

A Central Limit Theorem, Loss Aversion and Multi-Armed Bandits*

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Abstract

This paper studies a multi-armed bandit problem where the decision-maker is loss averse, in particular she is risk averse in the domain of gains and risk loving in the domain of losses. The focus is on large horizons. Consequences of loss aversion for asymptotic (large horizon) properties are derived in a number of analytical results. The analysis is based on a new central limit theorem for a set of measures under which conditional variances can vary in a largely unstructured history-dependent way subject only to the restriction that they lie in a fixed interval.

Keywords: multi-armed bandit, loss aversion, sequential sampling, rectangular sets of measures, robustness, central limit theorem, oscillating Brownian motion

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1 Introduction

We study the following (multi-armed bandit) sequential choice problem.¹ There are finitely many arms (or actions), each yielding a random payoff. Probability distributions have a common mean but differ otherwise and may not be known to the decision-maker (DM). At each stage $i = 1, 2, \dots, n$, DM chooses one arm, knowing the realized outcomes from previous choices. Ex ante she chooses a strategy to maximize expected utility, where the utility index is a function of the (suitably weighted) average payoff. Because we are interested in varying horizons, it is convenient to define a strategy for an infinite horizon, and then to use its truncation for any given finite horizon. Refer to a strategy as *asymptotically optimal* if the expected utility it implies in the limit as horizon $n \rightarrow \infty$ is at least as large as that implied by any other strategy. We study large-horizon approximations to the value (indirect utility) of the bandit problem and corresponding asymptotically optimal strategies.

A second novelty in our model is the assumption that DM is loss averse (global risk aversion is a limiting special case). Loss aversion was introduced via cumulative prospect theory by Tversky and Kahneman (1992), and has since been well-established empirically and widely applied in economics and finance (see for example, Kahneman and Tversky 2000, Köbberling and Wakker 2005, Barberis 2013, and the references therein). Its essential elements are (i) a reference point; (ii) utility depends only on gains and losses relative to that reference point rather than on the total payoff (or total wealth); (iii) risk aversion (concavity) for gains and risk loving (convexity) for losses; and (iv) greater sensitivity to losses than to gains. Our interest in this paper is the effect of loss aversion in the sequential context defined by a bandit problem. To our knowledge, this is the first study of loss aversion in bandit problems.²

We have two related reasons for studying asymptotics. First, it promotes

¹Overviews and textbook treatments of the bandit model can be found in Berry and Fristadt (1985) and Slivkins (2019), for example. The first application to economics is Rothschild (1974). See Bergemann and Valimaki (2008) for references to a range of other economic applications.

²Xu and Zhou (2013) and Ebert and Strack (2015) study optimal stopping problems assuming prospect theory. Their focus is on the probability-weighting aspect of prospect theory and loss aversion plays no role in their analyses. Two studies of loss aversion in a sequential context are Easley and Yang (2015) and Shi et al (2015). The former uses numerical analysis to study the wealth and price effects of loss aversion in the equilibrium of a dynamic heterogeneous-agent economy. The latter models dynamic portfolio choice with loss aversion, where the reference point varies endogenously in response to prior wealth outcomes. In both cases, analysis is largely numerical and there is little overlap with the bandit literature in general, and with our model, in particular. Guasoni et al (2020) study shortfall aversion, which shares the spirit of loss aversion but which is more directly relevant to preference over deterministic consumption streams rather than over lotteries.

tractability and the derivation of analytical results. Though the literature on bandit problems is enormous, theoretical analysis of Bayesian models is, to the best of our knowledge, restricted to the special case of risk neutrality (see section 2.1.3 for elaboration and a qualification).³ Besides its obvious limitations, risk neutrality also imposes the invariance of risk attitude as some outcomes are realized, and this invariance is key to well-known sequential properties of optimal strategies derived in the literature.⁴ In contrast, endogenously varying risk attitude is inherent in loss aversion. Moreover, in our setting where means are known and common to all arms, risk neutrality would trivialize the problem.

Our second reason for studying asymptotics is that tractability is plausibly a concern not only for the modeler but also for the decision-maker within the model. We view her as struggling to comprehend an extremely complicated finite-horizon optimization problem, and adopting instead the simplifying assumption of an infinite horizon. She does so with the recognition that an asymptotically optimal strategy is approximately optimal if her horizon is sufficiently long.

Here is an informal outline of some of our analytical results, which obtain as stated in the infinite-horizon limit and approximately for sufficiently large finite horizons.

1. Maximum ex ante expected utility depends on the distributions describing each arm *only through their variances*. Moreover, it depends only on the largest and smallest variance. Consequently, at each history, arms with intermediate conditional variances can be ignored.
2. Depending on the reference point, it is possible to achieve a level of ex ante expected utility that is equal to, or strictly greater than, the level when the payoff to each arm is riskless. In that sense, *risk may be desirable* in the sequential context, even though "comparable" risks would be rejected in a one-shot choice setting.
3. Suppose that the distributions describing every arm are known. Then, in spite of the absence of learning, an asymptotically optimal strategy switches indefinitely between two fixed extreme arms (those with the smallest and largest variances) as the decision-maker moves between cumulative gains and cumulative losses. Given two arms that exhibit the two extreme variances, all other arms are redundant.

³Two studies of bandit problems that explicitly address risk are Sani, Lazaric and Munos (2013) and Huo and Fu (2017). They assume regret minimization rather than expected utility maximization, and focus on computational algorithms rather than on qualitative theoretical results.

⁴For example, in an infinite-horizon setting where means can differ, and with one unknown arm and one arm whose distribution is known, then once the known arm is chosen it will continue to be chosen thereafter (Rothschild 1974, pp. 190-191).

4. Suppose there are two arms and that the pair of variances is known, but there is prior uncertainty about which arm has which variance. Then it is asymptotically optimal to choose myopically at each stage, that is, as though there are no subsequent choices to be made.
5. None of the above results rely on assumptions about the nature of risk aversion in the domain of gains or about the nature of risk loving in the domain of losses. They depend only on preference over "mixed" lotteries.

Finally, we turn to the proofs of these and other results about bandits and loss aversion. It is not surprising that asymptotic results may be approached via limit theorems. However, classic limit results do not apply, and the key to our proofs is a new central limit theorem (CLT). The martingale version of the central limit theorem considers a sequence (X_i) of random variables having zero conditional mean and constant conditional variance σ^2 , and shows that (under suitable additional conditions) the distribution of $\sum_{i=1}^n X_i / \sqrt{n}$ converges to the normal $\mathbb{N}(0, \sigma^2)$ as $n \rightarrow \infty$. (The classic result for identically and independently distributed random variables is an immediate special case). This paper establishes a CLT under the relaxed assumption on variance according to which conditional variances can vary in a largely unstructured history-dependent way subject only to the restriction that they lie in a fixed interval $[\underline{\sigma}^2, \bar{\sigma}^2]$, in which case limits take a novel and tractable form. This CLT is the main technical contribution of the paper. One well-known motivation for generalizing from a single probability distribution (hence single variance) to a set of probability distributions (hence set of variances) is robustness to model uncertainty or ambiguity. However, model uncertainty plays no role in our bandit problem - DM is a Bayesian agent, perfectly confident in her understanding of the environment - thus highlighting the usefulness of sets of measures even for Bayesian models.

We proceed as follows. The bandit model and the results outlined above are described in detail in the next section. Proofs for these results must await the CLT which is presented next in section 3.3. Proofs of the CLT and related results are presented in Appendix A and proofs for the bandit application are in Appendix B.

2 Multi-Armed Bandits

2.1 Beliefs, utility and optimization

Let \mathcal{A} be a finite set of arms (or actions). The outcome of any action lies in the finite set $\bar{\Omega} \subset \mathbb{R}$. Thus outcome sequences lie in $\Omega = \Pi_1^\infty \Omega_i$, where $\Omega_i = \bar{\Omega}$ for each i . The timing is as follows: At each $i \geq 1$, the history $\omega^{(i-1)} = (\omega_1, \dots, \omega_{i-1})$

is known, ($\omega^{(0)} = \emptyset$), an action $a_i \in \mathcal{A}$ is chosen, and then the resulting outcome ω_i is realized. Define $X_i(\omega) = \omega_i$, the outcome at stage i .

Let \mathcal{G}_{i-1} be the σ -algebra representing information at stage i , ($\mathcal{G}_0 = \{\Omega, \emptyset\}$), and let $\mathcal{G} = \sigma(\cup_1^\infty \mathcal{G}_i)$ be the corresponding σ -algebra on Ω .

The outcome resulting from any action is uncertain and the choice of a contingent plan, or strategy, is determined by expected utility maximization. The remaining primitives of the model - strategies, beliefs and the vNM utility index - are described next.

2.1.1 Strategies and beliefs

The contingent choice of action at stage i depends on (conditional) beliefs about the next outcome, which generally depend on the arm being considered and also on what is learned from previous choices and their outcomes. Importantly, the inference to be drawn from the history $\omega^{(i-1)}$ of outcomes depends on which arms produced them. Thus, the choice of action at stage i is expressed as

$$a_i = s_i(a^{(i-1)}, \omega^{(i-1)}), \quad (2.1)$$

where $a^{(i-1)} = (a_1, \dots, a_{i-1})$ denotes the history of past actions ($a^0 = \emptyset$). Refer to $s_i : \mathcal{A}^{i-1} \times \prod_{j=1}^{i-1} \Omega_j \rightarrow \mathcal{A}$ as the *strategy at stage i* , and denote the set of all such s_i by \mathcal{S}_i . The infinite sequence $s = (s_i)_1^\infty$ is called simply a *strategy*. The corresponding set of strategies is \mathcal{S} .⁵

Turn to beliefs. For the reasons noted above, beliefs about the next outcome depend on both the action being considered, hence on the strategy for the current stage, and on the history of past actions. Thus we model these beliefs for stage i by the conditional probability measure

$$P_i^{s_i} = P_i^{s_i}(\cdot | a^{(i-1)}, \omega^{(i-1)}) \in \Delta(\Omega_i). \quad (2.2)$$

The set of 1-step-ahead conditionals $\{P_i^{s_i}\}_{i \geq 1, s_i \in \mathcal{S}_i}$ is a *primitive* that represents beliefs (which may be taken to be subjective or objective).

Given a (fixed) strategy $s = (s_1, \dots, s_i, \dots)$, we can combine the primitive conditionals into a measure P^s on $\prod_1^\infty \Omega_i$. To describe why and how, suppose that DM is considering the strategy s . Then she knows that the action history $a^{(i-1)}$ at any future stage i is uniquely determined by the corresponding outcome history $\omega^{(i-1)}$, and by the given stage strategies s_1, \dots, s_{i-1} , via repeated iteration of the relation

$$a_j = s_j(a^{(j-1)}, \omega^{(j-1)}), \quad j = 1, 2, \dots, i-1. \quad (2.3)$$

⁵For any given n , $s \in \mathcal{S}$ induces the contingent plan $(s_i)_1^n$, which is adequate if one is interested only in the n -horizon case. Because we will be interested in varying horizons, it is convenient to define a strategy to apply to all finite horizons.

Consequently, DM faces uncertainty only about future outcomes and she seeks a probability measure over all outcome sequences that is consistent with the primitive 1-step-ahead conditionals $\{P_i^{s_i}\}_{i \geq 1, s_i \in \mathcal{S}_i}$. As noted, the action history appearing in (2.2) can be substituted out and the conditioning information becomes a function of $\omega^{(i-1)}$ alone. Therefore, the 1-step-ahead conditionals can be pasted together in the usual fashion: By the Ionescu-Tulcea extension theorem, one obtains a (unique) measure P^s ,

$$P^s \in \Delta(\Pi_1^\infty \Omega_i, \mathcal{G}). \quad (2.4)$$

Moreover, its 1-step-ahead conditional $P_i^s(\cdot | \mathcal{G}_i) \in \Delta(\Omega_i)$ “agrees” with the primitive conditional $P_i^{s_i}$ in the sense that

$$P_i^s(\cdot | \mathcal{G}_i)(\omega^{(i-1)}) = P_i^{s_i}(\cdot | a^{(i-1)}, \omega^{(i-1)}), \quad (2.5)$$

where $a^{(i-1)}$ is obtained from (2.3).

We assume that each P_i^s has *full support* on $\bar{\Omega}$. We assume also that mean outcomes are common to all arms (hence also strategies) and fixed:

$$E_{P^s}[X_i | \mathcal{G}_{i-1}] = m = 0 \quad \text{for all } i \geq 1 \text{ and all } s \in \mathcal{S}, \quad (2.6)$$

where setting $m = 0$ is without loss of generality. Finally, the extreme (largest and smallest) variances across all arms are assumed constant:

$$\text{ess sup}_{s \in \mathcal{S}} E_{P^s}[X_i^2 | \mathcal{G}_{i-1}] = \bar{\sigma}^2 \quad \text{and} \quad \text{ess inf}_{s \in \mathcal{S}} E_{P^s}[X_i^2 | \mathcal{G}_{i-1}] = \underline{\sigma}^2 \quad \text{for all } i \geq 1, \quad (2.7)$$

for some $\bar{\sigma}^2 > \underline{\sigma}^2 > 0$.

Apart from these conditions, the history-dependence of any primitive conditional $P_i^{s_i}$ is unrestricted, and consequently so is the implied model of learning. For example, the model does not rule out that past outcomes of arm a may inform beliefs about future outcomes of arm $a' \neq a$. Independence across arms, or other hypotheses about the nature of learning, can be accommodated by suitably specializing beliefs, but are not assumed in the general model. (Sections 2.2.2 and 2.2.3 describe two such specializations.)

For readers who find the strategy-dependence of probability measures unorthodox we add that it is readily understood in the following terms. Consider a generic static choice problem of the form $\sup_{a \in \mathcal{A}} E_\mu[u(X^a)]$, where X^a is the random variable outcome associated with action a and μ is a prior over the underlying state space Ω .⁶ Then each X^a induces a probability distribution, denoted p^a , over Ω , and the preceding optimization problem can be written as $\sup_{a \in \mathcal{A}} E_{p^a}[u(X)]$,

⁶Here Ω is an abstract state space, not necessarily related to the product state space used in the bandit model. Similarly, for \mathcal{A} and for X below.

where $X(\omega) = \omega$. Thus the choice between actions, modeled as the choice between random variables, can be expressed alternatively as the choice between action-dependent probability distributions over outcomes (that is, lotteries). The analogue of this reformulation for our sequential choice context leads to strategy-dependent probability measures.⁷

2.1.2 Utility

We assume that, at each stage i , outcomes for each action are evaluated according to whether they produce gains or losses relative to a reference point, which we take to be their common mean (taken to be zero for convenience). Then X_i gives the gain/loss at stage i . Since gains/losses are incurred at each stage, they must be aggregated. We posit that, for any horizon length n , utility depends on their \sqrt{n} -weighted average. Consequently, given the strategy s , the implied stream of gains/losses has expected utility given by

$$U_n(s) = E_{P^s} [\varphi (\Sigma_1^n X_i / \sqrt{n})], \quad (2.8)$$

where φ is the vNM utility index, which will be described shortly.

The \sqrt{n} -weighted averaging calls for some discussion. Consider a setting (such as a casino, where trials correspond to playing one or another slot machine or gambling device) where the time between trials is so small as to preclude discounting, and where the monetary payoffs at different trials are perfect substitutes. We are not aware of any axiomatic (or empirical) guidance for how a decision-maker does or should aggregate or average money streams in this context given that arbitrarily large horizons are relevant. The unweighted arithmetic average might be slightly simpler to contemplate and calculate, but significantly, it also reflects a specific and possibly inappropriate weighting to finite sets of trials. Indeed, as is familiar from discussions of the classic law of large numbers (LLN) and CLT, one might argue that scaling by $\frac{1}{n}$ implies "too little" weight for finite sets of trials, particularly when considering volatility. That will be argued also in our setting (Remark 2.3) and is our practical modelers' rationale for preferring (2.8).

Remark 2.1. *To be perfectly clear, the utility functions U_n rank strategies for any given horizon n . They do not rank horizons. That is, statements such as $U_n(s) \geq U_n(s')$ are meaningful, but statements such as $U_n(s) \geq U_{n'}(s)$ are not and do not play a role below.*

⁷The use of action-dependent probabilities (or moral hazard) has been recognized in the decision theory literature (Dreze 1987, Kelsey and Milne 1999, and Karni 2011, for example). These papers are concerned primarily with axiomatic foundations, extending those for subjective expected utility, while our motivation in studying the bandit problem is more applied. We differ also in our focus on sequential choice.

The utility index φ appearing in (2.8) is defined by

$$\varphi(x) = \begin{cases} \varphi_1(x - c) & x \geq c \\ -\theta^{-1}\varphi_1(-\theta(x - c)) & x < c \end{cases} \quad (2.9)$$

where we assume:

Assumption-Utility: $\theta = \underline{\sigma}/\bar{\sigma} < 1$, $\varphi_1(0) = 0$, $\varphi_1 \in C_b^3(\mathbb{R}_+)$, and φ_1 is (strictly) increasing and (strictly) concave for $x > c$.⁸

Then, φ is increasing globally, concave for $x > c$ (corresponding to gains) and convex for $x < c$ (corresponding to losses), implying risk aversion for gains and risk seeking for losses. In addition,

$$x > y \geq 0 \implies (c + y, \frac{1}{2}; c - y, \frac{1}{2}) \succ (c + x, \frac{1}{2}; c - x, \frac{1}{2}), \quad (2.10)$$

indicating greater sensitivity to the increased loss ($-x < -y$) than to the increased gain ($x > y$). In differential form, it states that

$$\varphi'(c - x) > \varphi'(c + x), \quad \text{for all } x > 0. \quad (2.11)$$

We take these to be the defining properties of (strict) loss aversion, following Wakker and Tversky (1993, p. 164), for example. An implication is that $-\varphi(c - x) > \varphi(c + x)$, for all $x > 0$, that is, the lottery $(c + x, \frac{1}{2}; c - x, \frac{1}{2})$ is strictly inferior to receiving 0 for sure.

The following example will be useful in the sequel (see (2.21)) because of its tractability.

Example 1 (Exponential). *Let $\varphi_1(x) = 1 - \exp(-x)$, so that*

$$\varphi(x) = \begin{cases} 1 - \exp(-(x - c)) & x \geq c \\ \theta^{-1}(\exp(\theta(x - c)) - 1) & x < c \end{cases} \quad (2.12)$$

where $c \in \mathbb{R}$ and $\theta = \underline{\sigma}/\bar{\sigma}$.

Because of its origins in prospect theory, loss aversion is often viewed as tied to probability weighting or distortion, (which is absent in our expected utility model), and also to a kink in the utility index at the reference point (which is also absent here because φ defined above is continuously differentiable everywhere). However, neither is necessary mathematically or conceptually for the above behavioral properties that define loss aversion.⁹ Accordingly, consistent with common practice, we

⁸ $C_b^3(\mathbb{R}_+)$ is the set of functions on the non-negative real line with continuous and bounded third order derivatives.

⁹Kobberling and Wakker (2005) argue explicitly for a conceptual separation between loss aversion and probability weighting. They write (p. 124): “We have introduced utility, probability weighting and loss aversion as logically independent factors of risk attitude ... their (in)dependence empirically is more intricate.”

exclude probability distortions, hence Allais-type behavior, in order to isolate the effects of loss aversion on sequential decision-making. As for a kink, it has limited empirical content; for example, a finite set of pairwise rankings of lotteries, as is common in experimental investigations of loss aversion, cannot refute differentiability. Moreover, the theoretical connection of a kink to loss aversion is very much dependent on the choice of functional form. For example, suppose that, instead of (2.9), one posits that

$$\varphi(x) = \begin{cases} \varphi_1(x-c) & x \geq c \\ -\lambda\varphi_1(-(x-c)) & x < c \end{cases} \quad (2.13)$$

where $\lambda \geq 1$. Then (2.10) is satisfied if and only if $\lambda > 1$, which renders φ nondifferentiable at c . Thus a kink is necessary for loss aversion given (2.13), but not given (2.9).

We add some interpretation of the functional form (2.9). Take $c = 0$ for simplicity. Then, as observed above, loss aversion implies

$$(x, \frac{1}{2}; -x, \frac{1}{2}) \prec 0.$$

How might one measure the degree of loss aversion expressed thereby? One possibility is to use the reduction in the loss needed to imply indifference, but then the new lottery would have nonzero mean which would obfuscate the determination of "greater sensitivity to losses". Similarly if one were to increase the odds of a gain with prizes unchanged. Thus we adjust both so as to keep the zero mean. Specifically, we look for $\lambda > 1$ such that

$$(x, \lambda p; -\lambda x, p) \sim 0 \text{ for all } x > 0 \text{ and } 0 < p < 1. \quad (2.14)$$

(For probabilities to sum to 1, one needs $p(1 + \lambda) = 1$, but that can be safely ignored for present purposes given expected utility theory.) The above condition states that when both the odds of a gain and the size of the loss are increased by the factor λ , then (the zero mean condition is satisfied and) indifference with 0 is restored. In contrast, when $\lambda = 1$, then the strictly inferior $\frac{1}{2}/\frac{1}{2}$ lottery is obtained. This suggests using $\lambda - 1$ to measure loss aversion. Such a measure is well-defined for our model, using (2.9), since (2.14) is satisfied (uniquely) with $\lambda = \theta^{-1}$. Thus $\theta^{-1} - 1$ gives a measure of loss aversion that is *behavioral* (defined by the preference condition (2.14)), and *global* (the same λ works for all x and p as indicated).¹⁰ Alternatively, in our model $(\alpha x, p; -x, \alpha p) \sim 0$ is satisfied (uniquely)

¹⁰In fact, existence of λ satisfying (2.14) is *equivalent* to our specification with $\lambda = \theta^{-1}$. More generally, one might weaken (2.14) by allowing λ to depend on x and/or p . From that perspective, our model yields a constant measure of loss aversion, perhaps suggesting a partial analogue to CARA utility functions.

by $\alpha = \theta < 1$, suggesting $1 - \theta$ as a measure of loss aversion. In either case, the parameter θ admits a simple behavioral interpretation.

The results below, and the CLT underlying them, are limited to the case $\theta = \underline{\sigma}/\bar{\sigma}$. However, they are robust to the specification of φ_1 , which is unrestricted except for nonparametric monotonicity and concavity assumptions and technical (smoothness and boundedness) conditions. In particular, what follows makes no assumption about the nature of risk aversion in the domain of gains or alternatively, or about the nature of risk loving in the domain of losses. The only relevant restriction, imposed by (2.9) and expected utility theory, is on preference over “mixed” lotteries.

2.1.3 Optimization

The preceding leads finally to the optimization problem (for each n)

$$V_n \equiv \sup_{s \in \mathcal{S}} E_{P^s} [\varphi (\Sigma_1^n X_i / \sqrt{n})]. \quad (2.15)$$

(Recall that $X_i(\omega) = \omega_i$ gives the outcome at stage i .)

The finite horizon problem is not tractable (for us). For reasons of tractability, Bayesian models in the literature typically take φ to be linear. In that case, given the fixed finite horizon n , the $\frac{1}{\sqrt{n}}$ factor is irrelevant and the objective is to maximize the expected value of the sum $\Sigma_1^n X_i$. If outcomes are monetary prizes that are perfect substitutes, which is the way we think of our model, then a linear φ implies risk neutrality as remarked in the introduction. An alternative is that outcomes are measured in utils, as in the common expected-additive-utility model of preference over risky consumption streams. Then the underlying prizes (consumption levels, for example) at different stages are not perfect substitutes, and also the ranking of the risky consumption at stage i is independent of the risks involved at other stages (implying indifference to correlation in consumption risks). In applications where these features are appropriate, indifference to risk in consumption (or other underlying prizes) is not implied by a linear φ . However, for the settings we have in mind, *tractability comes at the cost of assuming risk neutrality*.

Consider briefly a common approach to solving bandit problems analytically which is to establish the optimality of index-based strategies, most commonly using the Gittins index (Gittins and Jones 1974). When arms can be valued separately, then at each stage and history an index summarizes each arm and comparison of these indices determines which arm to pull. This approach does not work in our model because arms cannot be delinked for at least two reasons: (i) outcomes from one arm may be informative about the distribution describing other arms because of common unknown parameters (see section 2.2.3); (ii) because of loss aversion risk attitude depends on the sign of the sum of past payoffs from *all* arms.

Our approach to analysing (2.15) for the loss averse utility index (2.9) is to study large-horizon approximations to the value (indirect utility) of the bandit problem and corresponding approximately optimal strategies. More precisely, define, conditional on showing below that the following limit exists,

$$V \equiv \lim_{n \rightarrow \infty} V_n. \quad (2.16)$$

Below we derive results for V , which therefore imply approximate results for V_n when n is sufficiently large. Secondly, say that the strategy s^* is *asymptotically optimal* if

$$\lim_{n \rightarrow \infty} U_n(s^*) = \lim_{n \rightarrow \infty} V_n; \quad (2.17)$$

or, equivalently, if, for every $\epsilon > 0$, there exists n^* such that

$$|U_n(s^*) - V_n| < \epsilon \text{ if } n > n^*.$$

Thus asymptotic optimality of s^* is a more concise way to say that " s^* is approximately optimal for problems with sufficiently long horizon."¹¹

2.2 Results

In all our results for the bandits model, the assumptions specified above are adopted: conditional beliefs satisfy full support, (2.6) and (2.7), and the utility index φ is given by (2.9) and satisfies Assumption-Utility. Though the latter requires $\bar{\sigma} > \underline{\sigma}$, all the results that follow are trivially valid, by the classic martingale CLT, also when $\bar{\sigma} = \underline{\sigma}$. Then all arms have a common variance and are equivalent in the large horizon limit, making the (asymptotic) choice between arms trivial. It simplifies discussions below to exclude that case.

2.2.1 Value

Our first result concerns the limiting value V . We emphasize the surprising (to us) degree to which this result is *robust* to specifications of φ_1 and the primitives $\{P_i^{s_i}\}_{i \geq 1, s_i \in \mathcal{S}_i}$, and therefore also to assumptions about the nature of learning.

Theorem 2.2. (i) *Let V_n be the value of the n -horizon problem (2.15). Then $\lim_{n \rightarrow \infty} V_n$ exists. Moreover,*

$$V = \lim_{n \rightarrow \infty} V_n = \int_{-\infty}^{\infty} \varphi(y) q(y) dy, \quad (2.18)$$

¹¹An implication is that, for any s , $\lim_{n \rightarrow \infty} U_n(s) \leq \lim_{n \rightarrow \infty} U_n(s^*)$. This follows from (2.17) and $U_n(s) \leq V_n$ for all n .

where q is the pdf in (B.1)-(B.2), which, for $c = 0$ yields the simple form

$$q(y) = \begin{cases} q^*(y; \underline{\sigma}) \left[\frac{2\bar{\sigma}}{\underline{\sigma} + \bar{\sigma}} \right] & y \geq 0 \\ q^*(y; \bar{\sigma}) \left[\frac{2\underline{\sigma}}{\underline{\sigma} + \bar{\sigma}} \right] & y < 0 \end{cases} \quad (2.19)$$

Here $q^*(y; \sigma) = \frac{1}{\sqrt{2\pi\sigma}} \exp(-(y/\sigma)^2/2)$ is the pdf for $\mathbb{N}(0, \sigma^2)$.

(ii) Let primitive beliefs be modified to $\{\widehat{P}_i^{s_i}\}_{i \geq 1, s_i \in \mathcal{S}_i}$, another set satisfying our assumptions, including counterparts of (2.6) and (2.7), and where the latter is satisfied by the identical variance extremes $\underline{\sigma}^2$ and $\bar{\sigma}^2$. Then $\widehat{V} = V$.

(iii) The limiting value V satisfies

$$V = \begin{cases} = \varphi(0) & c = 0 \\ > \varphi(0) & c > 0 \\ < \varphi(0) & c < 0 \end{cases} \quad (2.20)$$

(i) not only proves that the large-horizon limit V is well-defined, but also gives an explicit description of V . Moreover, for some functions φ the integral in (2.18) can be expressed in closed form yielding a closed form expression for V for each c . For example, if φ is taken to be the exponential example (2.12), then, using the density in (B.1)-(B.2),

$$V = \begin{cases} \Phi(-\frac{c}{\underline{\sigma}}) - \Phi(\frac{c}{\underline{\sigma}}) + e^{\frac{\sigma^2}{2}} \left(e^{-c} \Phi(-\underline{\sigma} + \frac{c}{\underline{\sigma}}) - e^c \Phi(-\underline{\sigma} - \frac{c}{\underline{\sigma}}) \right) & c \leq 0 \\ \frac{\bar{\sigma}}{\underline{\sigma}} \left[\Phi(-\frac{c}{\bar{\sigma}}) - \Phi(\frac{c}{\bar{\sigma}}) + e^{\frac{\sigma^2}{2}} \left(e^{-\frac{c}{\bar{\sigma}}} \Phi(-\underline{\sigma} + \frac{c}{\bar{\sigma}}) - e^{\frac{c}{\bar{\sigma}}} \Phi(-\underline{\sigma} - \frac{c}{\bar{\sigma}}) \right) \right] & c > 0, \end{cases} \quad (2.21)$$

where Φ is the standard normal cdf.

The density q in (2.19) yields a zero mean and variance equal to $\underline{\sigma}\bar{\sigma}$, the geometric average of the two extreme variances. Incorporation of the low (high) variance normal density for positive (negative) arguments reflects risk aversion and loving on the two subdomains respectively. Evidently, q reduces to the normal density if $\underline{\sigma} = \bar{\sigma}$, for example, there is a single arm. Then (2.18) is an immediate implication of the classic CLT. In the same way, (i) follows directly from the new CLT in section 3.3. Moreover, (i) is the main content of the theorem - the other parts follow immediately from it. (ii) follows by inspection of the density and (iii) follows from a simple calculation (see details in Appendix B).

Part (ii) supports our hypothesis that the long-horizon heuristic reduces the cognitive burden of the decision-maker. She need only know the variances of arms, and even then, only for arms that have extreme variances.¹² Here is some

¹²The identity of the extreme arms can vary with history. Therefore, there may not exist two arms that are adequate for the entire horizon.

rough intuition: Let the horizon be n and consider the choice of arm at the last stage given past realizations x_i of X_i , $i < n$. It can be thought of as maximizing $E_{P_n^s} [\varphi ((\sum_1^{n-1} x_i + X_n)/\sqrt{n})]$ by choice of s_n (P_n^s is the 1-step-ahead conditional in (2.5)). The incremental payoff X_n/\sqrt{n} is small if n is large. Thus a second-order Taylor series expansion in X_n can be used to approximate the objective function, implying that the latter can be approximated (for each s) by a linear function of both the mean (equal to zero by (2.6)) and the (conditional) variance. Finally, maximization of a linear function of variance is necessarily achieved at an arm associated with either $\underline{\sigma}$ or $\bar{\sigma}$.

To interpret (iii), consider first the case $c = 0$. Thus, for large n , maximum expected utility is approximately equal to that achievable when the payoff to each action is riskless, hence identically equal to the common mean, implying zero gains and losses for sure. In other words, *risk is a matter of indifference in the limit*. The freedom to switch between arms in response to experience is critical. If one arm must be chosen ex ante for all trials, then maximum expected utility is negative, hence less than $\varphi(0) = 0$. (The classic CLT applies to each arm separately and, by loss aversion, $\varphi(-x) < -\varphi(x)$ for all $x > 0$; hence $\varphi(\cdot)$ has negative expected value under the normal $\mathbb{N}(0, \sigma^2)$ for any positive variance.) For further perspective, consider the following lottery: Toss a fair coin. If Heads, then receive a positive prize according to $\mathbb{N}(0, \underline{\sigma}^2)$ conditioned on \mathbb{R}_+ and if Tails receive a negative prize according to $\mathbb{N}(0, \bar{\sigma}^2)$ conditioned on \mathbb{R}_- . This lottery has negative expected utility using φ . It is less attractive because the ability to choose actions sequentially affords some influence over positive versus negative outcomes, while in the lottery that influence belongs to nature alone.

Finally, (iii) implies that, *in the limit $n \rightarrow \infty$, a decision-maker with a positive reference point ($c > 0$) strictly prefers the risky sequential choice problem to receiving zero gain/loss for sure*. The intuition is that zero for sure is a certain loss relative to a positive reference point, which makes it unattractive. A positive reference point c also reduces the limit value V , because it reduces all gains and increases all losses ($\varphi(x) \overset{c}{\searrow}$ for all x), but to a lesser degree because of the flexibility afforded by switching actions. Similarly, a negative reference point implies the preference for the certain zero outcome. In this sense, a higher benchmark or aspiration level leads to more participation in risky endeavors.

Remark 2.3. *Suppose that DM uses the unweighted arithmetic average and maximizes $E_{P_n^s} [\varphi ((\sum_1^n X_i)/n)]$. Then a LLN would replace the CLT underlying (2.18) and would yield, by the LLN in Peng (2019, Theorem 2.4.1),*

$$\lim_{n \rightarrow \infty} V_n = \varphi(0) = 0. \quad (2.22)$$

To reflect, consider the special case where there is independence across trials of a single arm and across arms. Then by the classic LLN, the expected utility of

playing any $a \in \mathcal{A}$ at every stage and history converges to 0 as $n \rightarrow \infty$. Consequently, for large n , DM is approximately indifferent between repeated plays of a and repeated plays of any other a' , because their means are identical. The implication of (2.22) is that all such single-arm strategies are asymptotically optimal, from which we conclude that, (in our setting, where only variances differ), the LLN cannot serve as the basis for usefully approximating optimal strategies for finite horizon problems. Furthermore, under the LLN, (2.22) is valid not only for the loss averse functions φ that we assume throughout, but also for all (suitably bounded and continuous) φ satisfying $\varphi(0) = 0$. In contrast, in our model using the \sqrt{n} -weighted average, such asymptotic risk neutrality is satisfied only in the knife-edge case $c = 0$, and risk is even strictly desirable for $c > 0$.

2.2.2 Strategies and the absence of learning

We describe an asymptotically optimal strategy for the special case where there is no learning. The latter corresponds to the following restriction on the primitive conditionals $\{P_i^{s_i}\}_{i \geq 1, s_i \in \mathcal{S}_i}$: For all $i \geq 1$, $s_i \in \mathcal{S}_i$ and histories $(a^{(i-1)}, \omega^{(i-1)})$,

$$P_i^{s_i}(\cdot \mid a^{(i-1)}, \omega^{(i-1)}) = P_1^{s_1} \quad \text{if } s_i(a^{(i-1)}, \omega^{(i-1)}) = s_1. \quad (2.23)$$

Recall that at stage 1, history is null. Thus s_1 is simply an action and $P_1^{s_1}$ gives (unconditional or) prior beliefs about the outcome of action s_1 . Thus (2.23) stipulates that for each given action (s_1 above), subsequent *beliefs about the next outcome of that action do not change with history* (where history includes past outcomes associated with any, possibly different, action). An implication is that for each fixed arm a , the joint probability distribution over outcomes given repeated choice of a is i.i.d. However, for other strategies s , the induced measure P^s (recall (2.4)) need not be a product measure. (For example, if ω_1 and ω'_1 are distinct outcomes, and if s specifies different actions at the histories (a_1, ω_1) and (a_1, ω'_1) , then the two conditional probability distributions for stage 2 outcomes generally differ. This reflects a difference in the choice of action at stage 2 rather than updating or learning.)

Define

$$\sigma_a^2 = E_{P_1^{s_1}} [X_1^2], \quad \text{if } s_1 = a \in \mathcal{A}.$$

Then

$$\bar{\sigma} = \max_{a \in \mathcal{A}} \sigma_a \quad \text{and} \quad \underline{\sigma} = \min_{a \in \mathcal{A}} \sigma_a.$$

For simplicity, we focus first on $c = 0$ and then indicate at the end of this subsection how to accommodate $c \neq 0$.

Theorem 2.4. Let $c = 0$. Define strategy s^* by $s_1^* = \bar{a}$ and, for $n > 1$,

$$s_n^* = \begin{cases} \bar{a} & \text{if } \sum_1^{n-1} X_i \leq 0 \\ \underline{a} & \text{if } \sum_1^{n-1} X_i > 0 \end{cases} \quad (2.24)$$

where $\sigma_{\bar{a}} = \bar{\sigma}$ and $\sigma_{\underline{a}} = \underline{\sigma}$. Then: (i) s^* is asymptotically optimal.
(ii) For every $N > 0$,

$$P^{s^*}(\cap_{n=N}^{\infty} \{\sum_1^n X_i \leq 0\}) \leq \frac{\bar{\sigma}}{\bar{\sigma} + \underline{\sigma}} < 1 \quad \text{and}$$

$$P^{s^*}(\cap_{n=N}^{\infty} \{\sum_1^n X_i > 0\}) \leq \frac{\underline{\sigma}}{\bar{\sigma} + \underline{\sigma}} < 1.$$

(iii) The high variance action is chosen less frequently in the limit. In fact,

$$\lim_{n \rightarrow \infty} \frac{P^{s^*}(\sigma_{s_n^*} = \bar{\sigma})}{P^{s^*}(\sigma_{s_n^*} = \underline{\sigma})} = \frac{\sigma}{\bar{\sigma}} < 1. \quad (2.25)$$

(i) identifies an asymptotically optimal s^* , while (ii) states that s^* exhibits switching between actions indefinitely with positive probability according to the measure P^{s^*} induced by s^* . The latter fact indicates a difference between our model with loss aversion and many bandit models. Commonly in the bandit literature, learning (or exploration) provides the reason for switching, and eventually it is decided that one arm is superior and experimentation ceases. Here, in contrast, switching is optimal even in the absence of learning and (with positive probability) persists indefinitely. This is because loss aversion implies that the identity of the more attractive action or arm depends on whether one is in a region of cumulative gains ($\sum_1^n X_i > 0$) or cumulative losses ($\sum_1^n X_i < 0$).¹³ Finally, (iii) gives explicitly the limiting relative frequencies induced by s^* .

We have emphasized the robustness of our results to the specification of beliefs. An implication such robustness is given by comparing the present no-learning model with a more general model where learning is admitted. Take beliefs to be objective and consider DM being offered the choice between two regimes, and suppose she chooses the one with the higher limiting value. One regime corresponds to the conditional probability laws assumed to satisfy the assumptions in section 2.1.1. This regime accommodates many different correlation patterns across trials. The other regime also satisfies (2.23). Suppose further that the initial probability laws $\{P_1^{s_1}\}_{s_1 \in \mathcal{A}}$ are common to the two regimes. Then these two specifications imply the same variance bounds $\bar{\sigma}$ and $\underline{\sigma}$. Therefore, by Theorem 2.2, they imply

¹³A global risk averter would choose the low variance action \underline{a} at every stage.

the same limiting value V . Consequently, *the correlation (or lack thereof) between outcomes in distinct trials of an action is a matter of indifference in the limit.*¹⁴

Remark 2.5. *It does not follow that there is an asymptotically optimal strategy common to both regimes. Theorem 2.7 in the next section describes an optimal strategy under learning for a special case.*

Finally, we describe how the theorem can be extended to accommodate $c \neq 0$. For that purpose, instead of using a single strategy to approximate finite-horizon problems, consider a *sequence* $s^n = (s_i^n)$ of *strategies*, where, for each n , $s^n \in \mathcal{S}$ is thought of as a strategy used in the n -horizon problem (2.15). (Accordingly, components s_i^n with $i > n$ are irrelevant.) The counterpart of (2.17) is

$$\lim_{n \rightarrow \infty} E_{P^{s^n}}[\varphi(\sum_1^n X_i/\sqrt{n})] = \lim_{n \rightarrow \infty} V_n = V \quad (2.26)$$

Then, arguing as in the proof of Theorem 2.4, one can show that (2.26) is satisfied by s^n , where, for each $n \geq 1$ and $1 \leq i \leq n$,

$$s_i^n = \begin{cases} \bar{a} & \text{if } \sum_1^{i-1} X_j/\sqrt{n} \leq c \\ \underline{a} & \text{if } \sum_1^{i-1} X_j/\sqrt{n} > c. \end{cases}$$

s_i^n can be defined arbitrarily if either $n = 1$ or $i > n$.

2.2.3 A classic two-armed bandit problem revisited

There are two arms, a and b , hence $\mathcal{A} = \{a, b\}$. Thus the set of possible outcomes for each arm and stage is $\bar{\Omega} = \{1, -1, 0\}$, and outcomes are governed, both *ex ante* and *for any history*, by the following probabilities:

$$\begin{aligned} \text{arm } a: & \Pr(1) = \Pr(-1) = p_a/2 \\ \text{arm } b: & \Pr(1) = \Pr(-1) = p_b/2. \end{aligned}$$

For each arm, outcomes follow a random walk with zero mean and with variance equal to the appropriate value of p . It is known that

$$\{p_a, p_b\} = \{\underline{p}, \bar{p}\}, \quad (2.27)$$

where $0 < \underline{p} < \bar{p} < 1$ are known; equivalently, the set of variances $\{\underline{\sigma}^2, \bar{\sigma}^2\}$ is known, where

$$\underline{\sigma}^2 = \underline{p} \text{ and } \bar{\sigma}^2 = \bar{p}.$$

¹⁴In the case of a single arm, the above reduces to the following familiar property of the classic martingale CLT. Let $P \in \Delta(\Pi_1^\infty \Omega_i)$ satisfy the counterparts of (2.6) and (2.7). Then they are also satisfied by Q , the i.i.d. product of the marginal $P_1 \in \Delta(\Omega_1)$. Moreover, P and Q imply the same limiting probability distribution for $\sum_1^n X_i/\sqrt{n}$.

However, there is uncertainty about which of \underline{p} and \bar{p} describes arm a and which describes arm b , that is, there is uncertainty about which arm has the higher variance. DM has prior beliefs about which arm is which, and forms Bayesian posteriors as experience accumulates. At each stage, she chooses which arm to pull taking into account what she has learned about the arms from past experience.

Remark 2.6. *Uncertainty about "which arm is which" in a 2-arm setting is a classic version of the bandit problem (Bradt, Johnson and Karlin 1956; Feldman 1962); indeed, the former refer to it (p. 1060) as "the Two-armed Bandit." These and subsequent papers typically assume a finite horizon and maximization of the expected value of the sum of payoffs, (in particular, means rather than variances are the focus).*

Our framework accommodates the above learning process. The set of primitive conditionals $\{P_n^{s_n}\}_{n \geq 1, s_n \in \mathcal{S}_n}$ is defined as follows. DM's prior beliefs about which arm is which are completely specified by μ_1 , the probability she assigns initially to $p_a = \underline{p}$. Thus, prior probabilities of the outcomes from choosing arm α , $\alpha = a, b$, are given by

$$\begin{aligned} P_1^a(1) &= \mu_1 \underline{p}/2 + (1 - \mu_1) \bar{p}/2 = P_1^a(-1) \\ P_1^b(1) &= (1 - \mu_1) \underline{p}/2 + \mu_1 \bar{p}/2 = P_1^b(-1), \end{aligned}$$

which can be expressed in terms of our formalism by

$$\begin{aligned} P_1^{s_1}(\omega_1) &= I_{\{s_1=a, \omega_1 \neq 0\}} [\mu_1 \underline{p}/2 + (1 - \mu_1) \bar{p}/2] \\ &\quad + I_{\{s_1=a, \omega_1=0\}} [\mu_1(1 - \underline{p}) + (1 - \mu_1)(1 - \bar{p})] \\ &\quad + I_{\{s_1=b, \omega_1 \neq 0\}} [(1 - \mu_1) \underline{p}/2 + \mu_1 \bar{p}/2] \\ &\quad + I_{\{s_1=b, \omega_1=0\}} [(1 - \mu_1)(1 - \underline{p}) + \mu_1(1 - \bar{p})]. \end{aligned}$$

For later stages, DM updates her prior probability that $p_a = \underline{p}$ to the Bayesian posterior μ_n , $n > 1$, defined inductively by

$$\begin{aligned} &\log \left(\frac{\mu_{n+1}/(1 - \mu_{n+1})}{\mu_n/(1 - \mu_n)} \right) \tag{2.28} \\ &= [I_a(a_n) - I_b(a_n)] \left((1 - I_0(\omega_n)) \log \left(\frac{\underline{p}}{\bar{p}} \right) + I_0(\omega_n) \log \left(\frac{1 - \underline{p}}{1 - \bar{p}} \right) \right). \end{aligned}$$

Then the conditional probability $P_n^{s_n}$, for each $n > 1$ and stage strategy s_n , is given by

$$\begin{aligned} P_n^{s_n}(\omega_n | a^{(n-1)}, \omega^{(n-1)}) &= I_{\{s_n=a, \omega_n \neq 0\}} [\mu_n \underline{p}/2 + (1 - \mu_n) \bar{p}/2] \\ &\quad + I_{\{s_n=a, \omega_n=0\}} [\mu_n(1 - \underline{p}) + (1 - \mu_n)(1 - \bar{p})] \tag{2.29} \end{aligned}$$

$$\begin{aligned}
& + I_{\{s_n=b, \omega_n \neq 0\}} \left[(1 - \mu_n) \underline{p}/2 + \mu_n \bar{p}/2 \right] \\
& + I_{\{s_n=b, \omega_n = 0\}} \left[(1 - \mu_n)(1 - \underline{p}) + \mu_n(1 - \bar{p}) \right].
\end{aligned}$$

Consider also the probability measure P^s , for $s \in \mathcal{S}$, constructed as in (2.4) by pasting the above conditionals. It is completely described by its restriction to finite dimensional cylinders, and thus view P^s as a measure on $\Pi_1^n \Omega_i$. For any $\omega^{(n)} = (\omega_1, \dots, \omega_n)$, the outcomes of the first n trials, and the given s , define the induced frequency vector $f^s(\omega^{(n)})$,

$$f^s(\omega^{(n)}) = (f_a^s(\omega^{(n)}), f_b^s(\omega^{(n)}), f_{a,0}^s(\omega^{(n)}), f_{b,0}^s(\omega^{(n)})), \quad (2.30)$$

where: for $\alpha \in \{a, b\}$, $f_\alpha^s(\omega^{(n)})$ and $f_{\alpha,0}^s(\omega^{(n)})$ give, respectively, the number of trials of arm α and the number of those that yield the outcome 0. Then the ex ante probability of the above outcomes are given by¹⁵

$$\begin{aligned}
P^s(\omega_1, \dots, \omega_n) &= \mu_1 \left[(\underline{p}/2)^{f_a^s - f_{a,0}^s} (\bar{p}/2)^{f_b^s - f_{b,0}^s} (1 - \underline{p})^{f_{a,0}^s} (1 - \bar{p})^{f_{b,0}^s} \right] \\
&+ (1 - \mu_1) \left[(\bar{p}/2)^{f_a^s - f_{a,0}^s} (\underline{p}/2)^{f_b^s - f_{b,0}^s} (1 - \bar{p})^{f_{a,0}^s} (1 - \underline{p})^{f_{b,0}^s} \right].
\end{aligned} \quad (2.31)$$

The two terms on the right correspond to the two possible scenarios, $p_a = \underline{p}$ or \bar{p} , weighted by their prior probabilities. Conditional on each scenario the expression reflects two assumptions: (i) independence between distinct trials, whether conducted with the same arm or with different arms; and (ii) all trials with a given arm are viewed as similar (or interchangeable) so that the probability of any (finite) sequence of outcomes for that arm is invariant to any reordering (accordingly, for each arm, the probability of a set of outcomes depends only on the number of occurrences of 0 and $\{1, -1\}$). This latter assumption of "symmetry" within each arm is known as *partial exchangeability*, a property introduced by de Finetti (1938), who also showed that it implies conditional independence as in (i), and, in fact, that it characterizes a representation such as in (2.31).¹⁶

The preceding satisfies all the assumptions of our general model and hence Theorem 2.2 applies. Moreover, with the added structure assumed herein we can also address strategies and what is learned asymptotically. Below we assume $c = 0$.

¹⁵The proof is elementary and is omitted.

¹⁶The stronger property of exchangeability, which is better known, assumes interchangeability also across distinct arms and thus views the two arms as being identical, which is excluded in our case because of (2.27) and $\underline{p} \neq \bar{p}$. See Link (1980) and Diaconis and Freedman (1982) for more on partial exchangeability and Kallenberg (2005) for a comprehensive treatment of probabilistic symmetries.

Define the strategy s^* by $s_1^* = a$ and, for $n > 1$,

$$s_n^* = \begin{cases} a & \text{if } \begin{array}{l} \Sigma_1^{n-1} X_j \leq 0, \mu_n < \frac{1}{2} \text{ OR} \\ \Sigma_1^{n-1} X_j > 0, \mu_n > \frac{1}{2} \end{array} \\ b & \text{if } \text{otherwise} \end{cases}$$

According to s^* , arm a is used at stage $n > 1$ if (and only if) there are cumulative losses and it is more likely that a has higher variance ($\mu_n < \frac{1}{2}$), or there are cumulative gains and it is more likely that a has lower variance ($\mu_n > \frac{1}{2}$). Intuition argues for this choice of arm at stage n if there are no later trials remaining, but may seem myopic more generally. Nevertheless, we show that s^* is approximately optimal for large horizons. (For other instances where myopic strategies are optimal in bandit problems see, for example, Banks and Sundaram (1992) and the papers cited in Remark 2.6.)

Theorem 2.7. *Let $c = 0$ and $\mu_1 \in [0, 1]$. Then s^* is asymptotically optimal.*

When $\mu_1 \in \{0, 1\}$, we are back in the no-learning case of the last section and Theorem 2.4 applies.

Conclude with observations about the process of posteriors $\{\mu_n\}$ that confirm for our setting properties familiar from Bayesian learning theory.

Remark 2.8. *Let $s \in \mathcal{S}$ be any strategy. Then:*

(i) *Posteriors converge to certainty, that is, for any prior μ_1 ,*¹⁷

$$\lim_{n \rightarrow \infty} \mu_n \in \{0, 1\} \quad P^s\text{-a.s.} \quad (2.32)$$

(ii) *Suppose that, unknown to the decision-maker, the truth is that $p_a = \underline{p}$. Consequently, given any strategy s , outcomes are governed by the probability law $Q^s \in \Delta(\Pi_1^\infty \Omega_i, \mathcal{G})$, whose 1-step-ahead conditionals are Q_i^s , $i \geq 1$, given by*

$$Q_i^s(1) = Q_i^s(-1) = \begin{cases} \underline{p}/2 & \text{if } s_i = a \\ \overline{p}/2 & \text{if } s_i = b \end{cases}$$

Then, for every $\mu_1 > 0$,

$$\lim_{n \rightarrow \infty} \mu_n = 1 \quad Q^s\text{-a.s.} \quad (2.33)$$

Think of $\{\mu_n\}$ as representing subjective beliefs. Then (2.32) expresses the decision-maker's ex ante complete confidence that asymptotically she will know "which arm is which." In (ii), Q^s is the true probability law over outcome sequences when strategy s is adopted, and hence (2.33) is an expression of "Bayesian consistency". Both results are valid for any strategy, and thus reflect Bayesian updating alone and not asymptotic optimality.

¹⁷In fact, convergence to certainty is valid for every P^s , $s \in \mathcal{S}$.

3 A Central Limit Theorem

3.1 Preliminaries

The mathematical basis for our analysis of the bandit problem is a central limit theorem about sets of measures that will be provided here. To smooth the transition for the reader, we begin with a few remarks about connect the bandit model to sets of measures.

In section 2.1.1, we introduced the primitive set of one-step-ahead conditionals $\{P_i^{s_i}\}_{i \geq 1, s_i \in \mathcal{S}_i}$, and then pointed out that, for each $s = (s_1, \dots, s_i, \dots)$, these conditionals can be pasted together to obtain a measure $P^s \in \Delta(\Pi_1^\infty \Omega_i, \mathcal{G})$. Now we collect all these measures and define the set $\mathcal{P} \subset \Delta(\Pi_1^\infty \Omega_i, \mathcal{G})$ by

$$\mathcal{P} = \{P^s : s \in \mathcal{S}\}. \quad (3.1)$$

Our CLT will be applied to this set. However, in order to better reveal its underlying structure and to facilitate other potential applications, (for example, to models concerned with robustness to model uncertainty), the CLT will be formulated and proven more generally. In particular, rather than restricting ourselves to the sets associated with the bandit problem, the CLT will take as a primitive an abstract set $\mathcal{P} \subset \Delta(\Pi_1^\infty \Omega_i, \mathcal{G})$ satisfying "rectangularity," defined in the next section, which we suggest is key to the CLT, and which we show (Lemma 3.2) is satisfied by the set defined in (3.1).

One more observation is helpful for the transition. For the set \mathcal{P} defined by (3.1), it is immediate that, for each n ,

$$V_n = \sup_{s \in \mathcal{S}} E_{P^s}[\varphi(\Sigma_1^n X_i / \sqrt{n})] = \sup_{Q \in \mathcal{P}} E_Q[\varphi(\Sigma_1^n X_i / \sqrt{n})]. \quad (3.2)$$

The CLT will involve expressions such as that on the right in (3.2). However, we can think of the supremum over measures as equivalent to optimization over strategies.

To proceed, adopt the mathematical primitives $(\Pi_1^\infty \Omega_i, \{\mathcal{G}_n\}_{n=1}^\infty)$ and \mathcal{G} , though with possibly different interpretations.¹⁸ For each $i \geq 1$, $X_i : \Pi_1^\infty \Omega_j \rightarrow \mathbb{R}$ is \mathcal{G}_i -measurable. Another primitive is a set $\mathcal{P} \subset \Delta(\Pi_1^\infty \Omega_i, \mathcal{G})$, not to be confused with the set in (3.1). The following assumptions are adopted throughout: All measures in \mathcal{P} are equivalent on each \mathcal{G}_n ,

$$E_Q[X_i | \mathcal{G}_{i-1}] = 0 \text{ for all } Q \in \mathcal{P} \text{ and all } i \geq 1, \quad (3.3)$$

¹⁸In fact, we do not need the previous assumptions that Ω_i is identical for all i and finite. Here the Ω_i s are arbitrary.

and conditional variances satisfy, for some $\bar{\sigma} \geq \underline{\sigma} > 0$,

$$\operatorname{ess\,sup}_{Q \in \mathcal{P}} E_Q [X_i^2 | \mathcal{G}_{i-1}] = \bar{\sigma}^2 \text{ and } \operatorname{ess\,inf}_{Q \in \mathcal{P}} E_Q [X_i^2 | \mathcal{G}_{i-1}] = \underline{\sigma}^2 \text{ for all } i \geq 1. \quad (3.4)$$

Assume also that (X_i) satisfies the *Lindeberg condition*:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \sup_{Q \in \mathcal{P}} E_Q [|X_i|^2 I_{\{|X_i| > \sqrt{n}\epsilon\}}] = 0, \quad \forall \epsilon > 0. \quad (3.5)$$

When \mathcal{P} is a singleton, these conditions reduce to standard assumptions in the classic martingale CLT.

The following notation is convenient in the sequel: \mathcal{H} denotes the set of all random variables X on $(\Pi_1^\infty \Omega_i, \mathcal{G})$ satisfying $\sup_{Q \in \mathcal{P}} E_Q[|X|] < \infty$. For any X in \mathcal{H} , its (conditional) *upper expectations* are defined by

$$\mathbb{E}[X] \equiv \sup_{Q \in \mathcal{P}} E_Q[X], \quad \mathbb{E}[X | \mathcal{G}_n] \equiv \operatorname{ess\,sup}_{Q \in \mathcal{P}} E_Q[X | \mathcal{G}_n].$$

3.2 Rectangularity

The final assumption underlying our CLT is that the set \mathcal{P} is "rectangular", which means that it is closed with respect to the pasting of alien 1-step-ahead conditionals. Rectangularity was introduced in Epstein and Schneider (2003) in the context of recursive utility theory, where an axiomatic analysis demonstrated its role in modeling dynamic behavior for an ambiguity-averse decision-maker.¹⁹ When \mathcal{P} is the singleton $\{P\}$, rectangularity is trivially implied by Bayesian updating, specifically by the fact that after decomposing P into its 1-step-ahead conditionals, these can be pasted together to recover P . More generally, rectangularity requires that the set \mathcal{P} is closed also with respect to pasting together 1-step-ahead conditionals that are alien, that is, induced by possibly different measures in \mathcal{P} . This property endows \mathcal{P} with a recursive structure that yields a form of the law of iterated expectations (see below).

For a formal definition, we introduce some additional notation. Write

$$\begin{aligned} \omega_{(n)} &= (\omega_n, \dots), \quad \omega^{(n)} = (\omega_1, \dots, \omega_n), \\ \mathcal{P}_{0,n} &= \{P|_{\mathcal{G}_n} : P \in \mathcal{P}\} \text{ and} \end{aligned}$$

¹⁹It has been studied and applied also in robust stochastic dynamic optimization (Iyengar, 2005; Shapiro, 2016), in the literature on dynamic risk measures (Riedel, 2004; Cheridito, Delbaen and Kupper, 2006; Acciaio and Penner, 2011), and in continuous-time modeling in finance (Chen and Epstein, 2002).

$$\mathcal{G}_{(n+1)} = \{A \subset \Pi_{n+1}^\infty \Omega_i : \Pi_1^n \Omega_i \times A \in \mathcal{G}\}.$$

A probability kernel from $(\Pi_1^n \Omega_i, \mathcal{G}_n)$ to $(\Pi_{n+1}^\infty \Omega_i, \mathcal{G}_{(n+1)})$ is a function $\lambda : \Pi_1^n \Omega_i \times \mathcal{G}_{(n+1)} \rightarrow [0, 1]$ satisfying:

Kernel 1: $\forall \omega^{(n)} \in \Pi_1^n \Omega_i$, $\lambda(\omega^{(n)}, \cdot)$ is a probability measure on $(\Pi_{n+1}^\infty \Omega_i, \mathcal{G}_{(n+1)})$,

Kernel 2: $\forall A \in \mathcal{G}_{(n+1)}$, $\lambda(\cdot, A)$ is a \mathcal{G}_n -measurable function on $\Pi_1^n \Omega_i$.

Any pair (p_n, λ) consisting of a probability measure p_n on $(\Pi_1^n \Omega_i, \mathcal{G}_n)$ and a probability kernel λ as above, induces a unique probability measure P on $(\Pi_1^\infty \Omega_i, \mathcal{G})$ that coincides with p_n on \mathcal{G}_n . It is given by, $\forall A \in \mathcal{G}$,

$$P(A) = \int_{\Pi_1^n \Omega_i} \int_{\Pi_{n+1}^\infty \Omega_i} I_A(\omega^{(n)}, \omega_{(n+1)}) \lambda(\omega^{(n)}, d\omega_{(n+1)}) p_n(d\omega^{(n)}). \quad (3.6)$$

For $Q \in \mathcal{P}$, let $Q(\cdot | \mathcal{G}_n)$, denote its induced (regular) conditional. Then it defines a probability kernel λ by: $\forall \omega^{(n)} \in \Pi_1^n \Omega_i$,

$$\lambda(\omega^{(n)}, A) = Q(\Pi_1^n \Omega_i \times A | \mathcal{G}_n)(\omega^{(n)}), \forall A \in \mathcal{G}_{(n+1)}. \quad (3.7)$$

A feature of such a kernel is that the single measure Q is used to define the conditional at every $\omega^{(n)}$. We are interested in kernels for which the measure to be conditioned can vary with $\omega^{(n)}$. Thus say that the probability kernel λ is a \mathcal{P} -kernel if: $\forall \omega^{(n)} \in \Pi_1^n \Omega_i \exists Q \in \mathcal{P}$ such that (3.7) is satisfied.

Finally, say that \mathcal{P} is *rectangular* (with respect to the filtration $\{\mathcal{G}_n\}$) if: $\forall n \forall p_n \in \mathcal{P}_{0,n}$ and for every \mathcal{P} -kernel λ , if P is defined as in (3.6), then $P \in \mathcal{P}$.

The significance of rectangularity is illuminated by the following lemma. (Its proof can be found in Chen and Epstein (2020).)

Lemma 3.1. \mathcal{P} rectangular implies the following (for any $0 \leq m \leq n \in N$).

(i) **Stability by composition:** For any $Q, R \in \mathcal{P}$, $\exists P \in \mathcal{P}$ such that, for any $X \in \mathcal{H}$,

$$E_P[X | \mathcal{G}_m] = E_Q[E_R[X | \mathcal{G}_n] | \mathcal{G}_m].$$

(ii) **Stability by bifurcation:** For any $Q, R \in \mathcal{P}$, and any $A_n \in \mathcal{G}_n$, $\exists P \in \mathcal{P}$ such that, for any $X \in \mathcal{H}$,

$$E_P[X | \mathcal{G}_n] = I_{A_n} E_Q[X | \mathcal{G}_n] + I_{A_n^c} E_R[X | \mathcal{G}_n].$$

(iii) **Law of iterated upper expectations:** For any $X \in \mathcal{H}$,

$$\mathbb{E}[\mathbb{E}[X | \mathcal{G}_n] | \mathcal{G}_m] = \mathbb{E}[X | \mathcal{G}_m]. \quad (3.8)$$

(iv) Let $\{X_i\}$ be a sequence in \mathcal{H} . Then, for any continuous bounded functions f, h :

$$\mathbb{E} [f (\Sigma_1^{n-1} X_i) + h (\Sigma_1^{n-1} X_i) X_n^2] = \mathbb{E} [\mathbb{E} [f (\Sigma_1^{n-1} X_i) + h (\Sigma_1^{n-1} X_i) X_n^2 | \mathcal{G}_{n-1}]] .$$

If the conditional means and variances of X_n satisfy (3.3) and (3.4), then

$$\mathbb{E} [h (\Sigma_1^{n-1} X_i) X_n^2 | \mathcal{G}_{n-1}] = \bar{\sigma}^2 [h(\Sigma_1^{n-1} X_i)]^+ - \underline{\sigma}^2 [h(\Sigma_1^{n-1} X_i)]^- .$$

(Superscripts $+$ and $-$ denote the positive and negative parts respectively.)

Part (iii) gives the law of iterated expectations for upper expectations (a similar condition for lower expectations is implied). (iv) is an extension that is used in the proofs of our CLTs. Parts (i) and (ii) of the lemma describe direct implications of \mathcal{P} being "closed with respect to the pasting of alien marginals and conditionals."

Importantly, the set of measures generated in the bandit model is rectangular.

Lemma 3.2. *The set \mathcal{P} defined in (3.1) is rectangular.*

Proof: Because of the finiteness of each Ω_i in the bandit model, it suffices to consider the following. Let $P^s, P^{s'}$ and $P^{s''}$ be measures in \mathcal{P} . Take $n > 1$ and $A_n \in \mathcal{G}_n$, and paste the measures together by constructing the new measure P by,

$$P(B) = \int_{\Pi_1^n \Omega_i} \int_{\Pi_{n+1}^\infty \Omega_i} I_B(\omega^{(n)}, \omega_{(n+1)}) \lambda(\omega^{(n)}, d\omega_{(n+1)}) P_{|\mathcal{G}_n}^s(d\omega^{(n)}), \forall B \in \mathcal{G},$$

where, for all $(\omega^{(n)}, A) \in \Pi_{i=1}^n \Omega_i \times \mathcal{G}_{(n+1)}$,

$$\lambda(\omega^{(n)}, A) = I_{A_n}(\omega^{(n)}) P^{s'}(\Pi_1^n \Omega_i \times A | \mathcal{G}_n) + I_{A_n^c}(\omega^{(n)}) P^{s''}(\Pi_1^n \Omega_i \times A | \mathcal{G}_n) .$$

Then $P = P^{\hat{s}}$, where $\hat{s} = (s_1, \dots, s_n, \hat{s}_{n+1}, \hat{s}_{n+2}, \dots) \in \mathcal{S}$ is defined by, for any $i > n$ and $\omega^{(i-1)} = (\omega^{(n)}, \dots, \omega_{i-1})$,

$$\begin{aligned} \hat{s}_i(\omega^{(i-1)}) &= I_{A_n}(\omega^{(n)}) s'_i(a'^{(i-1)}, \omega^{(i-1)}) + I_{A_n^c}(\omega^{(n)}) s''_i(a''^{(i-1)}, \omega^{(i-1)}), \\ \text{and } a'_j &= s'_j(a'^{(j-1)}, \omega^{(j-1)}), \quad a''_j = s''_j(a''^{(j-1)}, \omega^{(j-1)}), \text{ for } 1 \leq j \leq i-1. \quad \blacksquare \end{aligned}$$

As a concrete example, consider again the special case of the bandit model with no-learning (section 2.2.2). In that model, for each arm a , the probability distribution over outcomes on a single trial is P_1^a , ($P_1^a \equiv P_1^{s_1}$ where $s_1 = a$), independent of history. Denote by \mathcal{L} the set of all such measures over outcomes as a

varies over all arms. Since any arm can be chosen at any history, the corresponding set of measures over outcome sequences is²⁰

$$\mathcal{P} = \{P \in \Delta(\Pi_1^\infty \Omega_i, \mathcal{G}) : P_i \in \mathcal{L} \text{ for every } i \text{ and history}\}.$$

(P_i is the 1-step-ahead conditional at stage i induced by P .) This set is obviously rectangular. Note that in spite of \mathcal{L} being common to all trials and histories, trial outcomes as modeled by \mathcal{P} are not necessarily identical. Indeed, any measure in \mathcal{L} can describe the i^{th} trial at a specific history in conjunction with any possibly different measure in \mathcal{L} being the law describing the j^{th} trial at any other history. As a result, besides the restriction imposed by \mathcal{L} , the set \mathcal{P} imposes no restrictions on the pattern of heterogeneity across trials. In particular, it *accommodates heteroscedasticity that is largely unstructured* apart from the restrictions imposed by the extreme variances $\underline{\sigma}$ and $\bar{\sigma}$. To varying degrees, the same is true for all rectangular sets associated with our bandit model.

3.3 The theorem

We extend (a version of) the classic martingale CLT to admit a set of variances while maintaining the assumption of a fixed zero mean. Throughout (B_t) denotes a standard Brownian motion under a probability space $(\Omega^*, \mathcal{F}^*, P^*)$ and $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration generated by (B_t) .

In the classic case, the limiting distribution is normal, which is the distribution of B_1 . In the more general case, the corresponding (upper) limit is not given by the normal distribution, but is described instead by the time 1 value of an *oscillating Brownian motion* (Keilson and Wellner 1978; Lejay and Pigato 2018), defined as follows: Given $\bar{\sigma} \geq \underline{\sigma} > 0$ and threshold $c \in \mathbb{R}$, let (W_t^c) denote the unique strong solution, (which exists by Le Gall (1984)), of the stochastic differential equation (SDE)

$$Y_t = \int_0^t \sigma(Y_s) dB_s, \quad t \geq 0, \quad (3.9)$$

where the diffusion coefficient σ is the positive two-valued function, discontinuous at the threshold c ,

$$\sigma(y) = \underline{\sigma} I_{[c, \infty)}(y) + \bar{\sigma} I_{(-\infty, c)}(y), \quad \forall y \in \mathbb{R}. \quad (3.10)$$

There is a seeming connection to the bandit model - lower volatility in the region (c, ∞) of gains where there is risk aversion, and large volatility in the region of losses $(-\infty, c)$ where there is risk loving.²¹ In fact, by Keilson and Wellner

²⁰Formally, it follows from (2.23) and (2.5).

²¹Reversing the roles of $\bar{\sigma}$ and $\underline{\sigma}$ also defines an oscillating Brownian motion, but one that is irrelevant here given the assumption of loss aversion.

(1978, Theorem 1), the time 1 value W_1^c of the oscillating Brownian motion has distribution given by the density q referred to in Theorem 2.2(i).

Theorem 3.3. *Let the sequence (X_i) be such that $X_i \in \mathcal{H}$ for each i , and where (X_i) satisfies (3.3) and (3.4), with conditional upper and lower variances $\bar{\sigma}^2 \geq \underline{\sigma}^2 > 0$. Assume also the Lindeberg condition (3.5), that measures in \mathcal{P} are equivalent on each \mathcal{G}_i , and that \mathcal{P} is rectangular. Set $\theta = \underline{\sigma}/\bar{\sigma}$. For any $c \in \mathbb{R}$ and $\varphi_1 \in C_b^3(\mathbb{R}_+)$, with $\varphi_1(0) = 0$, define φ by*

$$\varphi(x) = \begin{cases} \varphi_1(x - c) & x \geq c \\ -\frac{1}{\theta}\varphi_1(-\theta(x - c)) & x < c \end{cases} \quad (3.11)$$

If $\varphi_1''(x) \leq 0$ for $x \geq 0$, then

$$\lim_{n \rightarrow \infty} \sup_{Q \in \mathcal{P}} E_Q \left[\varphi \left(\frac{\sum_1^n X_i}{\sqrt{n}} \right) \right] = E_{P^*}[\varphi(W_1^c)]. \quad (3.12)$$

The most important point to make about the theorem is that all its assumptions are satisfied by the bandit model with \mathcal{P} defined by (3.1). (The Lindeberg condition (3.5) is satisfied because of the finiteness of $\Omega_i = \bar{\Omega}$.) Therefore, using also the noted density for W_1^c , the CLT implies Theorem 2.2(i). Though the bandit theorem is stated with reference only to a density and not to oscillating Brownian motions, we prefer to include the latter here because it is more revealing of what underlies the limit and, to a degree, how the limit result is proven.

For perspective, if instead of defining φ by (3.11), we took φ to be any (suitably bounded, smooth and) *globally concave* function, then the limit in (3.12) would equal the expected value of φ under $\mathbb{N}(0, \underline{\sigma}^2)$, as in the classic case with fixed variance $\underline{\sigma}^2$. Informally, this result is suggested by taking $c \rightarrow -\infty$ above. (For a rigorous argument, see Proposition 2.2.15 and Theorem 2.4.4 in Peng (2019).)

Some extensions of the CLT are possible. For example, one can obtain similar limits with any combination of the modifications $\theta = \frac{\bar{\sigma}}{\underline{\sigma}}$, $\varphi_1''(x) \geq 0$ on $(0, \infty)$, and/or one considers the limit of the lower expectation $\inf_{Q \in \mathcal{P}} E_Q [\varphi(\sum_1^n X_i/\sqrt{n})]$. These extensions do not seem relevant to the bandit problem, but the reader can find them in our working paper version listed in the bibliography. It is also possible to derive closed-form limiting results for other integrands (functions φ), for example, for some indicator functions (Appendix A.3). For many other functions φ , the corresponding expressions for the limit are more complex, less transparent and arguably intractable, and consequently are excluded.

We conclude with mention of related CLTs in the literature. Chen and Epstein (2020) establish CLTs assuming, contrary to (3.3)-(3.4), that conditional means lie in an interval $[\underline{\mu}, \bar{\mu}]$ while all conditional variances equal a constant σ^2 . In common with this paper, rectangularity is a key assumption. However, their theorems

are substantially different, for example, limits have a different form and proofs are much different. There exist other generalizations of the classic CLT that are motivated by robustness to ambiguity. In both Marinacci (1999, Theorem 16) and Epstein, Kaido and Seo (2016), experiments are not ordered and their analyses are better suited for a cross-sectional, rather than sequential, context. Another difference is that in both cases, limiting distributions are normal. Peng (2007, 2019) and Fang et al (2019) assume that experiments are ordered. Comparison with Theorem 3.2 of the latter is representative. It is more general than our results, for example, in permitting ambiguity about both mean and variance. For purposes of comparison, limit attention to the special case of their theorem where there is ambiguity about variance only. Even then, an important difference, particularly given the application developed here, is that greater generality comes arguably at the cost of reduced tractability. In particular, limits are much more complicated (they involve Peng’s (2007) notion of a "G-normal" distribution), and a counterpart of Theorem 3.3 is not apparent from their results.²² Finally, none of the above papers recognize the potential application to sequential decision problems such as the bandit problem.

A Appendix: Main Proofs

The notation and assumptions in Theorem 3.3 are adopted throughout this appendix. Let (B_t) be the standard Brownian motion under a probability space $(\Omega^*, \mathcal{F}^*, P^*)$, and let $(\mathcal{F}_t)_{t \geq 0}$ be the natural filtration generated by $(B_t)_{t \geq 0}$.

A.1 Lemmas

For a small fixed $h > 0$, and any fixed $(t, x, c) \in [0, 1 + h] \times \mathbb{R} \times \mathbb{R}$, $(Y_s^{t,x,c})_{s \in [t, 1+h]}$ denotes the solution of the SDE

$$\begin{cases} dY_s^{t,x,c} = \sigma(Y_s^{t,x,c}) dB_s, & s \in [t, 1+h] \\ Y_t^{t,x,c} = x, \end{cases} \quad (\text{A.1})$$

where $\sigma(y) = \underline{\sigma}I_{[c,\infty)}(y) + \bar{\sigma}I_{(-\infty,c)}(y)$, $\forall y \in \mathbb{R}$.

By Keilson and Wellner (1978, Theorem 1), (see also Chen and Zili (2015)), the transition probability density of $(Y_s^{t,x,c})_{s \in [t, 1+h]}$ is given by, for any $t < s \leq 1 + h$

²²Another difference, apart from applicability, is that Peng and coauthors take a nonlinear expectation operator as the core primitive and adopt the PDE approach, while our primitive is a set of probability measures. Notably, conditionals are central in our analysis, including in the notion of rectangularity, but are not easily accommodated in the operator approach. Thus their key assumption is similar in spirit to rectangularity, but is expressed in terms of nonlinear expectations and without recourse to conditionals.

and $y \in \mathbb{R}$,

$$q^c(t, x; s, y) = \frac{1}{\sqrt{2\pi(s-t)}} \frac{1}{\sigma(y)} \exp\left(-\frac{\left(\frac{x-c}{\sigma(x)} - \frac{y-c}{\sigma(y)}\right)^2}{2(s-t)}\right) + \frac{\bar{\sigma} - \underline{\sigma}}{\bar{\sigma} + \underline{\sigma}} \frac{1}{\sqrt{2\pi(s-t)}} \frac{\text{sgn}(y-c)}{\sigma(y)} \exp\left(-\frac{\left(\left|\frac{x-c}{\sigma(x)}\right| + \left|\frac{y-c}{\sigma(y)}\right|\right)^2}{2(s-t)}\right). \quad (\text{A.2})$$

Given $\varphi_1 \in C_b^3(\mathbb{R}_+)$, φ is defined by (3.11). Then

$$\varphi \in C_b^1(\mathbb{R}) \text{ and } \varphi''(z+c) = -\frac{\underline{\sigma}}{\bar{\sigma}} \varphi''\left(-\frac{\underline{\sigma}}{\bar{\sigma}}z+c\right), \quad \forall z < 0.$$

Define the set of functions $\{H_t\}_{t \in [0, 1+h]}$ by

$$H_t(x) = E_{P^*} [\varphi(Y_{1+h}^{t,x,c})], \quad \forall x \in \mathbb{R}. \quad (\text{A.3})$$

Then

$$H_{1+h}(x) = \varphi(x), \quad H_0(0) = E_{P^*}[\varphi(Y_{1+h}^{0,0,c})] = E_{P^*}[\varphi(W_{1+h}^c)].$$

The following lemma describes some properties of the functions $\{H_t\}_{t \in [0, 1+h]}$.

Lemma A.1. *The functions $\{H_t\}$ defined by (A.3) satisfy:*

- (1) *For any $t \in [0, 1]$, $H_t \in C_b^2(\mathbb{R})$, and the first and second derivatives of H_t are bounded uniformly in $t \in [0, 1]$.*
- (2) *There exists a constant L such that, for any $x_1, x_2 \in \mathbb{R}$ and $t \in [0, 1]$,*

$$|H_t''(x_1) - H_t''(x_2)| \leq L|x_1 - x_2|.$$

- (3) *If $\varphi''(x) \leq 0$ for $x > c$, then*

$$\begin{cases} H_t''(x) \leq 0 & \text{for } x \geq c \\ H_t''(x) \geq 0 & \text{for } x \leq c. \end{cases}$$

(4) For any $r \in [0, 1 + h - t]$,

$$H_t(x) = E_{P^*} [H_{t+r}(Y_{t+r}^{t,x,c})], \quad \forall x \in \mathbb{R}.$$

(5) If $\varphi''(x) \leq 0$ for $x > c$, then

$$\lim_{n \rightarrow \infty} \sum_{m=1}^n \sup_{x \in \mathbb{R}} \left| H_{\frac{m-1}{n}}(x) - H_{\frac{m}{n}}(x) - \frac{\bar{\sigma}^2}{2n} [H_{\frac{m}{n}}''(x)]^+ + \frac{\sigma^2}{2n} [H_{\frac{m}{n}}''(x)]^- \right| = 0.$$

(6) There exists a constant C_0 such that

$$\sup_{x \in \mathbb{R}} |H_1(x) - \varphi(x)| \leq C_0 \sqrt{\underline{\sigma}^2 + \bar{\sigma}^2} \sqrt{h}.$$

Proof: (1) Given the transition probability density in (A.2), we have, for $t \in [0, 1]$,

$$H_t(x) = \int_{-\infty}^{\infty} \varphi(y) q^c(t, x; 1 + h, y) dy, \quad \forall x \in \mathbb{R}.$$

For $T = 1 + h$, we have

$$H_t'(x) = \begin{cases} \frac{1}{\underline{\sigma} \sqrt{2\pi(T-t)}} \int_0^{\infty} \varphi_1'(y) \left[e^{-\frac{(x-c+y)^2}{2\underline{\sigma}^2(T-t)}} + e^{-\frac{(x-c-y)^2}{2\underline{\sigma}^2(T-t)}} \right] dy & \text{if } x \geq c \\ \frac{1}{\bar{\sigma} \sqrt{2\pi(T-t)}} \int_0^{\infty} \varphi_1'\left(\frac{\sigma}{\bar{\sigma}}y\right) \left[e^{-\frac{(x-c+y)^2}{2\bar{\sigma}^2(T-t)}} + e^{-\frac{(x-c-y)^2}{2\bar{\sigma}^2(T-t)}} \right] dy & \text{if } x \leq c \end{cases}$$

$$H_t''(x) = \begin{cases} \frac{1}{\underline{\sigma} \sqrt{2\pi(T-t)}} \int_0^{\infty} \varphi_1''(y) e^{-\frac{(x-c-y)^2}{2\underline{\sigma}^2(T-t)}} \left[1 - e^{-\frac{2y(x-c)}{\underline{\sigma}^2(T-t)}} \right] dy & \text{if } x \geq c \\ \frac{-1}{\bar{\sigma} \sqrt{2\pi(T-t)}} \int_0^{\infty} \frac{\sigma}{\bar{\sigma}} \varphi_1''\left(\frac{\sigma}{\bar{\sigma}}y\right) e^{-\frac{(x-c+y)^2}{2\bar{\sigma}^2(T-t)}} \left[1 - e^{-\frac{2y(x-c)}{\bar{\sigma}^2(T-t)}} \right] dy & \text{if } x \leq c. \end{cases}$$

The assertion follows from $\varphi_1 \in C_b^3(\mathbb{R}_+)$ and the definition of φ in (3.11).

(2) For any $x > c$, $H_t'''(x) =$

$$\frac{1}{\underline{\sigma} \sqrt{2\pi(T-t)}} \left[2\varphi_1''(0) e^{-\frac{(x-c)^2}{2\underline{\sigma}^2(T-t)}} + \int_0^{\infty} \varphi_1'''(y) \left(e^{-\frac{(x-c-y)^2}{2\underline{\sigma}^2(T-t)}} + e^{-\frac{(x-c+y)^2}{2\underline{\sigma}^2(T-t)}} \right) dy \right],$$

and, for $x < c$, $H_t'''(x) =$

$$\frac{1}{\bar{\sigma} \sqrt{2\pi(T-t)}} \left[-2\varphi_1''(0) e^{-\frac{(x-c)^2}{2\bar{\sigma}^2(T-t)}} + \int_0^{\infty} \frac{\sigma^2}{\bar{\sigma}^2} \varphi_1''' \left(\frac{\sigma}{\bar{\sigma}} y \right) \left(e^{-\frac{(x-c-y)^2}{2\bar{\sigma}^2(T-t)}} + e^{-\frac{(x-c+y)^2}{2\bar{\sigma}^2(T-t)}} \right) dy \right].$$

Since $\varphi_1 \in C_b^3(\mathbb{R}_+)$, there exists a constant L such that

$$\sup_{x \in \mathbb{R}, x \neq c} |H_t'''(x)| \leq L \text{ for all } t \in [0, 1].$$

The assertion follows by the Mean Value Theorem.

(3) It follows from the explicit form of $H_t''(x)$ given above.

(4) Since $(Y_s^{t,x,c})$ is a time-homogeneous Markov process, for any $r \in [0, 1 + h - t]$,

$$H_t(x) = E_{P^*}[\varphi(Y_{1+h}^{t,x,c})] = E_{P^*}[E_{P^*}[\varphi(Y_{1+h}^{t,x,c})|\mathcal{F}_{t+r}]] = E_{P^*}[H_{t+r}(Y_{t+r}^{t,x,c})].$$

(5) It follows from part (4) that, for any $1 \leq m \leq n$,

$$H_{\frac{m-1}{n}}(x) = E_{P^*}\left[H_{\frac{m}{n}}\left(Y_{\frac{m}{n}}^{\frac{m-1}{n},x,c}\right)\right].$$

Apply Itô's formula to $H_{\frac{m}{n}}\left(Y_{\frac{m}{n}}^{\frac{m-1}{n},x,c}\right)$ to derive

$$\begin{aligned} H_{\frac{m}{n}}\left(Y_{\frac{m}{n}}^{\frac{m-1}{n},x,c}\right) &= H_{\frac{m}{n}}(x) + \int_{\frac{m-1}{n}}^{\frac{m}{n}} H_{\frac{m}{n}}'\left(Y_s^{\frac{m-1}{n},x,c}\right) \sigma\left(Y_s^{\frac{m-1}{n},x,c}\right) dB_s \\ &\quad + \frac{1}{2} \int_{\frac{m-1}{n}}^{\frac{m}{n}} H_{\frac{m}{n}}''\left(Y_s^{\frac{m-1}{n},x,c}\right) \left(\sigma\left(Y_s^{\frac{m-1}{n},x,c}\right)\right)^2 ds \end{aligned}$$

Using parts (3) and (4), we have

$$\begin{aligned} H_{\frac{m-1}{n}}(x) &= E_{P^*}\left[H_{\frac{m}{n}}\left(Y_{\frac{m}{n}}^{\frac{m-1}{n},x,c}\right)\right] = \\ &E_{P^*}\left[H_{\frac{m}{n}}(x) + \frac{\bar{\sigma}^2}{2} \int_{\frac{m-1}{n}}^{\frac{m}{n}} \left[H_{\frac{m}{n}}''\left(Y_s^{\frac{m-1}{n},x,c}\right)\right]^+ ds - \frac{\sigma^2}{2} \int_{\frac{m-1}{n}}^{\frac{m}{n}} \left[H_{\frac{m}{n}}''\left(Y_s^{\frac{m-1}{n},x,c}\right)\right]^- ds\right] \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{m=1}^n \sup_{x \in \mathbb{R}} \left| H_{\frac{m-1}{n}}(x) - H_{\frac{m}{n}}(x) - \frac{\bar{\sigma}^2}{2n} \left[H_{\frac{m}{n}}''(x)\right]^+ + \frac{\sigma^2}{2n} \left[H_{\frac{m}{n}}''(x)\right]^- \right| \\ &\leq \sum_{m=1}^n \sup_{x \in \mathbb{R}} E_{P^*} \left[\frac{\sigma^2 + \bar{\sigma}^2}{2} \int_{\frac{m-1}{n}}^{\frac{m}{n}} \left| H_{\frac{m}{n}}''\left(Y_s^{\frac{m-1}{n},x,c}\right) - H_{\frac{m}{n}}''(x) \right| ds \right] \\ &\leq \sum_{m=1}^n \sup_{x \in \mathbb{R}} \frac{(\sigma^2 + \bar{\sigma}^2)L}{2n} E_{P^*} \left[\sup_{s \in [\frac{m-1}{n}, \frac{m}{n}]} \left| Y_s^{\frac{m-1}{n},x,c} - x \right| \right] \end{aligned}$$

$$\leq \sum_{m=1}^n \sup_{x \in \mathbb{R}} \frac{C}{n} \left(E_{P^*} \left[\int_{\frac{m-1}{n}}^{\frac{m}{n}} \left(\sigma \left(Y_s^{\frac{m-1}{n}, x, c} \right) \right)^2 dr \right] \right)^{\frac{1}{2}} \leq \frac{C \sqrt{\underline{\sigma}^2 + \bar{\sigma}^2}}{\sqrt{n}},$$

where C is a constant that depends only on $\underline{\sigma}, \bar{\sigma}, L$.

(6) Since $\varphi \in C_b^1(\mathbb{R})$, $C_0 \equiv \|\varphi'\| = \sup_{x \in \mathbb{R}} |\varphi'(x)| < \infty$, and

$$\begin{aligned} \sup_{x \in \mathbb{R}} |H_1(x) - \varphi(x)| &= \sup_{x \in \mathbb{R}} |E_{P^*}[\varphi(Y_{1+h}^{1,x,c})] - \varphi(x)| \\ &\leq \sup_{x \in \mathbb{R}} E_{P^*} [|\varphi(Y_{1+h}^{1,x,c}) - \varphi(x)|] \\ &\leq \sup_{x \in \mathbb{R}} C_0 E_{P^*} \left[\left| \int_1^{1+h} \sigma(Y_s^{1,x,c}) dB_s \right| \right] \\ &\leq \sup_{x \in \mathbb{R}} C_0 \left(E_{P^*} \left[\int_1^{1+h} (\sigma(Y_s^{1,x,c}))^2 ds \right] \right)^{\frac{1}{2}} \\ &\leq C_0 \sqrt{\underline{\sigma}^2 + \bar{\sigma}^2} \sqrt{h}. \quad \blacksquare \end{aligned}$$

Lemma A.2. Let $\{H_t\}_{t \in [0,1]}$ be the functions defined in (A.3), and define the family of functions $\{L_{m,n}\}_{m=1}^n$ by

$$L_{m,n}(x) = H_{\frac{m}{n}}(x) + \frac{\bar{\sigma}^2}{2n} \left[H_{\frac{m}{n}}''(x) \right]^+ - \frac{\underline{\sigma}^2}{2n} \left[H_{\frac{m}{n}}''(x) \right]^-. \quad (\text{A.4})$$

Then

$$\lim_{n \rightarrow \infty} \sum_{m=1}^n \left| \mathbb{E} \left[H_{\frac{m}{n}} \left(\frac{\sum_1^m X_i}{\sqrt{n}} \right) \right] - \mathbb{E} \left[L_{m,n} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right] \right| = 0. \quad (\text{A.5})$$

Proof: It suffices to prove

$$\lim_{n \rightarrow \infty} \sum_{m=1}^n \left| \mathbb{E} \left[H_{\frac{m}{n}} \left(\frac{\sum_1^m X_i}{\sqrt{n}} \right) \right] - f(m, n) \right| = 0 \text{ and} \quad (\text{A.6})$$

$$\lim_{n \rightarrow \infty} \sum_{m=1}^n \left| f(m, n) - \mathbb{E} \left[L_{m,n} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right] \right| = 0, \quad (\text{A.7})$$

where

$$f(m, n) = \mathbb{E} \left[H_{\frac{m}{n}} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) + H_{\frac{m}{n}}' \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \frac{X_m}{\sqrt{n}} + H_{\frac{m}{n}}'' \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \frac{X_m^2}{2n} \right].$$

By Lemma A.1, there exists $L > 0$ such that

$$\sup_{t \in [0,1]} \sup_{x \in \mathbb{R}} |H_t''(x)| \leq L, \quad \sup_{t \in [0,1]} \sup_{x, y \in \mathbb{R}, x \neq y} \frac{|H_t''(x) - H_t''(y)|}{|x - y|} \leq L.$$

By the Taylor expansion of $H_t \in C_b^2(\mathbb{R})$, $\forall \epsilon > 0 \exists \delta > 0$ (δ depends only on L and ϵ), such that, for any $x, y \in \mathbb{R}$ and $t \in [0, 1]$,

$$\left| H_t(x+y) - H_t(x) - H_t'(x)y - \frac{1}{2}H_t''(x)y^2 \right| \leq \epsilon|y|^2 I_{\{|y| < \delta\}} + L|y|^2 I_{\{|y| \geq \delta\}}. \quad (\text{A.8})$$

Let $x = \Sigma_1^{m-1} X_i / \sqrt{n}$ and $y = X_m / \sqrt{n}$ in (A.8) to derive

$$\sum_{m=1}^n \left| \mathbb{E} \left[H_{\frac{m}{n}} \left(\frac{\Sigma_1^m X_i}{\sqrt{n}} \right) \right] - f(m, n) \right| \leq \bar{\sigma}^2 \epsilon + \frac{L}{n} \sum_{m=1}^n \mathbb{E} [|X_m|^2 I_{\{|X_m| \geq \sqrt{n}\delta\}}].$$

By the arbitrariness of ϵ and the Lindeberg condition (3.5), we obtain (A.6).

By Lemma 3.1, we have

$$\begin{aligned} f(m, n) &= \mathbb{E} \left[H_{\frac{m}{n}} \left(\frac{\Sigma_1^{m-1} X_i}{\sqrt{n}} \right) + H'_{\frac{m}{n}} \left(\frac{\Sigma_1^{m-1} X_i}{\sqrt{n}} \right) \frac{X_m}{\sqrt{n}} + H''_{\frac{m}{n}} \left(\frac{\Sigma_1^{m-1} X_i}{\sqrt{n}} \right) \frac{X_m^2}{2n} \right] \\ &= \mathbb{E} \left[H_{\frac{m}{n}} \left(\frac{\Sigma_1^{m-1} X_i}{\sqrt{n}} \right) + H''_{\frac{m}{n}} \left(\frac{\Sigma_1^{m-1} X_i}{\sqrt{n}} \right) \frac{X_m^2}{2n} \right] \\ &= \mathbb{E} \left[H_{\frac{m}{n}} \left(\frac{\Sigma_1^{m-1} X_i}{\sqrt{n}} \right) + \frac{1}{2n} \mathbb{E} \left[H''_{\frac{m}{n}} \left(\frac{\Sigma_1^{m-1} X_i}{\sqrt{n}} \right) X_m^2 | \mathcal{G}_{m-1} \right] \right] \\ &= \mathbb{E} \left[L_{m,n} \left(\frac{\Sigma_1^{m-1} X_i}{\sqrt{n}} \right) \right]. \end{aligned}$$

This implies (A.7) and completes the proof. ■

A.2 Proof of the CLT (Theorem 3.3)

For $h > 0$ sufficiently small, let $\{H_t\}_{t \in [0, 1+h]}$ be the functions defined by (A.3). First prove

$$\lim_{n \rightarrow \infty} \left| \mathbb{E} \left[H_1 \left(\frac{\Sigma_1^n X_i}{\sqrt{n}} \right) \right] - E_{P^*} [\varphi(W_{1+h}^c)] \right| = 0.$$

We have

$$\mathbb{E} \left[H_1 \left(\frac{\Sigma_1^n X_i}{\sqrt{n}} \right) \right] - E_{P^*} [\varphi(W_{1+h}^c)]$$

$$\begin{aligned}
&= \mathbb{E} \left[H_1 \left(\frac{\sum_1^n X_i}{\sqrt{n}} \right) \right] - H_0(0) \\
&= \sum_{m=1}^n \left\{ \mathbb{E} \left[H_{\frac{m}{n}} \left(\frac{\sum_1^m X_i}{\sqrt{n}} \right) \right] - \mathbb{E} \left[H_{\frac{m-1}{n}} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right] \right\} \\
&= \sum_{m=1}^n \left\{ \mathbb{E} \left[H_{\frac{m}{n}} \left(\frac{\sum_1^m X_i}{\sqrt{n}} \right) \right] - \mathbb{E} \left[L_{m,n} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right] \right\} \\
&\quad + \sum_{m=1}^n \left\{ \mathbb{E} \left[L_{m,n} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right] - \mathbb{E} \left[H_{\frac{m-1}{n}} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right] \right\} \\
&=: I_{1n} + I_{2n},
\end{aligned}$$

where $L_{m,n}(x) = H_{\frac{m}{n}}(x) + \frac{\bar{\sigma}^2}{2n} [H_{\frac{m}{n}}''(x)]^+ - \frac{\sigma^2}{2n} [H_{\frac{m}{n}}''(x)]^-$, $1 \leq m \leq n$.

By Lemma A.2,

$$|I_{1n}| \leq \sum_{m=1}^n \left| \mathbb{E} \left[H_{\frac{m}{n}} \left(\frac{\sum_1^m X_i}{\sqrt{n}} \right) \right] - \mathbb{E} \left[L_{m,n} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right] \right| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Furthermore, by Lemma A.1(5), as $n \rightarrow \infty$,

$$\begin{aligned}
|I_{2n}| &\leq \sum_{m=1}^n \mathbb{E} \left[\left| L_{m,n} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) - H_{\frac{m-1}{n}} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right| \right] \\
&\leq \sum_{m=1}^n \sup_{x \in \mathbb{R}} \left| L_{m,n}(x) - H_{\frac{m-1}{n}}(x) \right| \\
&= \sum_{m=1}^n \sup_{x \in \mathbb{R}} \left| H_{\frac{m-1}{n}}(x) - H_{\frac{m}{n}}(x) - \frac{\bar{\sigma}^2}{2n} [H_{\frac{m}{n}}''(x)]^+ + \frac{\sigma^2}{2n} [H_{\frac{m}{n}}''(x)]^- \right| \rightarrow 0.
\end{aligned}$$

By Lemma A.1(6), $\lim_{n \rightarrow \infty} \left| \mathbb{E} \left[\varphi \left(\frac{\sum_1^n X_i}{\sqrt{n}} \right) \right] - E_{P^*} [\varphi(W_1^c)] \right| \leq$
 $\lim_{n \rightarrow \infty} \left| \mathbb{E} \left[\varphi \left(\frac{\sum_1^n X_i}{\sqrt{n}} \right) \right] - \mathbb{E} \left[H_1 \left(\frac{\sum_1^n X_i}{\sqrt{n}} \right) \right] \right| +$
 $\lim_{n \rightarrow \infty} \left| \mathbb{E} \left[H_1 \left(\frac{\sum_1^n X_i}{\sqrt{n}} \right) \right] - E_{P^*} [\varphi(W_{1+h}^c)] \right| + |E_{P^*} [\varphi(W_{1+h}^c)] - E_{P^*} [\varphi(W_1^c)]|$
 $\leq \sup_{x \in \mathbb{R}} |H_1(x) - \varphi(x)| + C_0 \sqrt{\bar{\sigma}^2 + \underline{\sigma}^2} \sqrt{h} \leq 2C_0 \sqrt{\bar{\sigma}^2 + \underline{\sigma}^2} \sqrt{h}.$
Since h is arbitrary, the proof is complete. ■

A.3 A corollary

Indicator functions for one-sided intervals $[c, \infty)$ can be suitably approximated by functions φ satisfying the conditions in Theorem 3.3, which suggests that the

limiting result (3.12) is valid also for such indicators. The following corollary confirms this, and is of interest also because it is used below in the proof of Theorem 2.4. See our working paper version (Corollary 3.4) for a more general result that considers also indicators for intervals of the form $(-\infty, c]$.

Corollary A.3. *Adopt the assumptions in Theorem 3.3. Then, for any $c \in \mathbb{R}$,*

$$\lim_{n \rightarrow \infty} \sup_{Q \in \mathcal{P}} Q \left(\frac{\sum_1^n X_i}{\sqrt{n}} \geq c \right) = P^*(W_1^c \geq c) \quad (\text{A.9})$$

and

$$P^*(W_1^c \geq c) = \begin{cases} \frac{2\bar{\sigma}}{\bar{\sigma} + \underline{\sigma}} \Phi\left(-\frac{c}{\bar{\sigma}}\right) & c > 0 \\ 1 - \frac{2\underline{\sigma}}{\bar{\sigma} + \underline{\sigma}} \Phi\left(\frac{c}{\underline{\sigma}}\right) & c \leq 0 \end{cases}, \quad (\text{A.10})$$

where Φ is the standard normal cdf.

Proof: For any $c \in \mathbb{R}$ and $\varepsilon > 0$, suppose that $f_1, g_1 \in C_b^3(\mathbb{R}_+)$ satisfy

$$\begin{cases} f_1(x) = 1 & \text{for } x \geq \frac{\underline{\sigma}}{\bar{\sigma}}\varepsilon \\ f_1''(x) \leq 0 & \text{for } x \geq 0 \\ f_1(0) = \frac{\bar{\sigma}}{\bar{\sigma} + \underline{\sigma}} \end{cases} \quad \begin{cases} g_1(x) = 1 & \text{for } x \geq \varepsilon \\ g_1''(x) \leq 0 & \text{for } x \geq 0 \\ g_1(0) = \frac{\bar{\sigma}}{\bar{\sigma} + \underline{\sigma}} \end{cases}$$

Define f_ε and g_ε by

$$f_\varepsilon(x) = \begin{cases} f_1(x - c - \varepsilon) & \text{for } x \geq c + \varepsilon \\ -\frac{\bar{\sigma}}{\underline{\sigma}} f_1\left(-\frac{\underline{\sigma}}{\bar{\sigma}}(x - c - \varepsilon)\right) + \frac{\bar{\sigma}}{\underline{\sigma}} & \text{for } x \leq c + \varepsilon \end{cases} \quad (\text{A.11})$$

$$g_\varepsilon(x) = \begin{cases} g_1(x - c + \varepsilon) & \text{for } x \geq c - \varepsilon \\ -\frac{\bar{\sigma}}{\underline{\sigma}} g_1\left(-\frac{\underline{\sigma}}{\bar{\sigma}}(x - c + \varepsilon)\right) + \frac{\bar{\sigma}}{\underline{\sigma}} & \text{for } x \leq c - \varepsilon \end{cases} \quad (\text{A.12})$$

It can be checked that

$$g_\varepsilon(x) \geq I_{[c, \infty)}(x) \geq f_\varepsilon(x) \quad \text{and} \\ |g_\varepsilon(x) - f_\varepsilon(x)| \leq I_{[c - (1 + \frac{\bar{\sigma}}{\underline{\sigma}})\varepsilon, c + (1 + \frac{\underline{\sigma}}{\bar{\sigma}})\varepsilon]}(x), \quad \forall x \in \mathbb{R}.$$

Consider the solution $(\widetilde{W}_t^x)_{t \geq 0}$ of the SDE

$$\begin{cases} d\widetilde{W}_t^x = \left(\underline{\sigma} I_{[0, \infty)}(\widetilde{W}_t^x) + \bar{\sigma} I_{(-\infty, 0)}(\widetilde{W}_t^x) \right) dB_t, & t \geq 0 \\ \widetilde{W}_0^x = x. \end{cases} \quad (\text{A.13})$$

Then W_1^c and $c + \widetilde{W}_1^{-c}$ are described by the same law, and

$$\left| \sup_{Q \in \mathcal{P}} Q \left(\frac{\sum_1^n X_i}{\sqrt{n}} \geq c \right) - P^*(W_1^c \geq c) \right|$$

$$\begin{aligned}
&\leq \left| \mathbb{E} \left[f_\varepsilon \left(\frac{\sum_1^n X_i}{\sqrt{n}} \right) \right] - E_{P^*} [g_\varepsilon (W_1^c)] \right| + \left| \mathbb{E} \left[g_\varepsilon \left(\frac{\sum_1^n X_i}{\sqrt{n}} \right) \right] - E_{P^*} [f_\varepsilon (W_1^c)] \right| \\
&\leq \left| \mathbb{E} \left[f_\varepsilon \left(\frac{\sum_1^n X_i}{\sqrt{n}} \right) \right] - E_{P^*} [f_\varepsilon (W_1^{c+\varepsilon})] \right| + \left| \mathbb{E} \left[g_\varepsilon \left(\frac{\sum_1^n X_i}{\sqrt{n}} \right) \right] - E_{P^*} [g_\varepsilon (W_1^{c-\varepsilon})] \right| \\
&\quad + |E_{P^*} [f_\varepsilon (W_1^{c+\varepsilon}) - f_\varepsilon (W_1^c)]| + |E_{P^*} [g_\varepsilon (W_1^{c-\varepsilon}) - g_\varepsilon (W_1^c)]| \\
&\quad + 2 |E_{P^*} [f_\varepsilon (W_1^c) - g_\varepsilon (W_1^c)]| \\
&\leq \left| \mathbb{E} \left[f_\varepsilon \left(\frac{\sum_1^n X_i}{\sqrt{n}} \right) \right] - E_{P^*} [f_\varepsilon (W_1^{c+\varepsilon})] \right| + \left| \mathbb{E} \left[g_\varepsilon \left(\frac{\sum_1^n X_i}{\sqrt{n}} \right) \right] - E_{P^*} [g_\varepsilon (W_1^{c-\varepsilon})] \right| \\
&\quad + E_{P^*} \left[\left| f_\varepsilon (c + \varepsilon + \widetilde{W}_1^{-c-\varepsilon}) - f_\varepsilon (c + \widetilde{W}_1^{-c}) \right| + \left| g_\varepsilon (c - \varepsilon + \widetilde{W}_1^{-c+\varepsilon}) - g_\varepsilon (c + \widetilde{W}_1^{-c}) \right| \right] \\
&\quad + 2 |E_{P^*} [f_\varepsilon (W_1^c) - g_\varepsilon (W_1^c)]| \\
&\leq \left| \mathbb{E} \left[f_\varepsilon \left(\frac{\sum_1^n X_i}{\sqrt{n}} \right) \right] - E_{P^*} [f_\varepsilon (W_1^{c+\varepsilon})] \right| + \left| \mathbb{E} \left[g_\varepsilon \left(\frac{\sum_1^n X_i}{\sqrt{n}} \right) \right] - E_{P^*} [g_\varepsilon (W_1^{c-\varepsilon})] \right| \\
&\quad + C_0 E_{P^*} \left[2\varepsilon + \left| \widetilde{W}_1^{-c-\varepsilon} - \widetilde{W}_1^{-c} \right| + \left| \widetilde{W}_1^{-c+\varepsilon} - \widetilde{W}_1^{-c} \right| \right] + 2P^* \left(c - \left(1 + \frac{\bar{\sigma}}{\underline{\sigma}} \right) \varepsilon \leq W_1^c \leq c + \left(1 + \frac{\underline{\sigma}}{\bar{\sigma}} \right) \varepsilon \right),
\end{aligned}$$

where C_0 is a constant that depends on $\|f'_\varepsilon\|, \|g'_\varepsilon\|$. With Le Gall (1984, Theorem 1.5) and Theorem 3.3, the upper probability equation in (A.9) is proven.

The expression (A.10) may be derived by integrating the pdf (B.1)-(B.2). ■

B Appendix: Proofs for bandits

B.1 An explicit density

Let W_1^c be the $t = 1$ value of the oscillating Brownian motion defined by (3.9)-(3.10). Keilson and Wellner (1978, Theorem 1) give the following expression for its pdf: For $c \geq 0$,

$$q(y) = \begin{cases} \frac{1}{\underline{\sigma}\sqrt{2\pi}} \left(e^{-\frac{(\frac{-c}{\underline{\sigma}} - \frac{y-c}{\underline{\sigma}})^2}{2}} + \frac{\bar{\sigma}-\underline{\sigma}}{\underline{\sigma}+\bar{\sigma}} e^{-\frac{(\frac{c}{\bar{\sigma}} + \frac{y-c}{\bar{\sigma}})^2}{2}} \right) & y \geq c \\ \frac{1}{\bar{\sigma}\sqrt{2\pi}} \left(e^{-\frac{(\frac{-c}{\bar{\sigma}} - \frac{y-c}{\bar{\sigma}})^2}{2}} - \frac{\bar{\sigma}-\underline{\sigma}}{\underline{\sigma}+\bar{\sigma}} e^{-\frac{(\frac{c}{\underline{\sigma}} + \frac{c-y}{\underline{\sigma}})^2}{2}} \right) & y < c \end{cases} \quad (\text{B.1})$$

and for $c < 0$,

$$q(y) = \begin{cases} \frac{1}{\underline{\sigma}\sqrt{2\pi}} \left(e^{-\frac{(\frac{-c}{\underline{\sigma}} - \frac{y-c}{\underline{\sigma}})^2}{2}} + \frac{\bar{\sigma}-\underline{\sigma}}{\underline{\sigma}+\bar{\sigma}} e^{-\frac{(\frac{-c}{\bar{\sigma}} + \frac{y-c}{\bar{\sigma}})^2}{2}} \right) & y \geq c \\ \frac{1}{\bar{\sigma}\sqrt{2\pi}} \left(e^{-\frac{(\frac{-c}{\bar{\sigma}} - \frac{y-c}{\bar{\sigma}})^2}{2}} - \frac{\bar{\sigma}-\underline{\sigma}}{\underline{\sigma}+\bar{\sigma}} e^{-\frac{(\frac{c}{\underline{\sigma}} + \frac{c-y}{\underline{\sigma}})^2}{2}} \right) & y < c \end{cases} \quad (\text{B.2})$$

These expressions are used to derive (2.21) and to prove Corollary A.3 and Theorem 2.2.

B.2 Proof of Theorem 2.2

As indicated in the text, (i) follows from Theorem 3.3 and the above density; and (ii) follows from (i) by inspection of the above density. It remains to prove (iii).

Take $c \geq 0$. The proof for $c < 0$ is similar. In light of (3.2) and (3.12), it suffices to compute $E_{P^*}[\varphi(W_1^c)]$. Use the pdf of W_1^c in (B.1), to deduce that, for $c \geq 0$,

$$\begin{aligned}
E_{P^*}[\varphi(W_1^c)] &= \int_c^\infty q(y) \varphi_1(y-c) dy + \int_{-\infty}^c q(y) \left[-\frac{\bar{\sigma}}{\underline{\sigma}} \varphi_1\left(-\frac{\underline{\sigma}}{\bar{\sigma}}(y-c)\right) \right] dy \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\underline{\sigma}} \int_c^\infty \varphi_1(y-c) \left[\frac{2\bar{\sigma}}{\bar{\sigma} + \underline{\sigma}} \right] e^{-\frac{(\frac{c}{\bar{\sigma}} + \frac{y-c}{\underline{\sigma}})^2}{2}} dy \\
&\quad - \frac{1}{\sqrt{2\pi}} \frac{1}{\underline{\sigma}} \int_{-\infty}^c \varphi_1\left(-\frac{\underline{\sigma}}{\bar{\sigma}}(y-c)\right) \left(e^{-\frac{(\frac{-c}{\bar{\sigma}} - \frac{y-c}{\underline{\sigma}})^2}{2}} - \frac{\bar{\sigma} - \underline{\sigma}}{\underline{\sigma} + \bar{\sigma}} e^{-\frac{(\frac{c}{\bar{\sigma}} + \frac{c-y}{\underline{\sigma}})^2}{2}} \right) dy \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\underline{\sigma}} \int_0^\infty \varphi_1(y) \left[\frac{2\bar{\sigma}}{\bar{\sigma} + \underline{\sigma}} \right] e^{-\frac{(\frac{c}{\bar{\sigma}} + \frac{y}{\underline{\sigma}})^2}{2}} dy \\
&\quad - \frac{1}{\sqrt{2\pi}} \frac{1}{\underline{\sigma}} \int_{-\infty}^0 \varphi_1\left(-\frac{\underline{\sigma}}{\bar{\sigma}}y\right) \left(e^{-\frac{(\frac{-c}{\bar{\sigma}} - \frac{y}{\underline{\sigma}})^2}{2}} - \frac{\bar{\sigma} - \underline{\sigma}}{\underline{\sigma} + \bar{\sigma}} e^{-\frac{(\frac{c}{\bar{\sigma}} - \frac{y}{\underline{\sigma}})^2}{2}} \right) dy \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\underline{\sigma}} \int_0^\infty \varphi_1(y) \left[\frac{2\bar{\sigma}}{\bar{\sigma} + \underline{\sigma}} \right] e^{-\frac{(\frac{c}{\bar{\sigma}} + \frac{y}{\underline{\sigma}})^2}{2}} dy \\
&\quad - \frac{1}{\sqrt{2\pi}} \frac{1}{\underline{\sigma}} \frac{\bar{\sigma}}{\underline{\sigma}} \int_0^\infty \varphi_1(z) \left(e^{-\frac{(\frac{-c}{\bar{\sigma}} + \frac{z}{\underline{\sigma}})^2}{2}} - \frac{\bar{\sigma} - \underline{\sigma}}{\underline{\sigma} + \bar{\sigma}} e^{-\frac{(\frac{c}{\bar{\sigma}} + \frac{z}{\underline{\sigma}})^2}{2}} \right) dz \\
&= \frac{1}{\sqrt{2\pi}\underline{\sigma}} \int_0^\infty \varphi_1(y) \frac{\bar{\sigma}}{\underline{\sigma}} \left[e^{-\frac{(y+m)^2}{2\underline{\sigma}^2}} - e^{-\frac{(y-m)^2}{2\underline{\sigma}^2}} \right] dy
\end{aligned}$$

where $m = \frac{\underline{\sigma}}{\bar{\sigma}}c$. Thus we want to prove that

$$\frac{1}{\sqrt{2\pi}\underline{\sigma}} \int_0^\infty \varphi_1(y) \frac{\bar{\sigma}}{\underline{\sigma}} \left[e^{-\frac{(y+m)^2}{2\underline{\sigma}^2}} - e^{-\frac{(y-m)^2}{2\underline{\sigma}^2}} \right] dy \geq -\frac{\bar{\sigma}}{\underline{\sigma}} \varphi_1\left(\frac{\underline{\sigma}}{\bar{\sigma}}c\right),$$

with equality if and only if $c = 0$.

It is evident that $E_{P^*}[\varphi(W_1^c)] = 0 = \varphi(0)$ if $c = 0$. Henceforth, take $c > 0$ and prove that

$$\int_0^\infty \varphi_1(y) \left[\left(e^{-\frac{(y-m)^2}{2\sigma^2}} - e^{-\frac{(y+m)^2}{2\sigma^2}} \right) / \sqrt{2\pi\sigma} \right] dy < \varphi_1(m).$$

Denote by $f(y)$ the expression in the square bracket, (thus $f(y) > 0$ for all $y > 0$), and let $F \equiv \int_0^\infty f(y) dy$, $0 < F < 1$. Then f/F is a density. If its mean is μ , then, by strict concavity of φ_1 ,

$$\int_0^\infty \varphi_1(y) f(y) dy < F\varphi_1(\mu). \quad (\text{B.3})$$

Next we prove that $F\mu = m$:

$$\begin{aligned} F\mu &= \int_0^\infty y \left[\left(e^{-\frac{(y-m)^2}{2\sigma^2}} - e^{-\frac{(y+m)^2}{2\sigma^2}} \right) / \sqrt{2\pi\sigma} \right] dy \\ &= \int_{-m}^\infty (z+m) \left[e^{-\frac{z^2}{2\sigma^2}} / \sqrt{2\pi\sigma} \right] dz - \int_m^\infty (z-m) \left[\left(e^{-\frac{z^2}{2\sigma^2}} \right) / \sqrt{2\pi\sigma} \right] dz \\ &= \int_{-m}^m z \left[e^{-\frac{z^2}{2\sigma^2}} / \sqrt{2\pi\sigma} \right] dz + m \int_{-m}^\infty \left[e^{-\frac{z^2}{2\sigma^2}} / \sqrt{2\pi\sigma} \right] dz + m \int_m^\infty \left[e^{-\frac{z^2}{2\sigma^2}} / \sqrt{2\pi\sigma} \right] dz \\ &= 0 + m [\Pr(Z > -m) + \Pr(Z > m)] = m, \end{aligned}$$

where probabilities are computed according to $\mathbb{N}(0, \sigma^2)$.

Finally, $F\mu = m \implies F\varphi_1(\mu) = F\varphi_1(m/F) \leq \varphi_1(m)$, by $F < 1$, $\varphi_1(0) = 0$, and the concavity of φ_1 . Combine with (B.3) to complete the proof. \blacksquare

B.3 Proof of Theorem 2.4

(i) We are given that $c = 0$. For small enough $h > 0$, let $\{H_t\}_{t \in [0, 1+h]}$ be the corresponding functions defined by (A.3).

First prove

$$\lim_{n \rightarrow \infty} \left| E_{P^{s^*}} \left[H_1 \left(\frac{\sum_1^n X_i}{\sqrt{n}} \right) \right] - E_{P^*} [\varphi(W_{1+h}^0)] \right| = 0 \quad (\text{B.4})$$

We have

$$E_{P^{s^*}} \left[H_1 \left(\frac{\sum_1^n X_i}{\sqrt{n}} \right) \right] - E_{P^*} [\varphi(W_{1+h}^0)]$$

$$\begin{aligned}
&= E_{P^{s^*}} \left[H_1 \left(\frac{\sum_1^n X_i}{\sqrt{n}} \right) \right] - H_0(0) \\
&= \sum_{m=1}^n \left\{ E_{P^{s^*}} \left[H_{\frac{m}{n}} \left(\frac{\sum_1^m X_i}{\sqrt{n}} \right) \right] - E_{P^{s^*}} \left[H_{\frac{m-1}{n}} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right] \right\} \\
&= \sum_{m=1}^n \left\{ E_{P^{s^*}} \left[H_{\frac{m}{n}} \left(\frac{\sum_1^m X_i}{\sqrt{n}} \right) \right] - E_{P^{s^*}} \left[L_{m,n} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right] \right\} \\
&\quad + \sum_{m=1}^n \left\{ E_{P^{s^*}} \left[L_{m,n} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right] - E_{P^{s^*}} \left[H_{\frac{m-1}{n}} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right] \right\} \\
&=: J_{1n} + J_{2n},
\end{aligned}$$

where $L_{m,n}(x) = H_{\frac{m}{n}}(x) + \frac{\bar{\sigma}^2}{2n} \left[H_{\frac{m}{n}}''(x) \right]^+ - \frac{\underline{\sigma}^2}{2n} \left[H_{\frac{m}{n}}''(x) \right]^-$, $1 \leq m \leq n$.

By a similar argument to that in the proof of Lemma A.2, (using Lemma A.1(3) and the fact that $E_{P^{s^*}}[X_m^2 | \mathcal{G}_{m-1}] = I_{\{\sum_1^{m-1} X_i \leq 0\}} \bar{\sigma}^2 + I_{\{\sum_1^{m-1} X_i > 0\}} \underline{\sigma}^2$), deduce that

$$\lim_{n \rightarrow \infty} |J_{1n}| = 0.$$

On the other hand, by Lemma A.1(5), (argue as in the proof that $|I_{2n}| \rightarrow 0$ in Appendix A.2), we have $\lim_{n \rightarrow \infty} |J_{2n}| = 0$. Thus we obtain (B.4).

By the definition of functions $\{H_t\}$ and Lemma A.1(6), and arguing as at the end of Appendix A.2, the proof of (i) is complete.

(ii) By Corollary A.3, we have that, for any $N > 0$,

$$P^{s^*}(\cap_{n=N}^{\infty} \{\sum_1^n X_i > 0\}) \leq \lim_{n \rightarrow \infty} \sup_{s \in \mathcal{S}} P^s(\sum_1^n X_i / \sqrt{n} > 0) = \frac{\bar{\sigma}}{\bar{\sigma} + \underline{\sigma}} < 1.$$

By the corresponding result for the indicator of $(-\infty, c]$, (see Corollary 3.4 in our working paper version),

$$P^{s^*}(\cap_{n=N}^{\infty} \{\sum_1^n X_i \leq 0\}) \leq \lim_{n \rightarrow \infty} \sup_{s \in \mathcal{S}} P^s(\sum_1^n X_i / \sqrt{n} \leq 0) = \frac{\bar{\sigma}}{\bar{\sigma} + \underline{\sigma}} < 1.$$

(iii) To derive (2.25), argue first, as in Corollary A.3, that the indicator for $[0, \infty)$ can be approximated by a function φ satisfying conditions of the CLT and the bandit application. Then it can be shown that (2.24) is asymptotically optimal also when the indicator replaces φ , that is, when DM solves $\sup_{s \in \mathcal{S}} P^s(\sum_1^n X_i / \sqrt{n} > d)$. Finally, apply the closed-form expression in the noted corollary. ■

B.4 Proof of Theorem 2.7 and Remark 2.8

Theorem 2.7: Bayesian updating implies that $\{\mu_n\}$ is a P^{s^*} -martingale adapted to $\{\mathcal{G}_n\}$. Since $\{\mu_n\}$ is uniformly bounded, there exists a random variable μ such that

$$\lim_{n \rightarrow \infty} \mu_n = \mu \quad P^{s^*}\text{-a.s.}$$

Step 1: $\mu = 0$ or 1 P^{s^*} -a.s., which implies (2.32): Purely for simplicity, we give the argument when $\underline{p} + \bar{p} = 1$; the proof for the general case will be evident.

We have $P^{s^*}(\widehat{\Omega}) = 1$, where $\widehat{\Omega} = \{\omega \in \Omega \mid \lim_{n \rightarrow \infty} \mu_n(\omega) = \mu(\omega)\}$. For any $\omega \in \widehat{\Omega}$,

$$\mu_n(\omega) = \frac{\underline{p}\mu_{n-1}(\omega)}{\underline{p}\mu_{n-1}(\omega) + \bar{p}(1 - \mu_{n-1}(\omega))} \quad \text{or} \quad \frac{\bar{p}\mu_{n-1}(\omega)}{\bar{p}\mu_{n-1}(\omega) + \underline{p}(1 - \mu_{n-1}(\omega))}.$$

Thus, without loss of generality, there exists a subsequence $\{\mu_{k_n}\}$ satisfying

$$\mu_{k_n}(\omega) = \frac{\underline{p}\mu_{k_n-1}(\omega)}{\underline{p}\mu_{k_n-1}(\omega) + \bar{p}(1 - \mu_{k_n-1}(\omega))},$$

which implies that

$$\mu(\omega) = \frac{\underline{p}\mu(\omega)}{\underline{p}\mu(\omega) + \bar{p}(1 - \mu(\omega))}.$$

Thus $\mu(\omega) = 0$ or 1 .

Step 2: For $n \geq 1$, define

$$\underline{M}_n = \min\{\mu_n, 1 - \mu_n\}, \quad \bar{M}_n = \max\{\mu_n, 1 - \mu_n\}$$

Then, by the dominated convergence theorem,

$$\lim_{n \rightarrow \infty} E_{P^{s^*}}[\underline{M}_n] = E_{P^{s^*}}\left[\lim_{n \rightarrow \infty} \underline{M}_n\right] = E_{P^{s^*}}[\min\{\mu, 1 - \mu\}] = 0.$$

For small enough $h > 0$, let $\{H_t\}_{t \in [0, 1+h]}$ be the functions defined in (A.3), and let $\{L_{m,n}\}_{m=1}^n$ be the functions defined in (A.4). We prove below that

$$\lim_{n \rightarrow \infty} \sum_{m=1}^n \left| E_{P^{s^*}} \left[H_{\frac{m}{n}} \left(\frac{\sum_1^m X_i}{\sqrt{n}} \right) \right] - E_{P^{s^*}} \left[L_{m,n} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right] \right| = 0. \quad (\text{B.5})$$

This is the counterpart for the present setting of the limit result (A.5) in the proof of our CLT (Lemma A.2), where instead of the expectation with respect to the

single measure P^{s^*} , one has the upper expectation \mathbb{E} corresponding to the set of measures \mathcal{P} . The proof of (B.5) roughly parallels the earlier arguments but the difference between $E_{P^{s^*}}$ and \mathbb{E} necessitates some adjustments (notably in Step 4).

Define

$$d(m, n) = E_{P^{s^*}} \left[H_{\frac{m}{n}} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) + H'_{\frac{m}{n}} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \frac{X_m}{\sqrt{n}} + H''_{\frac{m}{n}} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \frac{X_m^2}{2n} \right].$$

It suffices for (B.5) to prove that

$$\sum_{m=1}^n \left| E_{P^{s^*}} \left[H_{\frac{m}{n}} \left(\frac{\sum_1^m X_i}{\sqrt{n}} \right) \right] - d(m, n) \right| \rightarrow 0 \quad \text{and} \quad (\text{B.6})$$

$$\sum_{m=1}^n \left| d(m, n) - E_{P^{s^*}} \left[L_{m,n} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right] \right| \rightarrow 0. \quad (\text{B.7})$$

Step 3: Prove (B.6). The argument is similar to that for (A.6).

Step 4: Prove (B.7). By (2.29), for any $m \geq 1$, $E_{P^{s^*}}[X_m | \mathcal{G}_{m-1}] = 0$, and

$$E_{P^{s^*}}[X_m^2 | \mathcal{G}_{m-1}] = \begin{cases} \bar{\sigma}^2 \bar{M}_m + \underline{\sigma}^2 \underline{M}_m & \text{if } \sum_1^{m-1} X_i \leq 0 \\ \underline{\sigma}^2 \bar{M}_m + \bar{\sigma}^2 \underline{M}_m & \text{if } \sum_1^{m-1} X_i > 0 \end{cases} \quad (\text{B.8})$$

Therefore, for C_1 equal to the uniform bounded of $|H_t''(x)|$,

$$\begin{aligned} & \sum_{m=1}^n \left| d(m, n) - E_{P^{s^*}} \left[L_{m,n} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right] \right| \\ & \leq \sum_{m=1}^n E_{P^{s^*}} \left[\frac{1}{2n} \left[H''_{\frac{m}{n}} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right]^+ (\bar{\sigma}^2 - \bar{\sigma}^2 \bar{M}_m - \underline{\sigma}^2 \underline{M}_m) \right] \\ & \quad + \sum_{m=1}^n E_{P^{s^*}} \left[\frac{1}{2n} \left[H''_{\frac{m}{n}} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right]^- (\underline{\sigma}^2 \bar{M}_m + \bar{\sigma}^2 \underline{M}_m - \underline{\sigma}^2) \right] \\ & \leq \frac{C_1(\bar{\sigma}^2 - \underline{\sigma}^2)}{n} \sum_{m=1}^n E_{P^{s^*}} [\underline{M}_m] \rightarrow 0 \quad (\text{by Step 2}). \end{aligned}$$

Remark B.1. Step 4 involves a departure from the arguments of the CLT. In the latter, we had by assumption (3.4) that upper and lower conditional variances were constant and equal to $\bar{\sigma}^2$ and $\underline{\sigma}^2$ respectively, while here the relevant conditional variances are under P^{s^*} and are stochastic as shown in (B.8). Also noteworthy is that, while all other steps in the argument are valid for all strategies s , Step 4 relies explicitly on $s = s^*$.

Step 5: Complete the proof. It can be checked that,

$$\begin{aligned}
& E_{P^{s^*}} \left[H_1 \left(\frac{\sum_1^n X_i}{\sqrt{n}} \right) \right] - H_0(0) \\
&= \sum_{m=1}^n \left\{ E_{P^{s^*}} \left[H_{\frac{m}{n}} \left(\frac{\sum_1^m X_i}{\sqrt{n}} \right) \right] - E_{P^{s^*}} \left[H_{\frac{m-1}{n}} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right] \right\} \\
&= \sum_{m=1}^n \left\{ E_{P^{s^*}} \left[H_{\frac{m}{n}} \left(\frac{\sum_1^m X_i}{\sqrt{n}} \right) \right] - E_{P^{s^*}} \left[L_{m,n} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right] \right\} \\
&\quad + \sum_{m=1}^n \left\{ E_{P^{s^*}} \left[L_{m,n} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right] - E_{P^{s^*}} \left[H_{\frac{m-1}{n}} \left(\frac{\sum_1^{m-1} X_i}{\sqrt{n}} \right) \right] \right\} \\
&=: \hat{J}_{1n} + \hat{J}_{2n}.
\end{aligned}$$

By (B.5), we have $\lim_{n \rightarrow \infty} |\hat{J}_{1n}| = 0$. By Lemma A.1(5), (argue as in the proof that $|I_{2n}| \rightarrow 0$ in Appendix A.2), we have $\lim_{n \rightarrow \infty} |\hat{J}_{2n}| = 0$. Therefore,

$$\lim_{n \rightarrow \infty} \left| E_{P^{s^*}} \left[H_1 \left(\frac{\sum_1^n X_i}{\sqrt{n}} \right) \right] - H_0(0) \right| = 0.$$

By the definition of functions $\{H_t\}$, with arguments similar to those at the end of Appendix A.2, we have

$$\left| E_{P^{s^*}} \left[\varphi \left(\frac{\sum_1^n X_i}{\sqrt{n}} \right) \right] - E_{P^*}[\varphi(W_1^0)] \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad \blacksquare$$

Remark 2.8: (i) is proven in Step 1 above. It is assumed there that $s = s^*$, but the identical arguments apply to any s .

Consider (ii). Let $\nu_n = \mu_n/(1 - \mu_n)$ and apply (2.28) to derive, for any s ,

$$\begin{aligned}
& \log \nu_{n+1} - \log \nu_1 \\
&= [(f_a^s(\omega^{(n)}) - f_{a,0}^s(\omega^{(n)})) - (f_b^s(\omega^{(n)}) - f_{b,0}^s(\omega^{(n)}))] \log \left(\frac{p}{\bar{p}} \right) \\
&\quad + [f_{a,0}^s(\omega^{(n)}) - f_{b,0}^s(\omega^{(n)})] \log \left(\frac{1-p}{1-\bar{p}} \right).
\end{aligned}$$

Define the sets

$$N_a = \left\{ \omega : \lim_{n \rightarrow \infty} f_a^s(\omega^{(n)}) = \infty, \lim_{n \rightarrow \infty} f_b^s(\omega^{(n)}) < \infty \right\},$$

$$\begin{aligned}
N_b &= \left\{ \omega : \lim_{n \rightarrow \infty} f_a^s(\omega^{(n)}) < \infty, \lim_{n \rightarrow \infty} f_b^s(\omega^{(n)}) = \infty \right\}, \\
N_{a,b} &= \left\{ \omega : \lim_{n \rightarrow \infty} f_a^s(\omega^{(n)}) = \infty, \lim_{n \rightarrow \infty} f_b^s(\omega^{(n)}) = \infty \right\}, \\
M_a &= \left\{ \omega : \lim_{n \rightarrow \infty} \frac{f_{a,0}^s(\omega^{(n)})}{f_a^s(\omega^{(n)})} = 1 - \underline{p} \right\}, \\
M_b &= \left\{ \omega : \lim_{n \rightarrow \infty} \frac{f_{b,0}^s(\omega^{(n)})}{f_b^s(\omega^{(n)})} = 1 - \bar{p} \right\}.
\end{aligned}$$

Consider $\omega \in N_{a,b} \cap M_a \cap M_b$: Then $\log \nu_{n+1} - \log \nu_1 =$

$$\begin{aligned}
& -f_a^s \left[\underline{p} \log \left(\frac{\bar{p}}{\underline{p}} \right) + (1 - \underline{p}) \log \left(\frac{1 - \bar{p}}{1 - \underline{p}} \right) \right] \\
& -f_b^s \left[\bar{p} \log \left(\frac{\underline{p}}{\bar{p}} \right) + (1 - \bar{p}) \log \left(\frac{1 - \underline{p}}{1 - \bar{p}} \right) \right] \\
& \equiv -f_a^s H_1 - f_b^s H_2.
\end{aligned}$$

By the concavity of \log , $H_1, H_2 < 0$. Therefore, $\nu_n \rightarrow \infty$, equivalently $\mu_n \rightarrow 1$, on $N_{a,b} \cap M_a \cap M_b$. By the LLN, $Q^s(N_{a,b} \cap M_a \cap M_b) = Q^s(N_{a,b})$. Conclude that

$$Q^s(N_{a,b} \cap \{\omega : \mu_n \rightarrow 1\}) = Q^s(N_{a,b}).$$

Similar equations apply if $N_{a,b}$ is replaced by either N_a or N_b . Finally, since $\{N_a, N_b, N_{a,b}\}$ is a partition of Ω , conclude that $Q^s(\{\omega : \mu_n \rightarrow 1\}) = 1$.

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