Compressed sensing for real measurements of quaternion signals

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Abstract

The article concerns compressed sensing methods in the quaternion algebra. We prove that it is possible to uniquely reconstruct – by ℓ_1 norm minimization – a sparse quaternion signal from a limited number of its real linear measurements, provided the measurement matrix satisfies so-called restricted isometry property with a sufficiently small constant. We also provide error estimates for the reconstruction of a non-sparse quaternion signal in the noisy and noiseless cases.

1 Introduction

The idea of compressed sensing is to recover a sparse (supported on a set of small cardinality) finite dimensional signal $\mathbb{x}$ from a few number of its linear measurements $\mathbb{y} = \mathbf{\Phi}\mathbb{x}$, by solving the convex program of ℓ_1 norm minimization:

$$\min \|\mathbb{z}\|_1 \quad \text{subject to} \quad \mathbf{\Phi}\mathbb{z} = \mathbb{y}.$$

It is well known that the exact recovery is possible if the measurement matrix $\mathbf{\Phi}$ satisfies a condition known as the restricted isometry property (Definition 3.1), introduced in [4], with sufficiently small constant (see e.g. [2, 3, 4] and [7] for more references). Moreover, even if the original signal is not sparse but e.g. compressible (most of its entries close to zero), the same minimization provides a good sparse approximation of the signal and the procedure is stable in the sense that the error is bounded above by the ℓ_1 norm of the difference between the original signal and its best sparse approximation.

More general, one can assume that the observables are contaminated by a white noise,

$$\mathbb{y} = \mathbf{\Phi}\mathbb{x} + \mathbb{e}, \quad \text{where} \quad \|\mathbb{e}\|_2 \leq \eta.$$

The exact recovery is obviously impossible, however, if the signal $\mathbb{x}$ was sparse, we still are able to reconstruct it in a stable manner, i.e. with an error bounded in terms of η. To do so one solves a modified convex problem

$$\min \|\mathbb{z}\|_1 \quad \text{subject to} \quad \|\mathbb{y} - \mathbf{\Phi}\mathbb{z}\|_2 \leq \eta.$$

So far the attention of researchers in compressed sensing has mostly been focused on the case of real and complex signals and measurements. Our aim is to investigate if the compressed sensing methods can be successfully applied also to quaternion signals.

This generalization would be significant because of broad applications of the quaternion algebra. Apart from classical applications, e.g. in quantum mechanics, for the description of 3D solid body rotations, etc., quaternions have also been used in the field of signal processing. Their structure is suitable for description of a colour image – the imaginary part is interpreted in terms of three components of a colour image: red, green and blue. That is why quaternions have found numerous applications in image filtering, pattern recognition, edge detection and watermarking [8, 9, 11, 13]. There has also been proposed a dual-tree quaternion wavelet transform in a multiscale analysis of geometric image features [5]. For this purpose an alternative representation of quaternions is used – through its magnitude (norm) and three phase angles: two of them encode phase shifts while the third contains image texture information.

The motivation for this work was article [14], the authors of which performed a numerical experiment of a successful recovery of sparse quaternion signals from a limited amount of their random Gaussian quaternion measurements. There has also been proposed an algorithm for solving the ℓ_1 minimization problem in the algebra of quaternions (by using the second-order cone programming). However, to the authors' best knowledge, so far in the literature there has been no proof for any compressed sensing methods in the algebra of quaternions.

In this article we deal with the case of quaternion signals and their linear measurements with real coefficients. The main results, stated in Theorem 4.1 and Corollary 5.1, confirm the numerical experiments from [14] and provide estimates on the error for the problem of reconstruction of a quaternion signal (not necessarily sparse) from noisy and noiseless data by minimization of the ℓ_1 quaternion norm – under the condition that the real measurements matrix satisfies the restricted isometry property with a sufficiently small constant. This is a starting point for further research, i.e. investigating the case of quaternion measurements of quaternion signals, search for 'good' measurement quaternion matrices, etc.

The article is organized as follows. In the next section we recall basic properties of quaternions and provide two versions of polarization identity. Section 3 is devoted to the restricted isometry property and its consequences – we prove Lemma 3.3 which is an important tool in the proof of our main results. In the sections 4 and 5 we state and prove the main results – Theorem 4.1 and Corollary 5.1. Finally, section 6 presents results of a numerical experiment for the considered case, i.e. reconstruction (by ℓ_1 minimization) of sparse quaternion signals from their linear measurements with real coefficients and error estimation for non-sparse quaternion signals giving a lower bound on the constant C_0 from Corollary 5.1.

2 Algebra of quaternions

Denote by $\mathbb{H}$ the algebra of quaternions

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, \quad \text{where} \quad a, b, c, d \in \mathbb{R}$$

endowed with the standard norm

$$|q| = \sqrt{q\overline{q}} = \sqrt{a^2 + b^2 + c^2 + d^2},$$

where $\overline{q} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$ is the conjugate of q. The real part $\mathrm{Re}(q)$ of $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is the real number a while quaternion $b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is called the imaginary party of q and denoted by $\mathrm{Im}(q)$. The conjugate of q can also be expressed as

$$\overline{q} = -\frac{1}{2}\left(q + \mathbf{i}q\mathbf{i} + \mathbf{j}q\mathbf{j} + \mathbf{k}q\mathbf{k}\right).$$

Recall that multiplication is in general not commutative in the quaternion algebra and is defined by the following rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

and

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

However, we have the property that

$$\overline{q \cdot w} = \overline{w} \cdot \overline{q} \quad \text{for any} \quad q, w \in \mathbb{H}.$$

For any $n \in \mathbb{N}$ we introduce the following function $\langle \cdot, \cdot \rangle : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}$ with quaternion values:

$$\langle \mathbb{x}, \mathbb{y} \rangle = \sum_{i=1}^{n} x_i \overline{y_i}, \quad \text{where} \quad \mathbb{x} = (x_1, \dots, x_n)^T,\ \mathbb{y} = (y_1, \dots, y_n)^T \in \mathbb{H}^n$$

and T is the transpose. Denote also

$$\|\mathbb{x}\|_2 = \sqrt{\langle \mathbb{x}, \mathbb{x} \rangle} = \sqrt{\sum_{i=1}^{n} |x_i|^2}, \quad \text{for any } \mathbb{x} = (x_1, \dots, x_n)^T \in \mathbb{H}^n.$$

As a direct consequence of the following lemma, $\|\cdot\|_2$ is a norm in $\mathbb{H}^n$.

Lemma 2.1. *The function $\langle \cdot, \cdot \rangle$ satisfies axioms of the inner product.*

Proof. Let $\mathbb{x} = (x_1, \dots, x_n)^T, \mathbb{y} = (y_1, \dots, y_n)^T, \mathbb{z} = (z_1, \dots, z_n)^T \in \mathbb{H}^n$ and $\lambda \in \mathbb{H}$.

- $\overline{\langle \mathbb{x}, \mathbb{y} \rangle} = \overline{\sum_{i=1}^{n} x_i \overline{y_i}} = \sum_{i=1}^{n} \overline{x_i \overline{y_i}} = \sum_{i=1}^{n} y_i \overline{x_i} = \langle \mathbb{y}, \mathbb{x} \rangle$.
- $\langle \lambda \mathbb{x}, \mathbb{y} \rangle = \sum_{i=1}^{n} \lambda x_i \overline{y_i} = \lambda \sum_{i=1}^{n} x_i \overline{y_i} = \lambda \langle \mathbb{x}, \mathbb{y} \rangle.$
- $\langle \mathbb{x} + \mathbb{y}, \mathbb{z} \rangle = \sum_{i=1}^{n} (x_i + y_i) \overline{z_i} = \sum_{i=1}^{n} x_i \overline{z_i} + \sum_{i=1}^{n} y_i \overline{z_i} = \langle \mathbb{x}, \mathbb{z} \rangle + \langle \mathbb{y}, \mathbb{z} \rangle.$
- $\langle \mathbb{x}, \mathbb{x} \rangle = \sum_{i=1}^{n} x_i \overline{x_i} = \sum_{i=1}^{n} |x_i|^2 = \|\mathbb{x}\|_2^2 \geq 0.$
- $\langle \mathbb{x}, \mathbb{x} \rangle = \|\mathbb{x}\|_2^2 = 0 \iff \mathbb{x} = 0.$

□

We have also the Cauchy-Schwarz inequality.

Lemma 2.2. *For any $\mathbb{x}, \mathbb{y} \in \mathbb{H}^n$,*

$$|\langle \mathbb{x}, \mathbb{y} \rangle| \leq \|\mathbb{x}\|_2 \cdot \|\mathbb{y}\|_2 .$$

Proof. We carefully follow the classical steps of the proof, using the above properties and keeping the order of terms in multiplication. Take any $\mathbb{x}, \mathbb{y} \in \mathbb{H}^n$ and $q \in \mathbb{H}$. If $\mathbb{y} = 0$ we are done, hence assume that $\|\mathbb{y}\|_2 > 0$. By Lemma 2.1 we have that

$$\begin{aligned}
0 \le \langle \mathbb{x} - q\mathbb{y}, \mathbb{x} - q\mathbb{y} \rangle &= \langle \mathbb{x}, \mathbb{x} \rangle - q \langle \mathbb{y}, \mathbb{x} \rangle - \langle \mathbb{x}, \mathbb{y} \rangle \overline{q} + q \langle \mathbb{y}, \mathbb{y} \rangle \overline{q} \\
&= \|x\|_2^2 - q\overline{\langle \mathbb{x}, \mathbb{y} \rangle} - \langle \mathbb{x}, \mathbb{y} \rangle \overline{q} + |q|^2 \|\mathbb{y}\|_2^2 .
\end{aligned}$$

Putting $q = \frac{\langle \mathbb{x}, \mathbb{y} \rangle}{\|y\|^2}$ we get

$$0 \le \|x\|_2^2 - \frac{\langle \mathbb{x}, \mathbb{y} \rangle \overline{\langle \mathbb{x}, \mathbb{y} \rangle}}{\|y\|_2^2} - \frac{\langle \mathbb{x}, \mathbb{y} \rangle \overline{\langle \mathbb{x}, \mathbb{y} \rangle}}{\|y\|_2^2} + \frac{| \langle \mathbb{x}, \mathbb{y} \rangle |^2}{\|y\|_2^2} = \|x\|_2^2 - \frac{| \langle \mathbb{x}, \mathbb{y} \rangle |^2}{\|y\|_2^2},$$

which gives the result. □

The function $\langle \cdot, \cdot \rangle$ is not a standard inner product since its values are quaternions. However, we are able to obtain for it the following versions of polarization identity.

Theorem 2.3 (Polarization identity I)**.** *For any $\mathbb{x}, \mathbb{y} \in \mathbb{H}^n$ we have*

$$\begin{aligned}
\langle \mathbb{x}, \mathbb{y} \rangle = &\frac{1}{4} \left(\|\mathbb{x} + \mathbb{y}\|_2^2 - \|\mathbb{x} - \mathbb{y}\|_2^2 \right) \\
&+ \frac{\mathbf{i}}{4} \left(\|\mathbb{x} + \mathbf{i}\mathbb{y}\|_2^2 - \|\mathbb{x} - \mathbf{i}\mathbb{y}\|_2^2 \right) \\
&+ \frac{\mathbf{j}}{4} \left(\|\mathbb{x} + \mathbf{j}\mathbb{y}\|_2^2 - \|\mathbb{x} - \mathbf{j}\mathbb{y}\|_2^2 \right) \\
&+ \frac{\mathbf{k}}{4} \left(\|\mathbb{x} + \mathbf{k}\mathbb{y}\|_2^2 - \|\mathbb{x} - \mathbf{k}\mathbb{y}\|_2^2 \right) .
\end{aligned}$$

Proof. Denote $\mathbb{x} = (x_1, \ldots, x_n)^T$, $\mathbb{y} = (y_1, \ldots, y_n)^T$ and let us begin with the real part.

$$\begin{aligned}
\|\mathbb{x} + \mathbb{y}\|^2 - \|\mathbb{x} - \mathbb{y}\|^2 &= \sum_{i=1}^n \big((x_i + y_i)(\overline{x_i} + \overline{y_i}) - (x_i - y_i)(\overline{x_i} - \overline{y_i}) \big) \\
&= \sum_{i=1}^n \big(x_i\overline{x_i} + x_i\overline{y_i} + y_i\overline{x_i} + y_i\overline{y_i} - x_i\overline{x_i} + x_i\overline{y_i} + y_i\overline{x_i} - y_i\overline{y_i} \big) \\
&= 2\sum_{i=1}^n \big(x_i\overline{y_i} + y_i\overline{x_i} \big) \\
&= 2\big(\langle \mathbb{x}, \mathbb{y} \rangle + \langle \mathbb{y}, \mathbb{x} \rangle \big) = 2\left(\langle \mathbb{x}, \mathbb{y} \rangle + \overline{\langle \mathbb{x}, \mathbb{y} \rangle} \right) = 4 \operatorname{Re} \left(\langle \mathbb{x}, \mathbb{y} \rangle \right) .
\end{aligned}$$

Now, the term with the imaginary unit $\mathbf{i}$.

$$\begin{aligned}
\|\mathbb{x} + \mathbf{i}\mathbb{y}\|^2 - \|\mathbb{x} - \mathbf{i}\mathbb{y}\|^2 &= \sum_{i=1}^n \big((x_i + \mathbf{i}y_i) \left(\overline{x_i} + \overline{\mathbf{i}y_i}\right) - (x_i - \mathbf{i}y_i) \left(\overline{x_i} - \overline{\mathbf{i}y_i}\right) \big) \\
&= \sum_{i=1}^n \big(x_i\overline{x_i} + x_i\overline{\mathbf{i}y_i} + \mathbf{i}y_i\overline{x_i} + \mathbf{i}y_i\overline{\mathbf{i}y_i} - x_i\overline{x_i} + x_i\overline{\mathbf{i}y_i} + \mathbf{i}y_i\overline{x_i} - \mathbf{i}y_i\overline{\mathbf{i}y_i} \big) \\
&= 2\sum_{i=1}^n \big(- x_i\overline{y_i}\mathbf{i} + \mathbf{i}y_i\overline{x_i} \big) \\
&= -2\big(\langle \mathbb{x}, \mathbb{y} \rangle \mathbf{i} - \mathbf{i} \langle \mathbb{y}, \mathbb{x} \rangle \big) = -2\big(\langle \mathbb{x}, \mathbb{y} \rangle \mathbf{i} - \mathbf{i}\overline{\langle \mathbb{x}, \mathbb{y} \rangle} \big).
\end{aligned}$$

Analogously for the remaining imaginary units:

$$\|\mathbb{x}+\mathbf{j}\mathbb{y}\|_2^2-\|\mathbb{x}-\mathbf{j}\mathbb{y}\|_2^2=-2\big(\langle\mathbb{x},\mathbb{y}\rangle\,\mathbf{j}-\mathbf{j}\overline{\langle\mathbb{x},\mathbb{y}\rangle}\big)$$

and

$$\|\mathbb{x}+\mathbf{k}\mathbb{y}\|_2^2-\|\mathbb{x}-\mathbf{k}\mathbb{y}\|_2^2=-2\big(\langle\mathbb{x},\mathbb{y}\rangle\,\mathbf{k}-\mathbf{k}\overline{\langle\mathbb{x},\mathbb{y}\rangle}\big).$$

Multiplying the left hand sides by $\mathbf{i},\mathbf{j},\mathbf{k}$ respectively and summing up the identities we obtain that

$$\begin{aligned}\mathbf{i}\left(\|\mathbb{x}+\mathbf{i}\mathbb{y}\|_2^2-\|\mathbb{x}-\mathbf{i}\mathbb{y}\|_2^2\right)+\mathbf{j}\left(\|\mathbb{x}+\mathbf{j}\mathbb{y}\|_2^2-\|\mathbb{x}-\mathbf{j}\mathbb{y}\|_2^2\right)+\mathbf{k}\left(\|\mathbb{x}+\mathbf{k}\mathbb{y}\|_2^2-\|\mathbb{x}-\mathbf{k}\mathbb{y}\|_2^2\right)\\ =-2\left(\mathbf{i}\,\langle\mathbb{x},\mathbb{y}\rangle\,\mathbf{i}+\mathbf{j}\,\langle\mathbb{x},\mathbb{y}\rangle\,\mathbf{j}+\mathbf{k}\,\langle\mathbb{x},\mathbb{y}\rangle\,\mathbf{k}\right)-6\overline{\langle\mathbb{x},\mathbb{y}\rangle}\\ =-2\left(-2\overline{\langle\mathbb{x},\mathbb{y}\rangle}-\langle\mathbb{x},\mathbb{y}\rangle\right)-6\overline{\langle\mathbb{x},\mathbb{y}\rangle}\\ =2\,\langle\mathbb{x},\mathbb{y}\rangle-2\overline{\langle\mathbb{x},\mathbb{y}\rangle}=4\,\mathrm{Im}\left(\langle\mathbb{x},\mathbb{y}\rangle\right),\end{aligned}$$

which finishes the proof. □

In order to formulate the second version of the last result, let us introduce a different representation of a quaternion $q\in\mathbb{H}$:

$$q=x+uy,\quad\text{where}\quad x,y\in\mathbb{R},\ y\geq 0,\ u\in\mathbb{H},\ \mathrm{Re}(u)=0,\ |u|=1.$$

Then obviously

$$x=\mathrm{Re}(q),\quad uy=\mathrm{Im}(q)\quad\text{and}\quad |q|^2=x^2+y^2.$$

Since $\mathrm{Re}(u)=0$ and $|u|=1$, we also have

$$\overline{u}=-u\quad\text{and}\quad 1=u\,\overline{u}=u\,(-u)=-u^2.$$

Any quaternion with these properties can also be called an imaginary unit (cf. [12]).

Lemma 2.4. *For any $q\in\mathbb{H}$ with $q=x+uy$, where $x,y\in\mathbb{R}$, $y\geq 0$, $u\in\mathbb{H}$, $\mathrm{Re}(u)=0$, $|u|=1$, we have*

$$\overline{u}\,q\,u=q\quad\textit{and}\quad\overline{u}\,\overline{q}\,u=\overline{q}.$$

Proof. Using the fact that multiplying quaternions by real numbers is commutative, we get that

$$\overline{u}\,q\,u=\overline{u}(x+uy)u=\overline{u}\,x\,u+\overline{u}\,u\,y\,u=x|u|^2+|u|^2y\,u=x+uy=q.$$

And the second identity analogously. □

Theorem 2.5 (Polarization identity II)**.** *For any $\mathbb{x},\mathbb{y}\in\mathbb{H}^n$, if we denote $\langle\mathbb{x},\mathbb{y}\rangle=a+ub$, where $a,b\in\mathbb{R}$, $b\geq 0$, $u\in\mathbb{H}$, $\mathrm{Re}(u)=0$, $|u|=1$, we have*

$$\langle\mathbb{x},\mathbb{y}\rangle=\frac{1}{4}\left(\|\mathbb{x}+\mathbb{y}\|_2^2-\|\mathbb{x}-\mathbb{y}\|_2^2\right)+\frac{u}{4}\left(\|\mathbb{x}+u\mathbb{y}\|_2^2-\|\mathbb{x}-u\mathbb{y}\|_2^2\right).$$

Proof. The form of the real part was established in the previous theorem. Using the fact that $\overline{u}=-u$ we get that

$$\|\mathbb{x}+u\mathbb{y}\|_2^2=\|\mathbb{x}\|_2^2+u\overline{\langle\mathbb{x},\mathbb{y}\rangle}-\langle\mathbb{x},\mathbb{y}\rangle\,u+\|\mathbb{y}\|_2^2$$

and

$$\|\mathbb{x}-u\mathbb{y}\|_2^2=\|\mathbb{x}\|_2^2-u\overline{\langle\mathbb{x},\mathbb{y}\rangle}+\langle\mathbb{x},\mathbb{y}\rangle\,u+\|\mathbb{y}\|_2^2.$$

Hence, by Lemma 2.4,

$$\begin{aligned} u\left(\|\mathbb{x}+u\mathbb{y}\|_2^2-\|\mathbb{x}-u\mathbb{y}\|_2^2\right) &= 2u^2\,\overline{\langle \mathbb{x},\mathbb{y}\rangle}-2u\,\langle \mathbb{x},\mathbb{y}\rangle\,u=-2\overline{\langle \mathbb{x},\mathbb{y}\rangle}+2\,\overline{u}\,\langle \mathbb{x},\mathbb{y}\rangle\,u\\ &= -2\overline{\langle \mathbb{x},\mathbb{y}\rangle}+2\,\langle \mathbb{x},\mathbb{y}\rangle=2\left(\langle \mathbb{x},\mathbb{y}\rangle-\overline{\langle \mathbb{x},\mathbb{y}\rangle}\right)\\ &= 4\,\mathrm{Im}\left(\overline{\langle \mathbb{x},\mathbb{y}\rangle}\right). \end{aligned}$$

□

In what follows we will consider $\|\cdot\|_p$ norms for quaternion vectors $\mathbb{x}\in\mathbb{H}^n$ defined in the standard way:

$$\|\mathbb{x}\|_p=\left(\sum_{i=1}^n |x_i|^p\right)^{1/p}, \quad \text{for} \quad p\in[1,\infty)$$

and

$$\|\mathbb{x}\|_\infty=\max_{1\le i\le n}|x_i|,$$

where $\mathbb{x}=(x_1,\ldots,x_n)^T$. We will also apply the usual notation for the cardinality of the support of $\mathbb{x}$, i.e.

$$\|\mathbb{x}\|_0=\#\mathrm{supp}(\mathbb{x}), \quad \text{where} \quad \mathrm{supp}(\mathbb{x})=\{i\in\{1,\ldots,n\}: x_i\neq 0\}.$$

3 Restricted Isometry Property

Recall that we call a vector (signal) $\mathbb{x}\in\mathbb{H}^n$ s-sparse if it has at most s nonzero coefficients, i.e.

$$\|\mathbb{x}\|_0\le s.$$

As it was mentioned in the introduction, one of the conditions which guarantees exact reconstruction of a sparse real signal from a few number of its linear measurements is that the measurement matrix satisfies so-called restricted isometry property (RIP) with a sufficiently small constant. The notion of restricted isometry constants was introduced by Candès and Tao in [4].

Definition 3.1. *Let $s\in\mathbb{N}$ and $\mathbf{\Phi}\in\mathbb{R}^{m\times n}$. We say that $\mathbf{\Phi}$ satisfies the s-restricted isometry property (for real vectors) with a constant $\delta_s\ge 0$ if*

$$(1-\delta_s)\|\mathbb{x}\|_2^2\le\|\mathbf{\Phi}\mathbb{x}\|_2^2\le(1+\delta_s)\|\mathbb{x}\|_2^2$$

for all s-sparse real vectors $\mathbb{x}\in\mathbb{R}^n$. The smallest number $\delta_s\ge 0$ with this property is called the s-restricted isometry constant.

Note that we can define s-isometry constants for any matrix $\mathbf{\Phi}\in\mathbb{R}^{m\times n}$ and any number $s\in\{1,\ldots,n\}$. There are various examples of real matrices satisfying RIP (with overwhelming probability), e.g. Bernoulli/Gaussian random matrices with i.i.d. entries or, more general, random sampling matrices associated to bounded orthonormal systems (cf. [3, 4, 7]). The following observation allows us to use these matrices also for quaternion signals.

Lemma 3.2. *If a matrix $\mathbf{\Phi}\in\mathbb{R}^{m\times n}$ satisfies the s-restricted isometry property (for real vectors) with a constant $\delta_s\ge 0$, then it satisfies the s-restricted isometry property for quaternion vectors with the same constant, i.e.*

$$(1-\delta_s)\|\mathbb{x}\|_2^2\le\|\mathbf{\Phi}\mathbb{x}\|_2^2\le(1+\delta_s)\|\mathbb{x}\|_2^2$$

for all s-sparse quaternion vectors $\mathbb{x}\in\mathbb{H}^n$.

Proof. Take any s-sparse vector $\mathbb{x} \in \mathbb{H}^n$ and express it as

$$\mathbb{x} = \mathbb{x}_{\mathbf{r}} + \mathbf{i}\mathbb{x}_{\mathbf{i}} + \mathbf{j}\mathbb{x}_{\mathbf{j}} + \mathbf{k}\mathbb{x}_{\mathbf{k}}, \quad \text{where} \quad \mathbb{x}_i \in \mathbb{R}^n,\ i \in \{\mathbf{r}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}.$$

Notice that all vectors $\mathbb{x}_i \in \mathbb{R}^n$ are also s-sparse. Moreover,

$$\mathbf{\Phi}\mathbb{x} = \mathbf{\Phi}\mathbb{x}_{\mathbf{r}} + \mathbf{i}\mathbf{\Phi}\mathbb{x}_{\mathbf{i}} + \mathbf{j}\mathbf{\Phi}\mathbb{x}_{\mathbf{j}} + \mathbf{k}\mathbf{\Phi}\mathbb{x}_{\mathbf{k}}, \quad \text{where} \quad \mathbf{\Phi}\mathbb{x}_i \in \mathbb{R}^n,\ i \in \{\mathbf{r}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}.$$

Hence

$$\|\mathbf{\Phi}\mathbb{x}\|_2^2 = \|\mathbf{\Phi}\mathbb{x}_{\mathbf{r}}\|_2^2 + \|\mathbf{\Phi}\mathbb{x}_{\mathbf{i}}\|_2^2 + \|\mathbf{\Phi}\mathbb{x}_{\mathbf{j}}\|_2^2 + \|\mathbf{\Phi}\mathbb{x}_{\mathbf{k}}\|_2^2$$

and obviously

$$\|\mathbb{x}\|_2^2 = \|\mathbb{x}_{\mathbf{r}}\|_2^2 + \|\mathbb{x}_{\mathbf{i}}\|_2^2 + \|\mathbb{x}_{\mathbf{j}}\|_2^2 + \|\mathbb{x}_{\mathbf{k}}\|_2^2.$$

And the result follows after applying the s-restricted isometry inequalities to each vector $\mathbb{x}_i \in \mathbb{R}^n$ separately. $\square$

The next result is an important tool in the proof of Theorem 4.1. Note that for quaternion vectors we are not able to obtain the same estimate as in the real case. However, the enhanced version of the polarization identity (Theorem 2.5) – which is the key ingredient in the proof – allows us to decrease the multiplicative constant from 2 to $\sqrt{2}$.

Lemma 3.3. *Let δ_s be the s-isometry constant for a matrix $\mathbf{\Phi} \in \mathbb{R}^{m\times n}$ for $s \in \{1, \dots, n\}$. For any pair of $\mathbb{x}, \mathbb{y} \in \mathbb{H}^n$ with disjoint supports and such that $\|\mathbb{x}\|_0 \le s_1$ and $\|\mathbb{y}\|_0 \le s_2$, where $s_1 + s_2 \le n$, we have that*

$$|\langle \Phi\mathbb{x}, \Phi\mathbb{y}\rangle| \le \sqrt{2}\delta_{s_1+s_2} \|\mathbb{x}\|_2 \|\mathbb{y}\|_2 .$$

Proof. First take two unit vectors $\mathbb{x}, \mathbb{y} \in \mathbb{H}^n$ satisfying the assumptions of the lemma. Since $\mathbb{x}$ and $\mathbb{y}$ have disjoint supports and $\|\mathbb{x}\|_2 = \|\mathbb{y}\|_2 = 1$ we have that

$$\|\mathbb{x} \pm \mathbb{y}\|_2^2 = \|\mathbb{x}\|_2^2 + \|\mathbb{y}\|_2^2 = 2.$$

Moreover, for any quaternion $q \in \mathbb{H}$ with $|q| = 1$,

$$\|\mathbb{x} \pm q\mathbb{y}\|_2^2 = \|\mathbb{x}\|_2^2 + |q|^2 \|\mathbb{y}\|_2^2 = 2.$$

Denote $\langle \mathbf{\Phi}\mathbb{x}, \mathbf{\Phi}\mathbb{y}\rangle = a + ub$, where $a, b \in \mathbb{R}$, $b \ge 0$, $u \in \mathbb{H}$, $\mathrm{Re}(u) = 0$, $|u| = 1$. Applying the polarization identity from Theorem 2.5 we get that

$$\langle \mathbf{\Phi}\mathbb{x}, \mathbf{\Phi}\mathbb{y}\rangle = \frac{1}{4}\left(\|\mathbf{\Phi}\mathbb{x} + \mathbf{\Phi}\mathbb{y}\|_2^2 - \|\mathbf{\Phi}\mathbb{x} - \mathbf{\Phi}\mathbb{y}\|_2^2\right) + \frac{u}{4}\left(\|\mathbf{\Phi}\mathbb{x} + u\mathbf{\Phi}\mathbb{y}\|_2^2 - \|\mathbf{\Phi}\mathbb{x} - u\mathbf{\Phi}\mathbb{y}\|_2^2\right).$$

Since $\mathbb{x} \pm \mathbb{y}$ are $(s_1 + s_2)$-sparse,

$$(1 - \delta_{s_1+s_2}) \underbrace{\|\mathbb{x} \pm \mathbb{y}\|_2^2}_{=2} \le \|\mathbf{\Phi}\mathbb{x} \pm \mathbf{\Phi}\mathbb{y}\|_2^2 = \|\mathbf{\Phi}(\mathbb{x} \pm \mathbb{y})\|_2^2 \le (1 + \delta_{s_1+s_2}) \underbrace{\|\mathbb{x} \pm \mathbb{y}\|_2^2}_{=2}$$

and therefore

$$\|\mathbf{\Phi}\mathbb{x} + \mathbf{\Phi}\mathbb{y}\|_2^2 - \|\mathbf{\Phi}\mathbb{x} - \mathbf{\Phi}\mathbb{y}\|_2^2 \le 2\left(1 + \delta_{s_1+s_2}\right) - 2\left(1 - \delta_{s_1+s_2}\right) = 4\delta_{s_1+s_2}.$$

Similarly, since $\mathbf{\Phi}$ is real and $\mathbb{x} \pm u\mathbb{y}$ are also $(s_1 + s_2)$-sparse,

$$(1 - \delta_{s_1+s_2}) \underbrace{\|\mathbb{x} \pm u\mathbb{y}\|_2^2}_{=2} \le \|\mathbf{\Phi}\mathbb{x} \pm u\mathbf{\Phi}\mathbb{y}\|_2^2 = \|\mathbf{\Phi}(\mathbb{x} \pm u\mathbb{y})\|_2^2 \le (1 + \delta_{s_1+s_2}) \underbrace{\|\mathbb{x} \pm u\mathbb{y}\|_2^2}_{=2},$$

hence

$$\|\mathbf{\Phi}\mathbb{x} + u\mathbf{\Phi}\mathbb{y}\|_2^2 - \|\mathbf{\Phi}\mathbb{x} - u\mathbf{\Phi}\mathbb{y}\|_2^2 \leq 2\left(1+\delta_{s_1+s_2}\right) - 2\left(1-\delta_{s_1+s_2}\right) = 4\delta_{s_1+s_2}.$$

Finally

$$|\langle \mathbb{x}, \mathbb{y}\rangle| \leq \frac{1}{4}\sqrt{4^2\delta_{s_1+s_2}^2 + 4^2\delta_{s_1+s_2}^2} = \sqrt{2}\,\delta_{s_1+s_2}.$$

Now, if $\mathbb{x}, \mathbb{y} \in \mathbb{H}^n$ are any vectors satisfying assumptions of the lemma, applying the above estimate we conclude that

$$|\langle \mathbb{x}, \mathbb{y}\rangle| = \|\mathbb{x}\|_2 \|\mathbb{y}\|_2 \left|\left\langle \frac{\mathbb{x}}{\|\mathbb{x}\|_2}, \frac{\mathbb{y}}{\|\mathbb{y}\|_2}\right\rangle\right| \leq \sqrt{2}\,\delta_{s_1+s_2}\|\mathbb{x}\|_2 \|\mathbb{y}\|_2.$$

□

4 Stable reconstruction from noisy data

As we mentioned in the introduction, our aim is to reconstruct a quaternion signal from a limited amount of its linear measurements with real coefficients. We will also assume the presence of a white noise with bounded ℓ_2 quaternion norm. The observables are, therefore, given by

$$\mathbb{y} = \mathbf{\Phi}\mathbb{x} + \mathbb{e}, \quad \text{where} \quad \mathbb{x} \in \mathbb{H}^n,\ \mathbf{\Phi} \in \mathbb{R}^{m\times n},\ \mathbb{y} \in \mathbb{H}^m \text{ and } \mathbb{e} \in \mathbb{H}^n \text{ with } \|\mathbb{e}\|_2 \leq \eta$$

for some $m \leq n$ and $\eta \geq 0$.

We will use the following notation: for any $\mathbb{h} \in \mathbb{H}^n$ and a set of indices $T \subset \{1,\ldots,n\}$, the vector $\mathbb{h}_T \in \mathbb{H}^n$ is supported on T with entries

$$(\mathbb{h}_T)_i = \begin{cases} h_i & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases}, \quad \text{where} \quad \mathbb{h} = (h_1,\ldots,h_n)^T.$$

The complement of $T \subset \{1,\ldots,n\}$ will be denoted by $T^c = \{1,\ldots,n\}\setminus T$ and the symbol $\mathbb{x}|_s$ will be used for the best s-sparse approximation of the vector $\mathbb{x}$, i.e. $\mathbb{x}|_s = \mathbb{x}_{T_0}$, where T_0 is the set of indices of $\mathbb{x}$ coordinates with the biggest quaternion norms.

The following result is a generalization of [2, Theorem 1.3] to the case of quaternion signals.

Theorem 4.1. *Suppose that $\mathbf{\Phi} \in \mathbb{R}^{m\times n}$ satisfies the $2s$-restricted isometry property with a constant $\delta_{2s} < \frac{1}{3}$ and let $\eta \geq 0$. Then, for any $\mathbb{x} \in \mathbb{H}^n$ and $\mathbb{y} = \mathbf{\Phi}\mathbb{x} + \mathbb{e}$ with $\|\mathbb{e}\|_2 \leq \eta$, the solution $\mathbb{x}^\#$ of the problem*

$$\arg\min_{\mathbb{z}\in\mathbb{H}^n} \|\mathbb{z}\|_1 \quad \textit{subject to} \quad \|\mathbf{\Phi}\mathbb{z} - \mathbb{y}\|_2 \leq \eta \tag{4.1}$$

satisfies

$$\left\|\mathbb{x}^\# - \mathbb{x}\right\|_2 \leq \frac{C_0}{\sqrt{s}}\|\mathbb{x} - \mathbb{x}|_s\|_1 + C_1\eta \tag{4.2}$$

with constants

$$C_0 = \frac{2(1+\delta_{2s})}{1-3\delta_{2s}}, \quad C_1 = \frac{4\sqrt{1+\delta_{2s}}}{1-3\delta_{2s}},$$

where $\mathbb{x}|_s$ denotes the best s-sparse approximation of $\mathbb{x}$.

Proof. First, note that, since $\mathbb{x}^{\#}$ is the minimizer of (4.1) and $\mathbb{x}$ is feasible, we get that

$$\left\|\mathbf{\Phi}\left(\mathbb{x}^{\#}-\mathbb{x}\right)\right\|_2 \leq \left\|\mathbf{\Phi}\mathbb{x}^{\#}-\mathbb{y}\right\|_2+\|\mathbf{\Phi}\mathbb{x}-\mathbb{y}\|_2 \leq 2\eta. \tag{4.3}$$

Denote

$$\mathbb{h}=\mathbb{x}^{\#}-\mathbb{x}$$

and decompose $\mathbb{h}$ into a sum of vectors $\mathbb{h}_{T_0}, \mathbb{h}_{T_1}, \mathbb{h}_{T_2}\ldots$ in the following way: let T_0 be the set of indices of $\mathbb{x}$ coordinates with the biggest quaternion norms; T_1 is the set of indices of $\mathbb{h}_{T_0^c}$ coordinates with the biggest norms, T_1 is the set of indices of of $\mathbb{h}_{(T_0\cup T_1)^c}$ coordinates with the biggest norms, etc. Then obviously $\mathbb{h}_{T_j}$ are s-sparse and have disjoint supports.

Notice that for $j \geq 2$ we have that

$$\left\|\mathbb{h}_{T_j}\right\|_2^2=\sum_{i\in T_j}|h_i|^2 \leq \sum_{i\in T_j}\left\|\mathbb{h}_{T_j}\right\|_\infty^2 = s\left\|\mathbb{h}_{T_j}\right\|_\infty^2,$$

where h_i are the coordinates of $\mathbb{h}$ and $\left\|\mathbb{h}_{T_j}\right\|_\infty=\max_{i\in T_j}|h_i|$. Moreover, since all non-zero coordinates of $\mathbb{h}_{T_{j-1}}$ have norms not smaller than non-zero coordinates of $\mathbb{h}_{T_j}$,

$$\left\|\mathbb{h}_{T_j}\right\|_\infty \leq \frac{1}{s}\sum_{i\in T_{j-1}}|h_i| = \frac{1}{s}\left\|\mathbb{h}_{T_{j-1}}\right\|_1.$$

Hence, for $j \geq 2$ we get that

$$\left\|\mathbb{h}_{T_j}\right\|_2 \leq \sqrt{s}\left\|\mathbb{h}_{T_j}\right\|_\infty \leq \frac{1}{\sqrt{s}}\left\|\mathbb{h}_{T_{j-1}}\right\|_1,$$

which implies

$$\sum_{j\geq 2}\left\|\mathbb{h}_{T_j}\right\|_2 \leq \frac{1}{\sqrt{s}}\sum_{j\geq 1}\left\|\mathbb{h}_{T_j}\right\|_1 \leq \frac{1}{\sqrt{s}}\left\|\mathbb{h}_{T_0^c}\right\|_1. \tag{4.4}$$

Finally

$$\left\|\mathbb{h}_{(T_0\cup T_1)^c}\right\|_2 = \left\|\sum_{j\geq 2}\mathbb{h}_{T_j}\right\|_2 \leq \sum_{j\geq 2}\left\|\mathbb{h}_{T_j}\right\|_2 \leq \frac{1}{\sqrt{s}}\left\|\mathbb{h}_{T_0^c}\right\|_1. \tag{4.5}$$

Observe that $\left\|\mathbb{h}_{T_0^c}\right\|_1$ can not be to large since $\left\|\mathbb{x}^{\#}\right\|_1=\|\mathbb{x}+\mathbb{h}\|_1$ is minimal. Indeed,

$$\|\mathbb{x}\|_1 \geq \|\mathbb{x}+\mathbb{h}\|_1 = \|\mathbb{x}_{T_0}+\mathbb{h}_{T_0}\|_1+\left\|\mathbb{x}_{T_0^c}+\mathbb{h}_{T_0^c}\right\|_1 \geq \|\mathbb{x}_{T_0}\|_1-\|\mathbb{h}_{T_0}\|_1-\left\|\mathbb{x}_{T_0^c}\right\|_1+\left\|\mathbb{h}_{T_0^c}\right\|_1,$$

hence

$$\left\|\mathbb{x}_{T_0^c}\right\|_1=\|\mathbb{x}\|_1-\|\mathbb{x}_{T_0}\|_1 \geq -\|\mathbb{h}_{T_0}\|_1-\left\|\mathbb{x}_{T_0^c}\right\|_1+\left\|\mathbb{h}_{T_0^c}\right\|_1$$

and therefore

$$\left\|\mathbb{h}_{T_0^c}\right\|_1 \leq \|\mathbb{h}_{T_0}\|_1+2\left\|\mathbb{x}_{T_0^c}\right\|_1. \tag{4.6}$$

Now, the Cauchy-Schwarz inequality immediately implies that

$$\|h_{T_0}\|_1=\sum_{i\in T_0}|h_i|\cdot 1 \leq \sqrt{\sum_{i\in T_0}|h_i|^2}\cdot\sqrt{\sum_{i\in T_0}1^2}=\sqrt{s}\,\|\mathbb{h}_{T_0}\|_2$$

From this, (4.5) and (4.6) we conclude that

$$\left\|\mathbb{h}_{(T_0\cup T_1)^c}\right\|_2 \leq \frac{1}{\sqrt{s}}\left\|\mathbb{h}_{T_0^c}\right\|_1 \leq \frac{1}{\sqrt{s}}\|\mathbb{h}_{T_0}\|_1+\frac{2}{\sqrt{s}}\left\|\mathbb{x}_{T_0^c}\right\|_1 \leq \|\mathbb{h}_{T_0}\|_2+2\epsilon, \tag{4.7}$$

where $\epsilon = \frac{1}{\sqrt{s}} \left\|\mathbb{x}_{T_0^c}\right\|_1 = \frac{1}{\sqrt{s}} \left\|\mathbb{x} - \mathbb{x}|_s\right\|_1$. This is the first ingredient of the final estimate.

In what follows we are going to estimate the remaining component, i.e. $\left\|\mathbb{h}_{(T_0\cup T_1)^c}\right\|_2$. Since

$$\boldsymbol{\Phi}\mathbb{h}_{T_0\cup T_1} = \boldsymbol{\Phi}\left(\mathbb{h} - \sum_{j\geq 2}\mathbb{h}_{T_j}\right) = \boldsymbol{\Phi}\mathbb{h} - \sum_{j\geq 2}\boldsymbol{\Phi}\mathbb{h}_{T_j},$$

using linearity of $\langle\cdot,\cdot\rangle$, we get that

$$\begin{aligned}\left\|\boldsymbol{\Phi}\mathbb{h}_{T_0\cup T_1}\right\|_2^2 &= \langle\boldsymbol{\Phi}\mathbb{h}_{T_0\cup T_1}, \boldsymbol{\Phi}\mathbb{h}_{T_0\cup T_1}\rangle = \langle\boldsymbol{\Phi}\mathbb{h}_{T_0\cup T_1}, \boldsymbol{\Phi}\mathbb{h}\rangle - \sum_{j\geq 2}\left\langle\boldsymbol{\Phi}\mathbb{h}_{T_0\cup T_1}, \boldsymbol{\Phi}\mathbb{h}_{T_j}\right\rangle \\ &= \langle\boldsymbol{\Phi}\mathbb{h}_{T_0\cup T_1}, \boldsymbol{\Phi}\mathbb{h}\rangle - \sum_{j\geq 2}\left\langle\boldsymbol{\Phi}\mathbb{h}_{T_0}, \boldsymbol{\Phi}\mathbb{h}_{T_j}\right\rangle - \sum_{j\geq 2}\left\langle\boldsymbol{\Phi}\mathbb{h}_{T_1}, \boldsymbol{\Phi}\mathbb{h}_{T_j}\right\rangle.\end{aligned}$$

Estimate of the first element follows from Lemma 2.2, (4.3) and RIP

$$|\langle\boldsymbol{\Phi}\mathbb{h}_{T_0\cup T_1}, \boldsymbol{\Phi}\mathbb{h}\rangle| \leq \left\|\boldsymbol{\Phi}\mathbb{h}_{T_0\cup T_1}\right\|_2 \cdot \left\|\boldsymbol{\Phi}\mathbb{h}\right\|_2 \leq \sqrt{1+\delta_{2s}}\left\|\mathbb{h}_{T_0\cup T_1}\right\|_2 \cdot 2\eta. \tag{4.8}$$

For the remaining terms recall that $\mathbb{h}_{T_j}$ are s-sparse with disjoint supports and apply Lemma 3.3

$$\begin{aligned}\left|\left\langle\boldsymbol{\Phi}\mathbb{h}_{T_0}, \boldsymbol{\Phi}\mathbb{h}_{T_j}\right\rangle\right| &\leq \sqrt{2}\,\delta_{2s}\cdot\left\|\mathbb{h}_{T_0}\right\|_2\cdot\left\|\mathbb{h}_{T_j}\right\|_2, \quad j\geq 2,\\ \left|\left\langle\boldsymbol{\Phi}\mathbb{h}_{T_1}, \boldsymbol{\Phi}\mathbb{h}_{T_j}\right\rangle\right| &\leq \sqrt{2}\,\delta_{2s}\cdot\left\|\mathbb{h}_{T_1}\right\|_2\cdot\left\|\mathbb{h}_{T_j}\right\|_2, \quad j\geq 2.\end{aligned}$$

Since T_0 and T_1 are disjoint, $\left\|\mathbb{h}_{T_0\cup T_1}\right\|_2^2 = \left\|\mathbb{h}_{T_0}\right\|_2^2 + \left\|\mathbb{h}_{T_1}\right\|_2^2$ and therefore

$$\left\|\mathbb{h}_{T_0}\right\|_2 + \left\|\mathbb{h}_{T_1}\right\|_2 \leq \sqrt{2}\left\|\mathbb{h}_{T_0\cup T_1}\right\|_2$$

since for any $a, b\in\mathbb{R}$ we have $(a+b)^2 \leq 2(a^2+b^2)$. Hence, using the RIP, (4.8) and (4.4),

$$\begin{aligned}(1-\delta_{2s})\left\|\mathbb{h}_{T_0\cup T_1}\right\|_2^2 &\leq \left\|\boldsymbol{\Phi}\mathbb{h}_{T_0\cup T_1}\right\|_2^2 \\ &\leq \sqrt{1+\delta_{2s}}\left\|\mathbb{h}_{T_0\cup T_1}\right\|_2\cdot 2\eta + \sqrt{2}\,\delta_{2s}\cdot\left(\left\|\mathbb{h}_{T_0}\right\|_2 + \left\|\mathbb{h}_{T_1}\right\|_2\right)\sum_{j\geq 2}\left\|\mathbb{h}_{T_j}\right\|_2 \\ &\leq \left(2\sqrt{1+\delta_{2s}}\cdot\eta + \frac{2\delta_{2s}}{\sqrt{s}}\left\|\mathbb{h}_{T_0^c}\right\|_1\right)\left\|\mathbb{h}_{T_0\cup T_1}\right\|_2,\end{aligned}$$

which implies that

$$\left\|\mathbb{h}_{T_0\cup T_1}\right\|_2 \leq \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}\cdot\eta + \frac{2\delta_{2s}}{1-\delta_{2s}}\cdot\frac{\left\|\mathbb{h}_{T_0^c}\right\|_1}{\sqrt{s}}. \tag{4.9}$$

This, together with (4.6), gives the following estimate

$$\left\|\mathbb{h}_{T_0\cup T_1}\right\|_2 \leq \alpha\cdot\eta + \beta\cdot\frac{\left\|\mathbb{h}_{T_0}\right\|_1}{\sqrt{s}} + 2\beta\cdot\frac{\left\|\mathbb{x}_{T_0^c}\right\|_1}{\sqrt{s}},$$

where

$$\alpha = \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}} \qquad \text{and} \qquad \beta = \frac{2\delta_{2s}}{1-\delta_{2s}}.$$

Since $\left\|\mathbb{h}_{T_0}\right\|_1 \leq \sqrt{s}\left\|\mathbb{h}_{T_0}\right\|_2 \leq \sqrt{s}\left\|\mathbb{h}_{T_0\cup T_1}\right\|_2$, therefore

$$\left\|\mathbb{h}_{T_0\cup T_1}\right\|_2 \leq \alpha\cdot\eta + \beta\cdot\left\|\mathbb{h}_{T_0\cup T_1}\right\|_2 + 2\beta\cdot\epsilon,$$

where recall that $\epsilon = \frac{1}{\sqrt{s}} \left\| \mathbb{h}_{T_0^c} \right\|_2$, hence

$$\|\mathbb{h}_{T_0 \cup T_1}\|_2 \leq \frac{1}{1-\beta} \left(\alpha \cdot \eta + 2\beta \cdot \epsilon \right), \tag{4.10}$$

as long as $\beta < 1$ which is equivalent to $\delta_{2s} < \frac{1}{3}$.

Finally, (4.7) and (4.10) imply the main result

$$\begin{aligned}\|h\|_2 &\leq \|\mathbb{h}_{T_0 \cup T_1}\|_2 + \left\|\mathbb{h}_{(T_0 \cup T_1)^c}\right\|_2 \leq \|\mathbb{h}_{T_0 \cup T_1}\|_2 + \|\mathbb{h}_{T_0}\|_2 + 2\epsilon \\ &\leq 2\|\mathbb{h}_{T_0 \cup T_1}\|_2 + 2\epsilon \leq \frac{2\alpha}{1-\beta} \cdot \eta + \left(\frac{4\beta}{1-\beta} + 2 \right) \cdot \epsilon\end{aligned}$$

and the constants in the statement of the theorem equal

$$C_0 = \frac{4\beta}{1-\beta} = 2\frac{1+\beta}{1-\beta} = 2\frac{1+\delta_{2s}}{1-3\delta_{2s}} \quad \text{and} \quad C_1 = \frac{2\alpha}{1-\beta} = \frac{4\sqrt{1+\delta_{2s}}}{1-3\delta_{2s}}.$$

□

5 Stable reconstruction from noiseless data

In this section we will assume that our observables are exact, i.e.

$$\mathbb{y} = \mathbf{\Phi}\mathbb{x}, \quad \text{where} \quad \mathbb{x} \in \mathbb{H}^n, \ \mathbf{\Phi} \in \mathbb{R}^{m \times n}, \ \mathbb{y} \in \mathbb{H}^m.$$

The undermentioned result is a natural corollary of Theorem 4.1 for $\eta = 0$.

Corollary 5.1. *Let $\mathbf{\Phi} \in \mathbb{R}^{m\times n}$ satisfies the $2s$-restricted isometry property with a constant $\delta_{2s} < \frac{1}{3}$. Then for any $\mathbb{x} \in \mathbb{H}^n$ and $\mathbb{y} = \mathbf{\Phi}\mathbb{x} \in \mathbb{H}^m$, the solution $\mathbb{x}^{\#}$ of the problem*

$$\arg\min_{\mathbb{z} \in \mathbb{H}^n} \|\mathbb{z}\|_1 \quad \textit{subject to} \quad \mathbf{\Phi}\mathbb{z} = \mathbb{y} \tag{5.1}$$

satisfies

$$\left\|\mathbb{x}^{\#} - \mathbb{x}\right\|_1 \leq C_0 \left\|\mathbb{x} - \mathbb{x}|_s\right\|_1 \tag{5.2}$$

$$\left\|\mathbb{x}^{\#} - \mathbb{x}\right\|_2 \leq \frac{C_0}{\sqrt{s}} \left\|\mathbb{x} - \mathbb{x}|_s\right\|_1 \tag{5.3}$$

with constant C_0 as in the Theorem 4.1. In particular, if $\mathbb{x}$ is s-sparse and there is no noise, then the reconstruction by ℓ_1 minimization is exact.

Proof. Inequality (5.3) follows directly from Theorem 4.1 for $\eta = 0$. The result for sparse signals is obvious since in this case $\mathbb{x} = \mathbb{x}|_s$. We only need to prove (5.2).

We will use the same notation as in the proof of Theorem 4.1. Recall that

$$\|\mathbb{h}_{T_0}\|_1 \leq \sqrt{s}\,\|\mathbb{h}_{T_0}\|_2 \leq \sqrt{s}\,\|\mathbb{h}_{T_0 \cup T_1}\|_2$$

which together with (4.9) for $\eta = 0$ implies

$$\|\mathbb{h}_{T_0}\|_1 \leq \frac{2\delta_{2s}}{1-\delta_{2s}} \cdot \left\|\mathbb{h}_{T_0^c}\right\|_1$$

Using this and (4.6), denoting again $\beta = \frac{2\delta_{2s}}{1-\delta_{2s}}$, we get that

$$\left\|\mathbb{h}_{T_0^c}\right\|_1 \leq \beta \left\|\mathbb{h}_{T_0^c}\right\|_1 + 2\left\|\mathbb{x}_{T_0^c}\right\|_1, \qquad \text{hence} \qquad \left\|\mathbb{h}_{T_0^c}\right\|_1 \leq \frac{2}{1-\beta}\left\|\mathbb{x}_{T_0^c}\right\|_1.$$

Finally, we obtain the following estimate on the ℓ_1 norm of the vector $\mathbb{h} = \mathbb{x}^{\#} - \mathbb{x}$

$$\|\mathbb{h}\|_1 = \|\mathbb{h}_{T_0}\|_1 + \left\|\mathbb{h}_{T_0^c}\right\|_1 \leq (1+\beta) \left\|\mathbb{h}_{T_0^c}\right\|_1 \leq \underbrace{2\frac{1+\beta}{1-\beta}}_{=C_0} \left\|\mathbb{x}_{T_0^c}\right\|_1 ,$$

which finishes the proof. □

We conjecture that the requirement $\delta_{2s} < \frac{1}{3}$ is not optimal – there are known refinements of this condition for real signals (see e.g. [7, Chapter 6] for references). However, the authors of [1] constructed examples of s-sparse real signals which can not be uniquely reconstructed via ℓ_1 minimization for $\delta_s > \frac{1}{3}$. This gives an obvious upper bound for δ_s also for the general quaternion case.

6 Numerical experiment

Inspired by the article [14] we performed numerical experiments for the case considered in this paper, i.e. when the measurement matrix $\mathbf{\Phi}$ is real and satisfies (with overwhelming probability) the restricted isometry property and signals are quaternion vectors. We applied the algorithm described in [14] which express the ℓ_1 quaternion norm minimization problem in terms of the second-order cone programming (SOCP).

As it was also pointed out in [14], our problem is equivalent to

$$\arg\min_{t \in \mathbb{R}_+} t \quad \text{subject to} \quad \mathbb{y} = \mathbf{\Phi}\mathbb{x}, \quad \|\mathbb{x}\|_1 \leq t. \tag{6.1}$$

We decompose

$$t = \sum_{k=1}^{n} t_k, \quad \text{where} \quad t_k \in \mathbb{R}_+,$$

and

$$\mathbb{x} = \mathbb{x}_{\mathbf{r}} + \mathbf{i}\mathbb{x}_{\mathbf{i}} + \mathbf{j}\mathbb{x}_{\mathbf{j}} + \mathbf{k}\mathbb{x}_{\mathbf{k}}, \quad \text{where} \quad \mathbb{x}_{\mathbf{r}}, \mathbb{x}_{\mathbf{i}}, \mathbb{x}_{\mathbf{j}}, \mathbb{x}_{\mathbf{k}} \in \mathbb{R}^n,$$

and denote

$$\begin{aligned} \mathbb{x}_{\mathbf{r}} &= (x_{\mathbf{r},1}, x_{\mathbf{r},2}, \ldots, x_{\mathbf{r},n})^T, \\ \mathbb{x}_{\mathbf{i}} &= (x_{\mathbf{i},1}, x_{\mathbf{i},2}, \ldots, x_{\mathbf{i},n})^T, \\ \mathbb{x}_{\mathbf{j}} &= (x_{\mathbf{j},1}, x_{\mathbf{j},2}, \ldots, x_{\mathbf{j},n})^T, \\ \mathbb{x}_{\mathbf{k}} &= (x_{\mathbf{k},1}, x_{\mathbf{k},2}, \ldots, x_{\mathbf{k},n})^T. \end{aligned}$$

We also denote $\mathbf{\Phi} = (\phi_1, \ldots, \phi_n)$, where $\phi_k \in \mathbb{R}^m$ for $k \in \{1, 2, \ldots, n\}$. Then, we can write the second constraint from (6.1) as

$$\left\|(x_{\mathbf{r},k}, x_{\mathbf{i},k}, x_{\mathbf{j},k}, x_{\mathbf{k},k})^T\right\|_2 \leq t_k \quad \text{for} \quad k \in \{1, 2, \ldots, n\}. \tag{6.2}$$

This allows us to express our optimization problem (6.1) in the real-vector setup in the following way

$$\begin{aligned} &\arg\min_{\tilde{\mathbb{x}} \in \mathbb{R}^n} \mathbb{c}^T \tilde{x} \quad \text{subject to} \quad \tilde{\mathbb{y}} = \tilde{\mathbf{\Phi}}\tilde{\mathbb{x}} \\ &\qquad\qquad \text{and} \quad \left\|(x_{\mathbf{r},k}, x_{\mathbf{i},k}, x_{\mathbf{j},k}, x_{\mathbf{k},k})^T\right\|_2 \leq t_k, \quad k \in \{1, 2, \ldots, n\}, \end{aligned} \tag{6.3}$$

where

$$
\begin{aligned}
&\tilde{\mathbb{x}} = (t_1, x_{\mathbf{r},1}, x_{\mathbf{i},1}, x_{\mathbf{j},1}, x_{\mathbf{k},1}, \dots, t_n, x_{\mathbf{r},n}, x_{\mathbf{i},n}, x_{\mathbf{j},n}, x_{\mathbf{k},n})^T \in \mathbb{R}^{5n},\\
&\mathbb{c} = (1,0,0,0,0,\dots,1,0,0,0,0)^T \in \mathbb{R}^{5n},\\
&\tilde{\mathbb{y}} = (\mathbb{y}_{\mathbf{r}}^T, \mathbb{y}_{\mathbf{i}}^T, \mathbb{y}_{\mathbf{j}}^T, \mathbb{y}_{\mathbf{k}}^T)^T \in \mathbb{R}^{4m},\\
&\tilde{\mathbf{\Phi}} = \begin{pmatrix} \mathbf{0} & \phi_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \phi_n & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \phi_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \phi_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \phi_1 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \phi_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \phi_1 & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \phi_n \end{pmatrix} \in \mathbb{R}^{4m\times 5n}.
\end{aligned}
\tag{6.4}
$$

This is a standard form of the SOCP and we solved it using the Matlab toolbox SeDuMi [10]. Finally, the quaternion signal $\mathbb{x}^{\#}$, which is the solution of (6.1), can easily be obtained from $\tilde{\mathbb{x}}$.

Our program was carried out on a standard PC machine, with Microsoft Windows 8.1 Pro system with Intel(R) Core(TM) i7-4790 CPU (3.60GHz) and 8GB RAM. The following algorithm was implemented in Matlab R2011b:

1. Fix constants $n = 128$ (length of the signal $\mathbb{x}$) and m (number of observables, i.e. size of the vector $\mathbb{y}$) and generate the measurement matrix $\mathbf{\Phi} \in \mathbb{R}^{m\times n}$ with random entries from i.i.d. standard normal distribution $\mathcal{N}(0,1)$;

2. Choose the sparsity $s \le \frac{n}{2}$, select a support set T ($\#T = s$) uniformly at random and generate a vector $\mathbb{x} \in \mathbb{H}^n$ supported on T with i.i.d. standard normal distribution (in the quaternion ℓ_2-norm sense with independent components);

3. Compute observables $\mathbb{y} = \mathbf{\Phi}\mathbb{x}$, $\mathbb{y} \in \mathbb{H}^m$;

4. Construct vectors $\tilde{\mathbb{x}}$, $\mathbb{c}$, $\tilde{\mathbb{y}}$ and matrix $\tilde{\mathbf{\Phi}}$ as in (6.4);

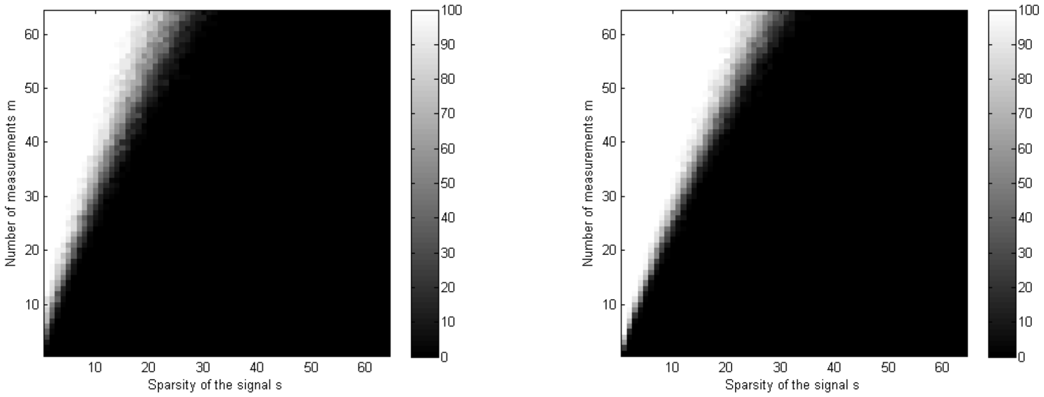
5. Call the SeDuMi toolbox to solve SOCP problem formulated in (6.3) and calculate the quaternion reconstructed vector $\mathbb{x}^{\#}$;

6. Compute the error of reconstruction, i.e. $\left\|\mathbb{x}^{\#} - \mathbb{x}\right\|_2$;

7. For each pair of n and s perform 100 experiments and save errors of each reconstruction and number of perfect reconstructions (we claim that the reconstruction is perfect if $\left\|\mathbb{x}^{\#} - \mathbb{x}\right\|_2 \le 10^{-7}$).

Next, we performed the same experiment for more general case, i.e. $\mathbf{\Phi} \in \mathbb{H}^{m\times n}$, as in [14]. Fig. 1(a) shows how the percentage of the perfect recovery depends on the number of measurements m and the sparsity s for $n = 128$. Fig. 1(b) shows the same results for the general case $\mathbf{\Phi} \in \mathbb{H}^{m\times n}$. We can see, for example, that for $s \le \frac{m}{4}$ and $m = 32$ the recovery rate is greater than 95% in the first experiment. Notice that this result is slightly worse than in the general case. So far we do not know the direct reason for this observation, however, it may be that the random quaternion matrix has better properties than the random real matrix (where three components are fixed to be zeros and only the real part is chosen at random). We plan to further study this issue in future research.

To further illustrate the results formulated in Theorem 4.1 and Corollary 5.1 we performed another experiment. We fixed constants $n = 256$ and $m = 32$ and generated the measurement matrix $\mathbf{\Phi} \in \mathbb{R}^{m\times n}$ with random entries from i.i.d. standard normal distribution and an arbitrary vector $\mathbb{x} \in \mathbb{H}^n$ (not assuming sparsity) with random quaternion entries from i.i.d. standard normal distribution (in the quaternion ℓ_2-norm sense with independent components). We performed the reconstruction of the vector $\mathbb{x}$, using the algorithm described above, and calculated errors of reconstruction $\left\|\mathbb{x}^{\#} - \mathbb{x}\right\|_1$ and

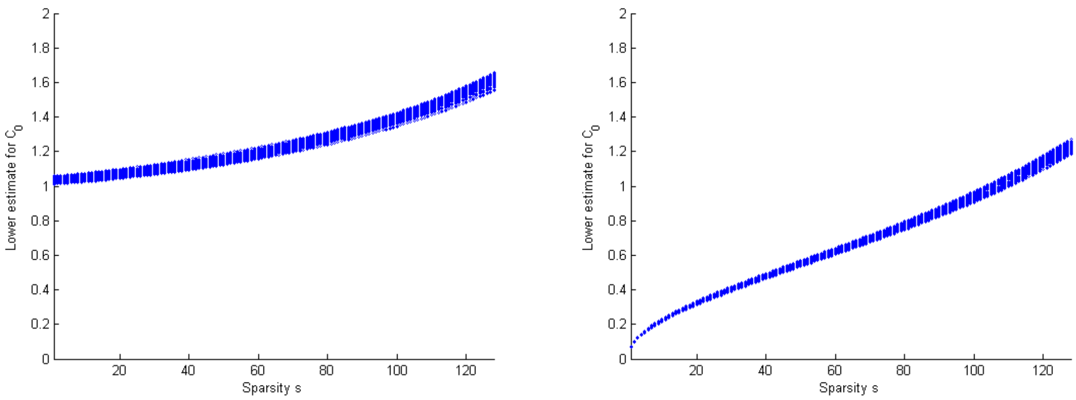


(a) (b)

Figure 1: Results of the recovery experiment for $n = 128$. The image intensity stands for the percentage of perfect reconstructions. (a) The case for $\mathbf{\Phi} \in \mathbb{R}^{m\times n}$. (b) The case for $\mathbf{\Phi} \in \mathbb{H}^{m\times n}$.

$\left\|\mathbb{x}^{\#} - \mathbb{x}\right\|_2$. We used the inequalities (5.2) and (5.3) to obtain a lower bound on the constant C_0 as a function of s (Fig. 2). We repeated the experiment for various choices of m and n but the results where comparable.

As we see the results slightly differ (which is not surprising since we use (5.3) to prove (5.2)), but we still obtain a lower bound for C_0 as the maximum of those two values (for arbitrary s). Observe that, as expected from the statement of Theorem 4.1 and Corollary 5.1, the dependence on s is monotone. Note also that the empirical bound on C_0 is smaller than 2 for all s, whereas the theoretical formula gives $C_0 > 2$. We suspect, therefore, that our result is not sharp and can be improved.



(a) (b)

Figure 2: Lower estimates for the constant C_0 in Corollary 5.1, experiment performed for $n = 256$ and $m = 32$, results for $s = 1, \ldots, 128$. (a) Results obtained from inequality (5.2). (b) Results obtained from inequality (5.3).

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