Many-Sorted Hybrid Modal Languages

Ioana Leuştean, Natalia Moangă and Traian Florin erbnu
Faculty of Mathematics and Computer Science, University of Bucharest,
Academiei nr.14, sector 1, C.P. 010014, Bucharest, Romania
ioana@fmi.unibuc.ro natalia.moanga@drd.unibuc.ro traian.serbanuta@fmi.unibuc.ro

Abstract

We continue our investigation into hybrid polyadic multi-sorted logic with a focus on expresivity related to the operational and axiomatic semantics of programming languages, and relations with first-order logic. We identify a fragment of the full logic, for which we prove sound and complete deduction and we show that it is powerful enough to represent both the programs and their semantics in an uniform way. Although weaker than other hybrid systems previously developed, this system is expected to have better computational properties. Finally, we provide a standard translation from full hybrid many-sorted logic to first-order logic.

Keywords: Hybrid modal logic, Many-sorted logic, Standard Translation, Operational semantics, Program verification

1 Introduction

This paper presents several hybrid modal logic systems based on the initial many-sorted structure we have developed [10] and progressively incorporating different operators and binders to it. These findings have enabled us to bridge the gap between the full many-sorted polyadic modal logic and First-Order Logic by developing a standard translation between them.

In Section 1 we recall our many-sorted polyadic modal logic, \mathcal{K}_{Σ} , introduced in [10], by presenting all the necessary information: the syntax, the semantics and the deductive system, in order for the reader to get familiarized with this logic. In Section 2, we propose and study $\mathcal{H}_{\Sigma}(\mathbb{Q}_z)$ a hybrid extension of \mathcal{K}_{Σ} and we prove its soundness and completeness. Moreover, we provide an example of using this system to axiomatically express operational semantics and to derive proofs for statements concerning program executions. Sections 3 and 4 recall two related hybrid systems introduced in [12]: $\mathcal{H}_{\Sigma}(\forall)$, an orthodogal extension of \mathcal{K}_{Σ} ; and $\mathcal{H}_{\Sigma}(\mathbb{Q}_z, \forall)$, a common extension of both $\mathcal{H}_{\Sigma}(\mathbb{Q}_z)$ and $\mathcal{H}_{\Sigma}(\forall)$. The paper concludes by providing a standard translation from $\mathcal{H}_{\Sigma}(\mathbb{Q}_z, \forall)$ to first order logic, showing that any many-sorted modal formula corresponds to a first-order formula from its corresponding first-order language.

1.1 Preliminaries: a many-sorted polyadic modal logic

For a general background on modal logic we refer to [5]. Basically, on top of modal logic we have added the sorts for each variable and the many-sorted polyadic operators σ together with the corresponding relation. The polyadic operators are defined also in [5], but in a mono-sorted version.

Our language is determined by a fixed, but arbitrary, many-sorted signature $\Sigma = (S, \Sigma)$ and an S-sorted set of propositional variables $P = \{P_s\}_{s \in S}$ such that $P_s \neq \emptyset$ for any $s \in S$ and $P_{s_1} \cap P_{s_2} = \emptyset$ for any $s_1 \neq s_2$ in S. For any $n \in \mathbb{N}$ and $s, s_1, \ldots, s_n \in S$ we denote $\Sigma_{s_1 \ldots s_n, s} = \{\sigma \in \Sigma \mid \sigma : s_1 \cdots s_n \to s\}$.

The set of formulas of \mathcal{K}_{Σ} , the many-sorted polyadic modal logic defined in [10], is an S-indexed family inductively defined by:

$$\phi_s ::= p \mid \neg \phi_s \mid \phi_s \vee \phi_s \mid \sigma(\phi_{s_1}, \dots, \phi_{s_n})$$

where $s \in S$, $p \in P_s$ and $\sigma \in \Sigma_{s_1 \cdots s_n, s}$.

We use the classical definitions of the derived logical connectors: for any $\sigma \in \Sigma_{s_1...s_n,s}$ the dual operation is $\sigma^{\square}(\phi_1,\ldots,\phi_n) := \neg \sigma(\neg \phi_1,\ldots,\neg \phi_n)$.

In the sequel, by ϕ_s we mean that ϕ is a formula of sort $s \in S$. Similarly, Γ_s means that Γ is a set of formulas of sort s. When the context uniquely determines the sort of a state symbol, we shall omit the subscript.

The deductive system is presented in Figure 1.

The system \mathcal{K}_{Σ}

- For any $s \in S$, if ϕ is a formula of sort s which is a theorem in propositional logic, then ϕ is an axiom.
- Axiom schemes: for any $\sigma \in \Sigma_{s_1 \cdots s_n, s}$ and for any formulas $\phi_1, \ldots, \phi_n, \phi, \chi$ of appropriate sorts, the following formulas are axioms:

$$(K_{\sigma}) \quad \sigma^{\square}(\dots,\phi_{i-1},\phi\to\chi,\phi_{i+1},\dots)\to \\ (\sigma^{\square}(\dots,\phi_{i-1},\phi,\phi_{i+1},\dots)\to\sigma^{\square}(\dots,\phi_{i-1},\chi,\phi_{i+1},\dots))$$

$$(Dual_{\sigma}) \quad \sigma(\psi_{1},\dots,\psi_{n})\leftrightarrow\neg\sigma^{\square}(\neg\psi_{1},\dots,\neg\psi_{n})$$

• Deduction rules: Modus Ponens and Universal Generalization

(MP) if
$$|\frac{s}{}\phi$$
 and $|\frac{s}{}\phi \to \psi$ then $|\frac{s}{}\psi$
(UG) if $|\frac{s_i}{}\phi$ then $|\frac{s}{}\sigma^{\square}(\phi_1,..,\phi,..\phi_n)$

Figure 1: (S, Σ) modal logic

In order to define the semantics we introduce (S, Σ) -frames and (S, Σ) -models. An (S, Σ) -frame is a tuple $\mathcal{F} = (W, (R_{\sigma})_{\sigma \in \Sigma})$ such that:

- $W = \{W_s\}_{s \in S}$ is an S-sorted set of worlds and $W_s \neq \emptyset$ for any $s \in S$,
- $R_{\sigma} \subseteq W_s \times W_{s_1} \times \ldots \times W_{s_n}$ for any $\sigma \in \Sigma_{s_1 \cdots s_n, s}$.

An (S, Σ) -model based on \mathcal{F} is a pair $\mathcal{M} = (\mathcal{F}, V)$ where $V = \{V_s\}_{s \in S}$ such that $V_s : P_s \to \mathcal{P}(W_s)$ for any $s \in S$. The model $\mathcal{M} = (\mathcal{F}, V)$ will be simply denoted as $\mathcal{M} = (W, (R_{\sigma})_{\sigma \in \Sigma}, V)$. For $s \in S$, $w \in W_s$ and ϕ a formula of sort s, the many-sorted satisfaction relation $\mathcal{M}, w \not\models \phi$ is inductively defined as follows:

- $\mathcal{M}, w \stackrel{s}{\models} p \text{ iff } w \in V_s(p)$
- $\mathcal{M}, w \stackrel{s}{\models} \neg \psi \text{ iff } \mathcal{M}, w \not\models \psi$
- $\mathcal{M}, w \stackrel{s}{\models} \psi_1 \vee \psi_2$ iff $\mathcal{M}, w \stackrel{s}{\models} \psi_1$ or $\mathcal{M}, w \stackrel{s}{\models} \psi_2$
- if $\sigma \in \Sigma_{s_1...s_n,s}$, then $\mathcal{M}, w \stackrel{s}{\models} \sigma(\phi_1, ..., \phi_n)$ iff for any $i \in [n]$ there exist $w_i \in W_{s_i}$ such that $R_{\sigma}ww_1...w_n$ and $\mathcal{M}, w_i \stackrel{s_i}{\models} \phi_i$.

Definition 1 (Validity and satisfiability). Let $s \in S$ and assume ϕ is a formula of sort s. Then ϕ is satisfiable if $\mathcal{M}, w \not\models \phi$ for some model \mathcal{M} and some $w \in W_s$. The formula ϕ is valid in a model \mathcal{M} if $\mathcal{M}, w \not\models \phi$ for any $w \in W_s$; in this case we write $\mathcal{M} \not\models \phi$. The formula ϕ is valid in a frame \mathcal{F} if ϕ is valid in all the models based on \mathcal{F} ; in this case we write $\mathcal{F} \not\models \phi$. Finally, the formula ϕ is valid if ϕ is valid in all frames; in this case we write $\not\models \phi$.

The set of theorems of \mathcal{K}_{Σ} is the least set of formulas that contains all the axioms and it is closed under deduction rules. Note that the set of theorems is obviously closed under *S*-sorted uniform substitution (i.e. propositional variables of sort s are uniformly replaced by formulas of the same sort). If ϕ is a theorem of sort s write $\frac{s}{\mathcal{K}_{\Sigma}}\phi$, or simply $\frac{s}{\mathcal{K}_{\Sigma}}\phi$. Obviously, \mathcal{K}_{Σ} is a generalization of the modal system \mathbf{K} (see [5] for the mono-sorted version). The completeness theorem of \mathcal{K}_{Σ} is proved in [10].

2 The many-sorted basic hybrid modal logic $\mathcal{H}_{\Sigma}(@_z)$

Let (S, Σ) be a many-sorted signature. A basic hybrid modal logic is defined on top of modal logic \mathcal{K}_{Σ} by adding nominals, states variables and specific operators. Nominals allow us to directly refer the worlds (states) of a model, since they are evaluated in singletons in any model. However, a nominal may refer different worlds in different models. The sorts will be denoted by s, t, \ldots and by $PROP = \{PROP_s\}_{s \in S}$, $NOM = \{NOM_s\}_{s \in S}$ and $SVAR = \{SVAR_s\}_{s \in S}$ we will denote some countable S-sorted sets. The elements of PROP are ordinary propositional variables and they will be denoted p, q, \ldots ; the elements of NOM are called nominals and they will be denoted by j, k, \ldots ; the elements of SVAR are called state variables and they are denoted x, y, \ldots . We shall assume that for any distinct sorts $s \neq t \in S$, the corresponding sets of propositional variables, nominals and state variables are distinct. A state symbol is a nominal or a state variable.

Recall that the satisfaction in modal logic is local, i.e. one analyzes what happens in a given point of the model. With respect to this, nominals can be

seen as local constants and, given a model (a frame and an evaluation), the value of a nominal is a fixed singleton set. State variables are variables that range over the individual points of a model, while the usual (propositional) variables range over arbitrary sets of points.

For this section we drew our inspiration mainly from [4]. As already announced, in this section we extend the system defined in Section 1 by adding the satisfaction operators $@_z^s$ where $s \in S$ and z is a *state symbol*. The formulas of $\mathcal{H}_{\Sigma}(@_z)$ are defined as follows:

$$\phi_s := p \mid j \mid x_s \mid \neg \phi_s \mid \phi_s \vee \phi_s \mid \sigma(\phi_{s_1}, \dots, \phi_{s_n})_s \mid @_z^s \psi_t$$

Here, $p \in PROP_s$, $j \in NOM_s$, $t \in S$, $x \in SVAR_s$, $\sigma \in \Sigma_{s_1 \cdots s_n, s}$, z is a state symbol of sort t and ψ is a formula of sort t.

In order to define the semantics for $\mathcal{H}_{\Sigma}(@_z)$ more is needed. Given an (S, Σ) -model $\mathcal{M} = (W, (R_{\sigma})_{\sigma \in \Sigma}, V)$, an assignment is an S-sorted function $g: \text{SVAR} \to W$, which evaluates states variables to singleton sets, and for any $s \in S$ we have $g_s: \text{SVAR}_s \to W_s$.

The satisfaction relation is defined similar with the one in \mathcal{K}_{Σ} , but we only need to add the definition for \mathbb{Q}_z :

$$\mathcal{M}, g, w \stackrel{s}{\models} @_{z}^{s} \phi$$
 if and only if $\mathcal{M}, g, Den_{g}(z) \stackrel{t}{\models} \phi$

where z is a state symbol of sort t and ϕ is a formula of the same sort t. Here, $Den_g(z)$ is the denotation of the state symbol z of sort s in an (S, Σ) -model \mathcal{M} with an assignment function g, where $Den_g(z) = V_s(z)$ if z is a nominal, and $Den_g(z) = g_s(z)$ if z is a state variable.

Let us remark that if z is a nominal, then the satisfaction relation is equivalent with the one in [11]:

$$\mathcal{M}, g, w \stackrel{s}{\models} @_z^s \phi$$
 if and only if $\mathcal{M}, g, Den_g(z) \stackrel{t}{\models} \phi$ if and only if $\mathcal{M}, g, v \stackrel{t}{\models} \phi$ where $Den_g(z) = V_t(z) = \{v\}$.

One important remark is the definition of the satisfaction modalities: if z and ϕ are a state symbol and a formula both of the sort $t \in S$, then we define a family of satisfaction operators $\{@_z^s\phi\}_{s\in S}$ such that $@_z^s\phi$ is a formula of sort s for any $s\in S$. This means that ϕ is true at the world denoted by z on the sort t and is acknowledged on any sort $s\in S$. For example, if we take j and k two nominals of sort t and $s\neq t$ the formula $@_j^s\neg k$ expresses the fact that at any world of sort s we know that the worlds of sort t named by t and t are different. So, our sorted worlds are not isolated any more, both from a syntactic and a semantic point of view.

Proposition 2 (Soundness). The deductive systems for $\mathcal{H}_{\Sigma}(\mathbb{Q}_z)$ from Figure 2 is sound.

Proof. Let \mathcal{M} be an arbitrary model and w any state of sort s.

 $(K_{@})$ Suppose $\mathcal{M}, g, w \stackrel{s}{\models} @_{z}^{s}(\phi_{t} \to \psi_{t})$ if and only if $\mathcal{M}, g, Den_{g}(z) \stackrel{t}{\models} \phi_{t} \to \psi_{t}$ if and only if $\mathcal{M}, g, Den_{g}(z) \stackrel{t}{\models} \phi_{t}$ implies $\mathcal{M}, g, Den_{g}(z) \stackrel{t}{\models} \psi_{t}$. Let us prove

The system $\mathcal{H}_{\Sigma}(@_z)$

- The axioms and the deduction rules of \mathcal{K}_{Σ}
- Axiom schemes: any formula of the following form is an axiom, where s, s', t are sorts, $\sigma \in \Sigma_{s_1 \dots s_n, s}, \phi, \psi, \phi_1, \dots, \phi_n$ are formulas (when necessary, their sort is marked as a subscript), and y, z are state symbols:

```
 \begin{array}{ll} (K@) & @_z^s(\phi_t \to \psi_t) \to (@_z^s\phi \to @_z^s\psi) \\ (SelfDual) & @_z^s\phi_t \leftrightarrow \neg @_z^s \neg \phi_t \\ (Intro) & z \to (\phi_s \leftrightarrow @_z^s\phi_s) \\ (Agree) & @_y^t @_z^t\phi_s \leftrightarrow @_z^t\phi_s \\ (Ref) & @_z^sz_t \\ (Back) & \sigma(\dots,\phi_{i-1}, @_z^{s_i}\psi_t,\phi_{i+1},\dots)_s \to @_z^s\psi_t \end{array}
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• Deduction rules:

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(BroadcastS) \quad \text{if } | \underline{s} \cdot @_z^s \phi_t \text{ then } | \underline{s}' \cdot @_z^{s'} \phi_t \\ (Gen@) \quad \text{if } | \underline{s}' \cdot \phi \text{ then } | \underline{s} \cdot @_z \phi, \text{ where } z \text{ and } \phi \text{ have the same sort } s' \\ (Paste0) \quad \text{if } | \underline{s} \cdot @_z^s (y \wedge \phi) \to \psi \text{ then } | \underline{s} \cdot @_z \phi \to \psi \\ \quad \text{where } z \text{ is distinct from } y \text{ that does not occur in } \phi \text{ or } \psi \\ (Paste1) \quad \text{if } | \underline{s} \cdot @_z^s \sigma(\ldots, y \wedge \phi, \ldots) \to \psi \text{ then } | \underline{s} \cdot @_z^s \sigma(\ldots, \phi, \ldots) \to \psi \\ \quad \text{where } z \text{ is distinct from } y \text{ that does not occur in } \phi \text{ or } \psi
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Figure 2: (S, Σ) basic hybrid modal logic

the non-trivial case: suppose that $\mathcal{M}, g, w \stackrel{s}{\models} @_{j}^{s} \phi_{t}$. Then $\mathcal{M}, g, Den_{g}(z) \stackrel{t}{\models} \phi_{t}$, but this implies that $\mathcal{M}, g, Den_{g}(z) \stackrel{t}{\models} \psi_{t}$ if and only if $\mathcal{M}, g, w \stackrel{s}{\models} @_{z}^{s} \psi_{t}$. Therefore, $\mathcal{M}, g, w \stackrel{s}{\models} @_{z}^{s} \phi_{t} \rightarrow @_{z}^{s} \psi_{t}$.

(Agree) Suppose $\mathcal{M}, g, w \models @_y^t @_z^{t'} \phi_s$ if and only if $\mathcal{M}, g, Den_g(y) \models @_z^t \phi_s$ implies $\mathcal{M}, g, Den_g(z) \models \phi_s$. It follows that $\mathcal{M}, g, w \models @_z^t \phi_s$.

(Self Dual) Suppose $\mathcal{M}, g, w \stackrel{s}{\models} \neg @_z^s \neg \phi_t$ if and only if $\mathcal{M}, g, w \not\stackrel{s}{\models} @_z^s \neg \phi_t$ if and only if $\mathcal{M}, g, Den_g(z) \not\models \neg \phi_t$ if and only if $\mathcal{M}, g, W \not\models @_z^s \phi_t$.

(Back) Suppose $\mathcal{M}, g, w \models \sigma(\dots, \phi_{i-1}, @_z^{s_i} \psi_t, \phi_{i+1}, \dots)_s$ if and only if there is $(w_1, \dots, w_n) \in W_{s_1} \times \dots \times W_{s_n}$ such that $R_{\sigma}ww_1 \dots w_n$ and $\mathcal{M}, g, w_i \models^{\underline{s_i}} \phi_i$ for any $i \in [n]$. This implies that there is $w_i \in W_{s_i}$ such that $\mathcal{M}, g, w_i \models^{\underline{s_i}} @_z^{s_i} \psi_t$, so $\mathcal{M}, g, Den_g(z) \models^{\underline{t}} \psi_t$. Hence, $\mathcal{M}, g, w \models^{\underline{s}} @_z^s \psi_t$

(Ref) Suppose $\mathcal{M}, g, w \not\models \mathbb{Q}_z^s z_t$. Then $\mathcal{M}, g, Den_g(z) \not\models z$, contradiction.

(Intro) Suppose $\mathcal{M}, g, w \stackrel{s}{\models} z$ and $\mathcal{M}, g, w \stackrel{s}{\models} \phi_s$. Then $w = Den_g(z)$, so we get that $\mathcal{M}, g, Den_g(z) \stackrel{s}{\models} z$ and $\mathcal{M}, g, Den_g(z) \stackrel{s}{\models} \phi_s$ implies that $\mathcal{M}, g, w \stackrel{s}{\models} @_z^s \phi_s$.

Now, suppose $\mathcal{M}, g, w \stackrel{s}{\models} z$ and $\mathcal{M}, g, w \stackrel{s}{\models} @_z^s \phi_s$. Because from the first assumption we have $Den_q(z) = \{w\}$, then, from the second one, we can conclude

that
$$\mathcal{M}, g, w \stackrel{s}{\models} \phi_s$$
.

The following lemma generalizes the results from [2], being essentially used in the proof of the completeness theorem.

Lemma 3. The following formulas are theorems:

$$(Nom_z) \qquad \underset{z}{@}_{s}y_t \to (\underset{z}{@}_{s}^s\phi_t \leftrightarrow \underset{y}{@}_{y}^s\phi_t)$$

$$for any \ s,t \in S, \ z_t,y_t \ state \ symbols \ of \ sort \ t \ and \ \phi_t \ a \ formula$$

$$of \ sort \ t.$$

$$(Sym) \qquad \underset{z}{@}_{s}^sy_t \to \underset{y}{@}_{s}^sz_t$$

$$where \ s,t \in S \ and \ z_t,y_t \ are \ state \ symbols \ of \ sort \ t,$$

$$(Bridge) \qquad \sigma(\ldots\phi_{i_1},z_{s_i},\phi_{i+1}\ldots) \wedge \underset{z}{@}_{s}^s\phi_{s_i} \to \sigma(\ldots\phi_{i-1},\phi_{s_i},\phi_{i+1},\ldots)$$

$$if \ \sigma \in \Sigma_{s_1...s_n,s}, \ z_{s_i} \ is \ a \ state \ symbol \ of \ sort \ s_i \ and \ \phi_{s_i} \ is \ a$$

$$formula \ of \ sort \ s_i.$$

Proof. In the sequel, by PL we mean classical propositional logic and by ML we mean the basic modal logic.

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(Nom_z)
(1) \vdash^{\underline{t}} y_t \to (\phi_t \leftrightarrow @_y^t \phi_t)
                                                                                                                                                                                                       (Intro)
(2) \stackrel{|s|}{=} @_z^s(y_t \to (\phi_t \leftrightarrow @_y^t \phi_t))
(3) \stackrel{|s|}{=} @_z^s(y_t \to (\phi_t \leftrightarrow @_y^t \phi_t)) \to (@_z^s y_t \to @_z^s(\phi_t \leftrightarrow @_y^t \phi_t))
(4) \stackrel{|s|}{=} @_z^s y_t \to @_z^s(\phi_t \leftrightarrow @_y^t \phi_t)
(5) \stackrel{|s|}{=} @_z^s(\phi_t \leftrightarrow @_y^t \phi_t) \leftrightarrow (@_z^s \phi_t \leftrightarrow @_z^s @_y^t \phi_t)
                                                                                                                                                                                                      (Gen@)
                                                                                                                                                                                                           (K@)
                                                                                                                                                                                  (MP):(2),(3)
                                                                                                                                                                                                                 ML
(6) \stackrel{s}{\models} @_z^s y_t \to (@_z^s \phi_t \leftrightarrow @_z^s @_y^t \phi_t)
                                                                                                                                                                                              PL:(4),(5)
(7) \mid \underline{}^s @_z^s @_y^t \phi_t \leftrightarrow @_y^s \phi_t
                                                                                                                                                                                                     (Agree)
(8) \xrightarrow{s} @_z^s y_t \to (@_z^s \phi_t \leftrightarrow @_y^s \phi_t)
                                                                                                                                                                                              PL:(6), (7)
          (Sym)
(1) \frac{s}{2} @_{y}^{s} z_{t} \wedge @_{z}^{s} y_{t} \rightarrow @_{z}^{s} y_{t}
                                                                                                                                                                                                             Taut
(2) \stackrel{s}{\models} (\overset{g}{@}_{y}^{s} z_{t} \wedge \overset{g}{@}_{z}^{s} y_{t} \rightarrow \overset{g}{@}_{z}^{s} y_{t}) \rightarrow (\overset{g}{@}_{y}^{s} z \rightarrow (\overset{g}{@}_{z}^{s} y_{t} \rightarrow \overset{g}{@}_{z}^{s} y_{t}))
                                                                                                                                                                                                             Taut
(3) \mid \stackrel{s}{=} @_y^s z \to (@_z^s y_t \to @_z^s y_t)
                                                                                                                                                                                   (MP):(1),(2)
(4) \mid \frac{s}{} (@_z^s y_t \to @_z^s y_t) \to @_z^s y_t
                                                                                                                                                                                                                   PL
 (5) \mid \underline{s} @_y^s z \to @_z^s y_t
                                                                                                                                                                                                                  PL
(6) \stackrel{s}{\models} @_z^s y_t \to @_y^s z
                                                                                                                                                                                                 Analogue
(7) \stackrel{s}{\models} @_z^s y_t \leftrightarrow @_u^s z
                                                                                                                                                                                              PL:(5),(6)
          (Bridge)
(1) \stackrel{s}{\models} \sigma(\dots \phi_{i-1}, z_{s_i}, \phi_{i+1} \dots) \wedge \sigma^{\square}(\dots, \neg \phi_{i-1}, \neg \phi_{s_i}, \neg \phi_{i+1}, \dots) \rightarrow \\ \rightarrow \sigma(\dots \phi_{i-1}, z_{s_i} \wedge \phi_{s_i}, \phi_{i+1}, \dots)
                                                                                                                                                                                                                 ML
(2) |s_i| z_{s_i} \wedge \neg \phi_{s_i} \rightarrow @_z^{s_i} \neg \phi_{s_i}
                                                                                                                                                                                                       (Intro)
(3) \quad \stackrel{s}{\models} \sigma(\ldots \phi_{i-1}, z_{s_i} \land \neg \phi_{s_i}, \phi_{i+1}, \ldots) \rightarrow \sigma(\ldots \phi_{i-1}, @_z^{s_i} \neg \phi_{s_i}, \phi_{i+1}, \ldots)
                                                                                                                                                                                                                 ML
(4) \mid \stackrel{s}{\longrightarrow} \sigma(\dots \phi_{i-1}, @_z^{s_i} \neg \phi_{s_i}, \phi_{i+1}, \dots) \rightarrow @_z^{s} \neg \phi_{s_i}
                                                                                                                                                                                                        (Back)
(5) \quad | \underline{s} \sigma(\dots \phi_{i-1}, z_{s_i} \wedge \neg \phi_{s_i}, \phi_{i+1}, \dots) \to @_z^s \neg \phi_{s_i}
                                                                                                                                                                                               PL:(3),(4)
(6) \quad \frac{s}{|s|} \sigma(\ldots \phi_{i_1}, z_{s_i}, \phi_{i+1} \ldots) \wedge \sigma^{\square}(\ldots, \neg \phi_{i-1}, \neg \phi_{s_i}, \neg \phi_{i+1}, \ldots) \to @_z^s \neg \phi_{s_i}
                                                                                                                                                                                               PL:(1),(5)
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- $(7) \quad | \stackrel{s}{-} \sigma(\dots \phi_{i_1}, z_{s_i}, \phi_{i+1} \dots) \to (\sigma^{\square}(\dots, \neg \phi_{i-1}, \neg \phi_{s_i}, \neg \phi_{i+1}, \dots) \to @_z^s \neg \phi_{s_i})$
- (8) $\models \sigma(\ldots \phi_{i_1}, z_{s_i}, \phi_{i+1} \ldots) \rightarrow (\neg @_z^s \neg \phi_{s_i} \rightarrow \neg \sigma^{\square}(\ldots, \neg \phi_{i-1}, \neg \phi_{s_i}, \neg \phi_{i+1}, \ldots))$
- $(9) \mid \frac{s}{-}\sigma(\dots\phi_{i_1}, z_{s_i}, \phi_{i+1}\dots) \to (@_z^s \phi_{s_i} \to \sigma(\dots, \phi_{i-1}, \phi_{s_i}, \phi_{i+1}, \dots))$ (Dual) (Self Dual)

$$(10) \mid \underline{s} \sigma(\dots \phi_{i_1}, z_{s_i}, \phi_{i+1} \dots) \wedge @_z^s \phi_{s_i} \to \sigma(\dots, \phi_{i-1}, \phi_{s_i}, \phi_{i+1}, \dots)$$
PL

Lemma 4. Let Γ_s be a maximal consistent set that contains a state symbol of sort s, and for all state symbols z, let $\Delta_z = \{\phi \mid \mathbb{Q}_z^s \phi \in \Gamma_s\}$. Then:

- 1) For every state symbol z of sort s, Δ_z is a maximal consistent set that contains z.
- 2) For all state symbols z and y of same sort, $@_z^s \phi \in \Delta_y$ if and only if $@_z^s \phi \in \Gamma_s$.
- 3) There is a state symbol z such that $\Gamma_s = \Delta_z$.
- 4) For all state symbols z and y of same sort, if $z \in \Delta_y$ then $\Delta_z = \Delta_y$.
- Proof. 1) Recall that for any state symbol z we have the (Ref) axiom, so $@_z^s z_t \in \Gamma_s$. Hence, $z \in \Delta_z$. But, is Δ_z a consistent set? Let us suppose that is not. So there are $\chi_1, \ldots, \chi_n \in \Delta_j$ such that $f(\chi_1, \ldots, \chi_n, \chi_n) \in \Lambda_j$ such that $f(\chi_1, \ldots, \chi_n, \chi_n) \in \Lambda_j$ such that $f(\chi_1, \ldots, \chi_n) \in \Lambda_j$ such that f(

Now, let us check if Δ_z is maximal. Assume it is not. Then there is a formula χ of sort t such that $\chi \not\in \Delta_z$ and $\neg \chi \not\in \Delta_z$. But then $@_z^s \chi \not\in \Gamma_s$ and $@_z^s \neg \chi \not\in \Gamma_s$. But also Γ_s is a maximal consistent set, then $\neg @_z^s \chi \in \Gamma_s$ and $\neg @_z^s \neg \chi \in \Gamma_s$. On the other hand, if $\neg @_z^s \neg \chi \in \Gamma_s$, then by (SelfDual) axiom we get that $@_z^s \chi \in \Gamma_s$, and this contradicts the consistency of Γ_s . Hence, we conclude that Δ_z is a maximal consistent set.

- 2) By definition of Δ_y , $@_z^t \phi \in \Delta_y$ if and only if $@_y^s @_z^t \phi \in \Gamma_s$. By (Agree) axiom we have that $@_y^s @_z^t \phi \in \Gamma_s$ if and only if $@_z^s \phi \in \Gamma_s$. This is called the @-agreement property, which it plays an important role in the completeness proof.
- 3) Let the state symbol z of sort s be contained in Γ_s . Suppose $\phi \in \Gamma_s$. Because $z \in \Gamma_s$, by (Intro) axiom we get $@_z^s \phi \in \Gamma_s$, and by definition of Δ_z , we have $\phi \in \Delta_z$. Conversely, if $\phi \in \Delta_z$, then by definition of Δ_z it follows that $@_z^s \phi \in \Gamma_s$. Moreover, $z \in \Gamma_s$ and using again the same axiom we get that $\phi \in \Gamma_s$.

4) Let $z \in \Delta_y$, then by definition of Δ_y we have that $@_y^s z \in \Gamma_s$ and by (Sym) we get that $@_z^s y \in \Gamma_s$. Firstly, let us prove that $\Delta_y \subseteq \Delta_z$. Let $\phi \in \Delta_y$, then by definition of Δ_y we have that $@_y^s \phi \in \Gamma_s$. Also, $@_z^s y \in \Gamma_s$, so by (Nom_z) it follows that $@_z^s \phi \in \Gamma_s$ and hence that $\phi \in \Delta_z$. Secondly, a similarly (Nom_z) -based proof shows that $\Delta_z \subseteq \Delta_y$.

This Lemma gives us the maximal consistent sets needed in the Existence Lemma. We build our models out of named sets, i.e. sets containing nominals. But more is needed in order for our model to support an Existential Lemma. Therefore, we add the *Paste* rules, as you can see in Figure 2. In this setting, the system is still sound as we prove in the following:

(BroadcastS) Suppose $\mathcal{M}, g, w \stackrel{s}{\models} @_z^s \phi_t$ if and only if $\mathcal{M}, g, Den_g(z) \stackrel{t}{\models} \phi_t$. Hence, for any $s' \in S$ we have $\mathcal{M}, g, w \stackrel{s'}{\models} @_z^{s'} \phi_t$.

Now, let \mathcal{M} be an arbitrary named model.

 $(Paste0) \text{ Suppose } \mathcal{M}, g, w \stackrel{s}{\models} @_z^s(y \land \phi) \to \psi \text{ if and only if } \mathcal{M}, g, w \stackrel{s}{\models} @_z^s(y \land \phi) \text{ implies } \mathcal{M}, g, w \stackrel{s}{\models} \psi. \text{ Hence, } (\mathcal{M}, g, v \stackrel{s'}{\models} y \land \phi \text{ where } Den_g(z) = \{v\} \text{ implies } \mathcal{M}, g, w \stackrel{s}{\models} \psi) \text{ if and only if } (\mathcal{M}, g, v \stackrel{s'}{\not\models} y \text{ and } \mathcal{M}, g, v \stackrel{s'}{\not\models} \phi, \text{ where } Den_g(z) = \{v\}, \text{ or } \mathcal{M}, g, w \stackrel{s}{\models} \psi). \text{ It follows that } (\mathcal{M}, g, v \stackrel{s'}{\not\models} y \text{ or } \mathcal{M}, g, w \stackrel{s}{\models} \psi) \text{ and } (\mathcal{M}, g, v \stackrel{s'}{\not\models} \phi \text{ or } \mathcal{M}, g, w \stackrel{s}{\models} \psi), \text{ where } Den_g(z) = \{v\}. \text{ Then, } (\mathcal{M}, g, v \stackrel{s'}{\not\models} \phi \text{ or } \mathcal{M}, g, w \stackrel{s}{\models} \psi), \text{ where } Den_g(z) = \{v\}. \text{ So, } \mathcal{M}, g, w \stackrel{s}{\models} @_z^s \phi \to \psi.$

 $(Paste1) \text{ Suppose } \mathcal{M}, g, w \overset{\hspace{0.1em} \models}{=} @_{z}^{s} \sigma(\psi_{1}, \ldots, \psi_{i-1}, y \wedge \phi, \psi_{i+1}, \ldots, \psi_{n}) \rightarrow \psi \text{ if and only if } \mathcal{M}, g, w \overset{\hspace{0.1em} \models}{=} @_{z}^{s} \sigma(\psi_{1}, \ldots, \psi_{i-1}, y \wedge \phi, \psi_{i+1}, \ldots, \psi_{n}) \text{ implies } \mathcal{M}, g, w \overset{\hspace{0.1em} \models}{=} \psi. \text{ Hence, } \mathcal{M}, g, v \overset{\hspace{0.1em} \models}{=} y \wedge \phi \text{ where } Den_{g}(z) = \{v\} \text{ if and only if exists } (v_{1}, \ldots, v_{n}) \in W_{s_{1}} \times \ldots \times W_{s_{n}} \text{ such that } R_{\sigma}vv_{1} \ldots v_{i} \ldots v_{n} \text{ where } Den_{g}(z) = \{v\} \text{ and } \mathcal{M}, g, v_{e} \overset{\hspace{0.1em} \models}{=} \psi \text{ for any } e \in [n], e \neq i \text{ and } \mathcal{M}, g, v_{i} \overset{\hspace{0.1em} \models}{=} y \wedge \phi. \text{ Hence, } \mathcal{M}, g, v_{i} \overset{\hspace{0.1em} \models}{=} y \text{ and } \mathcal{M}, g, v_{i} \overset{\hspace{0.1em} \models}{=} \phi, \text{ so } Den_{g}(y) = \{v_{i}\} \text{ and } \mathcal{M}, g, v_{i} \overset{\hspace{0.1em} \models}{=} \phi. \text{ Then, if there exists } (v_{1}, \ldots, v_{n}) \in W_{s_{1}} \times \ldots \times W_{s_{n}} \text{ such that } R_{\sigma}vv_{1} \ldots v_{i} \ldots v_{n} \text{ where } Den_{g}(z) = \{v\} \text{ and } \mathcal{M}, g, v_{e} \overset{\hspace{0.1em} \models}{=} \psi_{e} \text{ for any } e \in [n], e \neq i \text{ and } \mathcal{M}, g, v_{i} \overset{\hspace{0.1em} \models}{=} \phi, \text{ these imply } \mathcal{M}, g, w \overset{\hspace{0.1em} \models}{=} \psi. \text{ So, } \mathcal{M}, g, v \overset{\hspace{0.1em} \models}{=} \sigma(\psi_{1}, \ldots, \psi_{i-1}, \phi, \psi_{i+1}, \ldots, \psi_{n}) \text{ where } Den_{g}(z) = \{v\} \text{ implies } \mathcal{M}, g, w \overset{\hspace{0.1em} \models}{=} \psi. \text{ In conclusion, } \mathcal{M}, g, w \overset{\hspace{0.1em} \models}{=} 0 v_{2} \sigma(\psi_{1}, \ldots, \psi_{i-1}, \phi, \psi_{i+1}, \ldots, \psi_{n}) \rightarrow \psi.$

Definition 5 (Named and pasted). Let $s \in S$ and Γ_s be a set of formulas of sort s from $\mathcal{H}_{\Sigma}(\mathbb{Q}_z)$. We say that

- Γ_s is named if one of its elements is a nominal,
- ullet Γ_s is pasted if it is both 0-pasted and 1-pasted:
 - (-) Γ_s is 0-pasted if, for any $t \in S$, $\sigma \in \Sigma_{s_1 \cdots s_n, t}$, z a state symbol of sort t, and ϕ a formula of sort s_i , whenever $@_z^s \phi \in \Gamma_s$ there exists a nominal $j \in \text{NOM}_{s_i}$ such that $@_z^s \sigma(\ldots, \phi_{i-1}, j \land \phi, \phi_{i+1}, \ldots) \in \Gamma_s$.
 - (-) Γ_s is 1-pasted if, for any $t \in S$, $\sigma \in \Sigma_{s_1 \cdots s_n,t}$, z a state symbol of sort t, and ϕ a formula of sort s_i , whenever $@_z^s \sigma(\ldots,\phi_{i-1},\phi,\phi_{i+1},\ldots) \in$

 Γ_s there exists a nominal $j \in \text{NOM}_{s_i}$ such that $@_z^s \sigma(\ldots, \phi_{i-1}, j \land \phi, \phi_{i+1}, \ldots) \in \Gamma_s$.

Lemma 6 (Extended Lindenbaum Lemma). Let Λ be a set of formulas in the language of $\mathcal{H}_{\Sigma}(@_z)$ and $s \in S$. Then any consistent set Γ_s of formulas of sort s from $\mathcal{H}_{\Sigma}(@_z) + \Lambda$ can be extended to a named, pasted and @-maximal consistent set by adding countably many nominals to the language.

Proof. The proof generalizes to the S-sorted setting well-known proofs for the mono-sorted hybrid logic, see [5, Lemma 7.25], [2, Lemma 3, Lemma 4], [3, Lemma 3.9].

For each sort $s \in S$, we add a set of new nominals and enumerate this set. Given a set of formulas Γ_s , define Γ_s^k to be $\Gamma_s \cup \{k_s\}$, where k_s is the first new nominal of sort s in our enumeration. As showed in [11], Γ_s^k is consistent.

Now we enumerate on each sort $s \in S$ all the formulas of the new language obtained by adding the set of new nominals and define $\Gamma^0 := \Gamma^k_s$. Suppose we have defined Γ^m , where $m \geq 0$. Let ϕ_{m+1} be the m+1-th formula of sort s in the previous enumeration. We define Γ^{m+1} as follows. If $\Gamma^m \cup \{\phi_{m+1}\}$ is inconsistent, then $\Gamma^{m+1} = \Gamma^m$. Otherwise:

- (i) $\Gamma^{m+1} = \Gamma^m \cup \{\phi_{m+1}\}\$, if ϕ_{m+1} is not of the form $@_z\sigma(\ldots,\varphi,\ldots)$ or $@_xx$, where φ a formula of sort s'', $x \in SVAR_{s''}$ and z is a state symbol.
- (ii) $\Gamma^{m+1} = \Gamma^m \cup \{\phi_{m+1}\} \cup \{\mathbb{Q}_x(k \wedge x)\}$, if ϕ_{m+1} is of the form $\mathbb{Q}_x x$, where k is a new nominal that does not occur in Γ^m .
- (iii) $\Gamma^{m+1} = \Gamma^m \cup \{\phi_{m+1}\} \cup \{@_x\sigma(\dots, k \wedge \phi, \dots)\}$, if ϕ_{m+1} is of the form $@_x\sigma(\dots, \varphi, \dots)$ and k is a new nominal that does not occur in Γ^m or $@_x\sigma(\dots, \varphi, \dots)$.

In clauses (ii) and (iii), k is the first new nominal in the enumeration that does not occur in Γ^i for all $i \leq m$, nor in $@_x\sigma(\ldots,\varphi,\ldots)$.

Let $\Gamma^+ = \bigcup_{n\geq 0} \Gamma^n$. Because $k \in \Gamma^0 \subseteq \Gamma^+$, this set in named, maximal, pasted and @-witnessed by construction. We will check if it is consistent for the expansion made in the second, third and fourth items.

Suppose $\Gamma^{m+1} = \Gamma^m \cup \{\phi_{m+1}\} \cup \{\mathbb{Q}_x(k \wedge x)\}$ is an inconsistent set, where ϕ_{m+1} is $\mathbb{Q}_x x$. Then there is a conjunction of formulas $\chi \in \Gamma^m \cup \{\phi_{m+1}\}$ such that $|\underline{s} \chi \to \neg \mathbb{Q}_x(k \wedge x)|$ and so $|\underline{s} \mathbb{Q}_x(k \wedge x) \to \neg \chi$. But k is the first new nominal in the enumeration that does not occur neither in Γ^m , nor in $\mathbb{Q}_x x$ and by Paste0 rule we get $|\underline{s} \mathbb{Q}_x x \to \neg \chi$. Then $|\underline{s} \chi \to \neg \mathbb{Q}_x x|$, which contradicts the consistency of $\Gamma^m \cup \{\phi_{m+1}\}$.

Suppose $\Gamma^{m+1} = \Gamma^m \cup \{\phi_{m+1}\} \cup \{@_x\sigma(\ldots,k \wedge \varphi,\ldots)\}$ is an inconsistent set, where ϕ_{m+1} has the form $@_x\sigma(\ldots,\varphi,\ldots)$. Then there is a conjunction of formulas $\chi \in \Gamma^m \cup \{\phi_{m+1}\}$ such that $| \stackrel{s}{=} \chi \to \neg @_x\sigma(\ldots,k \wedge \varphi,\ldots)$ and so $| \stackrel{s}{=} @_x\sigma(\ldots,k \wedge \varphi,\ldots) \to \neg \chi$. But k is the first new nominal in the enumeration that does not occur neither in Γ^m , nor in $@_x\sigma(\ldots,\varphi,\ldots)$, therefore, by Paste1 rule we get $| \stackrel{s}{=} @_x\sigma(\ldots,\varphi,\ldots) \to \neg \chi$. It follows that $| \stackrel{s}{=} \chi \to \neg @_x\sigma(\ldots,\varphi,\ldots)$, which contradicts the consistency of $\Gamma^m \cup \{\phi_{m+1}\}$.

Definition 7 (Named models and natural assignments). For any $s \in S$, let Γ_s be a named, pasted and witnessed maximal consistent set and for all state symbols z, let $\Delta_z = \{\varphi \mid @_z^s \varphi \in \Gamma_s\}$. Define $W_s = \{\Delta_z \mid z \text{ a state symbol of sort } s \}$. Then, we define $\mathcal{M} = (W, \{R_\sigma\}_{\sigma \in \Sigma}, V)$, the named model generated by the S-sorted set $\Gamma = \{\Gamma_s\}_{s \in S}$, where R_σ and V are the restriction of the canonical relation and the canonical valuation. We define the natural assignment $g_s : \text{SVAR}_s \to W_s$ by $g_s(x) = \{w \in W_s \mid x \in w\}$.

Lemma 8 (Existence Lemma). Let $\mathcal{M} = (W, \{R_{\sigma}\}_{{\sigma} \in \Sigma}, V)$ be a named model generated by a named and pasted S-sorted set Γ and let w be a witnessed maximal consistent set. If $\sigma(\phi_1, \ldots, \phi_n) \in w$ then there exist witnessed maximal consistent sets u_i such that $R_{\sigma}wu_1 \ldots u_n$ and $\phi_i \in u_i$ for any $i \in [n]$.

Proof. Let $\sigma(\phi_1, \ldots, \phi_n) \in w$, then $@_j^s \sigma(\phi_1, \ldots, \phi_n) \in \Gamma_s$, but Γ_s is pasted(then 1 - pasted), so there exists k_1 a nominal of sort s_1 such that $@_j^s \sigma(\phi_1 \wedge k_1, \ldots, \phi_n) \in \Gamma_s$, so $\sigma(\phi_1 \wedge k_1, \ldots, \phi_n) \in \Delta_j = w$. We want to prove that $\Delta_{k_1}, \ldots, \Delta_{k_n}$ are suitable choices for u_1, \ldots, u_n .

Let $\psi_1 \in \Delta_{k_1}$. Then $@_{k_1}\psi_1 \in \Gamma_s$ and by agreement property we get $@_{k_1}\psi_1 \in \Delta_j$. But $|\frac{s}{k_1} \wedge \psi_1 \rightarrow @_{k_1}\psi_1$ (instance of (Intro) axiom), and by modal reasoning we get $\sigma(@_{k_1}\psi_1, \phi_2, \ldots, \phi_n) \in \Delta_j$. From (Back) axiom, $@_{k_1}\psi_1 \in \Delta_j$ and by using the agreement property, $@_{k_1}\psi_1 \in \Gamma_s$. Hence, $\psi_1 \in \Delta_{k_1}$.

Now, $\sigma(\psi_1, \phi_2, \dots, \phi_n) \in \Delta_j$, then $@_j \sigma(\psi_1, \phi_2, \dots, \phi_n) \in \Gamma_s$, but the set is pasted, then exists k_2 a nominal of sort s_2 such that $@_j \sigma(\psi_1, k_2 \land \phi_2, \phi_3, \dots, \phi_n) \in \Gamma_s$. Then $\sigma(\psi_1, k_2 \land \phi_2, \phi_3, \dots, \phi_n) \in \Delta_j$.

Let $\psi_2 \in \Delta_{k_2}$. Then $@_{k_2}\psi_2 \in \Gamma_s$ and by agreement property we get $@_{k_2}\psi_2 \in \Delta_j$. But $|\underline{s} k_2 \wedge \psi_2 \to @_{k_2}\psi_2|$ (instance of (Intro) axiom), and by modal reasoning we get $\sigma(\psi_1, @_{k_2}\psi_2, \phi_3, \ldots, \phi_n) \in \Delta_j$. From (Back) axiom, $@_{k_2}\psi_2 \in \Delta_j$ and by using the agreement property, $@_{k_2}\psi_2 \in \Gamma_s$. Hence, $\psi_2 \in \Delta_{k_2}$. Therefore, by induction, we get that $\psi_i \in \Delta_{k_i}$ for any $i \in [n]$. Then $@_{k_i}\psi_i \in \Gamma_s$ if and only if, by agreement property, $@_{k_i}\psi_i \in \Delta_j$. But $\sigma(k_1, \ldots, k_n) \in \Delta_j$ and by using (Bridge), it follows that $\sigma(\psi_1, \ldots, \psi_n) \in \Delta_j$. We proved that for any $i \in [n]$, $\psi_i \in \Delta_{k_i}$ we have $\sigma(\psi_1, \ldots, \psi_n) \in \Delta_j$ and by Definition 7, it follows that $R_{\sigma}\Delta_j\Delta_{k_1}\ldots\Delta_{k_n}$.

Lemma 9 (Truth Lemma). Let \mathcal{M} be an (S, Σ) -model, g an \mathcal{M} -assignment function and w a maximal consistent set. For any sort $s \in S$ and any formula ϕ of sort s, we have:

 $\phi \in w \text{ if and only if } \mathcal{M}, g, w \stackrel{s}{\models} \phi.$

Proof. We make the proof by structural induction on ϕ .

- $\mathcal{M}, g, w \stackrel{s}{\models} a$, where $a \in PROP_s \cup NOM_s$, if and only if $w \in V_s(a)$ if and only if $a \in w$;
- $\mathcal{M}, g, w \models^s x$, where $x \in \text{SVAR}_s$, if and only if $w = g_s(x)$, if and only if $x \in w$;

- $\mathcal{M}, g, w \stackrel{s}{\models} \neg \phi$ if and only if $\mathcal{M}, g, w \not\models \phi$ if and only if $\phi \not\in w$ (inductive hypothesis) if and only if $\neg \phi \in w$ (maximal consistent set);
- $\mathcal{M}, g, w \stackrel{s}{\models} \phi \lor \psi$ if and only if $\mathcal{M}, g, w \stackrel{s}{\models} \phi$ or $\mathcal{M}, g, w \stackrel{s}{\models} \psi$ if and only if $\phi \in w$ or $\psi \in w$ (inductive hypothesis) if and only if $\phi \lor \psi \in w$;
- let $\sigma \in \Sigma_{s_1...s_n,s}$ and $\phi = \sigma(\phi_1, \ldots, \phi_n)$; then $\mathcal{M}, g, w \not\models \sigma(\phi_1, \ldots, \phi_n)$, if and only if for any $i \in [n]$ there exist $u_i \in W_{s_i}$ such that $R_{\sigma}wu_1 \ldots u_n$ and $\mathcal{M}, g, u_i \not\models \phi_i$ if and only if for any $i \in [n]$ there exist $u_i \in W_{s_i}$ such that $\phi_i \in u_i$ and $R_{\sigma}wu_1 \ldots u_n$ (induction hypothesis) if and only if $\sigma(\phi_1, \ldots, \phi_n) \in w$ (using Existence Lemma 8).
- $\mathcal{M}, g, w \models @_z^s \phi$ if and only if $\mathcal{M}, g, \Delta_z \models @_z^s \phi \in \Delta_z$ (by Lemma 4.(3)) if and only if $\phi \in \Delta_z$ (inductive hypothesis) if and only if $@_z^s \phi$ (by Intro axiom together with $z \in \Delta_z$) if and only if $@_z^s \phi \in w$ (by Lemma 4.(2)).

Theorem 10 (Completeness). Every consistent set of formulas is satisfied.

Proof. Let Γ_s be an s-sorted set of formulas. By the Extended Lindenbaum Lemma 6 we can expand it to a named and pasted set Γ_s^+ . By the Truth Lemma 9, the named and natural assignment that Γ_s^+ give rise to satisfy Γ_s at Γ_s^+ .

2.1 Example

Modal logic has traditionally been used for program verification, one of the most remarkable examples being Propositional Dynamic Logic (PDL), which can represent Hoare Logics. In both Hoare Logics and Dynamic Logic programs are verified using axiomatic semantics, while the state transition system is only semantically defined. For a general discussion we refer to [8].

Our many-sorted setting allows us to define both the syntax of a programming language and its evaluation context in the syntactic layer of our logic, and consequently to define its operational semantics. The change of a configuration after the execution of a program is represented as an implication in our logic, the configuration and the programs being formulas of appropriate sorts.

Our goal is to express operational semantics of languages as axioms in this logic, and to make use of such semantics in program verification. We consider here the SMC Machine described by Plotkin [13], we derive a Dynamic Logic set of axioms from its proposed transition semantics, and we argue that this set of axioms can be used to derive Hoare-like assertions regarding functional correctness of programs written in the SMC machine language.

The semantics of the *SMC machine* as laid out by Plotkin consists of a set of transition rules defined between configurations of the form $\langle S, M, C \rangle$, where S is a value stack of intermediate results, M represents the memory, mapping

Syntax Semantics Val::= $Nat \mid Bool$ Natnatural numbers ValStack::= nilVarprogram variables |Val.ValStack|Bool $true \mid false$ AExp $Nat \mid Var$ Mem::= $empty \mid set(Mem, x, n)$ |AExp + AExp|| qet(x,n) |CtrlStackc(AExp)BExp $AExp \leftarrow AExp$ c(BExp)Stmt::= x := AExpc(Stmt) \mid if BExpasgn(x)then Stmt $plus \mid leq$ else StmtVal?while BExp do Stmtc1; c2skip config(ValStack, Mem)::= | Stmt ; Stmt |

Figure 3: Signature

program identifiers to concrete values, and C is the control stack of commands representing the control flow of the program.

Inspired by the *Propositional Dynamic Logic* (PDL) [8], we identify a command from the control stack with a "program" from PDL, and use the ";" operator from PDL to denote stack composition. We define our formulas to stand for configurations of the form config(vs, mem) comprising only a value stack and a memory.

Similarly to PDL, we use the modal operator [_]_: $CtrlStack \times Config \rightarrow Config$ to assert that a configuration formula must hold after executing the commands in the control stack. The axioms defining the dynamic logic semantics of the SMC machine are then formulas of the form $cfg \rightarrow [ctrl]cfg'$ saying that a configuration satisfying cfg must change to one satisfying cfg' after executing ctrl.

In Figure 3, we introduce the signature of our logic as a context-free grammar (CFG) in a BNF-like form. We make use of the established equivalence between CFGs and algebraic signatures (see, e.g., [9]), mapping non-terminals to sorts and CFG productions to operation symbols. Note that, due to non-terminal renamings (e.g., Exp ::= Int), it may seem that our syntax relies on sub-sorting. However, this is done for readability reasons only. The renaming of non-terminals in syntax can be thought of as syntactic sugar for defining injection functions. For example, Exp ::= Int can be thought of as Exp ::= int2Exp(Int), and all occurrences of an integer term in a context in which an expression is expected could be wrapped by the int2Exp function.

The sorts CtrlStack and Config correspond to "programs" and "formulas" from PDL, respectively. Therefore the usual operations of dynamic logic; (composition), \cup (reunion), * (repetition), [_]_ are defined accordingly [8, Chapter 5]. We depart from PDL with the definition of "?" (test): in our setting, in order to take a decision, we test the top value of the value stack. Consequently, the

signature of the test operator is $?: Val \rightarrow CtrlStack$.

We are ready to define our axioms. For the rest of the paper, whenever ϕ is a theorem of sort s, i.e. $\vdash^s \phi$, we will simply write $\vdash \phi$, since the sort s can be easily inferred.

PDL-inspired axioms. The first group of axioms is inspired by the axioms of PDL [8, Chapter 5.5]. π , π' are formulas of sort CtrlStack ("programs"), γ is a formula of sort Config (the analogue of "formulas" from PDL), v and v' are variables of sort Var, vs has the sort ValStack and mem has the sort Mem.

```
(A \cup) \qquad [\pi \cup \pi']\gamma \leftrightarrow [\pi]\gamma \wedge [\pi']\gamma
(A;) \qquad [\pi; \pi']\gamma \leftrightarrow [\pi][\pi']\gamma
(A^*) \qquad [\pi^*]\gamma \leftrightarrow \gamma \wedge [\pi][\pi^*]\gamma
(A?) \qquad config(v \cdot vs, mem) \rightarrow [v?]config(vs, mem)
(A \neg ?) \qquad config(v \cdot vs, mem) \rightarrow [v'?]\gamma \text{ where } v \text{ and } v' \text{ are distinct.}
```

SMC-inspired axioms. Next, we encode the transition system of the SMC machine as a set of axioms. Apart from the axioms for memory (which are straight-forward), we follow the rules of the SMC machine as closely as allowed by the formalism, using the same notation as in [13]. The sort of each variable can be easily deduced.

```
(CStmt)
                c(s1;s2) \leftrightarrow c(s1); c(s2)
(AMem0)
               empty \rightarrow qet(x,0)
(AMem1)
                set(mem, x, n) \rightarrow get(x, n)
(AMem2)
                set(set(mem, x, n), y, m) \leftrightarrow set(set(mem, y, m), x, n)
                where x and y are distinct
(AMem3)
                set(set(mem, x, n), x, m) \rightarrow set(mem, x, m)
(Aint)
                config(vs, mem) \rightarrow [c(n)]config(n \cdot vs, mem)
                where n is an integer
                config(vs, set(mem, x, n)) \rightarrow [c(x)]config(n \cdot vs, set(mem, x, n))
(Aid)
(Dplus)
                c(a1+a2) \leftrightarrow c(a1); c(a2); plus
                config(n2 \cdot n1 \cdot vs, mem) \rightarrow [plus]config(n \cdot vs, mem)
(Aplus)
                where n is n1 + n2
(Dleq)
                c(a1 \le a2) \leftrightarrow c(a2); c(a1); leq
(Aleq)
                config(n1 \cdot n2 \cdot vs, mem) \rightarrow [leq]config(t \cdot vs, mem)
                where t is the truth value of n1 \le n2
(Askip)
                \gamma \to [c(\mathtt{skip})]\gamma
(Dasgn)
                c(x := a) \leftrightarrow c(a); asgn(x)
(Aasgn)
                config(n \cdot vs, mem) \rightarrow [asgn(x)]config(vs, set(mem, x, n))
                c(\texttt{if }b\texttt{ then }s1\texttt{ else }s2)\leftrightarrow c(b);((true?;c(s1))\cup(false?;c(s2)))
(Dif)
(Dwhile)
                c(\text{while } b \text{ do } s) \leftrightarrow c(b); (true?; c(s); c(b))^*; false?
```

The system $\mathcal{H}_{\Sigma}(\mathbb{Q}_z)$ presented in this paper can be used to certify executions, but we still cannot perform symbolic verification similarly with the system presented in [10].

We conclude by a simple example formalizing and stating a formula which can be proven by deduction in our logic. Let pgm be the following program

```
i1:= 1; i2:= 2; if i1<=i2 then m:= i1 else m:= i2
```

Note that pgm is a formula of sort Stmt in our logic, m is a formula of sort Var and 1 is a formula of sort Nat. For this formula we have proved in [10] the following property:

```
(P_{pgm}) \vdash config(vs, mem) \rightarrow [c(pgm)]config(vs, mem') \text{ implies}
\stackrel{Mem}{=} mem' \rightarrow get(m, 1)
```

for any mem, mem' of sort Mem and vs of sort ValStack.

Which, can be read in plain English as: after executing pgm the value of the program variable m (in memory) will be 1, and the value stack will be the same as before the execution.

But $\mathcal{H}_{\Sigma}(\mathbb{Q}_z)$ is an enriched system with the satisfaction operator and we will show that for this system we can prove the following property:

```
(P') config(vs, mem) \rightarrow [c(pgm)]@_{mem'} get(m, 1)
```

In [10] we have already proved that:

```
\vdash config(vs, mem) \rightarrow [c(pgm)]config(vs, set(set(set(mem, i2, 2), i1, 1), m, 1))
```

But in order to carry on with the proof of the new property, we need to add a new axiom for the constructor config in order to perform unification:

(NoConfusion)
$$config(\phi_1, \psi_1) \wedge config(\phi_2, \psi_2) \rightarrow config(\phi_1 \wedge \phi_2, \psi_1 \wedge \psi_2)$$

We refer to [14] for a general discussion.

Due to lack of space and in order to ease understanding, from this point on we will use the following notation: mf = set(set(set(mem, i2, 2), i1, 1), m, 1)

```
Proof of (P'):
```

```
(1) \ config(vs, mem) \rightarrow [c(pgm)] config(vs, mf)
```

- (2) $config(vs, mem) \rightarrow [c(pgm)]config(vs, mem')$
- (3) $config(vs, mem) \rightarrow ([c(pgm)]config(vs, mf) \land [c(pgm)]config(vs, mem'))$ PL:(1),(2)

```
(4) ([c(pgm)]config(vs, mf) \land [c(pgm)]config(vs, mem')) \rightarrow [c(pgm)](config(vs, mf) \land config(vs, mem')) ML
```

(5)
$$(config(vs, mf) \land config(vs, mem')) \rightarrow config(vs \land vs, mf \land mem')$$

 $(NoConfusion)$

(6) $config(vs \land vs, mf \land mem') \rightarrow config(vs \land vs, @_{mem'} mf)$

(Intro), ML

$$(7) config(vs \land vs, @_{mem'} mf) \rightarrow @_{mem'} mf$$

$$(Back)$$

$$(8) [c(pgm)](config(vs \land vs, @_{mem'} mf) \to @_{mem'} mf)$$
 (UG)

$$(9) [c(pgm)] config(vs \land vs, @_{mem'} \ mf) \rightarrow [c(pgm)] @_{mem'} \ mf \qquad (K_{\sigma}), (MP)$$

(10)
$$mf \to get(m,1)$$
 (AMem2)

(11)
$$@_{mem'} mf \rightarrow @_{mem'} get(m,1)$$

$$ML:(10)$$

$$(12) [c(pgm)] @_{mem'} mf \rightarrow [c(pgm)] @_{mem'} get(m,1)$$
 (UG),(K_{\sigma}),(MP)

- (13) $[c(pgm)](config(vs, mf) \land config(vs, mem')) \rightarrow [c(pgm)]config(vs \land vs, mf \land mem')$ (UG):(5), $(K_{\sigma}), (MP)$
- (14) $[c(pgm)]config(vs \land vs, mf \land mem') \rightarrow [c(pgm)]config(vs \land vs, @_{mem'} mf)$ (UG):(6), (K_{σ}) , (MP)
- $(15) \ config(vs, mem) \rightarrow [c(pgm)]@_{mem'} \ get(m, 1) \ PL:(3),(4),(13),(14),(9),(12)$

3 The many-sorted hybrid modal logic $\mathcal{H}_{\Sigma}(\forall)$

The hybridization of our many-sorted modal logic is developed using a combination of ideas and techniques from [1, 2, 3, 5, 6, 7], but for this section we drew our inspiration mainly from [3].

Hybrid logic is defined on top of modal logic by adding *nominals*, *states variables* and specific binders. This is a first step towards employing the procedure of hybridization on top of the many-sorted polyadic modal logic. The main idea was to define a general logical system that is powerful enough to represent both the programs and their semantics in an uniform way.

Once again , the sorts will be denoted by s,t,\ldots and by PROP = $\{PROP_s\}_{s\in S}$, NOM = $\{NOM_s\}_{s\in S}$ and SVAR = $\{SVAR_s\}_{s\in S}$ we will denote the same countable S-sorted sets presented in Section 2.

Definition 11 $(\mathcal{H}_{\Sigma}(\forall))$ formulas). For any sort $s \in S$ we define the formulas of sort s:

$$\phi_s := p \mid j \mid y_s \mid \neg \phi_s \mid \phi_s \vee \phi_s \mid \sigma(\phi_{s_1}, \dots, \phi_{s_n})_s \mid \forall x_t \phi_s$$

Here, $p \in PROP_s$, $j \in NOM_s$, $t \in S$, $x \in SVAR_t$, $y \in SVAR_s$ and $\sigma \in \Sigma_{s_1 \dots s_n, s}$. We also define the dual binder \exists . For any $s, t \in S$, if ϕ is a formula of sort s and x is a state variable of sort t, then

$$\exists x \, \phi := \neg \forall x \, \neg \phi \text{ is a formula of sort } s.$$

The notions of free state variables and bound state variables are defined as usual

Given a model $\mathcal{M}=(W,(R_{\sigma})_{\sigma\in\Sigma},V)$, then $g: \mathrm{SVAR}\to W$ is an assignment is an S-sorted function. If g and g' are assignment functions $s\in S$ and $x\in \mathrm{SVAR}_s$ then we say that g' is an x-variant of g (and we write $g'\overset{x}{\sim}g$) if $g_t=g'_t$ for $t\neq s\in S$ and $g_s(y)=g'_s(y)$ for any $y\in \mathrm{SVAR}_s,\ y\neq x$.

The satisfaction relation is defined similar with the one in \mathcal{K}_{Σ} , but we only need to add the definition for binders:

$$\mathcal{M}, g, w \stackrel{s}{\models} \forall x \phi$$
, if and only if $\mathcal{M}, g', w \stackrel{s}{\models} \phi$ for all $g' \stackrel{x}{\sim} g$.

Consequently, $\mathcal{M}, g, w \models \exists x \phi$, if and only if $\exists g'(g' \stackrel{x}{\sim} g \text{ and } \mathcal{M}, g', w \models \phi)$. In order to define the axioms of our system, one more definition is needed. We assume #_be a new propositional variable of sort s and we inductively

We assume $\#_s$ be a new propositional variable of sort s and we inductively define $NC = \{NC_s\}_s$ by

•
$$\#_s, \top_s \in NC_s$$
 for any $s \in S$

• if $\sigma \in \Sigma_{s_1 \cdots s_n, s}$ and $\eta_i \in NC_{s_i}$ for any $i \in [n]$ then $\sigma(\eta_1, \dots, \eta_n) \in NC_s$.

We further define $NomC = \{NomC_s\}_{s \in S}$ such that $\eta \in NomC_s$ iff $\eta \in NC_s$ and $|\{\#_s \mid s \in S, \#_s \in \eta\}| = 1$. If $\eta \in NomC_s$ then η^{\square} is its dual and $\eta(\varphi) := \eta[\varphi/\#_{s'}]$.

Remark 12. If $\eta \in NomC_s$ and $\varphi \in Form_{s'}$ then $\mathcal{M}, g, w \models \eta(\varphi)$ iff $\mathcal{M}, h, w' \models \varphi$ for some w' in the submodel generated by \mathcal{X} where $\mathcal{X}_s = \{w\}$ and $\mathcal{X}_t = \emptyset$ for $t \neq s$. Dually, $\mathcal{M}, g, w \models \eta^{\square}(\varphi)$ iff $\mathcal{M}, h, w' \models \varphi$ for any w' in the submodel generated by \mathcal{X} .

The deductive system is presented in Figure 4.

The system $\mathcal{H}_{\Sigma}(\forall)$

- The axioms and the deduction rules of \mathcal{K}_{Σ}
- Axiom schemes: for any $\sigma \in \Sigma_{s_1 \dots s_n, s}$ and for any formulas $\phi_1, \dots, \phi_n, \phi, \psi$ of appropriate sorts, the following formulas are axioms:

```
 \begin{array}{ll} (Q1) & \forall x \, (\phi \to \psi) \to (\phi \to \forall x \, \psi) \text{ where } \phi \text{ contains no free occurrences of } \mathbf{x} \\ (Q2) & \forall x \, \phi \to \phi[y/x] \text{ where } y \text{ is substitutable for } x \text{ in } \phi \\ (Name) & \exists x \, x \\ (Barcan) & \forall x \, \sigma^{\square}(\phi_1, \ldots, \phi_n) \to \sigma^{\square}(\phi_1, \ldots, \forall x \phi_i, \ldots, \phi_n) \\ (Nom) & \forall x \, [\eta(x \land \phi) \to \theta^{\square}(x \to \phi)], \\ & \text{for any } s \in S, \, \eta \text{ and } \theta \in NomC_s, \, x \in \text{SVAR}_{s'} \\ \end{array}
```

• Deduction rules:

(Gen) if $|s| \phi$ then $|s| \forall x \phi$, where $\phi \in Form_s$ and $x \in SVAR_t$ for some $t \in S$.

Figure 4: (S, Σ) hybrid logic

In [12] we have proved the soundness and completeness of the $\mathcal{H}_{\Sigma}(\forall)$ system.

4 The many-sorted hybrid modal logic $\mathcal{H}_{\Sigma}(@_{z}, \forall)$

In [11], given a concrete language with a concrete SMC-inspired operational semantics, we showed how to define a corresponding (sound and complete) logical system and we also proved (rather general) results that allow us to perform Hoare-style verification. Our approach was to define the weakest system that allowed us to reach our goals. For that, we needed to define the satisfaction operator only on nominals.

Furthermore, in [12], in order to establish the connection with Matching logic, we have introduced the $\mathcal{H}_{\Sigma}(@_z, \forall)$ system which allows the satisfaction operators $@_z$ to also range over state variables, not just over nominals.

Therefore, let (S, Σ) be a many-sorted signature. As already announced, in this section we extend the system $\mathcal{H}_{\Sigma}(\forall)$ previously defined by adding the

satisfaction operators $@_z^s$ where $s \in S$ and z is a *state symbol*, i.e. a nominal or a state variable.

The formulas of $\mathcal{H}_{\Sigma}(\mathbb{Q}_z, \forall)$ are defined as follows:

$$\phi_s := p \mid j \mid y_s \mid \neg \phi_s \mid \phi_s \vee \phi_s \mid \sigma(\phi_{s_1}, \dots, \phi_{s_n})_s \mid \forall x_t \phi_s \mid @_z^s \psi_t$$

Here, $p \in PROP_s$, $j \in NOM_s$, $t \in S$, $x \in SVAR_t$, $y \in SVAR_s$, $\sigma \in \Sigma_{s_1 \cdots s_n, s}$, z is a state symbol of sort t and ψ is a formula of sort t.

The satisfaction relation is defined similar with the one in $\mathcal{H}_{\Sigma}(\forall)$, but we only need to add the definition for $@_z$: $\mathcal{M}, g, w \models @_z^s \phi$ if and only if $\mathcal{M}, g, Den_g(z) \models \phi$ where z is a state symbol of sort t and ϕ is a formula of the same sort t. Here, $Den_g(z)$ is the denotation of the state symbol z of sort s in a model \mathcal{M} with an assignment function g, where $Den_g(z) = V_s(z)$ if z is a nominal, and $Den_g(z) = g_s(z)$ if z is a state variable.

The deductive system is presented in Figure 5.

We have proved the soundness and completeness of the $\mathcal{H}_{\Sigma}(@_z, \forall)$ -system in [12].

5 Standard Translation

Next, we will talk about the relationship between modal and classical logic. We first specify our correspondence language, more precisely, the language we will translate our modal formulas to.

Recall that in our many-sorted polyadic modal logic we have $\tau = (S, \Sigma)$ a many-sorted signature, where the sorts are denoted by s, t, \ldots and by PROP = $\{PROP_s\}_{s \in S}$, NOM = $\{NOM_s\}_{s \in S}$ and SVAR = $\{SVAR_s\}_{s \in S}$ the well known S-sorted sets.

We introduce the notation $ar(\sigma)$ which denotes not just the arity of the many-sorted modal operator σ , but also the sort of the arguments, where $ar(\sigma) = \langle s_1 \dots s_n, s \rangle$.

Let us take a look at an (S, Σ) -model $\mathcal{M} = (W, \{R_{\sigma}\}_{\sigma \in \Sigma}, V)$ which is a relational structure where W can be seen as a domain of quantification, each R_{σ} a relation over this domain, and $V_s(p)$ is a unary relation for each $p \in \operatorname{PROP}_s$. On the other hand, if we talk about this model $\mathcal{M} = (W, \{R_{\sigma}\}_{\sigma \in \Sigma}, V)$ using first-order logic we will make use of a first-order language with a relation symbol R_{σ} for each $\sigma \in \Sigma$, and a unary relation symbol (predicate) P_p for every $p \in \operatorname{PROP}_s$.

For the correspondence language in First-Order Logic (FOL) we will define:

$$\mathcal{L}_{\tau} := \mathcal{L}_{\tau}(PROP, NOM, SVAR) := \{=\} \cup \{P_p \mid p \in PROP_s\}_{s \in S} \cup \{R_{\sigma} \mid \sigma \in \Sigma\}.$$

Therefore, we consider \mathcal{L}_{τ} the first-order language with equality which has unary predicates P_p corresponding to the propositional letters $p \in PROP_s$ where $ar(P_p) = \langle s \rangle$ if and only if $p \in PROP_s$. We add the (n+1)-ary relation symbol R_{σ} for each n-ary many-sorted modal operator σ , and we consider that $ar(R_{\sigma}) = \langle ss_1 \dots s_n \rangle$ if and only if $ar(\sigma) = \langle s_1 \dots s_n, s \rangle$.

The system $\mathcal{H}_{\Sigma}(@_z, \forall)$

- The axioms and the deduction rules of \mathcal{K}_{Σ}
- Axiom schemes: any formula of the following form is an axiom, where s, s', t are sorts, $\sigma \in \Sigma_{s_1 \dots s_n, s}, \phi, \psi, \phi_1, \dots, \phi_n$ are formulas (when necessary, their sort is marked as a subscript), x is state variable and y, z are state symbols:

```
(K@) \quad @_z^s(\phi_t \to \psi_t) \to (@_z^s\phi \to @_z^s\psi)
(SelfDual) \quad @_z^s\phi_t \leftrightarrow \neg @_z^s \neg \phi_t
(Intro) \quad z \to (\phi_s \leftrightarrow @_z^s\phi_s)
(Agree) \quad @_y^t @_z^t \phi_s \leftrightarrow @_z^t \phi_s
(Ref) \quad @_z^s z_t
(Back) \quad \sigma(\dots, \phi_{i-1}, @_z^{s_i} \psi_t, \phi_{i+1}, \dots)_s \to @_z^s \psi_t
(Q1) \quad \forall x \, (\phi \to \psi) \to (\phi \to \forall x \, \psi) \text{ where } \phi \text{ contains no free occurrences of } x
(Q2) \quad \forall x \, \phi \to \phi[y/x] \text{ where } y \text{ is substitutable for } x \text{ in } \phi
(Name) \quad \exists x \, x
(Barcan) \quad \forall x \, \sigma^{\square}(\phi_1, \dots, \phi_n) \to \sigma^{\square}(\phi_1, \dots, \forall x \phi_i, \dots, \phi_n)
(Barcan@) \quad \forall x \, @_z\phi \to @_z \forall x \, \phi, \text{ where } x \neq z
(Nom x) \quad @_zx \land @_yx \to @_zy
```

• Deduction rules:

```
(BroadcastS) \quad \text{if } | \overset{s}{=} @_z^s \phi_t \text{ then } | \overset{s'}{=} @_z^{s'} \phi_t \\ (Gen@) \quad \text{if } | \overset{s'}{=} \phi \text{ then } | \overset{s}{=} @_z \phi, \text{ where } z \text{ and } \phi \text{ have the same sort } s' \\ (Paste0) \quad \text{if } | \overset{s}{=} @_z^s (y \wedge \phi) \to \psi \text{ then } | \overset{s}{=} @_z \phi \to \psi \\ \quad \text{where } z \text{ is distinct from } y \text{ that does not occur in } \phi \text{ or } \psi \\ (Paste1) \quad \text{if } | \overset{s}{=} @_z^s \sigma(\dots, y \wedge \phi, \dots) \to \psi \text{ then } | \overset{s}{=} @_z^s \sigma(\dots, \phi, \dots) \to \psi \\ \quad \text{where } z \text{ is distinct from } y \text{ that does not occur in } \phi \text{ or } \psi \\ (Gen) \quad \text{if } | \overset{s}{=} \phi \text{ then } | \overset{s}{=} \forall x \phi, \\ \quad \text{where } \phi \in Form_s \text{ and } x \in \text{SVAR}_t \text{ for some } t \in S.
```

Figure 5: (S, Σ) hybrid logic

Recall that a model in many-sorted polyadic modal logic is defined by $\mathcal{M} = (W, R_{\sigma}, V)$ where $V: PROP \to W$. For each model in our logic we define the corresponding one by $\mathcal{M} = (W, R_{\sigma}, P_p)$. We use the same modal relation R_{σ} to interpret the relation symbol R_{σ} in FOL, and the set $V_s(p)$ to interpret the unary predicates P_p . As emphasized in [5], there is no mathematical distinction between modal and first-order models; because both modal and first-order models are relational structures. Given the construction of out logic on top of modal logic, we can also transfer this feature when talking about the relation between our logic and FOL. Moreover, we use the S-sorted set

 $VFOL = \{VFOL_s\}_{s \in S}$ for the set of first-order variables.

Definition 13. Let x be a first-order variable. The standard translation ST_x taking modal formulas to first-order formulas in \mathcal{M}_{τ} is defined as follows:

- $ST_x(p) = P_p(x)$, where $p \in PROP_s$
- $ST_x(y) = (x = y)$, where $y \in SVAR_s$
- $ST_x(j) = (x = c_j)$, where $j \in NOM_s$
- $ST_x(\sigma(\phi_1,\ldots,\phi_n)) = \exists y_1\ldots\exists y_n(R_\sigma xy1\ldots y_n \wedge ST_{y_1}(\phi_1)\wedge\ldots ST_{y_n}(\phi_n)),$ where y_1,\ldots,y_n are fresh variables, that is, variables that have not been used so far in the translation.
- $ST_x(@_y^s\phi) = ST_y(\phi)$
- $ST_x(\exists y\phi) = \exists yST_x(\phi)$

That is, the standard translation maps proposition symbols to unary predicates (that is $P_p(x)$ is true when p holds in world x), commutes with booleans, and handles σ by explicit first-order quantification over R_{σ} -accessible points. The variables y_1, \ldots, y_n that are used in the clauses for σ are chosen to be any new variables, ones that has not been used so far in the translation. Please notice that we are using the same set of symbols for state variables and first-order variables. Moreover, for each nominal $j \in NOM_s$, we introduce a corresponding constant c_j in the first-order language in order to translate the nominals into. Also, the satisfaction operators are translated by substituting the relevant first-order constant for the free-variable x. Note that this translation returns first-order formulas with at most one free variable x, not exactly one. This is because a constant may be substituted for the free occurrence of x. For example, the hybrid formula $\mathbb{Q}_s^s j$ translates into the first-order sentence j = j.

The truth of a formula of \mathcal{L}_{τ} in a structure \mathcal{M} , relative to an assignment function $g: SVAR \to W$ is given in the classical way. We can write $\mathcal{M} \models_{FOL} ST_x(\phi)[x \leftarrow w]$ which means that the first-order formula $ST_x(\phi)$ is satisfied in the usual sense of first-order logic in the model \mathcal{M} when w is assigned to the free variable x. By assigning a value to the free variable , which gives the internal perspective representative for modal logic, we can evaluate a formula inside a model at a certain point.

Proposition 14 (Local and Global Correspondence on Models). Let (S, Σ) be a many-sorted signature and ϕ a formula of sort $s \in S$.

- 1) For all (S, Σ) -models \mathcal{M} and all states w of \mathcal{M} : $\mathcal{M}, w \stackrel{s}{\models} \phi$ if and only if $\mathcal{M} \models_{FOL} ST_x(\phi)[x \leftarrow w]$
- 2) For all (S, Σ) -models \mathcal{M} : $\mathcal{M} \stackrel{s}{\models} \phi \text{ if and only if } \mathcal{M} \models_{FOL} \forall x ST_x(\phi).$

Proof. 1)By structural induction over ϕ .

Let $\mathcal{M} \models_{FOL} ST_x(p)[x \leftarrow w]$ if and only if $\mathcal{M} \models_{FOL} P_p(x)[x \leftarrow w]$ if and only if $w \in P_p$ if and only if $w \in V_s(p)$ if and only if $\mathcal{M}, g, w \not\models p$ for any assignment function g if and only if $\mathcal{M}, w \not\models p$.

Let $\mathcal{M} \models_{FOL} ST_x(y)[x \leftarrow w]$ if and only if $\mathcal{M} \models_{FOL} (x = y)[x \leftarrow w]$ for any g' where g'(y) = w and g'(z') = g(z') for any $z' \neq z$ state variables of sort t and $g_s(z') = g'_s(z')$ for any $s \neq t \in S$ if and only if $\mathcal{M}, g, w \not\models y$ for any g if and only if $\mathcal{M}, w \not\models y$.

Let $\mathcal{M} \models_{FOL} ST_x(j)[x \leftarrow w]$ if and only if $\mathcal{M} \models_{FOL} (x = c_j)[x \leftarrow w]$ if and only if $w = c_j$ for any assignment function g if and only if $w \in V_s(j)$ for any g if and only if $\mathcal{M}, g, w \models j$ for any g if and only if $\mathcal{M}, w \models j$.

Let $\sigma \in \Sigma_{s_1...s_n,s}$. Then $\mathcal{M} \models_{FOL} ST_x(\sigma(\phi_1,\ldots,\phi_n))[x \leftarrow w]$ if and only if $\mathcal{M} \models_{FOL} \exists y_1 \ldots \exists y_n (R_{\sigma}xy_1 \ldots y_n \wedge ST_{y_1}(\phi_1) \wedge \ldots \wedge ST_{y_n}(\phi_n))[x \leftarrow w]$ if and only if there exists $(u_1,\ldots,u_n) \in W_{s_1} \times \cdots \times W_{s_n}$ such that $\mathcal{M} \models_{FOL} (R_{\sigma}xy_1 \ldots y_n \wedge ST_{y_1}(\phi_1) \wedge \ldots \wedge ST_{y_n}(\phi_n))[x \leftarrow w, y_1 \leftarrow u_1,\ldots,y_n \leftarrow u_n]$

if and only if there exists $(u_1, \ldots, u_n) \in W_{s_1} \times \cdots \times W_{s_n}$ such that $R_{\sigma}wu_1 \ldots u_n$ and $\mathcal{M} \models_{FOL} ST_{y_i}(\phi_i)[y_i \leftarrow u_i]$ for any $i \in [n]$ if and only if there exists $(u_1, \ldots, u_n) \in W_{s_1} \times \cdots \times W_{s_n}$ such that $R_{\sigma}wu_1 \ldots u_n$ and $\mathcal{M}, u_i \stackrel{s_i}{\models} \phi_i$ for any $i \in [n]$ (induction hypothesis) if and only if there exists $(u_1, \ldots, u_n) \in W_{s_1} \times \cdots \times W_{s_n}$ such that $R_{\sigma}wu_1 \ldots u_n$ and $\mathcal{M}, g, u_i \stackrel{s_i}{\models} \phi_i$ for any $i \in [n]$ and any g if and only if $\mathcal{M}, g, w \stackrel{s}{\models} \sigma(\phi_1, \ldots, \phi_n)$.

Let z be a state variable of sort $t \in S$. Then $\mathcal{M}, w \models^{s} @_{z}^{s} \phi_{t}$ if and only if $\mathcal{M}, g, w \models^{s} @_{z}^{s} \phi_{t}$ for any g if and only if $\mathcal{M}, g', u \models^{t} \phi_{t}$ for any g' where $g'_{t}(z) = u$ and $g'_{t}(z') = g_{t}(z')$ for any $z' \neq z$ state variables of sort t and $g_{s}(z') = g'_{s}(z')$ for any $s \neq t \in S$ if and only if $\mathcal{M}, u \models^{t} \phi_{t}$ for any u if and only if $\mathcal{M} \models_{FOL} ST_{z}(\phi_{t})[z \leftarrow w]$ for any u if and only if $\mathcal{M} \models_{FOL} ST_{z}(\phi_{t})[x \leftarrow w]$ if and only if $\mathcal{M} \models_{FOL} ST_{z}(@_{z}^{s} \phi_{t})[x \leftarrow w]$.

Let j be a nominal of sort t. Then $\mathcal{M}, w \models @_j^s \phi_t$ if and only if $\mathcal{M}, g, w \models @_j^s \phi_t$ for any g if and only if $\mathcal{M}, g, u \models \phi_t$ for any g and $u \in V_t(j)$ if and only if $\mathcal{M}, u \models \phi_t$ where $u \in V_t(j)$ if and only if $\mathcal{M} \models_{FOL} ST_j(\phi_t)$ if and only if $\mathcal{M} \models_{FOL} ST_j(\phi_t)[x \leftarrow w]$ if and only if $\mathcal{M} \models_{FOL} ST_x(@_j^s \phi_t)[x \leftarrow w]$.

Let $\mathcal{M} \models_{FOL} ST_x(\forall y\phi)[x \leftarrow w]$ if and only if $\mathcal{M} \models_{FOL} (\forall y \ ST_z(\phi))[x \leftarrow w]$ if and only if for any $a \in W_t$, $\mathcal{M} \models_{FOL} ST_x(\phi)[x \leftarrow w, y \leftarrow a]$

2) Let $x \in VFOL_s$. Then $\mathcal{M} \models_{FOL} \forall xST_x(phi)$ if and only if for any $w \in W_s$, $\mathcal{M} \models_{FOL} ST_x(\phi)[x \leftarrow w]$ if and only if for any $w \in W_s$, $\mathcal{M}, w \models^s \phi($ use item 1) of this proposition) if and only if $\mathcal{M} \models^s \phi$.

Thus the standard translation gives us a bridge between many-sorted modal logic and classical logic.

6 Conclusions

Improving over previous work [10, 11, 12], this paper makes the following contributions: (1) We study the @-only fragment of the more general hybrid modal logic proposed in [11, 12], and provide a sound and complete deduction system for it. This logic is important as it is weaker than the full hybrid modal logic and thus it is expected to have better computational properties. Nevertheless, although weaker, we show it can be used to axiomatically express operational semantics and to derive proofs for statements concerning program executions. (2) We provide a standard translation from full hybrid modal logic to first-order logic and prove that it induces both local and global correspondence on models.

Future Work Although the use of quantifiers (particularly existentials) makes for easier to write and express statements about programs, the @ operator can suplement the need for quantification in many cases. Exploring the limits of this capacity seems like an interesting path to follow.

The promise of giving up quantification in favor of just @ is that we sacrifice expresiveness for better computational properties. We would like to find out if that indeed is the case, by investigating decidability results for the @-only fragment of the logic.

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