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ZERO DYNAMICS AND FUNNEL CONTROL FOR LINEAR ELECTRICAL CIRCUITS

THOMAS BERGER* AND TIMO REIS†

Abstract. We consider electrical circuits containing linear resistances, capacitances, inductances. The circuits can be described by differential-algebraic input-output systems, where the input consists of voltages of voltage sources and currents of current sources and the output consists of currents of voltage sources and voltages of current sources. We generalize a characterization of asymptotic stability of the circuit and give sufficient topological criteria for its invariant zeros being located in the open left half-plane. We show that asymptotic stability of the zero dynamics can be characterized by means of the interconnectivity of the circuit and that it implies that the circuit is high-gain stabilizable with any positive high-gain factor. Thereafter we consider the output regulation problem for electrical circuits by funnel control. We show that for circuits with asymptotically stable zero dynamics, the funnel controller achieves tracking of a class of reference signals within a pre-specified funnel; this means in particular that the transient behaviour of the output error can be prescribed and the funnel controller does neither incorporate any internal model for the reference signals nor any identification mechanism, it is simple in its design. The results are illustrated by a simulation of a discretized transmission line.

Key words. electrical circuits, passivity, differential-algebraic equations, zero dynamics, invariant zeros, high-gain stabilization, funnel control

AMS subject calssifications. 34A09, 15A22, 93B25, 93B52, 93C40

1. Introduction.

We consider linear differential-algebraic systems of the form

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) &= Ax(t)+Bu(t) \\ y(t) &= Cx(t)\,, \end{aligned} \tag{1.1}$$

where $E,A \in \mathbb{R}^{n,n}$, $B,C^\top \in \mathbb{R}^{n,m}$; the set of these square systems (i.e., same number of inputs and outputs) is denoted by $\Sigma_{n,m}$ and we write $[E,A,B,C] \in \Sigma_{n,m}$.

The functions $u,y : \mathbb{R} \to \mathbb{R}^m$ are called *input* and *output* of the system, respectively. A trajectory $(x,u,y) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ is said to be a *solution* of (1.1) if, and only if, it belongs to the *behaviour* of (1.1):

$$\mathfrak{B}_{[E,A,B,C]} := \left\{ (x,u,y) \in \mathscr{C}(\mathbb{R}_{\geq 0};\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m) \;\middle|\; \begin{array}{l} Ex \in \mathscr{C}^1(\mathbb{R}_{\geq 0};\mathbb{R}^n) \text{ and } (x,u,y) \\ \text{solves (1.1) for all } t \geq 0 \end{array} \right\}.$$

Particular emphasis is placed on the *zero dynamics* of (1.1). These are, for $[E,A,B,C] \in \Sigma_{n,m}$, defined by

$$\mathscr{Z}\mathscr{D}_{[E,A,B,C]} \;:=\; \left\{ (x,u,y) \in \mathfrak{B}_{[E,A,B,C]} \;\middle|\; y=0 \right\}.$$

By linearity of (1.1), $\mathscr{Z}\mathscr{D}_{[E,A,B,C]}$ is a real vector space.

The zero dynamics of (1.1) are called *autonomous* if, and only if,

$$\forall w_1,w_2 \in \mathscr{Z}\mathscr{D}_{[E,A,B,C]}\; \forall I \subseteq \mathbb{R}_{\geq 0} \text{ open interval}: \quad w_1|_I = w_2|_I \implies w_1 = w_2\,; \tag{1.2}$$

and *asymptotically stable* if, and only if,

$$\forall (x,u,y) \in \mathscr{Z}\mathscr{D}_{[E,A,B,C]}: \; \lim_{t\to\infty} \big(x(t),u(t)\big) = 0.$$

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Note that the above definitions are within the spirit of the *behavioural approach* [10] and take into account that the zero dynamics $\mathscr{ZD}_{[E,A,B,C]}$ are a linear behaviour. In this framework the definition for autonomy of a general behavior was given in [10, Sec. 3.2] and the definition of asymptotic stability in [10, Def. 7.2.1]. (Asymptotically stable) zero dynamics are the vector space of those trajectories of the system which are, loosely speaking, not visible at the output (and tend to zero).

In the present paper, we are interested in systems of the form (1.1), which arise from modified nodal analysis (MNA) models of electrical circuits, i.e.,

$$sE - A = \begin{bmatrix} s\mathrm{A}_{\mathcal{C}}\,\mathcal{C}\,\mathrm{A}_{\mathcal{C}}^\top + \mathrm{A}_{\mathcal{R}}\,\mathcal{G}\,\mathrm{A}_{\mathcal{R}}^\top & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{V}} \\ -\mathrm{A}_{\mathcal{L}}^\top & s\mathcal{L} & 0 \\ -\mathrm{A}_{\mathcal{V}}^\top & 0 & 0 \end{bmatrix}, \qquad B = C^\top = \begin{bmatrix} -\mathrm{A}_{\mathscr{I}} & 0 \\ 0 & 0 \\ 0 & -I_{n_{\mathcal{V}}} \end{bmatrix}, \tag{1.3}$$

$$x = (\eta^\top, i_{\mathcal{L}}^\top, i_{\mathcal{V}}^\top)^\top, \quad u = (i_{\mathscr{I}}^\top, v_{\mathcal{V}}^\top)^\top, \quad y = (-v_{\mathscr{I}}^\top, -i_{\mathcal{V}}^\top)^\top, \tag{1.4}$$

where

$$\left.\begin{array}{l} \mathcal{C} \in \mathbb{R}^{n_{\mathcal{C}}, n_{\mathcal{C}}}, \mathcal{G} \in \mathbb{R}^{n_{\mathcal{G}}, n_{\mathcal{G}}}, \mathcal{L} \in \mathbb{R}^{n_{\mathcal{L}}, n_{\mathcal{L}}}, \\ \mathrm{A}_{\mathcal{C}} \in \mathbb{R}^{n_e, n_{\mathcal{C}}}, \mathrm{A}_{\mathcal{R}} \in \mathbb{R}^{n_e, n_{\mathcal{G}}}, \mathrm{A}_{\mathcal{L}} \in \mathbb{R}^{n_e, n_{\mathcal{L}}}, \mathrm{A}_{\mathcal{V}} \in \mathbb{R}^{n_e, n_{\mathcal{V}}}, \mathrm{A}_{\mathscr{I}} \in \mathbb{R}^{n_e, n_{\mathscr{I}}}, \\ n = n_e + n_{\mathcal{L}} + n_{\mathcal{V}}, \quad m = n_{\mathscr{I}} + n_{\mathcal{V}}. \end{array}\right\} \tag{1.5}$$

Here $\mathrm{A}_{\mathcal{C}}$, $\mathrm{A}_{\mathcal{R}}$, $\mathrm{A}_{\mathcal{L}}$, $\mathrm{A}_{\mathcal{V}}$ and $\mathrm{A}_{\mathscr{I}}$ denote the element-related incidence matrices, $\mathcal{C}$, $\mathcal{G}$ and $\mathcal{L}$ are the matrices expressing the consecutive relations of capacitances, resistances and inductances, $\eta(t)$ is the vector of node potentials, $i_{\mathcal{L}}(t)$, $i_{\mathcal{V}}(t)$, $i_{\mathscr{I}}(t)$ are the vectors of currents through inductances, voltage and current sources, and $v_{\mathcal{V}}(t)$, $v_{\mathscr{I}}(t)$ are the voltages of voltage and current sources.

We show that, for models of electrical circuits (1.3), asymptotic stability of the zero dynamics is a structural property. That is, this property can be guaranteed if the circuit has certain interconnectivity properties. These criteria do not incorporate any parameter values. In this context, we also characterize the absence of invariant zeros in the close right half-plane and stabilization by high-gain output-feedback. For systems with asymptotically stable zero dynamics, we prove that funnel control is feasible.

We close the introduction with the nomenclature used in this paper.

$\mathbb{N}$, $\mathbb{N}_0$		set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, set of all integers, resp.
$\mathbb{R}_{\geq 0}$, $(\mathbb{R}_{\geq 0})$	=	$[0,\infty)$, $((0,\infty))$
$\mathbb{C}_+ (\mathbb{C}_-)$		open set of complex numbers with positive (negative) real part, resp.
$\mathbb{R}[s]$		the ring of polynomials with coefficients in $\mathbb{R}$
$\mathbb{R}(s)$		the quotient field of $\mathbb{R}[s]$
$R^{n,m}$		the set of $n \times m$ matrices with entries in a ring R
$\mathbf{Gl}_n(R)$		the group of invertible matrices in $R^{n,n}$
$\mathscr{O}_n(\mathbb{R})$		the group of orthogonal matrices in $\mathbb{R}^{n,n}$
M^*	=	$\overline{M}^\top$, the conjugate transpose of $M \in \mathbb{C}^{n,m}$
$\|x\|$	=	$\sqrt{x^\top x}$, the Euclidean norm of $x \in \mathbb{R}^n$

$\lVert M\rVert$	=	$\max\{\lVert Mx\rVert \mid x\in\mathbb{R}^m,\ \lVert x\rVert=1\}$, induced norm of $M\in\mathbb{R}^{n,m}$
$\mathscr{C}^\ell(\mathscr{I};\mathbb{R}^n)$		the set of ℓ-times continuously differentiable functions $f:\mathscr{T}\to\mathbb{R}^n$, $\ell\in\mathbb{N}_0\cup\{\infty\}$, $\mathscr{T}\subseteq\mathbb{R}$ an interval
$\mathscr{B}^\ell(\mathscr{T};\mathbb{R}^n)$	=	$\{f\in\mathscr{C}^\ell(\mathscr{T};\mathbb{R}^n)\mid \frac{\mathrm{d}^i}{\mathrm{d}t^i}f$ is bounded for $i=0,\dots,\ell\}$, $\ell\in\mathbb{N}_0\cup\{\infty\}$, $\mathscr{T}\subseteq\mathbb{R}$ an interval

2. Matrix pencils and rational functions. Let $sE-A\in\mathbb{R}[s]^{k,n}$ be a matrix pencil. Then $sE-A$ is called *regular* if, and only if, $k=n$ and $\det(sE-A)\in\mathbb{R}[s]\setminus\{0\}$.

We introduce the following notation: For $k\in\mathbb{N}$, we define the matrices

$$N_k=\begin{bmatrix}0&&\\1&\ddots&\\&\ddots&\ddots\\&&1\ 0\end{bmatrix}\in\mathbb{R}^{k\times k},\quad K_k=\begin{bmatrix}1\ 0&&\\&\ddots\ \ddots&\\&&1\ 0\end{bmatrix},\ L_k=\begin{bmatrix}0\ 1&&\\&\ddots\ \ddots&\\&&0\ 1\end{bmatrix}\in\mathbb{R}^{(k-1)\times k}.$$

Many properties of a matrix pencil can be characterized in terms of the *Kronecker canonical form (KCF).*

LEMMA 2.1 (Kronecker canonical form [6]). *For a matrix pencil $sE-A\in\mathbb{C}[s]^{k,n}$, there exist matrices $W\in\mathbf{Gl}_k(\mathbb{C})$, $T\in\mathbf{Gl}_n(\mathbb{C})$, such that*

$$W(sE-A)T=\operatorname{diag}(\mathscr{C}_1(s),\dots,\mathscr{C}_k(s)),\tag{2.1}$$

where each of the pencils $\mathscr{C}_j(s)$ is of one of the types presented in Table 2.1.

The numbers λ appearing in the blocks of type W1 are called the generalized eigenvalues *of $sE-A$. A generalized eigenvalue is called* semi-simple, *if all blocks of type W1 corresponding to λ are of size 1×1.*

The index $\nu\in\mathbb{N}_0$ *of $sE-A$ is defined as*

$$\nu:=\max\left(\left\{\ k_j\ \middle|\ \mathscr{C}_j(s)\ \textit{is of type W2 or W4},\ j=1,\dots,k\ \right\}\cup\{0\}\right).$$

Type	Size	$\mathscr{C}_j(s)$	Parameters
W1	$k_j\times k_j$	$(s-\lambda)I_{k_j}-N_{k_j}$	$k_j\in\mathbb{N}$, $\lambda\in\mathbb{C}$
W2	$k_j\times k_j$	$sN_{k_j}-I_{k_j}$	$k_j\in\mathbb{N}$
W3	$(k_j-1)\times k_j$	$sK_{k_j}-L_{k_j}$	$k_j\in\mathbb{N}$
W4	$k_j\times(k_j-1)$	$sK_{k_j}^\top-L_{k_j}^\top$	$k_j\in\mathbb{N}$

Table 2.1: Block types in Kronecker canonical form

The following is immediate from the block structure of the KCF.

COROLLARY 2.2 (Generalized eigenvalues). *Let a pencil $sE-A\in\mathbb{R}[s]^{k,n}$ be given. Then $\lambda\in\mathbb{C}$ is a generalized eigenvalue of $sE-A$ if, and only if,*

$$\operatorname{rk}_{\mathbb{C}}(\lambda E-A)<\operatorname{rk}_{\mathbb{R}(s)}(sE-A).$$

It is shown in [6] that the KCF is unique up to permutation of the indices $j=1,\dots,k$. Since each block of type W3 (W4) leads to an additional column (resp. row) rank deficiency of 1, the regularity of a pencil is equivalent to the absence of blocks of type W3 and W4 in its KCF.

In the following we collect some facts on rational matrix functions. These concepts and findings will play an important role for the analysis of the MNA model (1.1), (1.3).

DEFINITION 2.3 (Positive real/proper rational function). *A rational matrix function $G(s) \in \mathbb{R}(s)^{m,m}$ is called* positive real *if, and only if, $G(s)$ does not have any poles in $\mathbb{C}_+$ and, for all $\lambda \in \mathbb{C}_+$, we have*

$$G(\lambda) + G^*(\lambda) \geq 0.$$

$G(s)$ is called proper *if, and only if, $\lim_{s\to\infty} G(s) \in \mathbb{R}^{m,m}$ exists.*

LEMMA 2.4 (Properties of positive real functions [1, Sec. 5.1]). *Let $G(s) \in \mathbb{R}(s)^{m,m}$ be positive real. Then there exist $\omega_1,\ldots,\omega_k \in \mathbb{R}$, Hermitian and positive semi-definite matrices $M_1,\ldots,M_k \in \mathbb{C}^{m,m}$, $M_0, M_\infty \in \mathbb{R}^{m,m}$ and some proper and positive real function $G_s(s) \in \mathbb{R}(s)^{m,m}$ which does not have any poles on $i\mathbb{R}$, such that*

$$G(s) = G_s(s) + sM_\infty + \frac{M_0}{s} + \sum_{j=1}^{k} \frac{M_j}{s - i\omega_j} + \frac{\overline{M}_j}{s + i\omega_j}.$$

In particular, we may characterize the positive realness of matrix pencils $sE - A \in \mathbb{R}[s]^{n,n}$ by means of certain definiteness properties of the matrices $E, A \in \mathbb{R}^{n,n}$.

LEMMA 2.5 (Positive real matrix pencils). *A matrix pencil $sE - A \in \mathbb{R}[s]^{n,n}$ is positive real if, and only if, $E = E^\top \geq 0$ and $A + A^\top \leq 0$.*

Proof. $\Rightarrow$: Since $E = E^\top \geq 0$ and $A + A^\top \leq 0$ we have that, for all $\lambda \in \mathbb{C}_+$,

$$(\lambda E - A) + (\lambda E - A)^* = \lambda E + \overline{\lambda} E^\top - A - A^\top = 2\,\mathrm{Re}(\lambda)E - (A + A^\top) \geq 0. \tag{2.2}$$

Therefore, $sE - A$ is positive real.

$\Leftarrow$: Since $sE - A$ is positive real, Lemma 2.4 implies existence of some additive decomposition

$$sE - A = sM_\infty + G_p(s),$$

where $G_p(s) \in \mathbb{R}(s)^{n,n}$ is proper and positive real, and $M_\infty \in \mathbb{R}^{n,n}$ is symmetric and positive semi-definite. Therefore, we obtain $E = E^\top = M_\infty \geq 0$, and the constant rational function $G_p(s) = -A$ is positive real. The latter implies, by definition of positive realness, that $A + A^\top \leq 0$. □

In the following we collect some further properties of positive real matrix pencils $sE - A$ with the additional assumption that the kernels of E and A intersect trivially. This in particular encompasses regular MNA models of passive electrical networks.

LEMMA 2.6 (Properties of positive real pencil). *Let a positive real pencil $sE - A \in \mathbb{R}[s]^{n,n}$ be such that $\ker E \cap \ker A = \{0\}$. Then the following holds true:*

(i) $sE - A$ is regular.
(ii) $(sE - A)^{-1} \in \mathbb{R}(s)^{n,n}$ is positive real.
(iii) All generalized eigenvalues of $sE - A$ have non-positive real part.
(iv) All generalized eigenvalues of $sE - A$ on the imaginary axis are semi-simple.
(v) The index of $sE - A$ is at most two.

Proof. Step 1: To prove that (i) and (iii) hold true, we show that $\ker(\lambda E - A) = \{0\}$ for all $\lambda \in \mathbb{C}_+$. Seeking a contradiction, assume that $\lambda \in \mathbb{C}_+$ and $x \in \mathbb{C}^n \setminus \{0\}$ are such that $(\lambda E - A)x = 0$. Then we obtain

$$0 = x^*\big((\lambda E - A) + (\lambda E - A)^*\big)x \overset{(2.2)}{=} 2\,\mathrm{Re}(\lambda)x^*Ex - x^*(A + A^\top)x.$$

Since, by Lemma 2.5, there holds $E \geq 0$, $A+A^\top \leq 0$ and $\mathrm{Re}(\lambda) > 0$, we have $x^*Ex = x^*(A+A^\top)x = 0$, whence, in particular, $Ex = 0$. Therefore, the equation $(\lambda E - A)x = 0$ gives also rise to $Ax = 0$ and consequently, $x \in \ker E \cap \ker A = \{0\}$, a contradiction.

Step 2: We show (ii). This is a consequence of

$$\begin{aligned}(\lambda E - A)^{-1} + (\lambda E - A)^{-*} &= (\lambda E - A)^{-1}((\lambda E - A)^* + (\lambda E - A))(\lambda E - A)^{-*}\\ &\overset{(2.2)}{=} (\lambda E - A)^{-1}(2\,\mathrm{Re}(\lambda)E - (A+A^\top))(\lambda E - A)^{-*},\end{aligned}$$

$E \geq 0$, $A+A^\top \leq 0$ and $\mathrm{Re}(\lambda) > 0$.

Step 3: It remains to show that (iv) and (v) are valid: Since $(sE-A)^{-1}$ is positive real by (ii), Lemma 2.4 gives rise to the fact that all poles on the imaginary axis are of order one and, moreover, $(sE-A)^{-1} = sM + G_p(s)$, where $G_p(s) \in \mathbb{R}[s]^{n,n}$ is proper and $M \in \mathbb{R}^{n,n}$. This in particular means that $s^{-1}(sE-A)^{-1}$ is proper. Let $W, T \in \mathbf{Gl}_n(\mathbb{C})$ be such that $W(sE-A)T$ is in KCF (2.1). Regularity of $sE-A$ then gives rise to

$$(sE-A)^{-1} = T^{-1}\,\mathrm{diag}(\mathscr{C}_1(s)^{-1}, \dots, \mathscr{C}_k(s)^{-1})W^{-1}. \tag{2.3}$$

Assuming that (iv) does not hold, i.e., there exists some $\omega \in \mathbb{R}$ such that $i\omega$ is a generalized eigenvalue of $sE-A$ which is not semi-simple. Then there exists some block

$$\mathscr{C}_j(s) = (s - i\omega)I_{k_j} - N_{k_j}$$

with $k_j > 1$ in the KCF of $sE-A$. Hence, due to

$$\mathscr{C}_j(s)^{-1} = \sum_{l=0}^{k_j-1} \frac{1}{(s-i\omega)^{l+1}} N_{k_j}^l,$$

the formula (2.3) implies that $(sE-A)^{-1}$ has a pole of order greater than one on the imaginary axis, a contradiction.

Assume that (v) does not hold, i.e., the index of $sE-A$ exceeds two. Then there exists some block

$$\mathscr{C}_j(s) = sN_{k_j} - I_{k_j}$$

with $k_j > 2$ in the KCF of $sE-A$. Then

$$\mathscr{C}_j(s)^{-1} = -\sum_{l=0}^{k_j-1} s^l N_{k_j}^l,$$

and this contradicts properness of $s^{-1}(sE-A)^{-1}$. □

3. Graph theoretical preliminaries. In this section we introduce the graph theoretical concepts which are crucial for the modified nodal analysis of electrical circuits. We derive some characterizations for the absence of cutsets and loops in a given subgraph. These characterizations will be given in terms of algebraic properties of the incidence matrices.

DEFINITION 3.1 (Graph theoretical concepts). *A* graph *is a triple $\mathscr{G} = (V, E, \varphi)$ consisting of a* node set V *and a* branch set E *together with an* incidence map

$$\varphi : E \to V \times V, \quad e \mapsto \varphi(e) = (\varphi_1(e), \varphi_2(e)).$$

If $\varphi(e) = (v_1, v_2)$, we call e to be directed from v_1 to v_2. *v_1 is called the* initial node *and v_2 the* terminal node *of e. Two graphs $\mathscr{G}_a = (V_a, E_a, \varphi_a)$, $\mathscr{G}_b = (V_b, E_b, \varphi_b)$ are called* isomorphic, *if* *there exist bijective mappings* $\iota_E : E_a \to E_b$, $\iota_V : V_a \to V_b$*, such that* $\varphi_{a,1} = \iota_V^{-1} \circ \varphi_{b,1} \circ \iota_E$ *and* $\varphi_{a,2} = \iota_V^{-1} \circ \varphi_{b,2} \circ \iota_E$.

Let $V' \subseteq V$ *and let* E' *be a set of branches satisfying*

$$E' \subseteq E|_{V'} := \left\{ e \in E \;\middle|\; \varphi_1(e) \in V' \text{ and } \varphi_2(e) \in V' \right\}.$$

Further let $\varphi|_{E'}$ *be the restriction of* φ *to* E'*. Then the triple* $\mathscr{K} := (V', E', \varphi|_{E'})$ *is called* subgraph of $\mathscr{G}$*. In the case where* $E' = E|_{V'}$*, we call* $\mathscr{K}$ *the* induced subgraph on V'*. If* $V' = V$*, then* $\mathscr{K}$ *is called a* spanning subgraph*. A* proper subgraph *is one with* $E \neq E'$.

$\mathscr{G}$ *is called* finite*, if both the node and the branch set are finite.*

For each branch e*, define an additional branch* $-e$ *being directed from the terminal to the initial node of* e*, that is* $\varphi(-e) = (\varphi_2(e), \varphi_1(e))$ *for* $e \in E$*. Now define the set* $\tilde{E} = \{ e \mid e \in E \text{ or } -e \in E \}$*. A tuple* $w = (w_1, \ldots, w_r) \in \tilde{E}^r$*, where for* $i = 1, \ldots, r-1$,

$$v_0 := \varphi_1(v_1), \quad v_i := \varphi_2(w_i) = \varphi_1(w_{i+1})$$

is called path from v_0 to v_r*;* w *is called* elementary path*, if* $v_1, \ldots, v_r$ *are distinct. A* loop *is an elementary path with* $v_0 = v_r$*. Two nodes* v, v' *are called* connected*, if there exists a path from* v *to* v'*. The graph itself is called connected, if any two nodes are connected. A subgraph* $\mathscr{K} = (V', E', \varphi|_{E'})$ *is called* component of connectivity*, if it is connected and* $\mathscr{K}^c := (V \setminus V', E \setminus E', \varphi|_{E \setminus E'})$ *is a subgraph.*

A spanning subgraph $\mathscr{K} = (V, E', \varphi|_{E'})$ *is called a* cutset *of* $\mathscr{G} = (V, E, \varphi)$*, if its branch set is non-empty,* $\mathscr{G} - \mathscr{K} := (V, E \setminus E', \varphi|_{E \setminus E'})$ *is a disconnected subgraph and* $\mathscr{G} - \mathscr{K}'$ *is a connected subgraph for any proper spanning subgraph* $\mathscr{K}'$ *of* $\mathscr{K}$.

For finite graphs we can set up special matrices which will be useful to describe Kirchhoff's laws.

DEFINITION 3.2 (Incidence matrix). *Let a finite graph* $\mathscr{G} = (V, E, \varphi)$ *with* l *branches* $E = \{e_1, \ldots, e_l\}$ *and* k *nodes* $V = \{v_1, \ldots, v_k\}$ *be given. Then the* all-node incidence matrix *of* $\mathscr{G}$ *is given by* $\mathrm{A}_0 = (a_{ij}) \in \mathbb{R}^{k,l}$*, where*

$$a_{ij} = \begin{cases} 1, & \text{if } \varphi_1(e_j) = v_i, \\ -1, & \text{if } \varphi_2(e_j) = v_i, \\ 0, & \text{otherwise.} \end{cases}$$

Since the rows of A_0 *sum up to the zero row vector, one might delete an arbitrary row of* A_0 *to obtain a matrix* A *having the same rank as* A_0*. We call* A *an* incidence matrix *of* $\mathscr{G}$.

This section continues with some results on the relation between properties of subgraphs and linear algebraic properties of corresponding submatrices of incidence matrices. First we declare some manners of speaking.

DEFINITION 3.3. *Let* $\mathscr{G}$ *be a graph,* $\mathscr{K}$ *be a spanning subgraph of* $\mathscr{G}$*,* $\mathscr{L}$ *be a subgraph of* $\mathscr{G}$*, and* ℓ *be a path of* $\mathscr{G}$.

(i) $\mathscr{L}$ *is called a* $\mathscr{K}$-cutset*, if* $\mathscr{L}$ *is a cutset of* $\mathscr{K}$.

(ii) ℓ *is called a* $\mathscr{K}$-loop*, if* ℓ *is a loop of* $\mathscr{K}$.

A spanning subgraph $\mathscr{K}$ of the finite graph $\mathscr{G}$ has an incidence matrix $\mathrm{A}_{\mathscr{K}}$ which is constructed by deleting rows of the incidence matrix A of $\mathscr{G}$ corresponding to the branches of the complementary spanning subgraph $\mathscr{G} - \mathscr{K}$. By a suitable reordering of the branches, the incidence matrix reads

$$\mathrm{A} = \begin{bmatrix} \mathrm{A}_{\mathscr{K}} & \mathrm{A}_{\mathscr{G}-\mathscr{K}} \end{bmatrix}. \tag{3.1}$$

LEMMA 3.4 (Subgraphs and incidence matrices [11, Lem. 2.1 & Lem. 2.3]). *Let $\mathscr{G}$ be a connected graph with incidence matrix $\mathrm{A} \in \mathbb{R}^{l-1,k}$. Further, let $\mathscr{K}$ be a spanning subgraph. Assume that the branches of $\mathscr{G}$ are sorted in a way that* (3.1) *is satisfied. Then the following holds true:*

(i) *The following two assertions are equivalent:*
a) $\mathscr{G}$ *does not contain* $\mathscr{K}$*-cutsets.*
b) $\ker \mathrm{A}_{\mathscr{G}-\mathscr{K}}^{\top} = \{0\}$.

(ii) *The following two assertions are equivalent:*
a) $\mathscr{G}$ *does not contain* $\mathscr{K}$*-loops.*
b) $\ker \mathrm{A}_{\mathscr{K}} = \{0\}$.

The following two auxiliary results are concerned with properties of subgraphs of subgraphs, and give some equivalent characterizations in terms of properties of their incidence matrices.

LEMMA 3.5 (Loops in subgraphs [11, Prop. 4.5]). *Let $\mathscr{G}$ be a connected graph with incidence matrix $\mathrm{A} \in \mathbb{R}^{k-1,l}$. Further, let $\mathscr{K}$ be a spanning subgraph of $\mathscr{G}$, and let $\mathscr{L}$ be a spanning subgraph of $\mathscr{K}$. Assume that the branches of $\mathscr{G}$ are sorted in a way that*

$$\mathrm{A} = \begin{bmatrix} \mathrm{A}_{\mathscr{L}} & \mathrm{A}_{\mathscr{K}-\mathscr{L}} & \mathrm{A}_{\mathscr{G}-\mathscr{K}} \end{bmatrix} \quad \textit{and} \quad \mathrm{A}_{\mathscr{K}} = \begin{bmatrix} \mathrm{A}_{\mathscr{L}} & \mathrm{A}_{\mathscr{K}-\mathscr{L}} \end{bmatrix}.$$

Then the following two assertions are equivalent:
a) $\mathscr{G}$ *does not contain* $\mathscr{K}$*-loops except for* $\mathscr{L}$*-loops.*
b) $\ker \mathrm{A}_{\mathscr{K}} = \ker \mathrm{A}_{\mathscr{L}} \times \{0\}$.

LEMMA 3.6 (Cutsets in subgraphs [11, Prop. 4.4]). *Let $\mathscr{G}$ be a connected graph with incidence matrix $\mathrm{A} \in \mathbb{R}^{k-1,l}$. Further, let $\mathscr{K}$ be a spanning subgraph of $\mathscr{G}$, and let $\mathscr{L}$ be a spanning subgraph of $\mathscr{K}$. Assume that the branches of $\mathscr{G}$ are sorted in a way that*

$$\mathrm{A} = \begin{bmatrix} \mathrm{A}_{\mathscr{L}} & \mathrm{A}_{\mathscr{K}-\mathscr{L}} & \mathrm{A}_{\mathscr{G}-\mathscr{K}} \end{bmatrix} \quad \textit{and} \quad \mathrm{A}_{\mathscr{G}-\mathscr{L}} = \begin{bmatrix} \mathrm{A}_{\mathscr{K}-\mathscr{L}} & \mathrm{A}_{\mathscr{G}-\mathscr{K}} \end{bmatrix}.$$

Then the following two assertions are equivalent:
a) $\mathscr{G}$ *does not contain* $\mathscr{K}$*-cutsets except for* $\mathscr{L}$*-cutsets.*
b) $\ker \mathrm{A}_{\mathscr{G}-\mathscr{K}}^{\top} = \ker \mathrm{A}_{\mathscr{G}-\mathscr{L}}^{\top}$.

4. Circuit equations. It is well-known [5, 7] that the graph underlying an electrical circuit can be described by an incidence matrix $\mathrm{A} \in \mathbb{R}^{k-1,l}$, which can be decomposed into submatrices

$$\mathrm{A} = \begin{bmatrix} \mathrm{A}_{\mathcal{C}} & \mathrm{A}_{\mathcal{R}} & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{V}} & \mathrm{A}_{\mathscr{I}} \end{bmatrix}$$

for the quantities in (1.5), where $n_e = k-1$ and $l = n_{\mathcal{C}} + n_{\mathcal{G}} + n_{\mathcal{L}} + n_{\mathcal{V}} + n_{\mathscr{I}}$. Each submatrix is the incidence matrix of a specific subgraph of the circuit graph. $\mathrm{A}_{\mathcal{C}}$ is the incidence matrix of the subgraph consisting of all circuit nodes and all branches corresponding to capacitors. Similarly, $\mathrm{A}_{\mathcal{R}}, \mathrm{A}_{\mathcal{L}}, \mathrm{A}_{\mathcal{V}}, \mathrm{A}_{\mathscr{I}}$ are the incidence matrices corresponding to the resistor, inductor, voltage source and current source subgraphs, resp. Then using the standard MNA modeling procedure [7], which is just a clever arrangement of Kirchhoff's laws together with the characteristic equations of the devices, results in a differential-algebraic system (1.1) with (1.3)–(1.5). $\mathcal{C}$, $\mathcal{G}$ and $\mathcal{L}$ are the matrices expressing the consecutive relations of capacitances, resistances and inductances, $\eta(t)$ is the vector of node potentials, $i_{\mathcal{L}}(t)$, $i_{\mathcal{V}}(t)$, $i_{\mathscr{I}}(t)$ are the vectors of currents through inductances, voltage and current sources, and $v_{\mathcal{V}}(t)$, $v_{\mathscr{I}}(t)$ are the voltages of voltage and current sources.

It is a reasonable assumption that an electrical circuit is connected; otherwise, since the components of connectivity do not physically interact, one might consider them separately.

Furthermore, in the present paper we consider circuits with *passive* devices. These assumptions lead to the following assumptions on the MNA model (1.3)–(1.5) of the circuit (compare Lemma 3.4).

$$\textbf{(A1)}\quad \operatorname{rk}\begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathcal{V}} & A_{\mathscr{I}} \end{bmatrix} = n_e,$$

$$\textbf{(A2)}\quad \mathcal{C} = \mathcal{C}^\top > 0, \mathcal{L} = \mathcal{L}^\top > 0, \mathcal{G} + \mathcal{G}^\top > 0.$$

It is possible that in the circuit equations (1.1) there are still redundant equations and superfluous variables, i.e., in general the pencil $sE - A$ arising from (1.3), (1.5) is not regular. In the following we show how this can be overcome by a simple transformation; the reduced circuit model is regular and positive real. This transformation is also important to show feasibility of funnel control in Section 7.

THEOREM 4.1 (Reduction of circuit pencil). *Let $sE - A \in \mathbb{R}[s]^{n,n}$ with E, A as in* (1.3), (1.5) *be given and suppose that* **(A1)** *and* **(A2)** *hold. Let $Z_{\mathcal{CRLV}}$, $Z'_{\mathcal{CRLV}}$, $\bar{Z}_{\mathcal{V}}$, $\bar{Z}'_{\mathcal{V}}$ be real matrices with full column rank such that*

$$\operatorname{im} Z_{\mathcal{CRLV}} = \ker\begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathcal{V}} \end{bmatrix}^\top, \qquad \operatorname{im} \bar{Z}_{\mathcal{V}} = \ker A_{\mathcal{V}},$$

$$\operatorname{im} Z'_{\mathcal{CRLV}} = \operatorname{im}\begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathcal{V}} \end{bmatrix}, \qquad \operatorname{im} \bar{Z}'_{\mathcal{V}} = \operatorname{im} A_{\mathcal{V}}^\top.$$

Then we have

$$T = \begin{bmatrix} Z'_{\mathcal{CRLV}} & 0 & 0 & Z_{\mathcal{CRLV}} & 0 \\ 0 & I_{n_{\mathcal{L}}} & 0 & 0 & 0 \\ 0 & 0 & \bar{Z}'_{\mathcal{V}} & 0 & \bar{Z}_{\mathcal{V}} \end{bmatrix} \in \mathbf{Gl}_n(\mathbb{R}), \tag{4.1}$$

and

$$T^\top (sE - A) T = \begin{bmatrix} s\widetilde{E} - \widetilde{A} & 0 \\ 0 & 0 \end{bmatrix},$$

where the pencil

$$s\widetilde{E} - \widetilde{A} = \begin{bmatrix} (Z'_{\mathcal{CRLV}})^\top \left(sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top\right) Z'_{\mathcal{CRLV}} & (Z'_{\mathcal{CRLV}})^\top A_{\mathcal{L}} & (Z'_{\mathcal{CRLV}})^\top A_{\mathcal{V}} \bar{Z}'_{\mathcal{V}} \\ -A_{\mathcal{L}}^\top Z'_{\mathcal{CRLV}} & s\mathcal{L} & 0 \\ -\bar{Z}'_{\mathcal{V}} A_{\mathcal{V}}^\top Z'_{\mathcal{CRLV}} & 0 & 0 \end{bmatrix} \tag{4.2}$$

is regular and satisfies $\ker \widetilde{E} \cap \ker \widetilde{A} = \{0\}$, $\widetilde{E} = \widetilde{E}^\top \geq 0$ *and* $\widetilde{A} + \widetilde{A}^\top \leq 0$.

Proof. The invertibility of T is a consequence of $\operatorname{im} Z_{\mathcal{CRLV}} \oplus \operatorname{im} Z'_{\mathcal{CRLV}} = \mathbb{R}^{n_e}$ and $\operatorname{im} \bar{Z}_{\mathcal{V}} \oplus \operatorname{im} \bar{Z}'_{\mathcal{V}} = \mathbb{R}^{n_{\mathcal{V}}}$. The properties $\widetilde{E} = \widetilde{E}^\top \geq 0$ and $\widetilde{A} + \widetilde{A}^\top \leq 0$ follow immediately from the construction of $\widetilde{E}$ and $\widetilde{A}$. To prove that $s\widetilde{E} - \widetilde{A}$ is regular, it suffices by Lemma 2.6 to show that $\ker \widetilde{E} \cap \ker \widetilde{A} = \{0\}$: Let $x \in \ker \widetilde{E} \cap \ker \widetilde{A}$. Partitioning according to the block structure of $\widetilde{E}$ and $\widetilde{A}$, i.e., $x = (x_1^\top, x_2^\top, x_3^\top)^\top$, and using that, by (A2), $\mathcal{C} > 0$, $\mathcal{L} > 0$ and $\mathcal{G} + \mathcal{G}^\top > 0$, we obtain from $x^\top \widetilde{E} x = x^\top \left(\widetilde{A} + \widetilde{A}^\top\right) x = 0$ that $x_2 = 0$ and

$$\begin{bmatrix} A_{\mathcal{C}}^\top \\ A_{\mathcal{R}}^\top \end{bmatrix} Z'_{\mathcal{CRLV}} x_1 = 0. \tag{4.3}$$

Furthermore, $\widetilde{A}x = 0$ gives rise to

$$\text{(a) } (\overline{Z}'_{\mathcal{V}})^\top \mathrm{A}_{\mathcal{V}}^\top Z'_{\mathcal{CRLV}} x_1 = 0, \quad \text{(b) } \mathrm{A}_{\mathcal{L}}^\top Z'_{\mathcal{CRLV}} x_1 = 0, \quad \text{and} \quad \text{(c) } (Z'_{\mathcal{CRLV}})^\top \mathrm{A}_{\mathcal{V}} \overline{Z}'_{\mathcal{V}} x_3 = 0.$$

(a) implies

$$\mathrm{A}_{\mathcal{V}}^\top Z'_{\mathcal{CRLV}} x_1 \in \ker(\overline{Z}'_{\mathcal{V}})^\top = (\operatorname{im} \overline{Z}'_{\mathcal{V}})^\perp = (\operatorname{im} \mathrm{A}_{\mathcal{V}}^\top)^\perp,$$

whence $\mathrm{A}_{\mathcal{V}}^\top Z'_{\mathcal{CRLV}} x_1 = 0$. Together with (4.3) and (b) this yields

$$Z'_{\mathcal{CRLV}} x_1 \in \ker \begin{bmatrix} \mathrm{A}_{\mathcal{C}} & \mathrm{A}_{\mathcal{R}} & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{V}} \end{bmatrix}^\top = \operatorname{im} Z_{\mathcal{CRLV}} = (\operatorname{im} Z'_{\mathcal{CRLV}})^\perp,$$

and therefore $x_1 = 0$. By (c) we find

$$\begin{aligned} \mathrm{A}_{\mathcal{V}} \overline{Z}'_{\mathcal{V}} x_3 \in \ker(Z'_{\mathcal{CRLV}})^\top = (\operatorname{im} Z'_{\mathcal{CRLV}})^\perp \\ = \ker \begin{bmatrix} \mathrm{A}_{\mathcal{C}} & \mathrm{A}_{\mathcal{R}} & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{V}} \end{bmatrix}^\top \subseteq \ker \mathrm{A}_{\mathcal{V}}^\top = (\operatorname{im} \mathrm{A}_{\mathcal{V}})^\perp, \end{aligned}$$

and thus $\mathrm{A}_{\mathcal{V}} \overline{Z}'_{\mathcal{V}} x_3 = 0$. From this, we obtain

$$\overline{Z}'_{\mathcal{V}} x_3 \in \ker \mathrm{A}_{\mathcal{V}} = (\operatorname{im} \mathrm{A}_{\mathcal{V}}^\top)^\perp = (\operatorname{im} \overline{Z}'_{\mathcal{V}})^\perp,$$

whence $x_3 = 0$. □

We may infer the following characterization of the presence of generalized eigenvalues from Theorem 4.1.

COROLLARY 4.2 (Kernel and generalized eigenvalues). *Let $sE - A \in \mathbb{R}[s]^{n,n}$ with E, A as in* (1.3), (1.5) *be given and suppose that* **(A1)** *and* **(A2)** *hold. Then*

$$\ker_{\mathbb{R}(s)} sE - A = \ker_{\mathbb{R}(s)} \begin{bmatrix} \mathrm{A}_{\mathcal{C}} & \mathrm{A}_{\mathcal{R}} & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{V}} \end{bmatrix}^\top \times \{0\} \times \ker_{\mathbb{R}(s)} \mathrm{A}_{\mathcal{V}}.$$

Furthermore, $\lambda \in \mathbb{C}$ is not a generalized eigenvalue of $sE - A$ if, and only if,

$$\ker_{\mathbb{C}} \lambda E - A = \ker_{\mathbb{C}} \begin{bmatrix} \mathrm{A}_{\mathcal{C}} & \mathrm{A}_{\mathcal{R}} & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{V}} \end{bmatrix}^\top \times \{0\} \times \ker_{\mathbb{C}} \mathrm{A}_{\mathcal{V}}.$$

Proof. Using the transformation matrix T in (4.1) and accompanying notation from Theorem 4.1, we obtain (denoting the number of columns of $Z_{\mathcal{CRLV}}$ by k_1 and the number of columns of $\overline{Z}_{\mathcal{V}}$ by k_2) that

$$\begin{aligned} \ker_{\mathbb{R}(s)} sE - A &= T\big(\underbrace{\ker_{\mathbb{R}(s)}(s\widetilde{E} - \widetilde{A})}_{=\{0\}} \times \mathbb{R}(s)^{k_1+k_2}\big) \\ &= \operatorname{im}_{\mathbb{R}(s)} Z_{\mathcal{CRLV}} \times \{0\} \times \operatorname{im}_{\mathbb{R}(s)} \overline{Z}_{\mathcal{V}} \\ &= \ker_{\mathbb{R}(s)} \begin{bmatrix} \mathrm{A}_{\mathcal{C}} & \mathrm{A}_{\mathcal{R}} & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{V}} \end{bmatrix}^\top \times \{0\} \times \ker_{\mathbb{R}(s)} \mathrm{A}_{\mathcal{V}}. \end{aligned}$$

Now let $\lambda \in \mathbb{C}$ and observe that

$$\ker_{\mathbb{C}} \lambda E - A = T\big(\ker_{\mathbb{C}} \lambda \widetilde{E} - \widetilde{A} \times \mathbb{C}^{k_1+k_2}\big).$$

By Corollary 2.2, λ is not a generalized eigenvalue of $sE - A$ if, and only if, $\operatorname{rk}_{\mathbb{C}} \lambda E - A = \operatorname{rk}_{\mathbb{R}(s)} sE - A$ or, equivalently, $\dim \ker_{\mathbb{C}} \lambda E - A = \dim \ker_{\mathbb{R}(s)} sE - A$. Therefore, λ is not a generalized eigenvalue of $sE-A$ if, and only if, $\ker_{\mathbb{C}} \lambda\widetilde{E}-\widetilde{A}=\{0\}$ and this implies the last statement of the corollary. □

In the following we will use expressions like $\mathscr{VL}$-loop for a loop in the circuit graph whose branch set consists only of branches corresponding to voltage sources and/or inductors. Likewise, a $\mathscr{IC}$-cutset is a cutset in the circuit graph whose branch set consist only of branches corresponding to current sources and/or capacitors.

COROLLARY 4.3 (Regularity of circuit pencil). *Let $sE-A \in \mathbb{R}[s]^{n,n}$ with E,A as in* (1.3), (1.5) *be the MNA model of an electrical circuit and suppose that* **(A1)** *and* **(A2)** *hold. Then the following statements are equivalent:*
a) $sE-A$ *is regular.*
b) $\ker\begin{bmatrix} \mathrm{A}_{\mathscr{C}} & \mathrm{A}_{\mathscr{R}} & \mathrm{A}_{\mathscr{L}} & \mathrm{A}_{\mathscr{V}} \end{bmatrix}^\top = \{0\}$ *and* $\ker \mathrm{A}_{\mathscr{V}} = \{0\}$.
c) The circuit neither contains $\mathscr{V}$-loops nor $\mathscr{I}$-cutsets.

Proof. The result follows immediately from Corollary 4.2 and Lemma 3.4. □

Next we give sufficient criteria for the absence of purely imaginary generalized eigenvalues of the pencil $sE-A$ as in (1.3), (1.5). This result can be seen as a generalization of the results in [11] to circuits which might contain $\mathscr{I}$-cutsets and/or $\mathscr{V}$-loops, i.e., where $sE-A$ is not necessarily regular.

THEOREM 4.4 (Absence of imaginary eigenvalues). *Let $sE-A \in \mathbb{R}[s]^{n,n}$ with E,A as in* (1.3), (1.5) *be the MNA model of an electrical circuit and suppose that* **(A1)** *and* **(A2)** *hold. Furthermore, suppose that at least one of the following two assertions holds:*

(i) *The circuit neither contains $\mathscr{VL}$-loops except for $\mathscr{V}$-loops, nor $\mathscr{ICL}$-cutsets except for $\mathscr{IL}$-cutsets; equivalently*

$$\begin{aligned} &\ker\begin{bmatrix} \mathrm{A}_{\mathscr{V}} & \mathrm{A}_{\mathscr{L}} \end{bmatrix} = \ker \mathrm{A}_{\mathscr{V}} \times \{0\} \\ \textit{and}\qquad &\ker\begin{bmatrix} \mathrm{A}_{\mathscr{R}} & \mathrm{A}_{\mathscr{V}} \end{bmatrix}^\top = \ker\begin{bmatrix} \mathrm{A}_{\mathscr{C}} & \mathrm{A}_{\mathscr{R}} & \mathrm{A}_{\mathscr{V}} \end{bmatrix}^\top. \end{aligned} \tag{4.4}$$

(ii) *The circuit neither contains $\mathscr{IC}$-cutsets except for $\mathscr{I}$-cutsets, nor $\mathscr{VCL}$-loops except for $\mathscr{VC}$-loops; equivalently*

$$\begin{aligned} &\ker\begin{bmatrix} \mathrm{A}_{\mathscr{R}} & \mathrm{A}_{\mathscr{L}} & \mathrm{A}_{\mathscr{V}} \end{bmatrix}^\top = \ker\begin{bmatrix} \mathrm{A}_{\mathscr{C}} & \mathrm{A}_{\mathscr{R}} & \mathrm{A}_{\mathscr{L}} & \mathrm{A}_{\mathscr{V}} \end{bmatrix}^\top \\ \textit{and}\qquad &\ker\begin{bmatrix} \mathrm{A}_{\mathscr{V}} & \mathrm{A}_{\mathscr{C}} & \mathrm{A}_{\mathscr{L}} \end{bmatrix} = \ker\begin{bmatrix} \mathrm{A}_{\mathscr{V}} & \mathrm{A}_{\mathscr{C}} \end{bmatrix} \times \{0\}. \end{aligned} \tag{4.5}$$

Then all generalized eigenvalues of $sE-A$ are in contained $\mathbb{C}_-$.

Proof. The equivalent characterizations of the absence of certain loops or cutsets in the circuit graph, resp., and kernel conditions on the element-related incidence matrices follow from Lemmas 3.5 and 3.6.

By Theorem 4.1 and Lemma 2.6 all generalized eigenvalues of $sE-A$ are contained in $\overline{\mathbb{C}_-}$. Then, using Corollary 4.2, we have to show that

$$\forall\, \omega \in \mathbb{R}: \ \ker_{\mathbb{C}}(i\omega E-A) = \ker_{\mathbb{C}}\begin{bmatrix} \mathrm{A}_{\mathscr{C}} & \mathrm{A}_{\mathscr{R}} & \mathrm{A}_{\mathscr{L}} & \mathrm{A}_{\mathscr{V}} \end{bmatrix}^\top \times \{0\} \times \ker_{\mathbb{C}} \mathrm{A}_{\mathscr{V}}. \tag{4.6}$$

Since "$\supseteq$" does always hold true, we show "$\subseteq$". Let $\omega \in \mathbb{R}$ and $x_1 \in \mathbb{C}^{n_e}$, $x_2 \in \mathbb{C}^{n_{\mathscr{L}}}$ and $x_3 \in \mathbb{C}^{n_{\mathscr{V}}}$ be such that

$$x := (x_1^\top, x_2^\top, x_3^\top)^\top \in \ker_{\mathbb{C}}(i\omega E-A). \tag{4.7}$$

By the structure of $sE-A$ as in (1.3), relation (4.7) implies $\mathrm{A}_{\mathscr{V}}^\top x_1 = 0$ and

$$0 = x^*\big((i\omega E-A)+(i\omega E-A)^*\big)x = -x^*(A+A^\top)x = -x_1^* \mathrm{A}_{\mathscr{R}}(\mathscr{G}+\mathscr{G}^\top)\mathrm{A}_{\mathscr{R}}^\top x_1,$$

hence $\mathrm{A}_\mathcal{R}^\top x_1 = 0$ since $\mathcal{G} + \mathcal{G}^\top > 0$ by (A2).

We show that (i) implies (4.6): Since $x_1 \in \ker_\mathbb{C} \begin{bmatrix} \mathrm{A}_\mathcal{R} & \mathrm{A}_\mathcal{V} \end{bmatrix}^\top$ we obtain from (4.4) that $x_1 \in \ker_\mathbb{C} \mathrm{A}_\mathcal{C}^\top$. Then (4.7) implies $\mathrm{A}_\mathcal{L} x_2 + \mathrm{A}_\mathcal{V} x_3 = 0$ and by (4.4) we find $\mathrm{A}_\mathcal{V} x_3 = 0$ and $x_2 = 0$. The latter implies that $x_1 \in \ker_\mathbb{C} \mathrm{A}_\mathcal{L}^\top$. Altogether, we have that (4.6) is valid.

We show that (ii) implies (4.6): From (4.7) we have

$$\mathrm{A}_\mathcal{C}(i\omega \mathcal{C} \mathrm{A}_\mathcal{C}^\top x_1) + \mathrm{A}_\mathcal{L} x_2 + \mathrm{A}_\mathcal{V} x_3 = 0, \tag{4.8}$$

and by (4.5) we obtain $x_2 = 0$. This implies $\mathrm{A}_\mathcal{L}^\top x_1 = 0$, hence $x_1 \in \ker_\mathbb{C} \begin{bmatrix} \mathrm{A}_\mathcal{R} & \mathrm{A}_\mathcal{L} & \mathrm{A}_\mathcal{V} \end{bmatrix}^\top$ which by (4.5) yields

$$\begin{bmatrix} \mathrm{A}_\mathcal{C} & \mathrm{A}_\mathcal{R} & \mathrm{A}_\mathcal{L} & \mathrm{A}_\mathcal{V} \end{bmatrix}^\top x_1 = 0.$$

Now, from (4.8) we have $\mathrm{A}_\mathcal{V} x_3 = 0$ and (4.6) is shown. ☐

5. Zero dynamics and invariant zeros. In this section we derive topological characterizations of autonomous and asymptotically stable zero dynamics of the circuit system. The latter is done by an investigation of the invariant zeros of the system.

Using a simple transformation of the system, properties of the zero dynamics can be led back to properties of a circuit pencil where voltage sources are replaced with current sources, and vice versa. To this end, consider $[E,A,B,C] \in \Sigma_{n,m}$ with (1.3), (1.5) and define the matrices $W, T \in \mathbf{Gl}_{n+m}(\mathbb{R})$ by

$$W = \begin{bmatrix} I_{n_e} & 0 & 0 & 0 & -\mathrm{A}_\mathcal{V} \\ 0 & I_{n_\mathcal{L}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{n_\mathcal{I}} & 0 \\ 0 & 0 & 0 & 0 & I_{n_\mathcal{V}} \\ 0 & 0 & I_{n_\mathcal{V}} & 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} I_{n_e} & 0 & 0 & 0 & 0 \\ 0 & I_{n_\mathcal{L}} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_\mathcal{V}} & 0 \\ 0 & 0 & I_{n_\mathcal{I}} & 0 & 0 \\ -\mathrm{A}_\mathcal{V}^\top & 0 & 0 & 0 & I_{n_\mathcal{V}} \end{bmatrix}.$$

Then we obtain

$$W \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} T = \begin{bmatrix} s\mathrm{A}_\mathcal{C} \mathcal{C} \mathrm{A}_\mathcal{C}^\top + \mathrm{A}_\mathcal{R} \mathcal{G} \mathrm{A}_\mathcal{R}^\top & \mathrm{A}_\mathcal{L} & \mathrm{A}_\mathcal{I} & 0 & 0 \\ -\mathrm{A}_\mathcal{L}^\top & s\mathcal{L} & 0 & 0 & 0 \\ -\mathrm{A}_\mathcal{I}^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_\mathcal{V}} & 0 \\ 0 & 0 & 0 & 0 & I_{n_\mathcal{V}} \end{bmatrix}. \tag{5.1}$$

As desired, the upper left part is a matrix pencil which is the MNA model of a circuit in which voltage sources are replaced with current sources, and vice versa. We may now derive the following important properties, which are immediate from Corollary 4.2 and (5.1).

COROLLARY 5.1 (Kernel and generalized eigenvalues of system pencil).
Let $[E,A,B,C] \in \Sigma_{n,m}$ *with* (1.3), (1.5) *be the MNA model of an electrical circuit and suppose*

that **(A1)** *and* **(A2)** *hold. Then*

$$\ker_{\mathbb{R}(s)} \begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix} = \left\{ \begin{bmatrix} x_1(s) \\ 0 \\ 0 \\ x_3(s) \\ -\mathrm{A}_{\mathscr{V}}^\top x_1(s) \end{bmatrix} \,\middle|\, \begin{array}{l} x_1(s) \in \ker_{\mathbb{R}(s)} \begin{bmatrix} \mathrm{A}_{\mathscr{C}} & \mathrm{A}_{\mathscr{R}} & \mathrm{A}_{\mathscr{L}} & \mathrm{A}_{\mathscr{I}} \end{bmatrix}^\top, \\ x_3(s) \in \ker_{\mathbb{R}(s)} \mathrm{A}_{\mathscr{I}} \end{array} \right\}.$$

Furthermore, $\lambda \in \mathbb{C}$ *is not a generalized eigenvalue of* $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ *if, and only if,*

$$\ker_{\mathbb{C}} \begin{bmatrix} \lambda E-A & -B \\ -C & 0 \end{bmatrix} = \left\{ \begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_3 \\ -\mathrm{A}_{\mathscr{V}}^\top x_1 \end{bmatrix} \,\middle|\, \begin{array}{l} x_1 \in \ker_{\mathbb{C}} \begin{bmatrix} \mathrm{A}_{\mathscr{C}} & \mathrm{A}_{\mathscr{R}} & \mathrm{A}_{\mathscr{L}} & \mathrm{A}_{\mathscr{I}} \end{bmatrix}^\top, \\ x_3 \in \ker_{\mathbb{C}} \mathrm{A}_{\mathscr{I}} \end{array} \right\}.$$

We now aim to characterize autonomous zero dynamics.

PROPOSITION 5.2 (Autonomous zero dynamics). *Let* $[E,A,B,C] \in \Sigma_{n,m}$ *with* (1.3), (1.5) *be the MNA model of an electrical circuit and suppose that* **(A1)** *and* **(A2)** *hold. Then the following statements are equivalent.*

(i) *The zero dynamics* $\mathscr{ZD}_{[E,A,B,C]}$ *are autonomous.*

(ii) $\operatorname{rk}_{\mathbb{R}(s)} \left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right] = n+m$.

(iii) $\ker_{\mathbb{R}(s)} \left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right] = \{0\}$.

(iv) $\ker \begin{bmatrix} \mathrm{A}_{\mathscr{C}} & \mathrm{A}_{\mathscr{R}} & \mathrm{A}_{\mathscr{L}} & \mathrm{A}_{\mathscr{I}} \end{bmatrix}^\top = \{0\}$ *and* $\ker \mathrm{A}_{\mathscr{I}} = \{0\}$.

(v) *The circuit neither contains* $\mathscr{I}$*-loops nor* $\mathscr{V}$*-cutsets.*

Proof. The equivalence of (i) and (ii) has been proved in [2, Prop. 3.6] (note that the rank over $\mathbb{R}[s]$ and over $\mathbb{R}(s)$ coincide). (ii)⇔(iii) is clear and (iii)⇔(iv) follows from Corollary 5.1. The equivalence of (iv) and (v) is then a consequence of Lemma 3.4. □

In order to characterize asymptotic stability of the zero dynamics we need the concept of invariant zeros. An invariant zero of $[E,A,B,C] \in \Sigma_{n,m}$ is defined as a generalized eigenvalue of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$, see e.g. [9].

DEFINITION 5.3 (Invariant zeros). *Let* $[E,A,B,C] \in \Sigma_{n,m}$. *Then* $\lambda \in \mathbb{C}$ *is called* invariant zero of $[E,A,B,C]$ *if, and only if,*

$$\operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E-A & -B \\ -C & 0 \end{bmatrix} < \operatorname{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}.$$

From Theorem 4.4 and (5.1) we get the following result on the location of invariant zeros.

COROLLARY 5.4 (Location of invariant zeros). *Let* $[E,A,B,C] \in \Sigma_{n,m}$ *with* (1.3), (1.5) *be the MNA model of an electrical circuit and suppose that* **(A1)** *and* **(A2)** *hold. Furthermore, suppose that at least one of the following two assertions holds:*

(i) *The circuit neither contains* $\mathscr{IL}$*-loops except for* $\mathscr{I}$*-loops, nor* $\mathscr{VCL}$*-cutsets except for* $\mathscr{VL}$*-cutsets.*

(ii) *The circuit neither contains $\mathscr{V}\mathscr{C}$-cutsets except for $\mathscr{V}$-cutsets, nor $\mathscr{I}\mathscr{C}\mathscr{L}$-loops except for $\mathscr{I}\mathscr{C}$-loops.*

Then all invariant zeros of $[E,A,B,C]$ are contained in $\mathbb{C}_-$.

We are now in the position to characterize asymptotically stable zero dynamics.

THEOREM 5.5 (Asymptotically stable zero dynamics). *Let $[E,A,B,C] \in \Sigma_{n,m}$ with* (1.3), (1.5) *be the MNA model of an electrical circuit and suppose that* **(A1)** *and* **(A2)** *hold. Then the zero dynamics $\mathscr{ZD}_{[E,A,B,C]}$ are asymptotically stable if, and only if,*

a) $\mathscr{ZD}_{[E,A,B,C]}$ are autonomous and

b) all invariant zeros of $[E,A,B,C]$ are contained in $\mathbb{C}_-$.

Furthermore, suppose that at least one of the following two assertions holds:

(i) *The circuit neither contains $\mathscr{I}\mathscr{L}$-loops, nor $\mathscr{V}\mathscr{C}\mathscr{L}$-cutsets except for $\mathscr{V}\mathscr{L}$-cutsets with at least one inductor.*

(ii) *The circuit neither contains $\mathscr{V}\mathscr{C}$-cutsets, nor $\mathscr{I}\mathscr{C}\mathscr{L}$-loops except for $\mathscr{I}\mathscr{C}$-loops with at least one capacitor.*

Then the zero dynamics $\mathscr{ZD}_{[E,A,B,C]}$ are asymptotically stable.

Proof. Step 1: We show that asymptotically stable zero dynamics imply $a)$ and $b)$. $a)$ follows from [2, Rem. 4.3] and $b)$ from [2, Lem. 4.2].

Step 2: We show that $a)$ and $b)$ imply asymptotically stable zero dynamics. By $a)$ and Proposition 5.2 we find that $\mathrm{rk}_{\mathbb{R}(s)} \left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right] = n+m$. Then $b)$ implies that

$$\forall \lambda \in \overline{\mathbb{C}}_+ : \ \mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A & -B \\ -C & 0 \end{bmatrix} = \mathrm{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} = n+m,$$

and therefore [2, Lem. 4.2] gives asymptotic stability of the zero dynamics.

Step 3: We show that (i) or (ii) implies asymptotically stable zero dynamics. In particular, we have "The circuit neither contains $\mathscr{I}$-loops nor $\mathscr{V}$-cutsets" and hence Proposition 5.2 implies $a)$. Furthermore, (i) or (ii) from Corollary 5.4 holds true and therefore $b)$ is valid. This yields the assertion of the theorem. □

6. High-gain stabilization. In this section we consider high-gain output feedback for a system $[E,A,B,C] \in \Sigma_{n,m}$, i.e., system (1.1) together with the feedback equation $u(t) = -k \cdot y(t)$, where $k > 0$. This gives rise to a differential-algebraic equation

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) = (A - kBC)x(t). \tag{6.1}$$

Usually (see e.g. [4, Def. 5.5]) a system is called *high-gain stabilizable* if the feedback interconnection leads to an asymptotically stable closed-loop system (6.1) (i.e., any solution tends to zero) for k large enough. In other words, there exists $\kappa > 0$ such that for all $k \geq \kappa$ the pencil $sE - (A - kBC)$ is regular and all of its generalized eigenvalues are contained in $\mathbb{C}_-$.

We will show that for electrical circuits, i.e., $[E,A,B,C]$ with (1.3), (1.5), the high-gain need not be high; any positive k is sufficient. In order to achieve this note that we have

$$sE - (A - kBC) = \begin{bmatrix} s\mathrm{A}_{\mathscr{C}}\mathscr{C}\mathrm{A}_{\mathscr{C}}^\top + \mathrm{A}_{\mathscr{R}}\mathscr{G}\mathrm{A}_{\mathscr{R}}^\top + k\mathrm{A}_{\mathscr{I}}\mathrm{A}_{\mathscr{I}}^\top & \mathrm{A}_{\mathscr{L}} & \mathrm{A}_{\mathscr{V}} \\ -\mathrm{A}_{\mathscr{L}}^\top & s\mathscr{L} & 0 \\ -\mathrm{A}_{\mathscr{V}}^\top & 0 & kI_{n_{\mathscr{V}}} \end{bmatrix}. \tag{6.2}$$

Then, for

$$W = \begin{bmatrix} I_{n_e} & 0 & -k^{-1}\mathrm{A}_{\mathcal{V}} \\ 0 & I_{n_{\mathcal{L}}} & 0 \\ 0 & 0 & k^{-1}I_{n_{\mathcal{V}}} \end{bmatrix}, \quad T = \begin{bmatrix} I_{n_e} & 0 & 0 \\ 0 & I_{n_{\mathcal{L}}} & 0 \\ k^{-1}\mathrm{A}_{\mathcal{V}}^\top & 0 & I_{n_{\mathcal{V}}} \end{bmatrix},$$

we find that

$$W(sE-(A-kBC))T = \begin{bmatrix} s\mathrm{A}_{\mathcal{C}}\mathcal{C}\mathrm{A}_{\mathcal{C}}^\top + \mathrm{A}_{\mathcal{R}}\mathcal{G}\mathrm{A}_{\mathcal{R}}^\top + k\mathrm{A}_{\mathscr{I}}\mathrm{A}_{\mathscr{I}}^\top + k^{-1}\mathrm{A}_{\mathcal{V}}\mathrm{A}_{\mathcal{V}}^\top & \mathrm{A}_{\mathcal{L}} & 0 \\ -\mathrm{A}_{\mathcal{L}}^\top & s\mathcal{L} & 0 \\ 0 & 0 & I_{n_{\mathcal{V}}} \end{bmatrix}. \quad (6.3)$$

The upper left part is a matrix pencil which is the MNA model of a circuit in which all current and voltage sources are replaced with resistances of values k^{-1} and k, resp. We may therefore conclude the following from Corollary 4.3.

COROLLARY 6.1 (Closed-loop pencil is regular). *Let $[E,A,B,C] \in \Sigma_{n,m}$ with* (1.3), (1.5) *be given and suppose that* **(A1)** *and* **(A2)** *hold true. Then, for all $k > 0$, the pencil $sE-(A-kBC)$ is regular.*

As a consequence of Theorem 4.4, we can furthermore analyze the asymptotic stability of the closed-loop system.

THEOREM 6.2 (Asymptotic stability of closed-loop pencil). *Let $[E,A,B,C] \in \Sigma_{n,m}$ with* (1.3), (1.5) *be the MNA model of an electrical circuit and suppose that* **(A1)** *and* **(A2)** *hold. Furthermore, suppose that at least one of the following two assertions holds true:*

(i) The circuit neither contains $\mathcal{L}$-loops, nor $\mathcal{CL}$-cutsets except for $\mathcal{L}$-cutsets.

(ii) The circuit neither contains $\mathcal{C}$-cutsets, nor $\mathcal{CL}$-loops except for $\mathcal{C}$-loops.

Then, for any $k > 0$, all generalized eigenvalues of $sE-(A-kBC)$ are contained in $\mathbb{C}_-$.

REMARK 6.3 (Asymptotically stable zero dynamics and high-gain). *Let $[E,A,B,C] \in \Sigma_{n,m}$ with* (1.3), (1.5) *be the MNA model of an electrical circuit and suppose that* **(A1)** *and* **(A2)** *hold. Then, under one of the assumptions (i) or (ii) from Theorem 5.5, the respective assumption from Theorem 6.2 holds true, but not vice versa. Therefore, the (topological condition for) asymptotic stability of the zero dynamics implies high-gain stabilizability, but in general not the other way round; this has already been observed for two important classes of DAEs in [3, Sec. 4].*

7. Funnel control. In this section we consider funnel control for systems $[E,A,B,C] \in \Sigma_{n,m}$ with (1.3), (1.5). The aim is to achieve tracking of a reference trajectory by the output signal with prescribed transient behavior. The funnel controller resolves several problems of other control strategies such as the classical adaptive high-gain controller; see the survey [8].

For any function φ belonging to

$$\Phi := \left\{ \varphi \in \mathcal{C}^\infty(\mathbb{R}_{\ge 0};\mathbb{R}) \cap \mathscr{B}^1(\mathbb{R}_{\ge 0};\mathbb{R}) \;\middle|\; \begin{array}{l} \varphi(0)=0, \quad \varphi(s)>0 \text{ for all } s>0 \\ \text{and } \liminf_{s\to\infty}\varphi(s)>0 \end{array} \right\}$$

we associate the *performance funnel*

$$\mathscr{F}_\varphi := \left\{ (t,e) \in \mathbb{R}_{\ge 0} \times \mathbb{R}^m \,\middle|\, \varphi(t)\|e\| < 1 \right\}, \quad (7.1)$$

see Figure 7.1. The control objective is feedback control so that the tracking error $e(\cdot) = y(\cdot) - y_{\mathrm{ref}}(\cdot)$, where $y_{\mathrm{ref}}(\cdot)$ is the reference signal, evolves within $\mathscr{F}_\varphi$ and all variables are

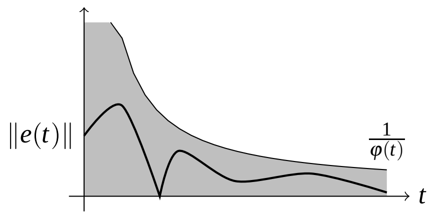


Fig. 7.1: Error evolution in a funnel $\mathscr{F}_\varphi$ with boundary $1/\varphi(t)$ for $t > 0$.

bounded. More specific, the transient behaviour is supposed to satisfy

$$\forall t > 0: \ \|e(t)\| < 1/\varphi(t),$$

and, moreover, if φ is chosen so that $\varphi(t) \geq 1/\lambda$ for all t sufficiently large, then the tracking error remains smaller than λ.

By choosing $\varphi(0) = 0$ we ensure that the width of the funnel is infinity at $t = 0$, see Figure 7.1. In the following we only treat "infinite" funnels for technical reasons, since if the funnel is finite, that is $\varphi(0) > 0$, then we need to assume that the initial error is within the funnel boundaries at $t = 0$, i.e., $\varphi(0)\|Cx^0 - y_{\text{ref}}(0)\| < 1$, and this assumption suffices.

As indicated in Figure 7.1, we do not assume that the funnel boundary decreases monotonically. Certainly, in most situations it is convenient to choose a monotone funnel, however there are situations where widening the funnel at some later time might be beneficial, e.g., when it is known that the reference signal varies strongly.

To ensure error evolution within the funnel, we introduce the *funnel controller*:

$$\boxed{\begin{aligned} &u(t) = -k(t)\,e(t), \qquad \text{where} \qquad e(t) = y(t) - y_{\text{ref}}(t) \\ &k(t) = \frac{1}{1 - \varphi(t)^2\|e(t)\|^2}\,. \end{aligned}} \tag{7.2}$$

If we assume asymptotically stable zero dynamics, we see intuitively that, in order to maintain the error evolution within the funnel, high gain values may only be required if the norm $\|e(t)\|$ of the error is close to the funnel boundary $\varphi(t)^{-1}$: $k(\cdot)$ increases if necessary to exploit the high-gain property of the system and decreases if a high gain is not necessary. This intuition underpins the choice of the gain $k(t)$ in (7.2). The control design (7.2) has two advantages: $k(\cdot)$ is non-monotone and (7.2) is a static time-varying proportional output feedback of striking simplicity.

Before we state and prove feasibility of funnel control for electrical circuits, we need to define consistency of the initial value of the closed-loop system and solutions of the latter. We also define what "feasibility of funnel control" will mean.

DEFINITION 7.1 (Consistent initial value). *Let* $[E,A,B,C] \in \Sigma_{n,m}$, $\varphi \in \Phi$ *and* $y_{\text{ref}} \in \mathscr{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$. *An initial value* $x^0 \in \mathbb{R}^n$ *is called* consistent *for the closed-loop system* (1.1), (7.2) *if, and only if, there exists a solution of the initial value problem* (1.1), (7.2), $x(0) = x^0$, *i.e., a function* $x \in \mathcal{C}^1([0,\omega); \mathbb{R}^n)$ *for some* $\omega \in (0,\infty]$, *such that* $x(0) = x^0$ *and* x *satisfies* (1.1), (7.2) *for all* $t \in [0,\omega)$.

Note that, in practice, consistency of the initial state of the "unknown" system should be satisfied as far as the DAE $[E,A,B,C]$ is the correct model.

In the following we define feasibility of funnel control for a system on a set of reference trajectories. For reference trajectories we allow signals in $\mathscr{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, whereas in [2] signals in $\mathscr{B}^{\nu}(\mathbb{R}_{\geq 0};\mathbb{R}^m)$ are allowed and $\nu \in \mathbb{N}$ is a number which can be calculated out of a certain system decomposition. To avoid the details of this calculation we restrict ourselves to the case of $\mathscr{B}^{\infty}(\mathbb{R}_{\geq 0};\mathbb{R}^m)$.

DEFINITION 7.2 (Feasibility of funnel control). *Let $[E,A,B,C] \in \Sigma_{n,m}$ and $\mathscr{S} \subseteq \mathscr{B}^{\infty}(\mathbb{R}_{\geq 0};\mathbb{R}^m)$ be a set of reference trajectories. We say that* funnel control is feasible for $[E,A,B,C]$ on $\mathscr{S}$ *if, and only if, for all $\varphi \in \Phi$, any reference signal $y_{\mathrm{ref}} \in \mathscr{S}$ and any consistent initial value $x^0 \in \mathbb{R}^n$ the application of the funnel controller* (7.2) *to* (1.1) *yields a closed-loop initial-value problem that has a solution and every solution can be extended to a global solution. Furthermore, for every global solution $x(\cdot)$,*

(i) *$x(\cdot)$ is bounded and the corresponding tracking error $e(\cdot) = Cx(\cdot) - y_{\mathrm{ref}}(\cdot)$ evolves uniformly within the performance funnel $\mathscr{F}_{\varphi}$; more precisely,*

$$\exists \varepsilon > 0\ \forall t > 0:\ \|e(t)\| \leq \varphi(t)^{-1} - \varepsilon. \tag{7.3}$$

(ii) *the corresponding gain function $k(\cdot)$ given by* (7.2) *is bounded.*

REMARK 7.3 (Bound for the gain). *If funnel control is feasible as stated in Definition 7.2, then the gain function k is bounded in a way that*

$$\forall t_0 > 0:\ \sup_{t \geq t_0} |k(t)| \leq \frac{1}{1-(1-\varepsilon\lambda_{t_0})^2},$$

where ε is given in (7.3) *and $\lambda_{t_0} := \inf_{t\geq t_0} \varphi(t) > 0$ for all $t_0 > 0$. A proof for this can be found in [2, Thm. 6.3].*

In the following we show that funnel control for systems $[E,A,B,C] \in \Sigma_{n,m}$ with (1.3), (1.5) is feasible provided that the invariant zeros have negative real part and the reference signal is sufficiently smooth and evolves in a certain subspace. The former means that the autonomous part of the zero dynamics has to be asymptotically stable, but autonomy of the whole zero dynamics is not required. As a preliminary result we derive that, for positive real systems $[E,A,B,C] \in \Sigma_{n,m}$ with asymptotically stable zero dynamics, funnel control will be feasible for any sufficiently smooth reference signal.

PROPOSITION 7.4 (Funnel control for systems with stable zero dynamics). *Let $[E,A,B,C] \in \Sigma_{n,m}$ be such that $E = E^{\top} \geq 0$, $A + A^{\top} \leq 0$, and $B = C^{\top}$. Further, assume that the zero dynamics of $[E,A,B,C]$ are asymptotically stable. Then funnel control is feasible for $[E,A,B,C]$ on $\mathscr{B}^{\infty}(\mathbb{R}_{\geq 0};\mathbb{R}^m)$.*

Proof. We aim to apply [2, Thm. 6.3] for $\hat{k} = 1$ and to this end verify its assumptions.

Step 1: The zero dynamics of $[E,A,B,C]$ are asymptotically stable by assumption.

Step 2: We show that for the inverse $L(s)$ of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$ the matrix

$$\Gamma = -\lim_{s\to\infty} s^{-1}[0, I_m]L(s)\begin{bmatrix} 0 \\ I_m \end{bmatrix} \in \mathbb{R}^{m,m}$$

exists and satisfies $\Gamma = \Gamma^{\top} \geq 0$. By Lemma 2.5, the pencil

$$\begin{bmatrix} sE-A & -B \\ C & 0 \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & -I_m \end{bmatrix}\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$$

is positive real. Then, for the inverse $L(s)$ of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$, $\tilde{L}(s) := L(s)\left[\begin{smallmatrix} I_n & 0 \\ 0 & -I_m \end{smallmatrix}\right]$ is the inverse of $\left[\begin{smallmatrix} sE-A & -B \\ C & 0 \end{smallmatrix}\right]$, and we have

$$\tilde{L}(\lambda) + \tilde{L}(\lambda)^* = \tilde{L}(\lambda)\left(\left[\begin{smallmatrix} \lambda E-A & -B \\ C & 0 \end{smallmatrix}\right]^* + \left[\begin{smallmatrix} \lambda E-A & -B \\ C & 0 \end{smallmatrix}\right]\right)\tilde{L}(\lambda)^* \geq 0$$

for all $\lambda \in \mathbb{C}_+$. Furthermore, since $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ does not have any invariant zeros in $\mathbb{C}_+$, $\tilde{L}(s)$ has no poles in $\mathbb{C}_+$. This shows that $\tilde{L}(s)$ is positive real. Hence, $H(s) := [0, I_m]\tilde{L}(s)[0, I_m]^\top$ is positive real as well and from Lemma 2.4 we obtain that

$$\Gamma = \lim_{s\to\infty} s^{-1}H(s) \in \mathbb{R}^{m,m}$$

exists and satisfies $\Gamma = \Gamma^\top \geq 0$.

Step 3: We show that $[E, A, B, C]$ is right-invertible in the sense of [2, Def. 5.1]. Since the zero dynamics of $[E, A, B, C]$ are in particular autonomous it follows from Proposition 5.2 (ii) that $\operatorname{rk} C = m$ and hence right-invertibility can be concluded from [2, Rem. 5.12].

Step 4: It remains to show that $\hat{k}$ in [2, Thm. 6.3] can be chosen as $\hat{k} = 1$ and funnel control is still feasible. A careful inspection of the proof of [2, Thm. 6.3] reveals that, in general, $\hat{k}$ large enough is needed in order to guarantee invertibility of $\tilde{A} - \tilde{k}(t)I_m$, where

$$\tilde{A} = \lim_{s\to\infty} \left([0, I_m]L(s) \begin{bmatrix} 0 \\ I_m \end{bmatrix} + s\Gamma \right)$$

and $\tilde{k}(t) = \hat{k} \cdot k(t)$, $t \geq 0$. Calculating

$$\tilde{A} = \lim_{s\to\infty} (s\Gamma - H(s)) = -H_0 - \lim_{s\to\infty} H_{\mathrm{sp}}(s)$$

where, since $H(s)$ is positive real, by Lemma 2.4 the rational function $H_0 + H_{\mathrm{sp}}(s)$ is positive real and $\lim_{s\to\infty} H_{\mathrm{sp}}(s) = 0$. Hence, it is easy to derive that $H_0 \geq 0$ (H_0 not necessarily symmetric) and hence

$$\tilde{A} - kI_m = -H_0 - kI_m < 0$$

for all $k > 0$ (again $\tilde{A} - kI_m$ not necessarily symmetric). The negative definiteness however implies that $\tilde{A} - kI_m$ is invertible for all $k > 0$ and therefore it is sufficient to assume $\hat{k} = 1$. ☐

Before we prove our main result we need to know how feasibility of funnel control behaves under transformation of the system.

LEMMA 7.5 (Funnel control under system transformation). *Let $E, A \in \mathbb{R}^{n,n}$, $B, C^\top \in \mathbb{R}^{n,m}$ and $\mathscr{S} \subseteq \mathscr{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$. Further, let $W, T \in \mathbf{Gl}_n(\mathbb{R})$, $U \in \mathscr{O}_m(\mathbb{R})$, and define*

$$[\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}] := [WET, WAT, WBU, U^\top CT].$$

Then funnel control is feasible for $[E, A, B, C]$ on $\mathscr{S}$ if, and only if, funnel control is feasible for $[\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}]$ on $U^\top \mathscr{S}$.

Proof. Observe that $(x, u, y) \in \mathfrak{B}_{[E,A,B,C]}$ and $y_{\mathrm{ref}} \in \mathscr{S}$ if, and only if,

$$(\widetilde{x}, \widetilde{u}, \widetilde{y}) = (T^{-1}x, U^\top u, U^\top y) \in \mathfrak{B}_{[\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}]} \quad \wedge \quad U^\top y_{\mathrm{ref}} \in U^\top \mathscr{S}.$$

Then the assertion follows from the observation that, for any $\varphi \in \Phi$, and tracking errors $e = y - y_{\mathrm{ref}}$, $\tilde{e} = \tilde{y} - \tilde{y}_{\mathrm{ref}}$ we have, for all $t \geq 0$,

$$\frac{1}{1 - \varphi(t)^2 \|e(t)\|^2} = \frac{1}{1 - \varphi(t)^2 \|\widetilde{e}(t)\|^2}.$$

☐

In the following, in order to show that funnel control is feasible for circuits where all invariant zeros are located in $\mathbb{C}_-$, but the zero dynamics are not necessarily autonomous, we derive a transformation of the circuit which decouples the "non-autonomous part" of the zero dynamics. This part, in particular, does not affect the input-output behavior of the system.

PROPOSITION 7.6 (Decoupling of circuit pencil). *Let* $[E,A,B,C] \in \Sigma_{n,m}$ *with* (1.3), (1.5) *be the MNA model of an electrical circuit and suppose that* **(A1)** *and* **(A2)** *hold. Let* $Z'_{\mathcal{CRL}\mathscr{I}} \in \mathbb{R}^{n_e,k_1}$, $Z_{\mathcal{CRL}\mathscr{I}} \in \mathbb{R}^{n_e,k_2}$ *with full column rank such that*

$$\operatorname{im} Z_{\mathcal{CRL}\mathscr{I}} = \ker \begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathscr{I}} \end{bmatrix}^\top, \quad \textit{and} \quad \operatorname{im} Z'_{\mathcal{CRL}\mathscr{I}} = \operatorname{im} \begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathscr{I}} \end{bmatrix}.$$

Further, let $Z_{\mathcal{V}-\mathcal{CRL}\mathscr{I}} \in \mathbb{R}^{n_{\mathcal{V}},k_3}$, $Z'_{\mathcal{V}-\mathcal{CRLV}} \in \mathbb{R}^{n_{\mathcal{V}},k_4}$, $\bar{Z}_{\mathscr{I}} \in \mathbb{R}^{n_{\mathscr{I}},k_5}$, $\bar{Z}'_{\mathscr{I}} \in \mathbb{R}^{n_{\mathscr{I}},k_6}$ *with orthonormal columns such that*

$$\begin{aligned} \operatorname{im} Z_{\mathcal{V}-\mathcal{CRL}\mathscr{I}} &= \ker Z_{\mathcal{CRL}\mathscr{I}}^\top A_{\mathcal{V}}, & \operatorname{im} \bar{Z}_{\mathscr{I}} &= \ker A_{\mathscr{I}}, \\ \operatorname{im} Z'_{\mathcal{V}-\mathcal{CRL}\mathscr{I}} &= \operatorname{im} A_{\mathcal{V}}^\top Z_{\mathcal{CRL}\mathscr{I}}, & \operatorname{im} \bar{Z}'_{\mathscr{I}} &= \operatorname{im} A_{\mathscr{I}}^\top. \end{aligned}$$

Then we have

$$W^\top := T := \begin{bmatrix} Z_{\mathcal{CRL}\mathscr{I}} & Z'_{\mathcal{CRL}\mathscr{I}} & 0 & 0 & 0 \\ 0 & 0 & I_{n_{\mathcal{L}}} & 0 & 0 \\ 0 & 0 & 0 & Z_{\mathcal{V}-\mathcal{RCL}\mathscr{I}} & Z'_{\mathcal{V}-\mathcal{RCL}\mathscr{I}} \end{bmatrix} \in \mathbf{Gl}_n(\mathbb{R}) \tag{7.4a}$$

and

$$U := \begin{bmatrix} 0 & \bar{Z}_{\mathscr{I}} & \bar{Z}'_{\mathscr{I}} & 0 \\ Z'_{\mathcal{V}-\mathcal{RCL}\mathscr{I}} & 0 & 0 & Z_{\mathcal{V}-\mathcal{RCL}\mathscr{I}} \end{bmatrix} \in \mathscr{O}_m(\mathbb{R}), \tag{7.4b}$$

and

$$W(sE-A)T = \begin{bmatrix} 0 & 0 & Z_{\mathcal{CRL}\mathscr{I}}^\top A_{\mathcal{V}} Z'_{\mathcal{V}-\mathcal{CRL}\mathscr{I}} \\ 0 & s\widetilde{E}_r - \widetilde{A}_r & \begin{bmatrix} (Z'_{\mathcal{CRL}\mathscr{I}})^\top A_{\mathcal{V}} Z'_{\mathcal{V}-\mathcal{CRL}\mathscr{I}} \\ 0 \\ 0 \end{bmatrix} \\ -(Z'_{\mathcal{V}-\mathcal{CRL}\mathscr{I}})^\top A_{\mathcal{V}}^\top Z_{\mathcal{CRL}\mathscr{I}} & \begin{bmatrix} -(Z'_{\mathcal{V}-\mathcal{CRL}\mathscr{I}})^\top A_{\mathcal{V}}^\top Z'_{\mathcal{CRL}\mathscr{I}}, & 0, & 0 \end{bmatrix} & 0 \end{bmatrix} \tag{7.5}$$

and

$$WBU = \left(U^\top C T\right)^\top = \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{B}_r \\ [-I_{k_4}, 0] & 0 \end{bmatrix}, \tag{7.6}$$

where

$$\begin{aligned} s\widetilde{E}_r - \widetilde{A}_r &= \begin{bmatrix} (Z'_{\mathcal{CRL}\mathscr{I}})^\top \left(s A_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^\top + A_{\mathcal{R}} \mathcal{G} A_{\mathcal{R}}^\top\right) Z'_{\mathcal{CRL}\mathscr{I}} & (Z'_{\mathcal{CRL}\mathscr{I}})^\top A_{\mathcal{L}} & Z_{\mathcal{CRL}\mathscr{I}}^\top A_{\mathcal{V}} Z_{\mathcal{V}-\mathcal{CRL}\mathscr{I}} \\ -A_{\mathcal{L}}^\top Z'_{\mathcal{CRL}\mathscr{I}} & s\mathcal{L} & 0 \\ -Z_{\mathcal{V}-\mathcal{CRL}\mathscr{I}}^\top A_{\mathcal{V}}^\top Z'_{\mathcal{CRL}\mathscr{I}} & 0 & 0 \end{bmatrix}, \\ \widetilde{B}_r = \widetilde{C}_r^\top &= \begin{bmatrix} -(Z'_{\mathcal{CRL}\mathscr{I}})^\top A_{\mathscr{I}} \bar{Z}'_{\mathscr{I}} & 0 \\ 0 & 0 \\ 0 & -I_{k_3} \end{bmatrix} \end{aligned} \tag{7.7}$$

Furthermore, the following holds true:

(a) $k_2 = k_4$ *and* $Z_{\mathscr{CRLI}}^\top \mathrm{A}_\mathscr{V} Z'_{\mathscr{V}-\mathscr{CRLI}} \in \mathbf{Gl}_{k_2}(\mathbb{R})$.
(b) *The zero dynamics of the system* $[\widetilde{E}_r, \widetilde{A}_r, \widetilde{B}_r, \widetilde{C}_r]$ *are autonomous.*
(c) $\lambda \in \mathbb{C}$ *is an invariant zero of* $[E,A,B,C]$ *if, and only if,* λ *is an invariant zero of* $[\widetilde{E}_r, \widetilde{A}_r, \widetilde{B}_r, \widetilde{C}_r]$.

Proof. The invertibility of W, T and U is a consequence of

$$\begin{aligned}
&\operatorname{im} Z'_{\mathscr{CRLI}} \oplus \operatorname{im} Z_{\mathscr{CRLI}} = \operatorname{im}\begin{bmatrix}\mathrm{A}_\mathscr{C} & \mathrm{A}_\mathscr{R} & \mathrm{A}_\mathscr{L} & \mathrm{A}_\mathscr{I}\end{bmatrix} \oplus \ker\begin{bmatrix}\mathrm{A}_\mathscr{C} & \mathrm{A}_\mathscr{R} & \mathrm{A}_\mathscr{L} & \mathrm{A}_\mathscr{I}\end{bmatrix}^\top = \mathbb{R}^{n_e},\\
&\operatorname{im} Z'_{\mathscr{V}-\mathscr{RCLI}} \oplus \operatorname{im} Z_{\mathscr{V}-\mathscr{RCLI}} = \operatorname{im} \mathrm{A}_\mathscr{V}^\top Z_{\mathscr{CRLI}} \oplus \ker Z_{\mathscr{CRLI}}^\top \mathrm{A}_\mathscr{V} = \mathbb{R}^{n_\mathscr{V}},\\
&\operatorname{im} \overline{Z}'_\mathscr{I} \oplus \operatorname{im} \overline{Z}_\mathscr{I} = \operatorname{im} \mathrm{A}_\mathscr{I}^\top \oplus \ker \mathrm{A}_\mathscr{I} = \mathbb{R}^{n_\mathscr{I}}.
\end{aligned}$$

Furthermore, by choice of $Z_{\mathscr{V}-\mathscr{CRLI}}$, $Z'_{\mathscr{V}-\mathscr{CRLV}}$, $\overline{Z}_\mathscr{I}$ and $\overline{Z}'_\mathscr{I}$ the matrix U is orthogonal. The representation of the transformed system in (7.5), (7.6) and (7.7) is then a simple calculation.

We prove assertions (a)–(c).

(a) The assertion will be inferred from the fact that both matrices $Z_{\mathscr{CRLI}}^\top \mathrm{A}_\mathscr{V} Z'_{\mathscr{V}-\mathscr{CRLI}}$ and $(Z_{\mathscr{CRLI}}^\top \mathrm{A}_\mathscr{V} Z'_{\mathscr{V}-\mathscr{CRLI}})^\top$ have trivial kernels. To prove the first assertion, assume let $z \in \ker Z_{\mathscr{CRLI}}^\top \mathrm{A}_\mathscr{V} Z'_{\mathscr{V}-\mathscr{CRLI}}$. Then

$$Z'_{\mathscr{V}-\mathscr{CRLI}} z \in \ker Z_{\mathscr{CRLI}}^\top \mathrm{A}_\mathscr{V} = (\operatorname{im} \mathrm{A}_\mathscr{V}^\top Z_{\mathscr{CRLI}})^\perp = (\operatorname{im} Z'_{\mathscr{V}-\mathscr{CRLI}})^\perp.$$

Therefore, $Z'_{\mathscr{V}-\mathscr{CRLI}} z = 0$, and the full column rank of $Z'_{\mathscr{V}-\mathscr{CRLI}}$ implies $z = 0$. Now let $z \in \ker (Z'_{\mathscr{V}-\mathscr{CRLI}})^\top \mathrm{A}_\mathscr{V}^\top Z_{\mathscr{CRLI}}$. Then

$$\mathrm{A}_\mathscr{V}^\top Z_{\mathscr{CRLI}} z \in \ker (Z'_{\mathscr{V}-\mathscr{CRLI}})^\top = (\operatorname{im} Z'_{\mathscr{V}-\mathscr{CRLI}})^\perp = (\operatorname{im} \mathrm{A}_\mathscr{V}^\top Z_{\mathscr{CRLI}})^\perp.$$

Thus, $Z_{\mathscr{CRLI}} z \in \ker \mathrm{A}_\mathscr{V}^\top$ and by choice of $Z_{\mathscr{CRLI}}$ we have

$$Z_{\mathscr{CRLI}} z \in \ker \begin{bmatrix}\mathrm{A}_\mathscr{C} & \mathrm{A}_\mathscr{R} & \mathrm{A}_\mathscr{L} & \mathrm{A}_\mathscr{I}\end{bmatrix}^\top \cap \ker \mathrm{A}_\mathscr{V}^\top \overset{\textbf{(A1)}}{=} \{0\},$$

Hence, we obtain $z = 0$ from the full column rank of $Z_{\mathscr{CRLI}}$.

(b) By Proposition 5.2 it is sufficient to show that the pencil

$$s\mathscr{E} - \mathscr{A} := \begin{bmatrix} s\widetilde{E}_r - \widetilde{A}_r & \widetilde{B}_r \\ -\widetilde{C}_r & 0 \end{bmatrix} = \begin{bmatrix} s\widetilde{E}_r - \widetilde{A}_r & -\widetilde{B}_r \\ -\widetilde{C}_r & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

is regular. Observing that $\mathscr{E} = \mathscr{E}^\top \geq 0$ and $\mathscr{A} + \mathscr{A}^\top \leq 0$, we can use Lemma 2.6 to further reduce the problem to showing that $\ker \mathscr{E} \cap \ker \mathscr{A} = \{0\}$:
Let $z = (z_1, z_2, z_3, z_4, z_5) \in \ker \mathscr{E} \cap \ker \mathscr{A}$ be suitably partitioned according to the block structure of $\widetilde{E}_r$, $\widetilde{A}_r$, $\widetilde{B}_r$ and $\widetilde{C}_r$ as in (7.7). Then, by **(A2)**, the equation $z^\top \mathscr{E} z = z^\top (\mathscr{A} + \mathscr{A}^\top) z = 0$ gives rise to $z_2 = 0$ and

$$z_1 \in \ker \begin{bmatrix}\mathrm{A}_\mathscr{C} & \mathrm{A}_\mathscr{R}\end{bmatrix}^\top Z'_{\mathscr{CRLI}}.$$

The equation $\mathscr{A} z = 0$ further implies $z_3 = 0$ and

$$z_1 \in \ker \mathrm{A}_\mathscr{L}^\top Z'_{\mathscr{CRLI}} \ \wedge \ z_1 \in \ker (\overline{Z}'_\mathscr{I})^\top \mathrm{A}_\mathscr{I}^\top Z'_{\mathscr{CRLI}}.$$

The latter implies

$$\mathrm{A}_\mathscr{I}^\top Z'_{\mathscr{CRLI}} z_1 \in \ker (\overline{Z}'_\mathscr{I})^\top = (\operatorname{im} \overline{Z}'_\mathscr{I})^\perp = (\operatorname{im} \mathrm{A}_\mathscr{I}^\top)^\perp,$$

whence

$$z_1 \in \ker \mathrm{A}_{\mathscr{I}}^\top Z'_{\mathcal{CRL}\mathscr{I}}.$$

Altogether, we have

$$\begin{aligned} Z'_{\mathcal{CRL}\mathscr{I}} z_1 \in \ker \begin{bmatrix} \mathrm{A}_{\mathcal{C}} & \mathrm{A}_{\mathcal{R}} & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathscr{I}} \end{bmatrix}^\top \\ = \left(\operatorname{im} \begin{bmatrix} \mathrm{A}_{\mathcal{C}} & \mathrm{A}_{\mathcal{R}} & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathscr{I}} \end{bmatrix}\right)^\perp = (\operatorname{im} Z'_{\mathcal{CRL}\mathscr{I}})^\perp. \end{aligned}$$

The full column rank of $Z'_{\mathcal{CRL}\mathscr{I}}$ now implies that $z_1 = 0$. Now using that $z_1 = 0$, $z_2 = 0$ and $z_3 = 0$, we can infer from $\mathscr{A} z = 0$ that $z_5 = 0$ and

$$(Z'_{\mathcal{CRL}\mathscr{I}})^\top \mathrm{A}_{\mathscr{I}} \overline{Z}'_{\mathscr{I}} z_4 = 0.$$

Thus,

$$\mathrm{A}_{\mathscr{I}} \overline{Z}'_{\mathscr{I}} z_4 \in \ker (Z'_{\mathcal{CRL}\mathscr{I}})^\top = (\operatorname{im} Z'_{\mathcal{CRL}\mathscr{I}})^\perp = \left(\operatorname{im} \begin{bmatrix} \mathrm{A}_{\mathcal{C}} & \mathrm{A}_{\mathcal{R}} & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathscr{I}} \end{bmatrix}\right)^\perp \subseteq (\operatorname{im} \mathrm{A}_{\mathscr{I}})^\perp.$$

Therefore, $\mathrm{A}_{\mathscr{I}} \overline{Z}'_{\mathscr{I}} z_4 = 0$ or, equivalently,

$$\overline{Z}'_{\mathscr{I}} z_4 \in \ker \mathrm{A}_{\mathscr{I}} = (\operatorname{im} \mathrm{A}_{\mathscr{I}}^\top)^\perp = (\operatorname{im} \overline{Z}'_{\mathscr{I}})^\perp.$$

This implies $\overline{Z}'_{\mathscr{I}} z_4 = 0$, and since $\overline{Z}'_{\mathscr{I}}$ has full column rank, we have that $z_4 = 0$.

(c) It can be obtained from simple row and column operations that for all $\lambda \in \mathbb{C}$ we have

$$\operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A & -B \\ -C & 0 \end{bmatrix} = \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda WET - WAT & -WBU \\ -U^T CT & 0 \end{bmatrix} = \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda \widetilde{E}_r - \widetilde{A}_r & -\widetilde{B}_r \\ -\widetilde{C}_r & 0 \end{bmatrix} + 2k_4$$

and, similarly,

$$\operatorname{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} = \operatorname{rk}_{\mathbb{R}(s)} \begin{bmatrix} s\widetilde{E}_r - \widetilde{A}_r & -\widetilde{B}_r \\ -\widetilde{C}_r & 0 \end{bmatrix} + 2k_4.$$

This implies that the generalized eigenvalues of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ coincide with those of $\left[\begin{smallmatrix} s\widetilde{E}-\widetilde{A}_r & -\widetilde{B}_r \\ -\widetilde{C}_r & 0 \end{smallmatrix}\right]$ and hence the assertion is proved.

This concludes the proof of the proposition. □

We are now in the position to prove the main result of this section.

THEOREM 7.7 (Funnel control for circuits). *Let* $[E,A,B,C] \in \Sigma_{n,m}$ *with* (1.3), (1.5) *be the MNA model of an electrical circuit and suppose that* **(A1)** *and* **(A2)** *hold. Assume that the system* $[E,A,B,C]$ *does not have any invariant zeros on the imaginary axis. Let* $Z_{\mathcal{CRL}\mathscr{I}}$ *be a matrix with full column rank such that*

$$\operatorname{im} Z_{\mathcal{CRL}\mathscr{I}} = \ker \begin{bmatrix} \mathrm{A}_{\mathcal{C}} & \mathrm{A}_{\mathcal{R}} & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathscr{I}} \end{bmatrix}^\top.$$

Then funnel control is feasible for $[E,A,B,C]$ *on*

$$\mathscr{B}^\infty \left(\mathbb{R}_{\geq 0}; \operatorname{im} \mathrm{A}_{\mathscr{I}}^\top \times \ker Z_{\mathcal{CRL}\mathscr{I}}^\top \mathrm{A}_{\mathcal{V}}\right).$$

Proof. Step 1: Use the notation from Proposition 7.6 and define

$$[\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C}] := [WET, WAT, WBU, U^\top CT].$$

Then, by Lemma 7.5, it suffices to prove that funnel control is feasible for $[\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C}]$ on

$$\mathscr{S} := U^\top \mathscr{B}^\infty \left(\mathbb{R}_{\geq 0}; \operatorname{im} \mathrm{A}_{\mathscr{I}}^\top \times \ker \mathrm{Z}_{\mathcal{CRL}\mathscr{I}}^\top \mathrm{A}_{\mathcal{V}}\right).$$

Step 2: We show that $[\widetilde{E}_r,\widetilde{A}_r,\widetilde{B}_r,\widetilde{C}_r]$ has asymptotically stable zero dynamics. By Proposition 7.6 (c), the zero dynamics of $[\widetilde{E}_r,\widetilde{A}_r,\widetilde{B}_r,\widetilde{C}_r]$ are autonomous. Furthermore, by Proposition 7.6 (d) and the fact that the invariant zeros of $[E,A,B,C]$ all have negative real part, we obtain from Theorem 5.5 that the zero dynamics of $[\widetilde{E}_r,\widetilde{A}_r,\widetilde{B}_r,\widetilde{C}_r]$ are asymptotically stable.

Step 3: We reduce the feasibility problem of funnel control to that of the system $[\widetilde{E}_r,\widetilde{A}_r,\widetilde{B}_r,\widetilde{C}_r]$. Let

$$(\widetilde{x},\widetilde{u},\widetilde{y}) \in \mathfrak{B}_{[\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C}]} \quad \text{and} \quad \widetilde{y}_{\text{ref}} = U^\top \begin{pmatrix} y_{\text{ref},1} \\ y_{\text{ref},2} \end{pmatrix} \in \mathscr{S}.$$

Since

$$y_{\text{ref},1} \in \operatorname{im} \mathrm{A}_{\mathscr{I}}^\top = \operatorname{im} \bar{Z}'_{\mathscr{I}} = (\operatorname{im} \bar{Z}_{\mathscr{I}})^\perp = \ker \bar{Z}_{\mathscr{I}}^\top$$

and

$$y_{\text{ref},2} \in \ker \mathrm{Z}_{\mathcal{CRL}\mathscr{I}}^\top \mathrm{A}_{\mathcal{V}} = \operatorname{im} \mathrm{Z}_{\mathcal{V}-\mathcal{CRL}\mathscr{I}} = \left(\operatorname{im} \mathrm{Z}'_{\mathcal{V}-\mathcal{CRL}\mathscr{I}}\right)^\perp = \ker\left(\mathrm{Z}'_{\mathcal{V}-\mathcal{CRL}\mathscr{I}}\right)^\top$$

we obtain that

$$\widetilde{y}_{\text{ref}} = \begin{bmatrix} 0, & 0, & \widetilde{y}_{\text{ref},1}, & \widetilde{y}_{\text{ref},2} \end{bmatrix}^\top,$$

where $\widetilde{y}_{\text{ref},1} = \left(\bar{Z}'_{\mathscr{I}}\right)^\top y_{\text{ref},1}$ and $\widetilde{y}_{\text{ref},2} = Z_{\mathcal{V}-\mathcal{CRL}\mathscr{I}}^\top y_{\text{ref},2}$. By suitably partitioning

$$\widetilde{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix}, \quad \widetilde{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix}, \quad \widetilde{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix}$$

according to the block structure of $s\widetilde{E} - \widetilde{A}$ as in (7.5), and $\widetilde{B}$, $\widetilde{C}$ as in (7.6), we obtain $\mathrm{Z}_{\mathcal{CRL}\mathscr{I}}^\top \mathrm{A}_{\mathcal{V}} Z'_{\mathcal{V}-\mathcal{CRL}\mathscr{I}} x_5 = 0$, whence, by Proposition 7.6 (b), we have $x_5 = 0$, and thus also $y_1 = 0$. Moreover, $y_2 = 0$ and

$$x_1 = -(\mathrm{Z}_{\mathcal{CRL}\mathscr{I}}^\top \mathrm{A}_{\mathcal{V}} Z'_{\mathcal{V}-\mathcal{CRL}\mathscr{I}})^{-1} (Z'_{\mathcal{V}-\mathcal{CRL}\mathscr{I}})^\top \mathrm{A}_{\mathcal{V}}^\top Z'_{\mathcal{CRL}\mathscr{I}} x_2 - u_1,$$

and, further

$$\widetilde{x}_r(t) = \begin{bmatrix} x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}, \quad \widetilde{u}_r(t) = \begin{bmatrix} u_3(t) \\ u_4(t) \end{bmatrix}, \quad \widetilde{y}_r(t) = \begin{bmatrix} y_3(t) \\ y_4(t) \end{bmatrix}$$

satisfy

$$(\widetilde{x}_r, \widetilde{u}_r, \widetilde{y}_r) \in \mathfrak{B}_{[\widetilde{E}_r, \widetilde{A}_r, \widetilde{B}_r, \widetilde{C}_r]}.$$

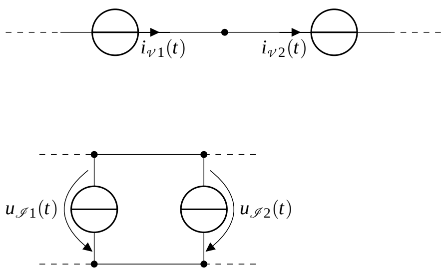
Application of the funnel controller (7.2) then yields $\widetilde{u} = -k(\widetilde{y} - \widetilde{y}_{\mathrm{ref}})$ and hence $u_1 = 0$ and $u_2 = 0$. Therefore, funnel control is feasible for $[\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}]$ on $\mathscr{S}$ if, and only if, funnel control is feasible for $[\widetilde{E}_r, \widetilde{A}_r, \widetilde{B}_r, \widetilde{C}_r]$ on $\mathscr{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^{k_3+k_6})$. The latter however follows from Step 2 and Proposition 7.4. This concludes the proof of the theorem. $\square$

REMARK 7.8 (Topological criteria for funnel control). *We analyze the constraints on the reference trajectories in Theorem 7.7.*

(a) The subspace restriction

$$y_{\mathrm{ref}}(t) \in \operatorname{im} A_{\mathscr{I}}^\top \times \ker Z_{\mathcal{CRL}\mathscr{I}}^\top A_{\mathcal{V}} \quad \forall t \geq 0 \tag{7.8}$$

on the reference signal can be interpreted as follows: If the circuit contains a $\mathcal{V}$-cutset, then, by Kirchhoff's current law, the currents of the voltage sources in the $\mathcal{V}$-cutset sum up to zero. Likewise, if the circuit contains an $\mathscr{I}$-loop, then Kirchhoff's voltage law implies that the voltages of the current sources in the $\mathscr{I}$-loop sum up to zero. Condition (7.8) *therefore means that, in a sense, the reference signal has to satisfy Kirchhoff's laws pointwise, see also Figure 7.2.*



$\Rightarrow i_{\mathcal{V}1}(t) = i_{\mathcal{V}2}(t)$

$\Rightarrow u_{\mathscr{I}1}(t) = u_{\mathscr{I}2}(t)$

Fig. 7.2: Interpretation of condition (7.8) in terms of Kirchhoff's laws

(b) Invoking that

$$\ker Z_{\mathcal{CRL}\mathscr{I}}^\top = (\operatorname{im} Z_{\mathcal{CRL}\mathscr{I}})^\perp = \left(\ker \begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathscr{I}} \end{bmatrix}^\top\right)^\perp$$
$$= \operatorname{im} \begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathscr{I}} \end{bmatrix},$$

we find

$$\ker Z_{\mathcal{CRL}\mathscr{I}}^T A_{\mathcal{V}} = \left\{ x \in \mathbb{R}^{n_{\mathcal{V}}} \;\middle|\; A_{\mathcal{V}} x \in \operatorname{im} \begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathscr{I}} \end{bmatrix} \right\}.$$

In particular, this space is independent of the choice of the matrix $Z_{\mathcal{CRL}\mathscr{I}}$ with $\operatorname{im} Z_{\mathcal{CRL}\mathscr{I}} = \ker \begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathscr{I}} \end{bmatrix}^\top$.

(c) We have that $\ker Z_{\mathcal{CRL}\mathscr{I}}^\top A_{\mathcal{V}} = \mathbb{R}^{n_{\mathcal{V}}}$ *if, and only if,*

$$\operatorname{im} A_{\mathcal{V}} \subseteq \ker Z_{\mathcal{CRL}\mathscr{I}}^\top = (\operatorname{im} Z_{\mathcal{CRL}\mathscr{I}})^\perp = \left(\ker \begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathscr{I}} \end{bmatrix}^\top\right)^\perp$$
$$= \operatorname{im} \begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathscr{I}} \end{bmatrix}.$$

Hence, by **(A1)**, $\ker Z_{\mathcal{CRLI}}^\top A_\mathcal{V} = \mathbb{R}^{n_\mathcal{V}}$ *is equivalent to*

$$\operatorname{im}\begin{bmatrix} A_\mathcal{C} & A_\mathcal{R} & A_\mathcal{L} & A_\mathcal{I} \end{bmatrix} = \mathbb{R}^{n_e}.$$

The latter is, by Lemma 3.4, equivalent to the absence of $\mathcal{V}$*-cutsets in the given electrical circuit.*
Furthermore, $\operatorname{im} A_\mathcal{I}^\top = \mathbb{R}^{n_\mathcal{I}}$ *if, and only if,* $\{0\} = \left(\operatorname{im} A_\mathcal{I}^\top\right)^\perp = \ker A_\mathcal{I}$*. By Lemma 3.4 the latter is equivalent to the absence of* $\mathcal{I}$*-loops in the given electrical circuit.*

(d) By virtue of Theorem 7.7 and Corollary 5.4, we see that funnel control is feasible for passive and connected electrical circuits (on a suitable set of reference trajectories) provided that at least one of the following two properties is satisfied:
  - *(i) The circuit neither contains* $\mathcal{IL}$*-loops except for* $\mathcal{I}$*-loops, nor* $\mathcal{VCL}$*-cutsets except for* $\mathcal{VL}$*-cutsets.*
  - *(ii) The circuit neither contains* $\mathcal{VC}$*-cutsets except for* $\mathcal{V}$*-cutsets, nor* $\mathcal{ICL}$*-loops except for* $\mathcal{IC}$*-loops.*

(e) By virtue of Proposition 7.4 and Theorem 5.5, we see that funnel control is feasible for passive and connected electrical circuits (on the set of all *sufficiently smooth reference trajectories) provided that at least one of the following two properties is satisfied:*
  - *(i) The circuit neither contains* $\mathcal{IL}$*-loops, nor* $\mathcal{VCL}$*-cutsets except for* $\mathcal{VL}$*-cutsets with at least one inductor.*
  - *(ii) The circuit neither contains* $\mathcal{VC}$*-cutsets, nor* $\mathcal{ICL}$*-loops except for* $\mathcal{IC}$*-loops with at least one capacitor.*

8. Simulation. For purposes of illustration we consider an example of a discretized transmission line. We derive the MNA model (1.3), (1.5) and show that the funnel controller (7.2) achievs tracking of a sinusoidal reference signal with prescribed transient behavior of the tracking error.

We consider a discretized transmission line as depicted in Figure 8.1, where n is the number of spacial discretization points.

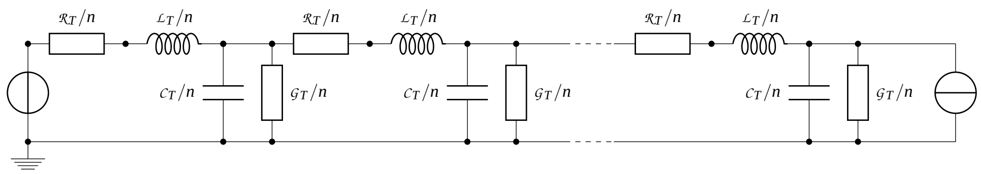


Fig. 8.1: Discretized transmission line

The element related incidence matrices of this circuit can be calculated as

$$\begin{aligned}
A_\mathcal{C} &= \operatorname{diag}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \ldots, \begin{bmatrix}0\\1\end{bmatrix}\right) \in \mathbb{R}^{2n+1,n},\\
A_\mathcal{R} &= \left[\operatorname{diag}\left(\begin{bmatrix}1\\-1\end{bmatrix}, \ldots, \begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}\right),\ A_\mathcal{C}\right] \in \mathbb{R}^{2n+1,2n},
\end{aligned}$$

$$
\begin{aligned}
\mathrm{A}_{\mathcal{L}} &= \operatorname{diag}\left(\begin{bmatrix}0\\1\\-1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix},\ldots,\begin{bmatrix}1\\-1\end{bmatrix}\right)\in\mathbb{R}^{2n+1,n},\\
\mathrm{A}_{\mathcal{V}} &= [1,0,\ldots,0]^\top\in\mathbb{R}^{2n+1,1},\\
\mathrm{A}_{\mathscr{I}} &= [0,\ldots,0,1]^\top\in\mathbb{R}^{2n+1,1}.
\end{aligned}
$$

The matrices expressing the consecutive relations of capacitances, resistances (and conductances, resp.) and inductances are given by

$$
\mathcal{C}=\frac{\mathcal{C}_T}{n}I_n,\quad \mathcal{G}=\operatorname{diag}\left(\frac{n}{\mathcal{R}_T}I_n,\frac{\mathcal{G}_T}{n}I_n\right),\quad \mathcal{L}=\frac{\mathcal{L}_T}{n}I_n.
$$

The differential-algebraic system (1.1) describing the discretized transmission line is then given by $[E,A,B,C]$ for the matrices in (1.3).

The circuit in Fig. 8.1 does not contain any $\mathscr{I}\mathcal{L}$-loops. Further, the only $\mathcal{V}\mathcal{C}\mathcal{L}$-cutset of the circuit is formed by the voltage source and the inductance of the left branch. We can therefore conclude from Theorem 5.5 that $[E,A,B,C]$ has asymptotically stable zero dynamics. Then, by Proposition 7.4, funnel control is feasible for $[E,A,B,C]$ on $\mathscr{B}^\infty(\mathbb{R}_{\geq 0};\mathbb{R}^2)$.

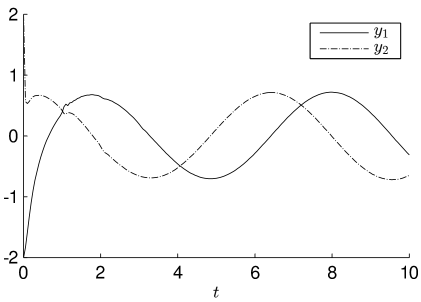


Fig. a: Solution components y_1 and y_2

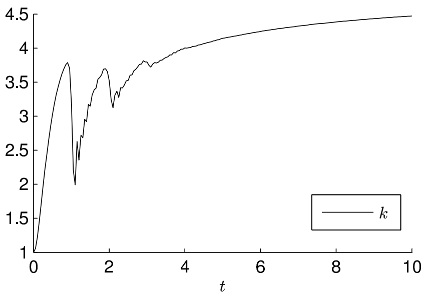


Fig. b: Gain k

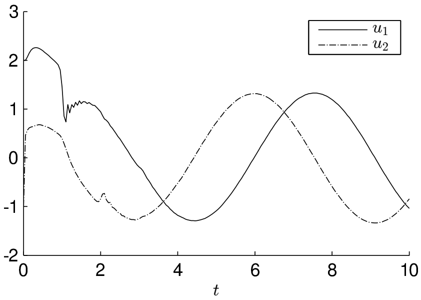


Fig. c: Input components u_1 and u_2

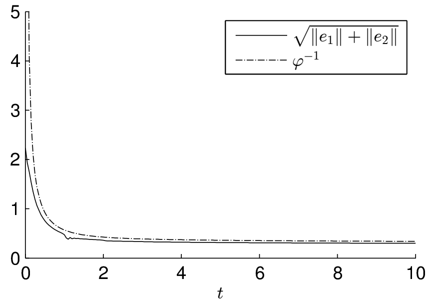


Fig. d: Norm of error $\|e(\cdot)\|$ and funnel boundary $\varphi(\cdot)^{-1}$

Fig. 8.2: Simulation of the funnel controller (7.2) with funnel boundary specified in (8.2) and reference signal $y_{\text{ref}}=(\sin,\cos)^\top$ applied to system $[E,A,B,C]$ with initial data (8.1).

For the simulation we chose the parameters

$$n = 50, \quad \mathcal{C}_T = \mathcal{R}_T = \mathcal{G}_T = \mathcal{L}_T = 1,$$

and the (consistent) initial value for the closed-loop system $[E,A,B,C]$, (7.2) by

$$x^0 = (-1, -1.04, \underbrace{2, 1.96, \ldots, 2, 1.96}_{(n-1)\text{-times}}, \underbrace{2, \ldots, 2}_{(n+1)\text{-times}}, -2) \in \mathbb{R}^{3n+2}. \tag{8.1}$$

As reference signal we take $y_{\text{ref}} = (\sin, \cos)^\top \in \mathscr{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^2)$. The funnel $\mathscr{F}_\varphi$ is determined by the function

$$\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad t \mapsto 0.5\, te^{-t} + 2 \arctan t\,. \tag{8.2}$$

Note that this prescribes an exponentially (exponent 1) decaying funnel in the transient phase $[0,T]$, where $T \approx 3$, and a tracking accuracy quantified by $\lambda = 1/\pi$ thereafter, see Fig. 8.2d.

Note further that the asymptotic stability of the zero dynamics can also be verified by a numerical test which shows that all invariant zeros of $[E,A,B,C]$ have real part -1.

The simulation has been performed in MATLAB. In Figure 8.2 the simulation, over the time interval $[0,10]$, of the funnel controller (7.2) with funnel boundary specified in (8.2) and reference signal $y_{\text{ref}} = (\sin, \cos)^\top$, applied to system $[E,A,B,C]$ with initial data (8.1) is depicted. Fig. 8.2a shows the output components y_1 and y_2 tracking the reference signal y_{ref} within the funnel shown in Fig. 8.2d. Note that an action of the input components u_1 and u_2 in Fig. 8.2c and the gain function k in Fig. 8.2b is required only if the error $\|e(t)\|$ is close to the funnel boundary $\varphi(t)^{-1}$. It can be seen that initially the error is very close to the funnel boundary and hence the gain rises sharply. Then, at approximately $t = 1$, the distance between error and funnel boundary gets larger and the gain drops accordingly. In particular we see that the gain function k is non-monotone.

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