economy at t. In light of (5.3), the labor share at t, defined as the share of wage incomes in total incomes, is

$$LS_t \equiv \frac{w_t H_t}{GDP_t} = \frac{\alpha}{1-\gamma_{A,t}(1-\alpha)}. \tag{5.6}$$

In Regime 0, $\gamma_{A,t} = 0$ and the labor share boils down to the share of wage income in the production sector, i. e., $LS_t = w_t H_t/(A_t p_t x_t) = \alpha$. In Regime 1, the labor share is higher the greater $\gamma_{A_t} > 0$. Intuitively, shifting a larger fraction of the total amount of hours worked into the research sector reduces the scale of output, hence, the value added generated in the production sector. Accordingly, GDP_t falls. As this shift in hours worked has no effect on wage incomes the labor share will increase.

With this mechanism at hand it is straightforward to establish how the labor share may evolve over two consecutive periods, t and $t+1$, for different equilibrium paths. If the economy stays in Regime 0 for these two periods, the labor share will not change, i. e., $LS_t = LS_{t+1} = \alpha$. If the economy stays in Regime 1 for two consecutive periods, then

$$LS_t \gtreqless LS_{t+1} \quad \Leftrightarrow \quad \gamma_{A,t} \gtreqless \gamma_{A,t+1},$$

i. e., if the evolution is such that the research sector gets bigger over time then $\gamma_{A,t+1} > \gamma_{A,t}$ and the labor share will increase (and vice versa). An analogous argument reveals that a switch from Regime 0 into Regime 1 will increase the labor share as $0 = \gamma_{A,t} < \gamma_{A,t+1}$. Finally, a regime switch from Regime 1 into Regime 0 will reduce the labor share as $\gamma_{A,t} > \gamma_{A,t+1} = 0$.

5.3 Stabilization Policy

If the economy exhibits fluctuations then a government may consider a stabilization policy to eliminate them. Since the steady state allocation is the only trajectory with this property such a policy has to implement the steady-state path. From Proposition 5.1 we know that the steady-state path has $\Delta V_t > 0$ for all overlapping cohorts. A stabilization policy may then be justifiable as a device to exclude the possibility of $\Delta V_t < 0$ that arises along fluctuating trajectories.[15]

We focus on the possibility of a one-time stabilization policy implemented at t such that from period $t+1$ onwards the economy will be in steady state. We show that such a policy exists if z_t is sufficiently large. Depending on whether $z_t < z$ or $z_t > z$ the policy involves a transfer from the old to the young or vice versa. The policy induces these transfers through a negative or a positive tax on the wage income of cohort t in conjunction with a negative or a positive tax on the asset income of cohort $t-1$ such that the budget of the government is balanced.

Let $\tau_{w,t} < 1$ denote the tax rate on the wage income of cohort t. Moreover, denote by $\tau_{R,t} < 1$ the tax rate on the asset income of cohort $t-1$. Then, a one-time stabilization policy implemented at t is a pair $\left(\tau^P_{w,t}, \tau^P_{R,t}\right) \in (-\infty, 1) \times (-\infty, 1)$ such that

$$z_{t+1} = z \quad \text{and} \quad \tau_{w,t} w_t h_t L_t = -\tau_{R,t} R_t s_{t-1} L_{t-1}. \tag{5.7}$$

Hence, from $t+1$ onwards taxes will be eliminated and the economy will be on its steady-state path. Moreover, the government's budget is balanced at t. Such a policy affects the budget constraint of cohort $t-1$ when old and the budget constraint of cohort t when young, i. e.,

$$c^o_t = \frac{\beta(1-\tau_{R,t}) R_t w_c^{\nu} w_{t-1}^{1-\nu}}{(1+\beta)(1-\nu)}, \quad h_t = w_c^{\nu}\left((1-\tau_{w,t})w_t\right)^{-\nu}, \quad s_t = \frac{\beta w_c^{\nu}\left((1-\tau_{w,t})w_t\right)^{1-\nu}}{(1+\beta)(1-\nu)}. \tag{5.8}$$

Imposing $(\tau_{w,t}, \tau_{R,t})$ changes the evolution of the state variable z_t. The arguments that lead to the dynamical system of Proposition 3.1 reveal here the dynamics of z_t between period t and $t+1$ as

$$z_{t+1} = \begin{cases} (1+g_L) z_t & \text{if} \quad z_t \le \frac{z_c}{(1-\tau_{w,t})^{1-\nu}}, \\ \\ (1+g_L) z_c \left(\frac{z_t}{z_c}\right)^{\psi - \nu(\sigma-1)} (1-\tau_{w,t})^{-(1-\nu)\eta} & \text{if} \quad z_t \ge \frac{z_c}{(1-\tau_{w,t})^{1-\nu}}. \end{cases} \tag{5.9}$$

Three remarks are in order. First, the government can only use the tax on wage income, $\tau_{w,t}$, as an instrument to manipulate the evolution between t and $t+1$. The role of $\tau_{R,t}$ is merely to balance the government's budget. Second, the critical value of z_t that separates Regime 0 from Regime 1 depends on $\tau_{w,t}$. In particular, the range of z_t for which the economy is in Regime 0 shrinks if $\tau_{w,t}$ becomes negative and declines. The intuition is the following. In Regime 0 the demand for shares is so weak that the equilibrium share price is below the level

[15] See Case 2 and 3 of Proposition 5.1. Observe that a justification of a stabilization policy on Pareto grounds is problematic. As compared to the steady-state path, a fluctuating evolution may imply a higher life-time utility for some cohorts. Moreover, as we show below, the implementation of the proposed stabilization policy creates winners and losers.

at which entering research firms break even. As a consequence, the research sector is inactive. A government that sets $\tau_{w,t} < 0$ effectively subsidizes wage incomes and boosts savings as well as the demand for shares.[16] Third, the stabilization policy is only effective if the economy at t is in Regime 1 where the research sector is active and the equilibrium share price is given by (3.9). Then, the choice of $\tau^P_{w,t}$ fine-tunes the demand for shares such that ΔA_t is such that $z_{t+1} = L_{t+1}/\left(A_t + \Delta A_t\right)^\eta = z$.

Denote

$$\bar{z} \equiv (1+g_L)^{1-\eta} z_c. \tag{5.10}$$

Then, the following holds.

Proposition 5.3 (Stabilization Policy)

Suppose $-\nu(\sigma-1) < -(1+\psi)$ and $z_t \geq \bar{z}$. Then, there is a unique stabilization policy, $\left(\tau^P_{w,t}, \tau^P_{R,t}\right)$, where

$$\tau^P_{w,t} = 1 - \left(\frac{z}{z_t}\right)^{\frac{\eta-1}{\eta(1-\nu)}} \quad \text{and} \quad \tau^P_{R,t} = -\frac{\alpha(1+\beta)(1-\nu)}{(1-\alpha)(1-\nu(1+\beta)) + \beta\frac{z_c}{z_t}} \frac{\tau^P_{w,t}}{(1-\tau^P_{w,t})^\nu}.$$

If $z_t < \bar{z}$ then a stabilization policy implemented at $t+\tau$ involving

$$\tau^P_{w,t+\tau} = 1 - \left(\frac{z}{z_{t+\tau}}\right)^{\frac{\eta-1}{\eta(1-\nu)}} \quad \text{and} \quad \tau^P_{R,t+\tau} = -\frac{\alpha(1+\beta)(1-\nu)}{(1-\alpha)(1-\nu(1+\beta)) + \beta\frac{z_c}{z_{t+\tau}}} \frac{\tau^P_{w,t+\tau}}{(1-\tau^P_{w,t+\tau})^\nu}$$

delivers $z_{t+\tau+1} = z$ for all $t = t+\tau+1, t+\tau+2, ...$ under a balanced government budget at $t+\tau$ where τ is the smallest integer greater than $\ln\left(\bar{z}/z\right)/\ln(1+g_L)$.

Hence, if z_t is sufficiently large then a unique stabilization policy exists. Moreover, if z_t is small then the economy may grow smoothly for a few periods before the stabilization becomes implemented. In both cases fluctuations involving switches back and force between the two growth regimes are avoided.

Observe that $-\nu(\sigma-1) < -(1+\psi)$ implies $\eta > 2$. Therefore, if $z_t > z$ then $\tau_{w,t} > 0$ and the stabilization policy induces a transfer from the young to the old. If $z_t < z$ then $\tau_{w,t} < 0$ and the intergenerational transfer has the opposite direction.

If $z_t > z$ then the old benefit in addition from a higher asset income. To see this, observe that the stabilization policy reduces the size of the research sector while leaving w_t and the aggregate amount of hours worked unaffected. As explained in the previous section, the induced

[16] Indeed, from (5.9) one readily verifies that the economy is in Regime 1 if $\tau_{w,t} < 1 - (z_c/z_t)^{\frac{1}{1-\nu}} < 0$. Whenever the latter inequality is satisfied the demand for shares will be sufficiently strong such that shares are traded at the break-even share price (3.9).

shift of hours worked into the production sector increases dividends. Moreover, as $z_t > z_c$ the policy leaves v_t unaffected so that the interest factor, hence, asset income, will increase. As the economy's wage bill remains constant, the increase in dividends also means a higher GDP_t. Hence, the labor share will decline.

If $z_t < z$ the policy increases the size of the research sector. The concomitant shift of hours worked from the production into the research sector reduces dividends and GDP_t. If $z_c < z_t < z$ then the stabilization policy leaves v_t unaffected. Hence, the interest factor declines and the old suffer from a loss in their asset income. However, if $\bar{z} < z_t < z_c < z$ then the stabilization policy increases v_t. Hence, there are two opposing effects on the level of the interest factor. In any case, as the stabilization policy leaves w_t unaffected the decline in GDP_t implies an increase in the labor share.

5.4 Empirical Plausibility of the Instability Condition

The aim of this section is to investigate the empirical plausibility of the condition for the instability of the steady state, $-\nu(\sigma - 1) < -(1 + \psi)$. To accomplish this, we first calibrate the steady state of our model. Then, we check whether the condition for instability is satisfied. The calibration exercise is conducted for given values of g_h, g_c, g_L, β, and γ_A where $g_c = g_{c^y} = g_{c^o}$. We approximate these values with real world data and use them to determine ν, σ, and ψ.

Boppart and Krusell (2020) estimate the average annual growth rate of hours worked per worker to equal -0.57%. Then, over 30 years we have $1 + g_h = 0.9943^{30}$. As to g_c, we stipulate an average annual growth rate of per capita consumption of 1.9% so that $1 + g_c = 1.019^{30}$.[17] Then, Proposition 2.1 implies for the steady state that

$$\nu = \left(1 - \frac{\ln(1 + g_c)}{\ln(1 + g_h)}\right)^{-1} = 0.233.$$

With equation (7.20) in the proof of Proposition 5.2, $z/z_c = 1 + g_A$, equation (3.5), and $1 + g_c = (1 + g_A)^{(\sigma-1)(1-\nu)}$ we obtain the steady-state fraction of hours worked in the research sector as

$$\gamma_A = \frac{\beta}{(1 + \beta)(1 - \nu)} \left[1 - (1 + g_c)^{\frac{-1}{(1-\nu)(\sigma-1)}}\right].$$

For γ_A we use the fraction of researchers as a percentage of the labor force in the United Kingdom in 2017. According to the OECD Main Science and Technology Indicators database, this figure is 0.87%. The annual discount factor is often estimated to be around 0.96 implying $\beta = 0.294$. Accordingly, given the values for g_c, ν, β and γ_A, the calibrated value of σ is 25.13.

Finally, using Proposition 2.1, (3.5) and (3.11) delivers the link between ψ and g_L through

$$1 + g_h = (1 + g_L)^{\frac{-\nu(\sigma-1)}{1-\psi+\nu(\sigma-1)}}.$$

[17] According to the Penn World Table 9.1, this is the average annual growth rate of per capita consumption in the United Kingdom over the period from 1950 to 2017.

$1+g_L$	ψ	$-\nu(\sigma-1)$
1.0058^{30}	0.9342	$-4.6871-\psi$
1.006^{30}	0.7387	$-4.8826-\psi$
1.0065^{30}	0.2500	$-5.3712-\psi$

Table 5.1: Calibration Results.

Table 5.1 presents the calibration results. Column 2 shows the calibrated values of ψ using the numerical values stated above for differing values of g_L. As required, for average annual population growth rates around 0.6% the calibrated values for ψ lie in the interval $(0, 1)$.[18] Column 3 reveals that $-\nu(\sigma-1) < -(1+\psi)$ holds for all cases. Hence, the corresponding steady states are unstable.

6 Concluding Remarks

In the literature on endogenous fluctuations between growth regimes, the labor-leisure choice of households plays no role. This omission is hard to justify on empirical grounds. More importantly, we show that it leaves our understanding of these fluctuations incomplete. The main point of this paper is that a negative and sufficiently strong equilibrium elasticity of the individual supply of hours worked to an expansion of the set of consumption-good varieties may be a cause of instability that opens up the possibility of fluctuations between growth cycles. These findings require a negative wage elasticity of the individual supply of hours worked that is consistent with the empirical evidence for many of today's industrialized countries.

Since the household sector of our model features two-period lived overlapping generations the evolutions that we describe and explain apply to periods with a length of roughly 30 years. This raises at least two interesting questions. The first asks whether our analytical framework can be adapted to capture fluctuations in the short run that the empirical literature detects (see, e. g., Wälde and Woitek (2004)). One route to accomplish this would be to stipulate a constant savings rate and an exogenous labor supply with a negative and constant wage elasticity. As an alternative, one may attempt to derive such functions endogenously from a representative Ramsey household equipped with Boppart-Krusell preferences. The second question starts with the observation that the actual working period of individuals extends to more than 30 years. Allowing individuals to chose their supply of working hours and their retirement date when old could capture this (see, e. g., Hu (1979), Reichlin (1986), or Matsuyama (2008)).

Finally, at the technical front, one may wonder whether our model harbors additional properties that are beyond the mathematical scope of the present paper. Contributions like Mukherji (2005), Gardini, Sushko, and Naimzada (2008), and Deng and Khan (2018) elicit such properties for the model of Matsuyama (1999). Studies of this kind applied to our model would certainly shed light on the role of the linear segment of our dynamical system for the global dynamics

[18] Observe that average annual population growth rates of roughly 0.6% is in line with the empirical evidence. For instance, for the U.K. and Sweden over the time span 1870 to 2010 this rate is equal to 0.59% and 0.58%, respectively (see UN Population Division (2019)).

and on the link between the possibility of chaos and the instability of the cycles we identify. At this stage, we leave these questions for future research.

7 Appendix A: Proofs

7.1 Proof of Proposition 2.1

In the first stage, the individual chooses the quantities $x_t^y(j)$ for given prices $p_t(j)$, $j \in [0, A_t]$, and the quantities $x_{t+1}^o(j)$ for given (perfect foresight) prices $p_{t+1}(j)$, $j \in [0, A_{t+1}]$, so as to minimize the costs of attaining aggregate consumption levels c_t^y and c_{t+1}^o, i. e., the two cost-minimization problems solved by cohort t are

$$\min_{\{x_t^y(j)\}_{j=0}^{A_t}} \int_0^{A_t} p_t(j) x_t^y(j) dj \qquad \text{s. t.} \qquad A_t^{\sigma - \frac{1}{\alpha}} \left[\int_0^{A_t} (x_t^y(j))^\alpha dj \right]^{\frac{1}{\alpha}} = c_t^y,$$

$$\min_{\{x_{t+1}^o(j)\}_{j=0}^{A_{t+1}}} \int_0^{A_{t+1}} p_{t+1}(j) x_{t+1}^o(j) dj \qquad \text{s. t.} \qquad A_{t+1}^{\sigma - \frac{1}{\alpha}} \left[\int_0^{A_{t+1}} (x_{t+1}^o(j))^\alpha dj \right]^{\frac{1}{\alpha}} = c_{t+1}^o.$$

Consolidating the two periodic budget constraints, the problem stated in (2.6) gives rise to the following Lagrangian

$$\mathscr{L} = \ln c_t^y + \ln\left(1 - \phi h_t (c_t^y)^{\frac{\nu}{1-\nu}}\right) + \beta \ln c_{t+1}^o + \lambda_t \left[w_t h_t - P_t c_t^y - \frac{P_{t+1} c_{t+1}^o}{R_{t+1}} \right].$$

Then, standard arguments following the algorithm developed in the proof of Proposition 1 in Irmen (2018) complete the proof. ■

7.2 Proof of Proposition 3.1

First, we derive algebraically the difference equation (3.8) in Section 7.2.1. Second, we prove the existence of a unique intertemporal general equilibrium in Section 7.2.2.

7.2.1 The Equilibrium Difference Equation

In Regime 0, inequality (2.16) is strict, and, accordingly, $H_{A,t} = \Delta A_t = 0$. Then, with s_t of Proposition 2.1 and (3.5), the asset market clearing condition (3.3) delivers

$$v_t = \frac{a \alpha L_t A_t^{(\sigma-1)(1-\nu)-1}}{z_c}. \tag{7.1}$$

Using the latter and (3.5) in inequality (2.16), gives

$$z_t \equiv \frac{L_t}{A_t^\eta} < z_c.$$

If this inequality holds, then the economy is in Regime 0, i. e., A_t is constant and L_t grows exogenously at rate g_L. Therefore,

$$z_{t+1} = (1 + g_L) z_t \quad \text{if} \quad z_t < z_c. \tag{7.2}$$

In Regime 1, (2.16) holds as equality, and, accordingly, $H_{A,t} \geq 0$ and $\Delta A_t \geq 0$. Then, with s_t of

Proposition 2.1, (3.5), and (3.9) the asset market equilibrium condition (3.3) becomes

$$\frac{A_t^{(1-\nu)(\sigma-1)}L_t}{z_c} = A_t^{\sigma-1-\psi}A_{t+1}.$$

Solving for A_{t+1}/A_t delivers

$$\frac{z_t}{z_c} = \frac{A_{t+1}}{A_t},$$

hence $z_t \geq z_c$. From the definition of z_t given in (3.6), we have

$$\frac{z_{t+1}}{z_t} = \frac{L_{t+1}A_{t+1}^{-\eta}}{L_t A_t^{-\eta}} \quad \text{or} \quad \frac{A_{t+1}}{A_t} = (1+g_L)^{\frac{1}{\eta}}\left(\frac{z_{t+1}}{z_t}\right)^{-\frac{1}{\eta}}. \tag{7.3}$$

Combining the latter two equations and solving for z_{t+1} delivers

$$z_{t+1} = (1+g_L)z_c\left(\frac{z_t}{z_c}\right)^{\psi-\nu(\sigma-1)} \quad \text{if} \quad z_t \geq z_c. \tag{7.4}$$

Hence, for $z_t < z_c$ the evolution of z_t is governed by (7.2), for $z_t \geq z_c$ the evolution of z_t is given by (7.4). It is straightforward to show from (7.2) that $\lim_{z_t \uparrow z_c} z_{t+1} = z_c$ and from (7.4) that $\lim_{z_t \downarrow z_c} z_{t+1} = z_c$. Hence, the piecewise defined difference equation stated in the proposition is continuous.

7.2.2 Existence and Uniqueness of the Intertemporal General Equilibrium

The proof has two steps. Step 1 shows that all elements of the equilibrium price system, $\{w_t, R_t, p_t, \pi_t, v_t\}_{t=1}^{\infty}$, and of the equilibrium allocation, $\{c_t^y, l_t, c_t^o, s_t, x_t^y, x_t^o, x_t, H_{x,t}, H_{A,t}, A_t\}_{t=1}^{\infty}$, can be expressed as a function of z_t. Step 2 proves that in equilibrium the shares are valued according to fundamentals.

Step 1 Given L_t, A_t can be pinned downed by z_t using (3.6), i. e.,

$$A_t = \left(\frac{L_t}{z_t}\right)^{\frac{1}{\eta}}. \tag{7.5}$$

The price of consumption goods and the wage rate are solely determined by A_t according to (3.4) and (3.5) respectively. Then using (7.5), we have

$$p_t = \left(\frac{L_t}{z_t}\right)^{\frac{\sigma-1}{\eta}} \quad \text{and} \quad w_t = \alpha\left(\frac{L_t}{z_t}\right)^{\frac{\sigma-1}{\eta}}. \tag{7.6}$$

The optimal plan of cohort t is given in Proposition 2.1, i. e., h_t, c_t^y, and s_t hinge on w_t. Then, using (7.6), we have

$$
\begin{aligned}
h_t &= \left(\frac{w_c}{\alpha}\right)^{\nu}\left(\frac{L_t}{z_t}\right)^{\frac{-\nu(\sigma-1)}{\eta}}, \quad s_t = \frac{\alpha\beta}{(1+\beta)(1-\nu)}\left(\frac{w_c}{\alpha}\right)^{\nu}\left(\frac{L_t}{z_t}\right)^{\frac{(1-\nu)(\sigma-1)}{\eta}}, \\
c_t^y &= \frac{1-\nu(1+\beta)}{(1+\beta)(1-\nu)} w_c^{\nu}\alpha^{1-\nu}\left(\frac{L_t}{z_t}\right)^{\frac{(1-\nu)(\sigma-1)}{\eta}}.
\end{aligned}
\tag{7.7}
$$

Using (7.7) the aggregate supply of hours worked becomes

$$
H_t = L_t h_t = \frac{a(1+\beta)(1-\nu)}{\beta}\frac{z_t}{z_c}\left(\frac{L_t}{z_t}\right)^{\frac{1-\psi}{\eta}}. \tag{7.8}
$$

Using (7.3) in (2.14) gives aggregate hours worked in research as

$$
H_{A,t} = \begin{cases} 0 & \text{if} \quad z_t \le z_c, \\ a\left(\frac{z_t}{z_c}-1\right)\left(\frac{L_t}{z_t}\right)^{\frac{1-\psi}{\eta}} & \text{if} \quad z_t \ge z_c. \end{cases} \tag{7.9}
$$

Accordingly, aggregate hours worked in the consumption-good sector obtain with (7.9) and (7.8) as

$$
H_{x,t} = \begin{cases} \frac{a(1+\beta)(1-\nu)}{\beta}\frac{z_t}{z_c}\left(\frac{L_t}{z_t}\right)^{\frac{1-\psi}{\eta}} & \text{if} \quad z_t \le z_c, \\ a\left[\frac{1-\nu(1+\beta)}{\beta}\frac{z_t}{z_c}+1\right]\left(\frac{L_t}{z_t}\right)^{\frac{1-\psi}{\eta}} & \text{if} \quad z_t \ge z_c. \end{cases} \tag{7.10}
$$

The supply of each consumption goods is given by $H_{x,t}/A_t$. With (7.5) and (7.10) this gives

$$
x_t = \begin{cases} \frac{a(1+\beta)(1-\nu)}{\beta}\frac{z_t}{z_c}\left(\frac{L_t}{z_t}\right)^{\frac{-\psi}{\eta}} & \text{if} \quad z_t \le z_c, \\ a\left[\frac{1-\nu(1+\beta)}{\beta}\frac{z_t}{z_c}+1\right]\left(\frac{L_t}{z_t}\right)^{\frac{-\psi}{\eta}} & \text{if} \quad z_t \ge z_c. \end{cases} \tag{7.11}
$$

Substituting (7.6) and (7.11) into (2.12) delivers the profit of each firm in the production sector as

$$
\pi_t = \begin{cases} (1-\alpha)\frac{a(1+\beta)(1-\nu)}{\beta}\frac{z_t}{z_c}\left(\frac{L_t}{z_t}\right)^{\frac{\sigma-1-\psi}{\eta}} & \text{if} \quad z_t \le z_c, \\ (1-\alpha)a\left[\frac{1-\nu(1+\beta)}{\beta}\frac{z_t}{z_c}+1\right]\left(\frac{L_t}{z_t}\right)^{\frac{\sigma-1-\psi}{\eta}} & \text{if} \quad z_t \ge z_c. \end{cases} \tag{7.12}
$$

The value of the shares of these firms in Regime 0 and Regime 1 result from the substitution of (7.5) into (7.1) and (3.9), respectively, as

$$
v_t = \begin{cases} a\alpha\frac{z_t}{z_c}\left(\frac{L_t}{z_t}\right)^{\frac{\sigma-1-\psi}{\eta}} & \text{if} \quad z_t \le z_c, \\ a\alpha\left(\frac{L_t}{z_t}\right)^{\frac{\sigma-1-\psi}{\eta}} & \text{if} \quad z_t \ge z_c. \end{cases} \tag{7.13}
$$

Consumption of the old can be obtained using (7.6), (7.7) and (7.11) in the market clearing condition for the consumption goods. This gives

$$c_t^o = \begin{cases} \frac{(1+\beta)(1-\nu)-\alpha(1-\nu(1+\beta))}{\beta} \frac{z_t}{z_c} a \left(\frac{L_t}{z_t}\right)^{\frac{\sigma-\psi}{\eta}} & \text{if} \quad z_t \leq z_c, \\ \left[\frac{(1+\beta)(1-\nu)-\alpha(1-\nu(1+\beta))}{\beta} \frac{z_t}{z_c} - \left(\frac{z_t}{z_c} - 1\right)\right] a \left(\frac{L_t}{z_t}\right)^{\frac{\sigma-\psi}{\eta}} & \text{if} \quad z_t \geq z_c. \end{cases} \tag{7.14}$$

One readily verifies that (2.3) in conjunction with (7.5) and, respectively, c_t^y of (7.7) and c_t^o of (7.14) delivers x_t^y and x_t^o as functions of z_t.

It remains to be shown that R_{t+1} can be expressed as a function of z_t. The following lemma accomplishes this.

Lemma 7.1 *(Perfect Foresight Interest Factor Along the Transition)*

Denote by R_{t+1} the perfect foresight interest factor at $t+1$. Then, the following holds.

1. *If $z_t < z_c/(1+g_L)$ then the economy is in Regime 0, stays there, and*

$$R_{t+1} = \frac{1+g_L}{\alpha\beta}\left((1-\alpha)(1-\nu(1+\beta))+\beta\right).$$

2. *If $z_c/(1+g_L) \leq z_t \leq z_c$ then the economy is in Regime 0, transits to Regime 1, and*

$$R_{t+1} = \frac{1+g_L}{\alpha\beta}\left((1-\alpha)(1-\nu(1+\beta)) + \frac{\beta}{1+g_L}\frac{z_c}{z_t}\right).$$

3. *If $z_c \leq z_t \leq z_c(1+g_L)^{\frac{-1}{\psi-\nu(\sigma-1)}}$ then the economy is in Regime 1, stays there, and*

$$R_{t+1} = \frac{1+g_L}{\alpha\beta}\left((1-\alpha)(1-\nu(1+\beta))\left(\frac{z_t}{z_c}\right)^{(1-\nu)(\sigma-1)} + \frac{\beta}{1+g_L}\left(\frac{z_t}{z_c}\right)^{\sigma-1-\psi}\right).$$

4. *If $z_c(1+g_L)^{\frac{-1}{\psi-\nu(\sigma-1)}} \leq z_t$ then the economy is in Regime 1, transits to Regime 0, and*

$$R_{t+1} = \frac{1+g_L}{\alpha\beta}\left((1-\alpha)(1-\nu(1+\beta))+\beta\right)\left(\frac{z_t}{z_c}\right)^{(1-\nu)(\sigma-1)}.$$

Proof of Lemma 7.1

The no-arbitrage condition (3.2) requires expressions for v_{t+1}/v_t and π_{t+1}/v_t. For all four cases, these are obtained from (7.12) and (7.13). Finally, using (3.8) delivers the expressions for R_{t+1} stated in the lemma. ■

Step 2 We use Lemma 7.1 to prove that in equilibrium condition (3.1) holds so that shares are indeed valued according to fundamentals. Lemma 7.2 accomplishes this.

Lemma 7.2 *(Equilibrium Share Valuation According to Fundamentals)*

The intertemporal general equilibrium satisfies condition (3.1).

Proof of Lemma 7.2

The no arbitrage condition (3.2) implies (3.1) if

$$\lim_{j\to\infty} \frac{v_{t+j}}{\Pi_{i=1}^{j} R_{t+j}} = 0. \tag{7.15}$$

Since

$$v_{t+j} = v_t \left(\frac{v_{t+1}}{v_t}\right)\left(\frac{v_{t+2}}{v_{t+1}}\right)\cdots\left(\frac{v_{t+j}}{v_{t+j-1}}\right) = v_t \Pi_{i=1}^{j} \frac{v_{t+i}}{v_{t+i-1}},$$

condition (7.15) can be written as

$$v_t \lim_{j\to\infty} \Pi_{i=1}^{j} \left(\frac{\frac{v_{t+i}}{v_{t+i-1}}}{R_{t+j}}\right) = 0. \tag{7.16}$$

Equations (7.12) and (7.13) imply that $v_{t+1}/v_t > 0$ and $\pi_{t+1}/v_t > 0$ for all t. Using this information in the no-arbitrage condition (3.2) reveals that $R_{t+1} > v_{t+1}/v_t$ for all t. As a consequence, the left-hand side of (7.16) has the product of infinitely many factors each being strictly smaller than unity. Hence, the limit of these factors vanishes and equation (7.15) holds. ■

7.3 Proof of Proposition 4.1

According to Proposition 1.9 of Galor (2007), z of the dynamical system (3.8) is locally stable iff

$$\left| \left.\frac{dz_{t+1}}{dz_t}\right|_{z_t=z} \right| = \left| \psi - \nu(\sigma-1) \right| < 1.$$

Then, the qualitative findings concerning the local stability are immediate. To prove the properties of the global dynamics we first state and prove two lemmata.

Lemma 7.3

Suppose $z_t \geq z_c$. As long as the economy stays in Regime 1 for s periods, the dynamics of z_t is given by

$$\frac{z_{t+s}}{z} = \left(\frac{z_t}{z}\right)^{[\psi-\nu(\sigma-1)]^s}. \tag{7.17}$$

Moreover, if $\mid \psi - \nu(\sigma-1) \mid < 1$, then

$$\lim_{s\to\infty} \frac{z_{t+s}}{z} = 1. \tag{7.18}$$

Proof of Lemma 7.3

From Proposition 3.10, the population growth factor satisfies $1 + g_L = (z/z_c)^{\eta}$. If $z_t > z_c$, then with (3.8) we have $z_{t+1}/z = (z_t/z)^{\psi-\nu(\sigma-1)}$. If $z_{t+s} \geq z_c$ for all $s = 0, 1, 2, \ldots$, then by successive iteration we obtain (7.17). If $\mid \psi - \nu(\sigma-1) \mid < 1$, then $\lim_{s\to\infty}[\psi - \nu(\sigma-1)]^s = 0$, i. e., (7.18) holds. ■

Lemma 7.4

Suppose $\psi - \nu(\sigma - 1) < 0$. Then, for any $z_1 \in \mathbb{R}_{++}$, there exist $\tau \geq 1$, such that $z_\tau \in [z_c, \Phi(z_c)]$.

Proof of Lemma 7.4

The piecewise defined difference equation (3.8) is strictly increasing on the subdomain $z_t \leq z_c$ and strictly decreasing on the subdomain $z_t \geq z_c$. Therefore, its maximum is $\Phi(z_c) = (1+g_L)z_c$. If $z_1 \leq z_c$, then $z_\tau \in [z_c, \Phi(z_c)]$ where $\tau = 1 + \lceil \ln(z_c/z_1)/\ln(1+g_L) \rceil$. If $z_1 > \Phi(z_c)$, then $z_2 = \Phi(z_c)(z_1/z_c)^{\psi-\nu(\sigma-1)} < \Phi(z_c)$. Then, either $z_2 < z_c$ or $z_2 \in [z_c, \Phi(z_c)]$. In the first case, $z_\tau \in [z_c, \Phi(z_c)]$ where $\tau = 2 + \lceil \ln(z_c/z_2)/\ln(1+g_L) \rceil$. ■

Proof of Case 1: Suppose $-\nu(\sigma-1) \in (-\psi, 0)$. Then, the difference equation (3.8) is increasing. If $z_1 \leq z_c$, then the economy transits to Regime 1 in finite time and stays there. If $z_c < z_1$, then the economy stays in Regime 1. With Lemma 7.3, the economy converges to z.

Proof of Case 2: Suppose $-\nu(\sigma-1) \in ((-(1+\psi), -\psi)$. Then, $z_c < \Phi^2(z_c) < z$ where $\Phi^2(z_c) = (1+g_L)^{1+\psi-\nu(\sigma-1)}z_c$. With Lemma 7.4, we have that $z_{\tau+1} \in [\Phi^2(z_c), \Phi(z_c)] \subset [z_c, \Phi(z_c)]$. Moreover, $z_{\tau+1} \in [\Phi^2(z_c), \Phi(z_c)]$ implies $z_{\tau+1+s} \in [\Phi^2(z_c), \Phi(z_c)]$ for all $s \geq 1$. Hence, for all $t \geq \tau$, $z_t \in [z_c, \Phi(z_c)]$. Finally, Lemma 7.3 implies $\lim_{t\to\infty} z_t = z$.

Proof of Case 3: First, we prove that $-\nu(\sigma-1) < -(1+\psi)$ implies for any initial value $z_1 \in \mathbb{R}_{++}$ that the economy enters the indicated absorbing interval in finite time. The argument is a follows.

Some straightforward algebra reveals that $\Phi^2(z_c) < z_c < \Phi(z_c)$. Since $\Phi'(z_t) > 0$ for all $z_t \in [\Phi^2(z_c), z_c]$ and $\Phi'(z_t) < 0$ for all $z_t \in [z_c, \Phi(z_c)]$, $\Phi(z_c) \geq \Phi(z_t)$ and $\Phi(z_t) \geq \Phi^2(z_c)$ for all $z_t \in [\Phi^2(z_c), \Phi(z_c)]$. Therefore, $[\Phi^2(z_c), \Phi(z_c)]$ is an absorbing interval in the sense that $z_t \in [\Phi^2(z_c), \Phi(z_c)]$ implies $z_{t+\tau} \in [\Phi^2(z_c), \Phi(z_c)]$ for all $\tau > 0$.

Second, we prove that inside the absorbing interval the evolution fluctuates between Regime 0 and 1. The argument is as follows.

If $z_t \in (\Phi^{-1}(z_c), \Phi(z_c)]$, then $z_{t+1} < z_c$. Moreover, since $\psi - \nu(\sigma-1) < (-1)$, if $z_t \in [z_c, \Phi^{-1}(z_c)]$ and $z_t \neq z$, then $z_{t+\tau} \in (\Phi^{-1}(z_c), \Phi(z_c)]$ where τ is finite. Therefore, the economy will move from Regime 1 to Regime 0 after a finite number of periods. Together with Lemma 7.4, it follows that inside the absorbing interval $[\Phi^2(z_c), \Phi(z_c)]$, the evolution of z_t fluctuates between Regime 0 and 1. ■

7.4 Proof of Corollary 4.1

Given in the main text. ■

7.5 Proof of Proposition 5.1

To assess the sign on ΔV_t we have to study the evolution of w_t and R_t. To accomplish this, we use equation (7.6) to express the equilibrium wage, w_t, as a function of (z_t, L_t) and Lemma 7.1 to express R_{t+1} as a function of z_t. One readily verifies that Cases 1 - 4 cover all starting values for z_{t-1}.

Cases 1 and 2 are explained in the main text. Case 3 follows from a comparison of the expressions of R_t and R_{t+1} stated in Case 2 and 3 of Lemma 7.1. The proof of Case 4 requires the distinction of six subcases. To simplify the notation define

$$\gamma_1 \equiv \frac{(1-\alpha)(1-\nu(1+\beta))(1+g_L)}{\alpha\beta} \quad \text{and} \quad \gamma_2 \equiv \psi - \nu(\sigma-1).$$

4.1 If $z_c(1+g_L)^{\frac{\psi-\nu(\sigma-1)}{1-\psi+\nu(\sigma-1)}} \leq z_{t-1} < z_c(1+g_L)^{\frac{\nu(\sigma-1)-\psi-1}{\psi-\nu(\sigma-1)}}$, then $z_{t-1} < z_c < z_{t+1} \leq z_t$ and $w_{t-1} = w_t$. Using (3.8) and Case 2 and Case 3 of Lemma 7.1, we have

$$\beta \ln\left(\frac{R_{t+1}}{R_t}\right) = \beta \ln \left[\frac{\gamma_1 \left(\frac{(1+g_L)z_{t-1}}{z_c}\right)^{(1-\nu)(\sigma-1)} + \frac{1}{\alpha}\left(\frac{(1+g_L)z_{t-1}}{z_c}\right)^{\sigma-1-\psi}}{\gamma_1 + \frac{1}{\alpha}\frac{z_c}{z_{t-1}}} \right].$$

Since $z_c(1+g_L)^{\frac{\psi-\nu(\sigma-1)}{1-\psi+\nu(\sigma-1)}} \leq z_{t-1}$, the second term in the numerator is greater than the denominator. Hence, $\ln(R_{t+1}/R_t) > 0$ and $\Delta V_t > 0$.

4.2 If $z_c(1+g_L)^{\frac{\nu(\sigma-1)-\psi-1}{\psi-\nu(\sigma-1)}} \leq z_{t-1} < z_c$, then $z_{t-1} < z_c < z_t$, $z_{t+1} < z_c$, and $w_{t-1} = w_t$. Moreover, Lemma 7.1 implies $R_t < R_{t+1}$, hence, $c_t^o < c_{t+1}^o$ and $\Delta V_t > 0$.

4.3 If $z_c < z_{t-1} \leq z_c(1+g_L)^{\frac{\nu(\sigma-1)-\psi-1}{(\psi-\nu(\sigma-1))^2}}$ then $z_{t+1} \leq z_c < z_{t-1} < z_t$. Here, $A_{t-1} < A_t < A_{t+1}$ and $w_{t-1} < w_t$. Moreover, Lemma 7.1 implies $R_t < R_{t+1}$, hence, $c_t^o < c_{t+1}^o$ and $\Delta V_t > 0$.

4.4 If $z_c(1+g_L)^{\frac{\nu(\sigma-1)-\psi-1}{(\psi-\nu(\sigma-1))^2}} < z_{t-1} \leq z_c(1+g_L)^{\frac{-1}{\psi-\nu(\sigma-1)}}$ then $\{z_{t-1}, z_t, z_{t+1}\} > z_c$ and the following sequences may obtain:

   a) If $z_c(1+g_L)^{\frac{\nu(\sigma-1)-\psi-1}{(\psi-\nu(\sigma-1))^2}} < z_{t-1} < z$ then $z_c < z_{t+1} < z_{t-1} < z_t$. It follows that $A_{t-1} < A_t < A_{t+1}$ and $w_{t-1} < w_t$. Moreover, Lemma 7.1 delivers $R_t < R_{t+1}$ as $z_{t-1} < z_t$, thus, $\Delta V_t > 0$.

   b) If $z_{t-1} = z$ then both cohorts live in steady state where $R_t = R_{t+1}$ and the real wage grows at a constant rate. Therefore, $\Delta V_t > 0$.

   c) If $z < z_{t-1} \leq z_c(1+g_L)^{\frac{-1}{\psi-\nu(\sigma-1)}}$ then $z_c < z_t < z < z_{t-1} < z_{t+1}$, $A_{t-1} < A_t$, and $w_{t-1} < w_t$. However, as $z_t < z_{t-1}$ Lemma 7.1 gives $R_t > R_{t+1}$. Nevertheless, $\Delta V_t > 0$ follows. To see this, observe that (7.6) and Proposition 3.1 allow us to write $w_t/w_{t-1} = (z_{t-1}/z_c)^{\sigma-1}$ so that

$$(1+\beta)(1-\nu)\ln\left(\frac{w_t}{w_{t-1}}\right) = (1+\beta)(1-\nu)(\sigma-1)\ln\left(\frac{z_{t-1}}{z_c}\right). \tag{7.19}$$

   Next, express R_{t+1}/R_t according to Case 3 of Lemma 7.1 and use the latter, (3.8), and

(7.19) in (5.2) to obtain ΔV_t as a function of z_{t-1}/z_c, i. e.,

$$\Delta V_t = \beta \ln \left(\frac{z_{t-1}}{z_c}\right)^{\frac{(\sigma-1)(1+\beta)(1-\nu)}{\beta}} + \beta \ln \left[\frac{\gamma_1 \left((1+g_L)\left(\frac{z_{t-1}}{z_c}\right)^{\gamma_2}\right)^{(1-\nu)(\sigma-1)} + \frac{1}{\alpha}\left((1+g_L)\left(\frac{z_{t-1}}{z_c}\right)^{\gamma_2}\right)^{\sigma-1-\psi}}{\gamma_1 \left(\frac{z_{t-1}}{z_c}\right)^{(1-\nu)(\sigma-1)} + \frac{1}{\alpha}\left(\frac{z_{t-1}}{z_c}\right)^{\sigma-1-\psi}}\right].$$

Then, using the boundary condition, $z_{t-1} \leq z_c(1+g_L)^{\frac{-1}{\psi-\nu(\sigma-1)}}$ in the latter, one finds that

$$\Delta V_t > \beta \ln \left[\frac{\gamma_1 \left(\frac{z_{t-1}}{z_c}\right)^{\frac{(\sigma-1)(1+\beta)(1-\nu)}{\beta}} + \frac{1}{\alpha}\left(\frac{z_{t-1}}{z_c}\right)^{\frac{(\sigma-1)(1+\beta)(1-\nu)}{\beta}}}{\gamma_1 \left(\frac{z_{t-1}}{z_c}\right)^{(1-\nu)(\sigma-1)} + \frac{1}{\alpha}\left(\frac{z_{t-1}}{z_c}\right)^{\sigma-1-\psi}}\right] > 0,$$

where the last inequality follows since $1 > \nu(1+\beta)$ is implied by (2.5) (see Irmen (2018)).

4.5 If $z_c(1+g_L)^{\frac{-1}{\psi-\nu(\sigma-1)}} < z_{t-1} < z_c(1+g_L)^{\frac{-2}{\psi-\nu(\sigma-1)}}$ then $z_{t-1} > z_c > z_t$ and $z_{t+1} > z_c$. Then, $A_{t-1} < A_t$ and $w_{t-1} < w_t$. Moreover, Lemma 7.1 delivers $R_t > R_{t+1}$. Nevertheless, $\Delta V_t > 0$. To see this, observe that $(1-\beta)(1-\nu)\ln(w_t/w_{t-1})$ is the same as in (7.19). Express R_{t+1}/R_t according to Case 2 and Case 4 of Lemma 7.1. Using the latter, (3.8), and (7.19) in (5.2) delivers ΔV_t as a function of z_{t-1}/z_c, i. e.,

$$\Delta V_t = \beta \ln \left[\frac{\gamma_1 \left(\frac{z_{t-1}}{z_c}\right)^{\frac{(1+\beta)(1-\nu)(\sigma-1)}{\beta}} + \frac{1}{\alpha}(1+g_L)^{-1}\left(\frac{z_{t-1}}{z_c}\right)^{\nu(\sigma-1)-\psi+\frac{(1+\beta)(1-\nu)(\sigma-1)}{\beta}}}{\gamma_1 \left(\frac{z_{t-1}}{z_t}\right)^{(1-\nu)(\sigma-1)} + \frac{1}{\alpha}(1+g_L)\left(\frac{z_{t-1}}{z_t}\right)^{(1-\nu)(\sigma-1)}}\right].$$

Since $1 > \nu(1+\beta)$, we have $\Delta V_t > 0$.

4.6 If $z_c(1+g_L)^{\frac{-2}{\psi-\nu(\sigma-1)}} \leq z_{t-1}$, then $z_{t-1} > z_c > z_{t+1} > z_t$. Then, $A_{t-1} < A_t$ and $w_{t-1} < w_t$. Again, Lemma 7.1 delivers $R_t > R_{t+1}$ but, nevertheless, $\Delta V_t > 0$. To see this, observe that $(1-\beta)(1-\nu)\ln(w_t/w_{t-1})$ is the same as (7.19). Express R_{t+1}/R_t according to Case 1 and Case 4 of Lemma 7.1. Using the latter and (7.19) in (5.2) delivers ΔV_t as a function of z_{t-1}/z_c, i. e.,

$$\Delta V_t = \beta \ln \left[\frac{\gamma_1 \left(\frac{z_{t-1}}{z_c}\right)^{\frac{(1+\beta)(1-\nu)(\sigma-1)}{\beta}} + \frac{1}{\alpha}(1+g_L)\left(\frac{z_{t-1}}{z_c}\right)^{\frac{(1+\beta)(1-\nu)(\sigma-1)}{\beta}}}{\gamma_1 \left(\frac{z_{t-1}}{z_t}\right)^{(1-\nu)(\sigma-1)} + \frac{1}{\alpha}(1+g_L)\left(\frac{z_{t-1}}{z_t}\right)^{(1-\nu)(\sigma-1)}}\right].$$

Since $\beta \in (0,1)$, we have $\Delta V_t > 0$. ∎

7.6 Proof of Proposition 5.2

First, we establish that the equilibrium share of aggregate hours worked in the research sector is given by

$$\gamma_{A,t} = \begin{cases} 0 & \text{if} \quad z_t \leq z_c, \\ \frac{\beta}{(1+\beta)(1-\nu)} \left(1 - \frac{z_c}{z_t}\right) & \text{if} \quad z_t \geq z_c. \end{cases} \tag{7.20}$$

To see this, express $H_{A,t}$ in terms of A_t and z_t. From the definition of z_t in (3.6), we have

$$\frac{z_{t+1}}{z_t} = (1 + g_L) \left(\frac{A_{t+1}}{A_t}\right)^{-\eta}.$$

For $z_t \geq z_c$, (3.8) can be written

$$\frac{z_{t+1}}{z_t} = (1 + g_L) \left(\frac{z_c}{z_t}\right)^{\eta}.$$

Equating the latter two equations delivers $A_{t+1}/A_t = z_t/z_c$, which upon substitution into (2.14) delivers

$$H_{A,t} = aA_t^{1-\psi} \left(\frac{z_t}{z_c} - 1\right).$$

Now, express H_t in terms of L_t and A_t. With h_t of Proposition 2.1 and (3.5), we obtain

$$H_t = \left(\frac{w_c}{\alpha}\right)^{\nu} L_t A_t^{-\nu(\sigma-1)}.$$

Then, $\gamma_{A,t}$ results as stated in (7.20) for $z_t \geq z_c$. For $z_t \leq z_c$, $H_{A,t} = 0$ and thus $\gamma_{A,t} = 0$.

With (7.20) one readily verifies that (5.3) can be expressed in terms of the state variable z_t as

$$GDP_t = \begin{cases} aA_t^{\sigma-\psi} \xi \left(\frac{z_t}{z_c}\right), & \text{if } z_t \leq z_c, \\ \\ aA_t^{\sigma-\psi} \left[(\xi - (1-\alpha)) \frac{z_t}{z_c} + (1-\alpha)\right], & \text{if } z_t \geq z_c, \end{cases} \tag{7.21}$$

where $\xi \equiv (1 + \beta)(1 - \nu)/\beta > 1$.

Case 1 is shown in the main text. As to Case 2, one finds that

$$GDP_{t+1} > GDP_t \quad \text{since} \quad (1 + g_{A,t})^{\sigma-\psi} \left[\frac{(\xi - (1-\alpha)) \frac{z_{t+1}}{z_c} + (1-\alpha)}{(\xi - (1-\alpha)) \frac{z_t}{z_c} + (1-\alpha)}\right] > 1.$$

The stated inequality clearly holds if $z_{t+1} \geq z_t$ since $\sigma > \psi$ and $\xi > 1$. If $z_{t+1} < z_t$ then it holds since $z_{t+1} = (1+g_L)z_c(z_t/z_c)^{\psi-\nu(\sigma-1)-1}$, $1+g_{A,t} = z_t/z_c$, and $(\sigma-1)(1-\nu) > 0$. For per-capita

GDP the result is not unequivocal as

$$\frac{gdp_{t+1}}{gdp_t} = \frac{\left(\xi-(1-\alpha)\right)\left(\frac{z_t}{z_c}\right)^{(\sigma-1)(1-\nu)}\frac{z_t}{z_c} + (1-\alpha)\left(\frac{z_t}{z_c}\right)^{\sigma-\psi}(1+g_L)^{-1}}{\left(\xi-(1-\alpha)\right)\frac{z_t}{z_c}+(1-\alpha)} \gtreqless 1.$$

Suppose $z_t > z = z_c(1+g_L)^{1/\eta}$. Then, the second term in the numerator satisfies

$$\left(\frac{z_t}{z_c}\right)^{\sigma-\psi} > (1+g_L)^{\frac{\sigma-\psi}{\eta}} \quad \text{or} \quad \left(\frac{z_t}{z_c}\right)^{\sigma-\psi}(1+g_L)^{-1} > (1+g_L)^{\frac{(\sigma-1)(1-\nu)}{\eta}} > 1.$$

Hence, since $(z_t/z_c)^{(\sigma-1)(1-\nu)} > 1$ and $(z_t/z_c)^{\sigma-\psi}(1+g_L)^{-1} > 1$, $gdp_{t+1} > gdp_t$.

For Case 3, (7.21) delivers

$$GDP_{t+1} > GDP_t \quad \text{since} \quad \frac{z_{t+1}}{z_t} - \frac{1-\alpha}{\xi}\left[\frac{z_{t+1}}{z_t} - \frac{z_c}{z_t}\right] > 1,$$

where the sign follows with $z_{t+1} = (1+g_L)z_t$ and $z_t \leq z_c$. As shown in the main text, $gdp_{t+1} < gdp_t$. The proof of Case 4 is given in the main text. ■

7.7 Proof of Proposition 5.3

If $\tau_{w,t} = \tau^P_{w,t}$ then (5.9) becomes

$$z_{t+1} = \begin{cases} (1+g_L)z_t & \text{if} \quad z_t \leq \bar{z}, \\ \\ (1+g_L)z_c\left(\frac{z_t}{z_c}\right)^{\psi-\nu(\sigma-1)}(1-\tau_{w,t})^{-(1-\nu)\eta} & \text{if} \quad z_t \geq \bar{z}. \end{cases} \tag{7.22}$$

If $z_t < \bar{z}$ then the proposed stabilization policy cannot induce $z_{t+1} = z$. Instead, the economy remains in Regime 0 until $z_{t+\tau} > \bar{z}$ where $\tau = 1, 2, ...$ As $z_{t+\tau} = (1+g_L)^\tau z_t$, the desired τ is the smallest integer such that $(1+g_L)^\tau z_t > \bar{z}$. Solving for τ delivers the expression stated in the proposition. Since $z_{t+\tau} > \bar{z}$, the reason why the taxes $\left(\tau^P_{w,t+\tau}, \tau^P_{R,t+\tau}\right)$ are part of a stabilization policy at $t+\tau$ is the same as for the taxes $\left(\tau^P_{w,t}, \tau^P_{R,t}\right)$ being part of a stabilization policy at t.

If $z_t \geq \bar{z}$. Then, $\tau^P_{w,t}$ solves $z_{t+1} = z$, i. e., from (7.22)

$$z = (1+g_L)z_c\left(\frac{z_t}{z_c}\right)^{\psi-\nu(\sigma-1)}(1-\tau^P_{w,t})^{-(1-\nu)\eta}.$$

The tax rate $\tau^P_{R,t}$ adjusts such that the second equation in (5.7) holds. Solving the latter using Proposition 2.1, (3.5), and (5.8) delivers

$$\tau^P_{R,t} = -\frac{(1+\beta)(1-\nu)(1+g_L)}{\beta}R_t^{-1}\left(\frac{A_t}{A_{t-1}}\right)^{(1-\nu)(\sigma-1)}\frac{\tau^P_{w,t}}{(1-\tau^P_{w,t})^\nu}. \tag{7.23}$$

From Claim 2 and Claim 3 of Lemma 7.1, the expression of R_t depends on whether $z_{t-1} \leq z_c$

or $z_{t-1} > z_c$. If $z_{t-1} \leq z_c$, then $A_{t-1} = A_t$,

$$R_t^{-1} = \frac{\alpha\beta}{1+g_L}\left((1-\alpha)(1-\nu(1+\beta)) + \frac{\beta}{1+g_L}\frac{z_c}{z_{t-1}}\right)^{-1},$$

and (7.23) delivers $\tau_{R,t}^P$ as stated in the proposition. If $z_{t-1} > z_c$, then $A_t/A_{t-1} = z_{t-1}/z_c$ and

$$R_t^{-1} = \frac{\alpha\beta}{1+g_L}\left((1-\alpha)(1-\nu(1+\beta))\left(\frac{z_{t-1}}{z_c}\right)^{(1-\nu)(\sigma-1)} + \frac{\beta}{1+g_L}\left(\frac{z_{t-1}}{z_c}\right)^{\sigma-1-\psi}\right)^{-1}.$$

Using $A_t/A_{t-1} = z_{t-1}/z_c$ and (3.8) in (7.23) delivers again the expression for $\tau_{R,t}^P$ stated in the proposition. ■

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8 Appendix B: Additional Results

This appendix collects additional results for an online appendix. Section 8.1 fully characterizes the steady-state paths of all endogenous variables. This includes an analysis of the comparative-static properties of these paths, the analysis of the steady-state allocation of hours worked as well as of the steady-state functional income distribution. Section 8.2 complements Section 5.1 of the main text and provides the analysis of the evolution of the intergenerational welfare for a fluctuating economy with a stable steady state. Finally, the focus of Section 8.3 is on period-n cycles. We first establish their existence. Then, we study the properties of such cycles and relate them to results obtained in Section 4 and 5 of the main text.

8.1 Structural Properties of the Steady State

The following proposition derives the steady-state growth rates of key variables.

Proposition 8.1 (Structural Properties of the Steady State)

Consider the steady state of Proposition 3.10. The steady-state growth factor of consumption-good varieties is

$$1 + g_A = (1 + g_L)^{\frac{1}{\eta}} > 1. \tag{8.1}$$

Moreover, it holds that

a) $1 + g_w = (1 + g_L)^{\frac{\sigma-1}{\eta}}$,

b) $1 + g_h = (1 + g_L)^{\frac{-\nu(\sigma-1)}{\eta}}$,

c) $1 + g_{c^y} = 1 + g_{c^o} = (1 + g_L)^{\frac{(1-\nu)(\sigma-1)}{\eta}}$,

d) $1 + g_H = (1 + g_L)^{\frac{1-\psi}{\eta}}$,

e) $1 + g_v = (1 + g_L)^{\frac{\sigma-1-\psi}{\eta}}$.

Proof of Proposition 8.1

The steady-state growth rate of A_t is derived in the main text. The growth rate of w_t follows from (3.5). The growth rates of h_t, c_t^y, and c_t^o follow with Proposition 2.1. Since $H_t = L_t h_t$, the growth rate of H_t equals to $(1 + g_L)(1 + g_h)$. Finally, g_v follows with (3.9). ■

According to Proposition 8.1 the number of available consumption-good varieties increases in steady state at a rate approximately equal to g_L/η. In addition to population growth and technology, this growth rate reflects preferences through the parameters ν and σ. The intuition is the following.

From the research technology (2.14) we have

$$g_A = \frac{\Delta A_t}{A_t} = \frac{H_{A,t}}{a} A_t^{\psi-1} \quad \Rightarrow \quad 1 + g_A = (1 + g_{H_A})^{\frac{1}{1-\psi}}. \tag{8.2}$$

Since increments in A_t improve the productivity of research at a decreasing rate, the amount of hours devoted to research, $H_{A,t}$, has to increase at a constant rate to support the steady-state growth rate g_A.

To detect the determinants of g_{H_A} consider the labor-market equilibrium which requires $g_{H_x} = g_{H_A} = g_H$. Hence, the fraction of the workforce allocated to research is constant over time. Accounting for the extensive and the intensive margin of the supply of hours worked, g_H satisfies

$$1 + g_H = (1 + g_L)(1 + g_h) = (1 + g_L)(1 + g_w)^{-\nu}, \tag{8.3}$$

where the last equality follows from Proposition 2.1. Combining (8.2) and (8.3) delivers

$$1 + g_A = \left[(1 + g_L)(1 + g_w)^{-\nu}\right]^{\frac{1}{1-\psi}}, \tag{8.4}$$

i. e., the research technology and the negative wage elasticity of the individual labor supply imply that faster wage growth induces a stronger decline in the amount of hours supplied by research workers and, hence, a smaller g_A.[19]

A second steady-state relationship between the growth rates g_A and g_w obtains from (3.5), i. e.,

$$1 + g_w = (1 + g_A)^{\sigma-1}. \tag{8.5}$$

It reflects the role of the taste for variety for the evolution of the real wage. As A_t increases so does the real price, p_t, charged by the monopolistically competitive firms. The constant mark-up rule (2.11) implies that p_t and w_t grow at the same rate. Solving (8.4) and (8.5) for the steady-state growth rates g_A and g_w gives the results stated in the proposition.

The steady-state growth rate of hours worked follows from Proposition 2.1 as $1 + g_h = (1 + g_w)^{-\nu}$. The corresponding growth rates of consumption when young and old coincide with the growth rate of the individual wage income, $w_t h_t$. The steady-state growth rate of aggregate hours worked satisfies $1 + g_H = (1 + g_L)(1 + g_h)$ and takes growth at the extensive and at the intensive margin into account. Finally, the growth rate of the steady-state value of ownership shares follows with (3.9). Accordingly, population growth speeds up this rate if $\sigma - 1 > \psi$.

8.1.1 The Comparative Statics of the Steady State

The steady state is illustrated in Figure 8.1 where the line doubted *Locus* I shows equation (8.4), and the line *Locus* II shows equation (8.5). The intersection of both loci determines the steady-state level of g_A and g_w as stated in Proposition 8.1.

The following proposition derives the determinants of g_A and g_w.

[19] Observe that (8.4) delivers the steady-state growth rate of the discrete-time version of Jones (1995) if $\nu = 0$. However, here, $\nu > 0$ which calls for an additional relationship to pin down g_A.

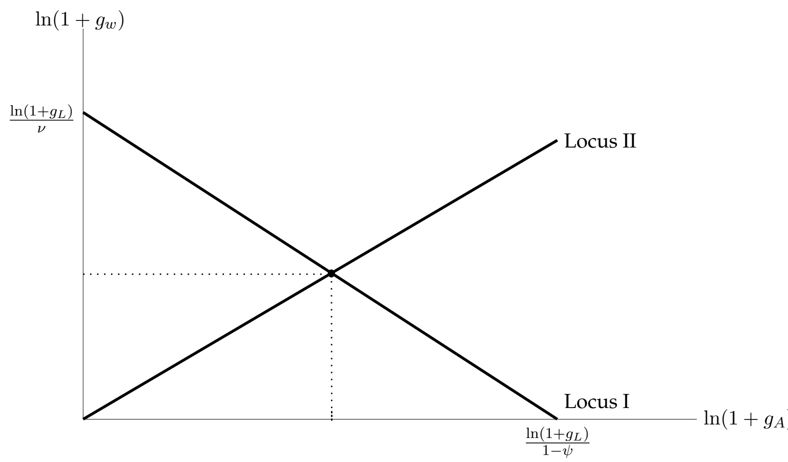

Figure 8.1: Steady-State Equilibrium.

Proposition 8.2 (Determinants of the Steady-State Growth Rates g_A and g_w)

The steady-state growth rate of the consumption-good varieties and the real wage satisfy

$$\frac{\partial g_A}{\partial g_L} > 0, \qquad \frac{\partial g_A}{\partial \psi} > 0, \qquad \frac{\partial g_A}{\partial \nu} < 0, \qquad \frac{\partial g_A}{\partial \sigma} < 0.$$
$$\frac{\partial g_w}{\partial g_L} > 0, \qquad \frac{\partial g_w}{\partial \psi} > 0, \qquad \frac{\partial g_w}{\partial \nu} < 0, \qquad \frac{\partial g_w}{\partial \sigma} > 0.$$

Proof of Proposition 8.2

The signs of all the partial derivatives can be inspected directly from Proposition 8.1. ■

To grasp the intuition behind these findings consider first the effect of a higher labor-force growth rate, $g_L' > g_L$. Intuitively, faster population growth means faster growth at the extensive margin of the labor supply, hence faster growth of the workforce in research. For all $g_w \geq 0$, this supports a higher g_A. Accordingly, in Figure 8.1 *Locus I* shifts outwards along *Locus II*, and both g_A and g_w increase.

Second, a stronger knowledge spillover, $\psi' > \psi$, pivots *Locus I* outwards and leads to a higher g_A and g_w. Intuitively, given g_w a higher ψ increases the labor productivity in the research sector, which supports a higher g_A.

Third, a lower wage elasticity, $-\nu' < -\nu$, pivots *Locus I* inwards and results in a lower value of both g_A and g_w. Given g_w, a higher ν speeds up the decline in the individual labor supply of research workers. This reduces the growth rate of consumption-good varieties that can be supported in steady state.

Finally, a stronger taste for variety, $\sigma' > \sigma$, pivots *Locus II* upwards since, given g_A, g_w will be higher. However, this speeds up the decline in hours worked in research, and, accordingly, g_A falls whereas g_w increases.

8.1.2 The Steady-State Allocation of Hours Worked

Consider the equilibrium allocation of hours worked as stated in (7.20). Clearly, in Regime 0 all labor is in the consumption-good sector. In Regime 1, the position of z_t relative to z_c is a summary statistic for the profit opportunities in the research sector. It determines how many new varieties will be invented, i. e., $z_t/z_c = A_{t+1}/A_t$.

The steady-state allocation of aggregate hours worked, γ_A, will depend on the position of z relative to z_c. In light of Proposition 3.10 one readily verifies that

$$\gamma_A = \frac{\beta}{(1+\beta)(1-\nu)}\left[1 - \frac{1}{1+g_A}\right] = \frac{\beta}{(1+\beta)(1-\nu)}\left[1 - \frac{1}{(1+g_L)^{\frac{1}{\eta}}}\right]. \tag{8.6}$$

As expected, faster steady-state growth of consumption-good varieties requires a larger share of aggregate hours allocated to the research sector. Therefore, all parameters with a positive effects on g_A will have a positive effects on γ_A, too. The following proposition makes this more precise.

Proposition 8.3 (Comparative Statics of the Steady-State Allocation of Labor)
It holds that

$$\frac{d\,\gamma_A}{d\,\beta} > 0, \quad \frac{d\,\gamma_A}{d\,g_L} > 0, \quad \frac{d\,\gamma_A}{d\,\alpha} = 0, \quad \frac{d\,\gamma_A}{d\,\psi} > 0, \quad \frac{d\,\gamma_A}{d\,\sigma} < 0, \quad \frac{d\,\gamma_A}{d\,\nu} \gtreqless 0.$$

Proof of Proposition 8.3

First, $d\gamma_A/d\beta = [1-(1+g_A)^{-1}][(1+\beta)(1-\nu)]^{-2} > 0$. Then observe that $d\gamma_A/dg_A > 0$. The parameters g_L, ψ, and σ affect γ_A indirectly via g_A. Hence, according to Proposition 8.2, we have $d\gamma_A/dg_L > 0$, $d\gamma_A/d\psi > 0$ and $d\gamma_A/d\sigma < 0$. Moreover, the parameter α does not appear in (8.6), so $d\gamma_A/d\alpha = 0$. Finally, the direct effect of ν on γ_A is positive but the indirect effect via g_A is negative, so the total effect is ambiguous. ■

The discount factor β has a positive level effect on individual saving. Therefore, a higher β "shrinks" Regime 0, i. e., it reduces z_c, and, accordingly, leads a higher γ_A. Furthermore, the parameters g_L, ψ, and σ affect γ_A only indirectly through g_A. The respective signs of the comparative statics follow from Proposition 8.2. Finally, the effect of a higher ν on γ_A are more intricate, as it has a positive level effect but a negative growth effect on individual saving (see Proposition 2.1). The former increases γ_A through a shrinking of Regime 0, the latter decreases the growth rate of the number of differentiated varieties, hence, γ_A.

8.1.3 The Steady-State Functional Income Distribution

To derive the equilibrium factor incomes recall that the economy has two factors of production, labor (in two uses), $H_t = H_{x,t} + H_{A,t}$, and technological knowledge, A_t. In equilibrium the wage rates in the production and the research sector are the same. Therefore, the wage income of labor is $w_t H_t$. The profits earned by the existing consumption-good firms are paid out as dividends to the current old who own their shares. Hence, the aggregate dividend income is $\pi_t A_t$. The production of the consumption-good varieties gives rise to a value added equal to the value of total output, i. e., $A_t p_t x_t$. Then, at any t factor incomes have the following properties.

Proposition 8.4 (Equilibrium Factor Incomes)

Equilibrium factor incomes satisfy

$$w_t H_{A,t} = \{0, v_t \Delta A_t\},$$

and

$$w_t H_{x,t} + \pi_t A_t = A_t p_t x_t,$$

where

$$\frac{w_t H_{x,t}}{A_t p_t x_t} = \alpha \quad \textit{and} \quad \frac{\pi_t A_t}{A_t p_t x_t} = 1 - \alpha.$$

Proof of Proposition 8.4

Using (2.14) and (2.15) gives $w_t H_{A,t} = \{0, v_t \Delta A_t\}$. Then, using (2.11) and (2.13) yields $w_t H_{x,t} = \alpha A_t p_t x_t$. Finally, (2.12) implies $\pi_t A_t = (1 - \alpha) A_t p_t x_t$. ∎

Next, we turn to the equilibrium factor shares. Let GDP_t denote the economy's equilibrium gross domestic product, i. e., its total value added, LS_t, its labor share, and CS_t its capital share at t. The total value added is the sum of the added values in the production and the research sector, i. e., $GDP_t = A_t p_t x_t + v_t \Delta A_t$. The labor share and the capital share relate, respectively, the economy's total wage bill and income from share holdings to its GDP_t, i. e.,

$$LS_t \equiv \frac{w_t H_t}{GDP_t} \quad \text{and} \quad CS_t \equiv \frac{\pi_t A_t}{GDP_t}.$$

Proposition 8.5 (GDP and Equilibrium Factor Shares)

In equilibrium GDP_t satisfies

$$GDP_t = A_t p_t x_t + v_t \Delta A_t = \begin{cases} \frac{1}{\alpha} w_t H_t & \text{if} \quad z_t \leq z_c, \\ \frac{1}{\alpha} w_t H_{x,t} + w_t H_{A,t} & \text{if} \quad z_t \geq z_c. \end{cases}$$

Moreover, the equilibrium factor shares satisfy

$$LS_t = \begin{cases} \alpha & \text{if} \quad z_t \leq z_c, \\ \frac{\alpha}{1-(1-\alpha)\gamma_{A,t}} & \text{if} \quad z_t \geq z_c. \end{cases}$$

and

$$CS_t = \begin{cases} 1-\alpha & \text{if} \quad z_t \leq z_c, \\ \frac{(1-\alpha)(1-\gamma_{A,t})}{1-(1-\alpha)\gamma_{A,t}} & \text{if} \quad z_t \geq z_c. \end{cases}$$

Proof of Proposition 8.5

In Regime 0, $\Delta A_t = 0$, so with (2.13) we have

$$GDP_t = A_t p_t x_t = p_t H_t = A_t^{\sigma-1} H_t. \tag{8.7}$$

In Regime 1, using (2.13), (2.14), (3.4) and (3.9) in GDP_t, we have

$$GDP_t = A_t p_t x_t + v_t \Delta A_t = A_t^{\sigma-1}(H_{x,t} + \alpha H_{A,t}). \tag{8.8}$$

Using (3.5) for the total wage bill yields

$$w_t H_t = \alpha A_t^{\sigma-1} H_t. \tag{8.9}$$

Then dividing (8.9) by (8.7) gives the labor share in Regime 0 as $LS_t = w_t H_t / GDP_t = \alpha$. Moreover, dividing (8.9) by (8.8) gives the labor share in Regime 1, i. e.,

$$LS_t = \frac{\alpha}{1-\gamma_{A,t} + \alpha\gamma_{A,t}} = \frac{\alpha}{1-(1-\alpha)\gamma_{A,t}}.$$

■

In Regime 0 there is no production in the research sector, $\Delta A = 0$, therefore GDP_t is the value added in the production sector, $A_t p_t x_t$. Due to monopolistic competition, the workers receive $w_t H_t$ which accounts for a fraction α of the value added while the shareholders of the firms receive the remaining fraction, $1-\alpha$. In Regime 1, the research sector also creates value added. Since the research sector is competitive with free entry, the researchers' income is equal to the value of their production, i. e., $w_t H_{A,t} = v_t \Delta A_t$. Accordingly, the labor share is higher when the share of aggregate hours worked in the research sector is higher, i. e., when γ_A is higher.

Since $\gamma_A \in [0,1]$, the labor share in Regime 1 is always greater than the one in Regime 0. The following proposition states the steady-state factor shares, LS and CS, and its comparative statics.

Proposition 8.6 (Steady-State Factor Shares and their Determinants)

Consider the steady state of Proposition 4.1. Then, the corresponding factor shares are

$$LS = \frac{\alpha}{1-(1-\alpha)\gamma_A} \quad \textit{and} \quad CS = \frac{(1-\alpha)(1-\gamma_A)}{1-(1-\alpha)\gamma_A},$$

where γ_A is given in (8.6). Moreover, it holds that

$$\frac{d\,LS}{d\,\beta} > 0, \quad \frac{d\,LS}{d\,g_L} > 0, \quad \frac{d\,LS}{d\,\alpha} > 0, \quad \frac{d\,LS}{d\,\psi} > 0, \quad \frac{d\,LS}{d\,\sigma} < 0, \quad \frac{d\,LS}{d\,\nu} \gtreqless 0.$$

Proof of Proposition 8.6

Since the steady-state equilibrium exists in Regime 1, we obtain LS and CS by substituting γ_A into the Proposition 8.5 for $z > z_c$. Then observe that LS is increasing in γ_A. Hence, except α, the effects of all other parameters on γ_A follow from Proposition 8.3. Finally, one readily finds that $dLS/d\alpha = (1-\gamma_A)[1-(1-\alpha)\gamma_A]^{-2} > 0$. ■

The rationale behind the positive relation between γ_A and LS is the same as that discussed in the context of Proposition 8.5. Due to the monopolistic power of the consumption-good firms, the steady-state labor share is higher if a larger portion of hours worked is allocated to the research sector.

With the exception of α, the effect of all parameters on LS operates through γ_A. Since the association between LS and γ_A is positive, the qualitative results obtained in Proposition 8.3 carry over. Recall that α is a measure of the substitutability between consumption-good varieties. Hence, the more substitutable the consumption-good varieties are, the higher is the steady-state labor share because the consumption-good firms enjoy less monopolistic power.

8.2 Intergeneration Welfare and Fluctuations for the Economy with a Stable Steady State

In Section 5.1 of main text, we provide a comprehensive analysis of the intergenerational welfare of consecutive cohorts under the assumption that $-\nu(\sigma-1) < -(1+\psi)$, i. e., the steady state is unstable. This section complements the analysis in the main text and studies the intergenerational welfare of consecutive cohorts under the assumption that the steady state is stable, i. e., $0 > -\nu(\sigma-1) \geq -(1+\psi)$. To structure the discussion, we first look at the parameter constellation $-\nu(\sigma-1) \in [-\psi, 0)$ in Section 8.2.1. Section 8.2.2 has the case where $-\nu(\sigma-1) \in [-(1+\psi), -\psi)$.

8.2.1 The Case $-\nu(\sigma-1) \in [-\psi, 0)$

From Proposition 4.1, the steady state is either globally stable with monotone convergence if $-\nu(\sigma-1) \in (-\psi, 0)$, or there is a finite τ such that $z_\tau = z$ for all $t > \tau$. The following proposition states and proves the implications for the welfare of two consecutive cohorts.

Proposition 8.7 (Intergenerational Welfare II)

Suppose $-\nu(\sigma-1) \in [-\psi, 0)$. Then the following holds:

1. *If $z_{t-1} \leq z_c(1+g_L)^{-2}$ then $\Delta V_t = 0$.*
2. *If $z_c(1+g_L)^{-2} < z_{t-1} \leq z_c(1+g_L)^{-1}$ then $\Delta V_t < 0$.*
3. *If $z_c(1+g_L)^{-1} < z_{t-1} < z_c$ then $\Delta V_t \gtreqless 0$.*
4. *If $z_c \leq z_{t-1}$ then $\Delta V_t > 0$.*

Proof of Proposition 8.7

1. If $z_{t-1} \leq z_c(1+g_L)^{-2}$, then $z_{t-1} < z_t < z_{t+1} \leq z_c$. Then, $A_{t-1} = A_t$, $w_{t-1} = w_t$, $h_{t-1} = h_t$, and $c^y_{t-1} = c^y_t$. Moreover, Lemma 7.1 implies $R_t = R_{t+1}$, hence, $c^o_t = c^o_{t+1}$ and $\Delta V_t = 0$.
2. If $z_c(1+g_L)^{-2} < z_{t-1} \leq z_c(1+g_L)^{-1}$, then $z_{t-1} < z_t \leq z_c < z_{t+1}$. Then, $A_{t-1} = A_t$, $w_{t-1} = w_t$, $h_{t-1} = h_t$, and $c^y_{t-1} = c^y_t$. Moreover, Lemma 7.1 implies $R_t > R_{t+1}$, hence, $c^o_t > c^o_{t+1}$ and $\Delta V_t < 0$.
3. If $z_c(1+g_L)^{-1} < z_{t-1} \leq z_c$, then $z_{t-1} \leq z_c < z_t \leq z_{t+1}$. Then, $A_{t-1} = A_t$, $w_{t-1} = w_t$, $h_{t-1} = h_t$, and $c^y_{t-1} = c^y_t$. Moreover, since $\sigma - 1 - \psi \gtreqless 0$, Lemma 7.1 implies $R_t \lesseqgtr R_{t+1} \Leftrightarrow c^o_t \lesseqgtr c^o_{t+1} \Leftrightarrow \Delta V_t \gtreqless 0$.
4. The following subcases arise:
   - 4.1 If $z_{t-1} = z_c$, then $z_c = z_{t-1} < z_t \leq z_{t+1}$. Then, $A_{t-1} = A_t$, $w_{t-1} = w_t$, $h_{t-1} = h_t$, and $c^y_{t-1} = c^y_t$. Moreover, Case 2 and Case 3 of Lemma 7.1 imply $R_t < R_{t+1}$, hence, $c^o_t < c^o_{t+1}$ and $\Delta V_t > 0$.
   - 4.2 If $z_c < z_{t-1} < z$, then $z_c < z_{t-1} < z_t \leq z_{t+1}$. Then, $A_{t-1} < A_t$, $w_{t-1} < w_t$, $h_{t-1} > h_t$, and $c^y_{t-1} < c^y_t$. Moreover, Case 3 of Lemma 7.1 with $z_{t-1} < z_t$ implies that $R_t < R_{t+1}$, hence, $c^o_t < c^o_{t+1}$ and $\Delta V_t > 0$.
   - 4.3 If $z_{t-1} = z$, then $z_{t-1} = z_t = z_{t+1} = z$. Then, $A_{t-1} < A_t$, $w_{t-1} < w_t$, $h_{t-1} > h_t$, and $c^y_{t-1} < c^y_t$. Moreover, Case 3 of Lemma 7.1 with $z_{t-1} = z_t$ implies that $R_t = R_{t+1}$, hence, $c^o_t < c^o_{t+1}$ and $\Delta V_t > 0$.
   - 4.4 If $z < z_{t-1}$, then $z_c < z_{t+1} \leq z_t < z_{t-1}$. Then, $A_{t-1} < A_t$, $w_{t-1} < w_t$, $h_{t-1} > h_t$, and $c^y_{t-1} < c^y_t$. Moreover, for $z_{t-1} > z_t$ Case 3 of Lemma 7.1 implies that $R_t > R_{t+1}$, hence, $c^o_t \lesseqgtr c^o_{t+1}$. Nevertheless, $\Delta V_t > 0$. To see this, observe that ΔV_t is given by (7.20). Since $\psi - \nu(\sigma-1) \geq 0$, we have $\Delta V_t > 0$. ■

8.2.2 The Case $-\nu(\sigma-1) \in [-(1+\psi), -\psi)$

From Proposition 4.1, the steady state is either globally stable with oscillating convergence if $-\nu(\sigma-1) \in (-(1+\psi), -\psi)$, or embarks on a period-2 cycle if $-\nu(\sigma-1) = -(1+\psi)$. The following proposition states and proves the implications for the welfare of two consecutive cohorts.

Proposition 8.8 *Suppose $-\nu(\sigma-1) \in [-(1+\psi), -\psi)$. Then the following holds:*

1. *If $z_{t-1} \leq z_c(1+g_L)^{-2}$ then $\Delta V_t = 0$.*
2. *If $z_c(1+g_L)^{-2} < z_{t-1} \leq z_c(1+g_L)^{-1}$ then $\Delta V_t < 0$.*
3. *If $z_c(1+g_L)^{-1} < z_{t-1} < z_c(1+g_L)^{\frac{\psi-\nu(\sigma-1)}{1-\psi+\nu(\sigma-1)}}$ then $\Delta V_t \gtreqless 0$.*
4. *If $z_c(1+g_L)^{\frac{\psi-\nu(\sigma-1)}{1-\psi+\nu(\sigma-1)}} \leq z_{t-1}$ then $\Delta V_t > 0$.*

Proof of Proposition 8.8

1. If $z_{t-1} \leq z_c(1+g_L)^{-2}$, then $z_{t-1} < z_t < z_{t+1} \leq z_c$. Then, $A_{t-1} = A_t$, $w_{t-1} = w_t$, $h_{t-1} = h_t$, and $c^y_{t-1} = c^y_t$. Moreover, Lemma 7.1 implies $R_t = R_{t+1}$, hence, $c^o_t = c^o_{t+1}$ and $\Delta V_t = 0$.
2. If $z_c(1+g_L)^{-2} < z_{t-1} \leq z_c(1+g_L)^{-1}$, then $z_{t-1} < z_t \leq z_c < z_{t+1}$. Then, $A_{t-1} = A_t$, $w_{t-1} = w_t$, $h_{t-1} = h_t$, and $c^y_{t-1} = c^y_t$. Moreover, Lemma 7.1 implies $R_t > R_{t+1}$, hence, $c^o_t > c^o_{t+1}$ and $\Delta V_t < 0$.
3. If $z_c(1+g_L)^{-1} < z_{t-1} < z_c(1+g_L)^{\frac{\psi-\nu(\sigma-1)}{1-\psi+\nu(\sigma-1)}}$, then $z_{t-1} \leq z_c < z_t < z_{t+1}$. Then, $A_{t-1} = A_t$, $w_{t-1} = w_t$, $h_{t-1} = h_t$, and $c^y_{t-1} = c^y_t$. Moreover, since $\sigma - 1 - \psi \gtreqless 0$ Lemma 7.1 implies $R_t \lesseqgtr R_{t+1} \Leftrightarrow c^o_t \lesseqgtr c^o_{t+1} \Leftrightarrow \Delta V_t \gtreqless 0$.
4. The following subcases arise:

   4.1 If $z_c(1+g_L)^{\frac{\psi-\nu(\sigma-1)}{1-\psi+\nu(\sigma-1)}} \leq z_{t-1} \leq z_c$, then $z_{t-1} < z_c \leq z_{t+1} \leq z_t$. Then, $A_{t-1} = A_t$, $w_{t-1} = w_t$, $h_{t-1} = h_t$, and $c^y_{t-1} = c^y_t$. Using Case 2 and Case 3 of Lemma 7.1, we have

   $$\beta \ln\left(\frac{R_{t+1}}{R_t}\right) = \beta \ln\left[\frac{\gamma_1\left(\frac{(1+g_L)z_{t-1}}{z_c}\right)^{(1-\nu)(\sigma-1)} + \frac{1}{\alpha}\left(\frac{(1+g_L)z_{t-1}}{z_c}\right)^{\sigma-1-\psi}}{\gamma_1 + \frac{1}{\alpha}\frac{z_c}{z_{t-1}}}\right].$$

   Since $z_c(1+g_L)^{\frac{\psi-\nu(\sigma-1)}{1-\psi+\nu(\sigma-1)}} \leq z_{t-1}$, the second term in the numerator is greater than that in the denominator. Hence, $\beta \ln(R_{t+1}/R_t) > 0$ and $\Delta V_t > 0$.

   4.2 If $z_c < z_{t-1} < z$, then $z_c < z_{t-1} \leq z_{t+1} < z_t$. Then, $A_{t-1} < A_t$, $w_{t-1} < w_t$, $h_{t-1} > h_t$, and $c^y_{t-1} < c^y_t$. Moreover, Case 3 of Lemma 7.1 with $z_{t-1} < z_t$ implies $R_t < R_{t+1}$, hence, $c^o_t < c^o_{t+1}$ and $\Delta V_t > 0$.

   4.3 If $z_{t-1} = z$, then $z_{t-1} = z_t = z_{t+1} = z$. Then, $A_{t-1} < A_t$, $w_{t-1} < w_t$, $h_{t-1} > h_t$, and $c^y_{t-1} < c^y_t$. Moreover, Case 3 of Lemma 7.1 with $z_{t-1} = z_t$ implies $R_t = R_{t+1}$, hence, $c^o_t < c^o_{t+1}$ and $\Delta V_t > 0$.

   4.4 If $z < z_{t-1} < z_c(1+g_L)^{\frac{1}{\nu(\sigma-1)-\psi}}$, then $z_c < z_t < z_{t+1} \leq z_{t-1}$. Then, $A_{t-1} < A_t$, $w_{t-1} < w_t$, $h_{t-1} > h_t$, and $c^y_{t-1} < c^y_t$. Moreover, Case 3 of Lemma 7.1 with $z_{t-1} > z_t$ implies $R_t > R_{t+1}$, hence, $c^o_t \gtreqless c^o_{t+1}$. Nevertheless, $\Delta V_t > 0$. To see this, observe that ΔV_t is given by (7.20). Since $z_c < z_t$ it follows with (3.8) that

   $$(1+g_L)^{\sigma-1-\psi} \quad > \quad \left(\frac{z_{t-1}}{z_c}\right)^{-\gamma_2(\sigma-1-\psi)}.$$

Using the latter in (7.20), one finds that

$$\Delta V_t > \beta \ln \left[\frac{\gamma_1 \left(\frac{z_{t-1}}{z_c}\right)^{\frac{(\sigma-1)(1+\beta)(1-\nu)}{\beta}} + \frac{1}{\alpha}\left(\frac{z_{t-1}}{z_c}\right)^{\frac{(\sigma-1)(1+\beta)(1-\nu)}{\beta}}}{\gamma_1 \left(\frac{z_{t-1}}{z_c}\right)^{(1-\nu)(\sigma-1)} + \frac{1}{\alpha}\left(\frac{z_{t-1}}{z_c}\right)^{\sigma-1-\psi}} \right] > 0,$$

where the last inequality follows since $1 > \nu(1+\beta)$ is implied by (2.5) (see Irmen (2018)).

4.5 If $z_c(1+g_L)^{\frac{1}{\nu(\sigma-1)-\psi}} < z_{t-1} < z_c(1+g_L)^{\frac{2}{\nu(\sigma-1)-\psi}}$ then $z_{t-1} > z_c > z_t$ and $z_{t+1} > z_c$. Then, since $A_{t-1} < A_t$, we have $w_{t-1} < w_t$, $h_{t-1} > h_t$, and $c^y_{t-1} < c^y_t$. Moreover, Lemma 7.1 delivers $R_t > R_{t+1}$. Hence, $c^o_t \gtreqless c^o_{t+1}$. Nevertheless, $\Delta V_t > 0$. To see this, observe that $(1-\beta)(1-\nu)\ln(w_t/w_{t-1})$ is the same as in (7.19). Using (3.8), Case 2 and Case 4 of Lemma 7.1 we write

$$\beta \ln\left(\frac{R_{t+1}}{R_t}\right) = \beta \ln \left[\frac{\gamma_1 + \frac{1}{\alpha}(1+g_L)^{-1}\left(\frac{z_{t-1}}{z_c}\right)^{\nu(\sigma-1)-\psi}}{\gamma_1\left(\frac{z_{t-1}}{z_t}\right)^{(1-\nu)(\sigma-1)} + \frac{1}{\alpha}(1+g_L)\left(\frac{z_{t-1}}{z_t}\right)^{(1-\nu)(\sigma-1)}} \right]. \tag{8.10}$$

The substitution of (7.19) and (8.10) into (5.2) delivers

$$\Delta V_t = \beta \ln \left[\frac{\gamma_1 \left(\frac{z_{t-1}}{z_c}\right)^{\frac{(1+\beta)(1-\nu)(\sigma-1)}{\beta}} + \frac{1}{\alpha}(1+g_L)^{-1}\left(\frac{z_{t-1}}{z_c}\right)^{\nu(\sigma-1)-\psi+\frac{(1+\beta)(1-\nu)(\sigma-1)}{\beta}}}{\gamma_1\left(\frac{z_{t-1}}{z_t}\right)^{(1-\nu)(\sigma-1)} + \frac{1}{\alpha}(1+g_L)\left(\frac{z_{t-1}}{z_t}\right)^{(1-\nu)(\sigma-1)}} \right].$$

Again, since $1 > \nu(1+\beta)$, we have $\Delta V_t > 0$.

4.6 If $z_c(1+g_L)^{\frac{2}{\nu(\sigma-1)-\psi}} \leq z_{t-1}$, then $z_{t-1} > z_c > z_{t+1} > z_t$. Then, we have $A_{t-1} < A_t$, $w_{t-1} < w_t$, $h_{t-1} > h_t$, and $c^y_{t-1} < c^y_t$. Again, Lemma 7.1 delivers $R_t > R_{t+1}$. Hence, $c^o_t \gtreqless c^o_{t+1}$. Nevertheless, $\Delta V_t > 0$. To see this, observe that $(1-\beta)(1-\nu)\ln(w_t/w_{t-1})$ is the same as in (7.19). Using (3.8), Case 1 and Case 4 of Lemma 7.1 we write

$$\beta \ln\left(\frac{R_{t+1}}{R_t}\right) = \beta \ln \left[\frac{\gamma_1 + \frac{1}{\alpha}(1+g_L)}{\gamma_1\left(\frac{z_{t-1}}{z_t}\right)^{(1-\nu)(\sigma-1)} + \frac{1}{\alpha}(1+g_L)\left(\frac{z_{t-1}}{z_t}\right)^{(1-\nu)(\sigma-1)}} \right]. \tag{8.11}$$

The substitution of (7.19) and (8.11) into (5.2) delivers

$$\Delta V_t = \beta \ln \left[\frac{\gamma_1 \left(\frac{z_{t-1}}{z_c}\right)^{\frac{(1+\beta)(1-\nu)(\sigma-1)}{\beta}} + \frac{1}{\alpha}(1+g_L)\left(\frac{z_{t-1}}{z_c}\right)^{\frac{(1+\beta)(1-\nu)(\sigma-1)}{\beta}}}{\gamma_1\left(\frac{z_{t-1}}{z_t}\right)^{(1-\nu)(\sigma-1)} + \frac{1}{\alpha}(1+g_L)\left(\frac{z_{t-1}}{z_t}\right)^{(1-\nu)(\sigma-1)}} \right].$$

Since $\beta \in (0,1)$, we have $\Delta V_t > 0$. ■

8.3 Period-n Cycles

This section shows that equilibrium paths may take the form of (unstable) period-n cycles. We first establish this possibility. Then, we discuss the economic implication of such cycles for the evolution of the number of available consumption good varieties, A_t, intergenerational welfare, ΔV_t, GDP_t, gdp_t, the functional income distribution, and for the stability policy devised above.

Proposition 8.9 (Existence of Unstable Period-n Cycles)

Let $n \in \{2, 3, \dots\}$ and suppose $-\nu(\sigma - 1) < -(n - 1 + \psi)$. Then, there exists an unstable period-n cycle with $n - 1$ consecutive points in Regime 0 followed by one period in Regime 1.

Moreover, suppose $z_1 < z_2 < \cdots < z_{n-1} < z_c < z_n$. Then for $t = 1, 2, \dots, n$,

$$z_t = z_c(1 + g_L)^{\frac{t+(n-t)[\psi-\nu(\sigma-1)]}{\eta}}. \tag{8.12}$$

Proof of Proposition 8.9

To establish the existence of a period-n cycle, we need to show that $\Phi^n(z_1) - z_1 = 0$ has a solution. Suppose $z_1 < z_2 < \cdots < z_{n-1} < z_c < z_n$. Then, the dynamical system in Proposition 3.1 gives, for all $t = 2, 3, \dots, n$,

$$\Phi^{t-1}(z_1) = (1 + g_L)^{t-1} z_1 \quad \text{and} \quad \Phi^n(z_1) = (1 + g_L) z_c \left[\frac{(1 + g_L)^{n-1} z_1}{z_c}\right]^{\psi-\nu(\sigma-1)}.$$

Then, solving $\Phi^n(z_1) - z_1 = 0$ for z_1 gives

$$z_1 = z_c(1 + g_L)^{\frac{1+(n-1)[\psi-\nu(\sigma-1)]}{\eta}}.$$

Moreover, $n - 1$ iterative substitutions of the latter equation into the equilibrium difference equation for Regime 0 gives for $t = 1, 2, \dots, n$, the expression for z_t of (8.12). To ensure that $z_1 < z_2 < \cdots < z_{n-1} < z_c$, it is sufficient to show $z_{n-1} < z_c$. Equation (8.12) implies that

$$z_{n-1} = z_c(1 + g_L)^{\frac{n-1+\psi-\nu(\sigma-1)}{\eta}}.$$

Simple manipulations show that the condition $-\nu(\sigma - 1) < -(n - 1 + \psi)$ ensures $z_{n-1} < z_c$.

To see that the period-n cycles are unstable observe that such cycles are asymptotically stable if $|\Phi'(z_1)\Phi'(z_2), \dots, \Phi'(z_n)| < 1$, and unstable if $|\Phi'(z_1)\Phi'(z_2), \dots, \Phi'(z_n)| > 1$ (see, e. g., Elaydi (2005), p. 39). Here, for all $t = 1, 2, \dots, n-1$, $\Phi'(z_t) = (1 + g_L)$ and $\Phi'(z_n) = [\psi - \nu(\sigma - 1)](1 + g_L)^{-(n-1)}$. Therefore, $|\Phi'(z_1)\Phi'(z_2), \dots, \Phi'(z_n)| = |\psi - \nu(\sigma - 1)|$. Since $-\nu(\sigma - 1) < -(n - 1 + \psi)$ and $n \geq 2$, $|\psi - \nu(\sigma - 1)| > 1$. ■

Hence, if the equilibrium elasticity of the individual supply of hours worked to changes in A_t is sufficiently small, then period-n cycles with $n - 1$ consecutive points in Regime 0 followed by one point in Regime 1 can be pinned down.

The following heuristic may be helpful to develop an intuition for this finding. Suppose all parameters except σ are fixed. Then, condition $-\nu(\sigma - 1) < -(n - 1 + \psi)$ can be satisfied for a larger n if σ is larger. What is the effect of increasing σ on the qualitative properties of the dynamical system of Proposition 3.1? First, observe that changing σ does not affect z_c. Hence, the rightward boundary of the absorbing interval does not change either as $\Phi(z_c) = (1+g_L)z_c$. However, the leftward boundary declines as $\Phi^2(z_c) = (1+g_L)^{1+\psi-\nu(\sigma-1)}z_c$. Hence, a higher σ allows for a smaller initial value, z_1, and, therefore, for more periods in Regime 0.

Economically, this means for a given value of A_1 that L_1 can be smaller for a higher σ. Then, through population growth it takes more periods before a sufficiently large cohort is alive willing to purchase $A_1 + \Delta A_n$ shares at a price $v_n = a\alpha A_1^{\sigma-1-\psi}$.

To elicit some further economic implications of these cycles denote by $\overline{g}_{A,n} \equiv [\Pi_{t=1}^{n}(1+g_{A,t})]^{\frac{1}{n}} - 1$ the average growth rate of A_t over a period-n cycle. Then, the evolution along a period-n cycles satisfies the following.

Corollary 8.1 (Economic Properties of Period-n Cycles)

Consider some period-n cycle of Proposition 8.9. Then, the following holds:

1. *The average growth rates of A_t over the period-n cycle coincide with the steady-state growth rate, i. e., $\overline{g}_{A,n} = g_A$;*
2. $\Delta V_2 = \dots = \Delta V_{n-2} = 0$, $\Delta V_{n-1} < 0$, $\Delta V_n > 0$, *and* $\Delta V_{n+1} > 0$;
3. $GDP_1 < \dots < GDP_{n-2} < GDP_{n-1} < GDP_n < GDP_{n+1}$;
4. $gdp_1 = \dots = gdp_{n-2} = gdp_{n-1}$, $gdp_{n-1} > gdp_n$, *and* $gdp_n < gdp_{n+1}$;
5. $LS_1 = \dots = LS_{n-1} = \alpha < LS_n$;
6. $\bar{z} < z_1 < \dots < z_n$ *and* $\tau_{w,1}^P < \dots < \tau_{w,n-1}^P < 0 < \tau_{w,n}^P$.

Proof of Corollary 8.1

Let $z_1 < \dots < z_{n-1} < z_c < z_n$ and $z_{n+1} = z_1$. We prove each claim in turn:

1. The period-n cycle begins without an active research sector for $n-1$ periods, i. e., $g_{A,t} = 0$ for all $t = 1, 2, \dots, n-1$. Then, in period n, the growth rate satisfies $1+g_{A,n} = z_n/z_c$. With Proposition 8.9, we obtain $1 + g_{A,n} = (1+g_L)^{\frac{n}{\eta}}$. Therefore, the average growth factor of A_t over a period-n cycle satisfies $(1+\overline{g}_{A,n}) = (1+g_L)^{\frac{1}{\eta}}$. Accordingly, $\overline{g}_{A,n}$ is equal to the steady-state growth rate, g_A, given in Proposition 8.1.

On readily verifies that

$$1+\overline{g}_{v,n}=\left[\left(\frac{v_2}{v_1}\right)\cdot\left(\frac{v_3}{v_2}\right)\cdot\dots\cdot\left(\frac{v_{n-1}}{v_{n-2}}\right)\cdot\left(\frac{v_n}{v_{n-1}}\right)\cdot\left(\frac{v_n}{v_{n-1}}\right)\right]^{\frac{1}{n}},$$

where the first $n-2$ factors are equal to $(1+g_L)$, $v_n/v_{n-1}=(1+g_L)^{1-n/\eta}$, and $v_{n+1}/v_n=(1+g_L)^{1+n(1-\nu)(\sigma-1)/\eta}$. Then, the result is immediate.

2. Case 1, Case 2, Case 4.2, and Case 4.6 of Proposition 5.1 imply, respectively, that $\Delta V_2=\cdots=\Delta V_{n-2}$, $\Delta V_{n-1}<0$, $\Delta V_n>0$, $\Delta V_{n+1}>0$.
3. Proposition 5.2 implies $GDP_1<\cdots<GDP_n<GDP_{n+1}$.
4. Case 1 of Proposition 5.2 implies $gdp_1=\cdots=gdp_{n-1}$ and $gdp_{n-1}>gdp_n$, Case 2 implies $gdp_n<gdp_{n+1}$.
5. Using $\gamma_{A,1}=\cdots=\gamma_{A,n-1}=0<\gamma_{A,n}$ in (5.6), we have $LS_1=\cdots=LS_{n-1}=\alpha<LS_n$.
6. Condition $-\nu(\sigma-1)<-(n-1+\psi)$ implies $\bar{z}<z_1$. Hence, we have $\bar{z}<z_1<\cdots<z_n$. Then, since $z_1<\cdots<z_{n-1}<z<z_n$, Proposition (5.3) implies $\tau^P_{w,1}<\cdots<\tau^P_{w,n-1}<0<\tau^P_{w,n}$.

■

According to Claim 1 of Corollary 8.1, the average growth rate over the entire period-n cycle, $\overline{g}_{A,n}$, with no growth of A_t for $n-1$ points, is equal to the steady-state growth rate, g_A, of equation (3.11). Intuitively, the mechanism behind this finding is the following. On a period-n cycle the only positive growth rate is $g_{A,n}$ which is given by $g_{A,n}=z_n/z_c-1$. Hence, it is greater the greater is z_n. From Proposition 8.9 we have $z_n=z_c(1+g_L)^{n/\eta}$ with $dz_n/dn>0$. Hence, $g_{A,n}$ increases in n. More precisely, combining the expressions for $g_{A,n}$ and z_n gives

$$g_{A,n}=\frac{z_c(1+g_L)^{\frac{n}{\eta}}}{z_c}-1=(1+g_L)^{\frac{n}{\eta}}-1.$$

Since $g_{A,t}=0$ for $t<n$ the average growth rate of A_t along a period-n cycle is indeed equal to the steady-state growth rate as $\overline{g}_{A,n}=(1+g_L)^{1/\eta}-1=g_A$.

Second, the evolution of intergenerational welfare reflects the pattern identified in Proposition 5.1. Cohorts $t=1,...,n-2$ live their life in Regime 0 so that the intergenerational welfare remains constant. As the economy switches into Regime 1 at $t=n$, it is cohort $t=n-1$ that lives its youth in Regime 0 and its old age in Regime 1. This cohort has a lower lifetime utility compared to its predecessor. The following two cohorts live at least one period of their lives in Regime 1. Hence, $\Delta V_n>0$, and $\Delta V_{n+1}>0$.

Claim 3 and 4 reflect the results of Proposition 5.2. Due to population growth GDP_t increases over time. However, gdp_t declines when the economy switches from Regime 0 into Regime 1 and increases when the economy switches back from Regime 1 into Regime 0. Claim 4 recalls that the labor share increases when the economy opens a research sector.

Finally, Claim 5 states that a period-n cycles is such that all points are to the right of the critical value $\bar{z}$. Therefore, at any point the stabilization policy $\left(\tau^P,w,t,\tau^P,R,t\right)$ of Proposition 5.3

can be implemented. Such a policy will involve a subsidy to wages (and a tax on asset income) when implemented at $t = 1, ..., t = n - 1$ and a tax on wage (and a subsidy to asset income) income when implemented at $t = n$.