# Metalevel transformation of strategies

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# Abstract

In the reflective Maude specification language, based on rewriting logic, a strategy language has been introduced to control rule rewriting while avoiding complex and verbose metalevel programs. However, just as multiple levels of reflection are required for some metaprogramming tasks, reflective manipulation and generation of strategies are convenient in multiple situations. Some examples of reflective strategy transformations are presented, which implement special forms of evaluation or extend the strategy language while preserving its advantages.

Keywords: Maude, Rewriting strategies, Reflection, Model checking

# 1. Introduction

Reflection can be intuitively defined as the capacity of a system for reasoning about itself, by representing and manipulating its objects in its own language. Classical examples of reflection can be seen in the coding of first-order arithmetic by Gödel and in universal Turing machines, but reflective metaprogramming features are also provided by many modern programming languages [25]. Rewriting logic [29] and its implementation Maude [10] are reflective languages where important aspects of its own metatheory can be represented [12]. As a result, manipulating, transforming, and analyzing rewriting logic theories specified in Maude can be easily done within Maude. Reflection has been extensively used throughout the history of Maude for specific metalinguistic applications, to extend and prototype new features of the language, and to design formal tools that reason about Maude programs. Significant examples are Full Maude [10, Part II] and the Maude Formal Environment [17]. The former is an extended Maude interpreter written in Maude itself, and the latter allows checking properties like confluence and termination on Maude specifications.

Rewriting systems are executed by successive and independent rule applications where rule and position are chosen nondeterministically. However, it is sometimes convenient to restrict and control how rules are applied either for semantic or efficiency purposes. This can be expressed at a higher level and without modifying the original system by means

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of rewriting strategies [1, 5]. Classical examples are the different reduction strategies of the  $\lambda$ -calculus [3] and those guiding deduction procedures and theorem provers [30, 24]. Moreover, strategies are a useful resource to write compositional rewriting specifications where the concerns of rules and their control are separate [24], so that the same rules can yield different algorithms depending on how they are applied. Some languages of programmable strategies have been developed to express rewriting strategies in an executable form like ELAN [4], Stratego [7] for program transformation, TOM [2],  $\rho \text{Log}$  [28], and Porgy [21] for graph-rewriting. In Maude, the metaprogramming features have been traditionally used to program the control of rules. Since programming metalevel computations is hard for beginners and verbose, an object-level strategy language has been proposed, tested, and finally made available in Maude 3 [10]. Although the strategy language has been introduced to avoid the need for the metalevel, the language itself and its operations have been metarepresented, and users may still resort to the metalevel to analyze strategy specifications and construct strategies depending on metatheoretic information. These transformed strategies can still be used at the object level thanks to the Maude support for interactive interfaces, and be analyzed using verification tools like the model checker for systems controlled by strategies [34, 37].

In this paper, we aim to show through three relatively simple examples the interesting applications and potential of the reflective manipulation and generation of strategies, and the resources offered by the Maude specification language to do so. Metaprogramming strategies is useful in multiple situations where they should be adapted to some input data specification or to the rewriting system being controlled itself. The first example in Section 3 should be understood as an introduction to the tools and the approach proposed in the paper, which can be applied to many other specific problems like this. In this case, given a Maude specification where operators are annotated with some restrictions, we generate a well-known normalization strategy in context-sensitive rewriting [26] that can be applied to any term. The second example in Section 4 describes a general procedure to extend the Maude strategy language with new combinators without losing any of its advantages. A skeleton is provided in order to simplify the task of the extension developer and facilitate the interaction of its users. This is illustrated with two families of operators available in other strategy languages. Finally, the third example in Section 5 presents a framework to specify compositional or agent-based strategy-controlled systems, whose control by a single strategy expression is cumbersome. Separate strategies are assigned to each agent or component, and a global strategy orchestrates their execution (concurrently, by turns, or as desired). Not only we think that these examples illustrate the possibilities of reflective transformations, but they are interesting by themselves. Although reflective languages featuring strategies are not common, this approach could be used in other tools apart from Maude, as it consists of applying the advantages of program transformation to this specific setting.

Section 2 begins by reviewing the basic of rewriting and Maude, before the aforementioned examples are introduced in Sections 3 to 5. Section 6 presents a discussion on related work and the conclusions. Maude 3 can be downloaded from maude.cs.illinois.edu, and its extension with the strategy-aware model checker is available at maude.ucm.es/strategies, as well as all the different examples appearing here.

# 2. Rewriting logic and Maude

Rewriting logic [29] was proposed as a unified model of concurrency extending membership equational logic with nondeterministic and possibly conditional rewriting rules. A rewrite theory  $\mathcal{R} = (\Sigma, E, R)$  consists of a signature  $\Sigma$  of order-sorted operators, i.e. their domains and values are typed with sorts and these sorts are related by a containment partial order relation, a set of equations E, and a set of rewriting rules R. Terms and rule applications are considered modulo equations and also structural axioms like commutativity, associativity, and identity that cannot be naively handled as regular equations due to their reversible nature. Maude [10] is a specification language based on rewriting logic, where rewrite systems can be specified compositionally, executed, and analyzed. Specifications are written in a mathematical-like notation and organized in modules of different kinds: functional modules (**fmod**) represent equational theories with declarations of **sort**s, **subsort** relations, and **op**erators. Beside their signatures, operator declarations may include some attributes between brackets that specify the structural axioms and other features applied to them. Moreover, functional modules may include equations of the form:

$$[\mathbf{c}]\mathbf{eq} \quad l = r \quad [\text{ if } \bigwedge_{i} l_{i} = r_{i} / \land \bigwedge_{i} l'_{i} := r'_{i} / \land \bigwedge_{i} t_{i} : s_{i}].$$

Equations are applied as if they were oriented from left to right on any position where they match.<sup>1</sup> Every variable in r and the condition must occur in the left-hand side lwith some exceptions. Conditions  $l_i = r_i$  are satisfied when these terms, instantiated by the matching substitution, coincide modulo equations and axioms. The same semantics operates on the  $l'_i := r'_i$  conditions, but the term  $l'_i$  may contain free variables that are instantiated by matching and can be used in the following condition clauses. Sortmembership condition fragments  $t_i : s_i$  hold when the instantiated term  $t_i$  belongs to the sort  $s_i$ . For example, the following functional module specifies a list of letters:

```
fmod LLIST-FM is
  protecting NAT .
  sorts Letter List .
  subsort Letter < List .
  ops a b c d e : -> Letter [ctor] .
  op nil : -> List [ctor] .
  op __ : List List -> List [ctor assoc comm id: nil] .
  var L : Letter . var LS : List .
  op length : List -> Nat .
  eq length(nil) = 0 .
  eq length(L LS) = 1 + length(LS) .
endfm
```

<sup>&</sup>lt;sup>1</sup>An unconditional oriented equation or rewriting rule  $l \to r$  is applied to a term t if there is a substitution  $\sigma$  assigning terms to the variables of l and a position p in t whose subterm  $t|_p = \sigma(l)$ . This subterm is then replaced by  $\sigma(r)$ . We say that l matches  $t_p$  and that  $\sigma$  is the matching substitution.

Underscores in operator names mark the holes where arguments are entered, although symbols in prefix notation may omit them. In LLIST-FM, Letter is made a subsort of List, the juxtaposition operator \_\_ is declared associative, commutative, and having nil as identity, and the module NAT is imported. The Maude prelude provides some modules like NAT that specify integer and floating-point numbers, strings, lists, sets, etc. Importation can be done with the keywords protecting, extending, or including that declare whether the definitions of the imported module will be kept unchanged, extended, or modified arbitrarily, respectively. Functional specifications can be executed with the reduce command, which applies the equations exhaustively on the given term.

```
Maude> reduce length(a b c) .
rewrites: 7
result NzNat: 3
```

System modules (mod) are rewrite theories with the addition of rules

$$[\mathbf{c}]\mathbf{r}\mathbf{l} \quad l \Rightarrow r \quad [ \text{ if } C \land \bigwedge_i l_i \Rightarrow r_i ] .$$

Conditional rules may include rewriting conditions in addition to those C available for equations, where terms matching  $r_i$  are searched by rewriting from  $l_i$  with the rules of the module. Every possible match of the left-hand side and the condition yields a different application of the rule. Like equations, all variables in the right-hand side r and the condition must occur in the left-hand side l, except those in the left-hand side of matching conditions and now also in the right-hand side  $r_i$  of rewriting conditions, since these are assigned by matching. Continuing with the example, the following module LLIST-M extends the previous LLIST-FM with two rules.

```
mod LLIST-M is
including LLIST-FM .
var LS : List . var L : Letter .

rl [pop] : LS L => LS .
rl [put] : LS => LS L [nonexec] .
endm
```

Note that the **put** rule contains an unbounded variable L in its right-hand side. What would be a syntax error without the **nonexec** attribute, which excludes the rule from being applied, can be useful when combined with a strategy language able to instantiate this variable. Rules can be executed modulo the equations with the **rewrite** command in the Maude interpreter, which will repeatedly apply the **pop** rule in the previous module.

```
Maude> rewrite a b c .
rewrites: 3
result List: nil
```

This command selects which rules to apply and where according to a fixed criterion described in the Maude manual [10]. This can be seen if the number of consecutive rewrites is limited to one with the [1] modifier.

```
Maude> rewrite [1] a b c .
rewrites: 1
result List: b c
```

The result is **b c** because the **pop** rule has been applied on the subterm **a**, extended by the identity axiom to **nil a**, but it could have also been applied on (**a b**) **c** yielding **a b** as a result. With the strategy language discussed in Section 2.2, strategy modules (**smod**) can be used to specify different ways of applying these rules.

Maude specifications are accurately executable under certain requirements [10], like the confluence and termination of its equations and the coherence of its rules with them, since rule rewrites take place when the term is exhaustively reduced to a normal form with the oriented equations. Confluence and coherence can be easily defined for abstract reduction systems  $(S, \rightarrow)$  where S is a set of states and  $(\rightarrow) \subseteq S \times S$  is a binary relation on them. This relation is *confluent* if for any states  $s, s_1, s_2 \in S$  satisfying  $s \to s_1^* s_1$ and  $s \to s_2$ , there is an  $s' \in S$  such that  $s_1 \to s'$  and  $s_2 \to s'$ , where  $\to s'$  denotes the transitive and reflexive closure of  $\rightarrow$ . A state s is *irreducible* if there is no  $s' \in S$ such that  $s \to s'$ , and we write  $s \to s''$  if  $s \to s''$  and s' is irreducible. Whenever the relation is confluent and  $s \rightarrow s'$ , we say that s' is a normal form of s and it is unique. Moreover, the relation is *terminating* if there is no infinite sequence of states  $(s_k)_{k=0}^{\infty}$ such that  $s_k \to s_{k+1}$ . For confluent and terminating relations, every state has a unique normal form. Given two relations  $\rightarrow_1$  and  $\rightarrow_2$  on S,  $\rightarrow_2$  is *coherent* with  $\rightarrow_1$  if for any  $s, s', u \in S$  such that  $s \to_1^! u$  and  $s \to_2 s'$ , then there are  $u', w \in S$  satisfying  $u \to_2 u'$ ,  $s' \rightarrow_1^! w$  and  $u' \rightarrow_1^! w$ . In other words,  $\rightarrow_2$  is coherent with  $\rightarrow_1$  if we compute steps of  $\rightarrow_2$  modulo  $\rightarrow_1$  by first reducing the state to its normal form by  $\rightarrow_1$  and then applying  $\rightarrow_2$ .

#### 2.1. Reflection and metalevel computations

Rewriting logic is a reflective logic, whose objects and operations can be consistently represented in itself. Maude offers a predefined *universal theory* [10, §17] to metatheoretically represent terms, equations, rules, modules, and so on. Operations like matching, reduction, and rule application can be programmed generically using regular operators and equations. However, Maude provides special operators backed by the object-level implementation in C++ to allow efficient reflective computations. Metarepresentations can in turn be metarepresented and terms be moved between different levels, thus yielding arbitrarily high reflective towers if needed.

This universal theory is specified in META-LEVEL and its imported modules, and it relies on the Qid sort of *quoted identifiers*, arbitrary words prefixed by an apostrophe. A variable X of sort Nat is metarepresented as the quoted identifier 'X:Nat, and the constant 'Nat of sort Qid is 'Nat.Qid. Terms with arguments are represented using the operator  $\_[\_]$  : Qid NeTermList -> Term, like '\_+\_['X:Nat, 's\_['0.Zero]] for X + s 0. Operator declarations, equations, rules, and similar statements are represented as terms with a syntax similar to the object-level reference. For example, the operator + may have a declaration op '\_+\_ : 'Nat 'Nat -> 'Nat [comm assoc] . and be involved in an equation eq '\_+\_['X:Nat, '0.Zero] = 'X:Nat [none] . where the trailing brackets enclose the set of operator or statement attributes, or none if there is none. Metamodules are terms with argument slots like fmod\_is\_sorts\_.\_\_\_\_endfm for each kind of module component. Auxiliary functions getOps, getEqs, getRls, etc., are defined to obtain these components.

Operations are accessible through some *descent functions* like metaMatch for matching, metaApply for rule application, metaReduce for equational reduction, metaRewrite, etc. For instance, metaReduce receives the metarepresentations of a module and a term, and produces a pair containing the metarepresentations of the normal form of the given term and its calculated sort. Other predefined functions allow obtaining the metarepresentation of a term (upTerm) or the object-level term from its metarepresentation (downTerm). The metarepresentation of loaded modules can be obtained with upModule given the module name and a Boolean flag indicating whether a flat version (with all imports resolved) of the module is required, like in upModule('NAT, false). The complete specification of the metalevel is in the Maude prelude and explained in [10, §17].

# 2.2. The Maude strategy language

Strategies have been specified since the beginnings of Maude using the reflective features just explained. Nevertheless, to control rewriting in a more accessible and understandable way, an object-level strategy language was designed, based on that experience with reflective strategies and other strategy languages like ELAN [4] and Stratego [7]. After being prototyped in Full Maude and tested, the language is finally implemented at the C++ level in Maude 3 with new features like compositional and parameterized strategy modules [35]. A strategy expression  $\alpha$  in the language restricts the possible next steps during the rewriting process, and it can be seen as a transformation from an initial term t to the set of terms that this controlled —but not necessarily deterministic rewriting yields as a result. This is what the command **srewrite** t using  $\alpha$  and its depth-first version **dsrewrite** show, by exploring all allowed execution paths.

The application of a rule  $rl[x_1 \leftarrow t_1, \ldots, x_n \leftarrow t_n] \{\alpha_1, \ldots, \alpha_m\}$  is the basic element of the strategy language, referred by its label rl and taking an optional initial substitution. If the rule to be applied includes rewriting conditions, a comma-separated list of strategies must be given between curly brackets to control all of them. Rules are applied anywhere within the term by default, but their application can be restricted to the top with the top( $\alpha$ ) combinator. Moreover, the strategy all applies any labeled or unlabeled rule of the module with the rewrite command semantics. Tests match P s.t. C discard executions when the subject term does not match the pattern term P or satisfy the equational condition C. The match keyword can be changed to amatch to match anywhere within the term, and the condition s.t. C can be omitted if not needed. These elements can be combined with the concatenation  $\alpha;\beta$  that executes  $\beta$  on the results of  $\alpha$ , the disjunction  $\alpha \mid \beta$  whose result comprises the executions allowed by any of its arguments, the iteration  $\alpha *$  and normalization  $\alpha !$  operators that iterate  $\alpha$  any number of times or until no more iterations are possible, and the conditional  $\alpha$ ? $\beta$ : $\gamma$  that evaluates  $\alpha$  and then  $\beta$  on its results, but if  $\alpha$  does not produce any, it executes  $\gamma$  on the initial term. Two constants **idle** and **fail** represent the strategy that produces the initial term as result and the strategy that does not produce any result, respectively. The combinator matchrew P s.t. C by  $x_1$  using  $\alpha_1, \ldots, x_n$  using  $\alpha_n$  allows rewriting selected subterms of the term where it is applied: those terms matched by the variables  $x_1, \ldots, x_n$  in the pattern are rewritten in parallel using  $\alpha_1, \ldots, \alpha_n$ , respectively, and their results replace the matched subterms to produce the **matchrew** results. This initial keyword can also be changed to **amatchrew** like for tests. Moreover, for efficiency reasons, the combinator **one**( $\alpha$ ) evaluates  $\alpha$  only until the first solution is found, if any, which is returned as its result. In addition to these core combinators, the language includes some others that can be defined in terms of the first, like try( $\alpha$ )  $\equiv \alpha$ ? idle : idle, not( $\alpha$ )  $\equiv \alpha$ ?

fail : idle, and test( $\alpha$ )  $\equiv$  not(not( $\alpha$ ). For example, using the LLIST-M module at the beginning of this section, we can execute the following strategy:

```
Maude> srewrite a b c using top(pop) ; top(put[L <- d]) .
Solution 1
rewrites: 2
result List: a b d
No more solutions.</pre>
```

Remember that L is the name of the unbounded variable of the put rule.

Strategy modules allow declaring and defining strategies with a name and any number of arguments. Delimited by the **smod** and **endsm** keywords, they may import modules of any kind and include any statement available in functional and system modules, although a clear separation of the model from its control encourages that only strategy declaration and definitions statements are used. Named strategies can be declared as **strat** name :  $s_1 \cdots s_n @ s$ . with the signature of its arguments and the sort s of the terms to which it is intended to be applied. They are defined with **sd** name  $(p_1, \ldots, p_n) := \alpha$  or **csd** name  $(p_1, \ldots, p_n) := \alpha$  **if** C. if they are conditional, which are only executed when their equational condition C is satisfied. Strategies can be called even recursively in strategy expressions as name  $(t_1, \cdots, t_n)$ . Extending the example module LLIST-M again, a strategy **seq** is defined to append a list of letters to the list on which it is applied:

```
smod LLIST is
protecting LLIST-M .
var LS : List . vars L L' : Letter .
strat seq : List @ List .
sd seq(nil) := idle .
sd seq(L' LS) := top(put[L <- L']) ; seq(LS) .
endsm</pre>
```

Note that L in put[L <- L'] refers to the variable L in the rule that is being instantiated, while L' refers to the strategy argument that decides its value.

The strategy language and strategy modules are also represented at the metalevel, faithfully reproducing the object-level syntax in most cases. Its combinators are specified as terms of the Strategy sort in the META-STRATEGY module. For instance, a simple rule application is denoted as 'label[none]{empty} and a strategy call as 'name[[TL]] with TL a possibly empty list of metarepresented terms. Strategy modules

```
op smod_is_sorts_.____endsm : Header ImportList SortSet
SubsortDeclSet OpDeclSet MembAxSet EquationSet RuleSet
StratDeclSet StratDefSet -> StratModule [ctor ...] .
```

as well as strategy declarations and definitions

```
op sd_:=_[_]. : CallStrategy Strategy AttrSet
                -> StratDefinition [ctor ...].
op csd_:=_if_[_]. : CallStrategy Strategy EqCondition
                AttrSet -> StratDefinition [ctor ...].
```

are specified too, and the commands **srewrite** and **dsrewrite** are accessible through the **metaSrewrite** descent function.

The last argument of the metaSrewrite operator is an index used to enumerate the potentially multiple solutions, until a failure term is obtained. These solutions are provided as pairs {\_,\_} of sort ResultPair containing the metarepresentation of the term and its calculated sort. Following a common notational pattern in Maude, the sort ResultPair? designates a supersort of ResultPair with the additional constant failure to indicate the absence of a result, as explained before.

Another useful descent function for building metalanguage interfaces is metaParse that parses terms on a given module and sort.

On success, it returns a pair with the metarepresentation of the parsed term and its least sort from a list of tokens of sort QidList, which can be obtained from a string using the tokenize function.

In previous prototypes of the Maude strategy language there was nothing like a metalevel of the strategy language, since it was specified within Maude and strategy expressions were directly Maude terms. The language was more easily extensible at the expense of efficiency, since the execution of strategies was implemented in Maude itself.

More details on the language can be found in [10, \$10].

# 2.3. Interactive interfaces

Writing interactive interfaces in Maude is relatively easy, and it is usually done to offer a convenient interface to the logic and semantic frameworks specified in the language. The archetype is Full Maude  $[10, \S15]$ , an extended interpreter written in Maude itself where many features later implemented in C++ have been first tested. The functionality of the Core Maude interpreter is replicated there along with additional features like tuple types and object-oriented modules. Users can also extend Full Maude to include their own features and commands. Moreover, since Maude 3 [14], the interactive capabilities of Maude have increased due to new external objects that allow reading and writing files as well as the standard input and output streams. External objects are an objectoriented mechanism that allows Maude programs to communicate with the outside world, already used in previous versions for Internet sockets. The standard CONFIGURATION module defines an extensible signature for defining objects and messages, which are held in a common soup or multiset where objects read and introduce messages by means of rewriting rules. The command erewrite conducts rewriting of these configurations following an object-fair strategy and handling the messages issued to and by the implicit external objects. In this case, the STD-STREAM module in the file.maude file of the Maude distribution declares the stdin and stdout objects, and the getLine/gotLine and write/wrote messages to read and write to the terminal.

# 2.4. Model checking

Model checking is an automated verification technique that explores all possible executions of a system to verify whether it meets a given specification. This umbrella term comprises different algorithms and multiple variations, but its models are essentially based on annotated transition systems  $\mathcal{K} = (S, \rightarrow, I, AP, \ell)$  known as Kripke structures, where  $\rightarrow$  is a (sometimes labeled) binary relation and AP is a finite set of atomic propositions associated to each state by a labeling function  $\ell : S \rightarrow \mathcal{P}(AP)$ . Properties are usually expressed in terms of these atomic propositions (and possibly on the labels of its transitions) using some temporal logics that include temporal operators describing how they occur in time. Examples of well-known logics are LTL [32], CTL [8], their superset CTL\* [20], and  $\mu$ -calculus [6].

Rewriting systems can be naturally seen as Kripke structures whose states are terms and whose transitions are one-step rule rewrites. Maude specifications can be model checked against LTL properties since its 2.0 version thanks to a builtin model checker [19, 10, §12]. We have extended it for systems controlled by strategies [34], and for the other logics mentioned in the previous paragraph [37]. Since strategies describe a subset or subtree of allowed executions of the model, properties are satisfied by a strategy-controlled model iff they are satisfied on this subset or subtree. The only question remaining is which are the executions described by a Maude strategy language expression. This is answered by a small-step operational semantics, respected by the model checker implementations. For checking properties other that LTL, some external model checkers can be used, including LTSmin [22], through an extensible model-checking interface umaudemc [33] that unifies the interaction and the syntax of the logics. This interface is built over a library that allows accessing Maude objects and operations from Python and other programming languages. We use this library to adapt umaudemc for the various examples in this paper.

#### 3. An introductory example

This section is intended as a preface through a simple example to the metaprogramming resources offered by Maude to manipulate and generate strategies from some data, like the metarepresentation of a Maude module. We will follow the same method that is applied to the more complex examples of the following sections and that can be applied in general for other reflective transformations. In this example, a strategy will be generated to normalize terms while respecting certain constraints that are included in the specification of a rewriting or functional program.

*Context-sensitive rewriting* [26] is a restricted form of term rewriting defined by simple constraints attached to the symbols of the signature that exclude some of their arguments from being rewritten. Maude has builtin support for this kind of restrictions by means of the **strat** and **frozen** attributes.

**op** 
$$f$$
 :  $s_1 \cdots s_n \rightarrow s$  [strat $(i_1 \cdots i_k \ 0)$  frozen $(j_1 \cdots j_l)$ ]

Regarding equational reduction, the evaluation strategy attribute strat specifies a zeroterminated list of argument indices  $i_m \in \{1, \ldots, n\}$  that fix the order in which arguments are reduced before applying equations to the top, while absent arguments are not reduced at all. By default, the evaluation strategy is  $1 \ 2 \ \cdots \ n \ 0$ . Regarding rules, the **frozen**  attribute inhibits rewriting with rules inside a given subset of arguments  $j_m \in \{1, \ldots, n\}$ . These restrictions may prevent non-terminating evaluations, but their direct application is not enough to obtain irreducible terms, for which strategies are needed, as we will see with a lazy programming example. Generating these strategies from the context-sensitive restrictions is the purpose of our introductory metalevel transformation. Let us present first the following functional module [15] that attempts to specify a lazy list of integers:<sup>2</sup>

```
fmod LAZY-LIST is
protecting INT .
sort LazyList .

op nil : -> LazyList [ctor] .
op _:_ : Int LazyList -> LazyList [ctor] .

var E : Int . var N : Nat . var L : LazyList .

op take : Nat LazyList -> LazyList .
eq take(0, L) = nil .
eq take(s(N), E : L) = E : take(N, L) .

op natsFrom : Nat -> LazyList .
eq natsFrom(N) = N : natsFrom(N + 1) .
endfm
```

Even though natsFrom(n) represents an infinite list, containing all natural numbers from n, we would expect that the lazy evaluation of a term like take(3, natsFrom(0)) leads to 0:1:2:nil. However, Maude's reduce command eagerly applies equations in an innermost leftmost manner, so the evaluation of this term will not terminate because of the continuous reduction of the tails in the natsFrom definition. Fortunately, the Maude strat attribute can be used on the \_:\_ operator to avoid reducing the tail of the list, by changing its [ctor] attribute to [ctor strat(1 0)]. However, this context-sensitive restriction limits rewriting too much, and no valid result is still produced:

```
Maude> reduce take(3, natsFrom(0)) .
rewrites: 2
result LazyList: 0 : take(2, natsFrom(0 + 1))
```

In the vocabulary of context-sensitive rewriting, strat and frozen annotations correspond to replacement maps  $\mu : \Sigma \to \mathcal{P}(\mathbb{N})$  where  $\mu(f) \subseteq \{1, \ldots, \operatorname{arity}(f)\}$  for all  $f \in \Sigma$ . Reduction is only allowed in the  $\mu$ -replacing positions of any term, defined recursively as

$$\operatorname{Pos}^{\mu}(f(t_1,\ldots,t_n)) = \{\varepsilon\} \cup \bigcup_{i \in \mu(f)} \{i\} \operatorname{Pos}^{\mu}(t_i),$$

where  $\varepsilon$  denotes the top position and the word *wi* the *i*-th argument of the subterm at position *w*. Exhaustively reducing in these positions yields  $\mu$ -normal forms, exactly what the previous command did for take(3, natsFrom(0)) and  $\mu(\_:\_) = \{1\}$ . As that

<sup>&</sup>lt;sup>2</sup>Natural numbers are represented in Maude using Peano notation with a constant 0 and a successor operator  $s_{-}$ , although numeric literals can also be written as syntactic sugar. Integers include an additional constructor  $-_{-}$ .

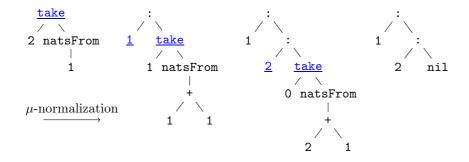


Figure 1: Layered normalization of take(2, natsFrom(1))

execution shows,  $\mu$ -normal forms are not necessarily normal forms of the unrestricted rewrite system, but  $\mu$ -normalization can be useful to build complete and lazy normalization procedures [15]. Normalization can be achieved via  $\mu$ -normalization using a *layered* evaluation that safely resumes reduction on the subterms of non-replacing positions [26, §9.3], as illustrated in Figure 1 for the term take(2, natsFrom(1)). At each step, the highlighted subterms at a given level of the term tree are applied  $\mu$ -normalization, and this continues to its arguments down to the leaves. This procedure is implemented by means of a strategy proposed by Salvador Lucas [27], which would need to traverse the term. Since the Maude strategy language does not offer any resource to do it generically, a signature-aware strategy must be produced.

The following function csrTransform implements a metalevel module transformation that extends the metarepresentation of the input module M with strategy declarations and definitions that normalize terms as described in the previous paragraph. Its global shape is given by the following equation.<sup>3</sup>

```
op csrTransform : Module -> StratModule
eq csrTransform(M) = smod append(getName(M),
                                              'CSR)
                                                      is
  getImports(M)
                                               *** module importation
  sorts 'AnyTerm ; getSorts(M) .
                                                          sort decls
  getSubsorts(M)
                                                        subsort
                                                                decls
  strat2frozen(getOps(M))
                                                                decls
                                                       operator
  getMbs(M)
                                                               axioms
                                              sort
                                                   membership
  none
                                                            equations
  getRls(M)
                                                                rules
  eqs2rls(getEqs(M))
  getStrats(M)
                                                   *** strategy decls
  (strat 'norm-via-munorm : nil @ 'AnyTerm [none] .)
  (strat 'munorm : nil @ 'AnyTerm [none] .)
  (strat 'decomp : nil @ 'AnyTerm [none] .)
                                                strategy definitions
  getSds(M)
  (sd 'norm-via-munorm[[empty]] :=
       'munorm[[empty]] ; 'decomp[[empty]] [none] .)
```

<sup>&</sup>lt;sup>3</sup>Strategy declarations must include the intended sort to which they are applied after the @ sign, although this is merely informative and reasonable candidates do not always exist. Since the strategies defined here are somehow polymorphic, we declare AnyTerm just to take its place.

```
(sd 'munorm[[empty]] := one(all) ! [none] .)
(sd 'decomp[[empty]] := makeDecomp(getOps(M)) [none] .)
endsm .
```

Except for the new strategies, the transformed module is essentially a copy of the original one. However, since the Maude strategy language can only control rule application, we translate all equations into rules,<sup>4</sup> and all strat attributes to frozen annotations.

The transformed module is always a strategy module, regardless of which type of module M is, where three strategies are declared. The entry point for the layered normalization procedure is norm-via-munorm, which executes two auxiliary strategies munorm for  $\mu$ -normalization, and then decomp for resuming normalization inside frozen arguments. munorm is implemented by exhaustively (!) applying the rules in the module respecting the frozen restrictions (all). Assuming the input system is  $\mu$ -confluent, i.e. confluent under the context-sensitive restrictions, the order in which rules are applied does not affect the result, so all is executed for efficiency under the one operator that discards alternative rewrite orders. For its part, the decomp strategy continues normalization on the symbol arguments. Strategies can be applied inside subterms in the Maude strategy language using the matchrew combinator, so one is generated for each  $f \in \Sigma$  to apply norm-via-munorm recursively to every subterm:

```
matchrew f(x_1, \ldots, x_n) by ..., x_i using norm-via-munorm, ...
```

The decomposition strategy decomp is defined as the disjunction of all these combinators. Only the one for the top symbol of the term where it is applied on each occasion will match. Since this definition depends on the signature of the module, decomp is reflectively generated by the makeDecomp function that walks through the operators declared in the module.

```
var Q : Qid . var Ops : OpDeclSet . var N : Nat .
var Ty : Type . var NeTyL : NeTypeList .

op makeDecomp ( opDeclSet -> Strategy .
eq makeDecomp (op Q : nil -> Ty [Attrs] . Ops) =
      (match qid(string(Q) + "." + string(Ty)) s.t. nil)
      | makeDecomp(Ops) .

eq makeDecomp(op Q : NeTyL -> Ty [Attrs] . Ops) =
   (matchrew Q[makeVarList(NeTyL, 1)] s.t. nil
   by makeUsingPart(NeTyL, 1)) | makeDecomp(Ops) .
```

 $<sup>{}^{4}</sup>$ Equations in Maude can be annotated with the **owise** attribute that cause them to be executed at a certain position only if equations without this attribute have failed. Respecting the **owise** semantics would require specifying strategies to apply their translation as rules likewise, but for simplicity we assume that there are no **owise** annotations.

Constants (operators with an empty list nil of arguments) do not have arguments in which normalization should be resumed, but they must also be matched by the decomp strategy so that it does not fail when any of them is encountered.<sup>5</sup> In this case, instead of a matchrew, a test match is used with the metarepresentation of that constant as described in Section 2.1. Auxiliary functions like makeVar and makeVarList are used to generate sequentially-numbered variable metarepresentations of the given sorts for the matchrew pattern. These variables are mapped to the norm-via-munorm strategy by the makeUsingPart function.

```
op makeUsingPart : NeTypeList Nat -> UsingPairSet .
eq makeUsingPart(Ty, N) =
    makeVar(N, Ty) using 'norm-via-munorm[[empty]] .
eq makeUsingPart(Ty NeTyL, N) = makeUsingPart(Ty, N),
    makeUsingPart(NeTyL, s(N)) .
op makeVar : Nat Type -> Variable .
eq makeVar(N, Ty) =
    qid("X" + string(N, 10) + ":" + string(Ty)) .
```

Finally, the term csrTransform(upModule('LAZY-LIST, true)) can be reduced to obtain the transformed 'LAZY-LIST module. Remember that upModule('name, true) evaluates to the flat metarepresentation of the module name where all importations have been resolved. Then, the norm-via-munorm strategy can be applied to a term using the metaSrewrite function, whose inputs and results are written at the metalevel:

Alternatively, Full Maude can be used with terms and strategies at the object level:<sup>6</sup>

```
(select CSR-TRANSFORM .)
(load csrTransform(upModule('LAZY-LIST, true)) .)
(select LAZY-LIST-CSR .)
(srewrite take(3, natsFrom(0)) using norm-via-munorm .)
Solution 1
result LazyList: 0 : 1 : 2 : nil
No more solutions
```

<sup>&</sup>lt;sup>5</sup>Instead of including constants in the disjunction of the **decomp** strategy, we could have surrounded the call to **decomp** with the **try** combinator so that it does not fail when no pattern matches.

<sup>&</sup>lt;sup>6</sup>Full Maude commands are typed between parentheses, once the full-maude.maude file is loaded. Its latest version can be downloaded from maude.cs.illinois.edu.

The evaluation now terminates with a meaningful result. When the rewrite system is confluent and terminating under the restrictions, as in this case, any search strategy, breadthFirst or depthFirst, srewrite or dsrewrite, would produce the same result since there is a single solution and a finite state space. The norm-via-munorm strategy can be used to normalize any term in the module without computing the transformation again. However, since it explicitly refers to the signature of that module, its definition is not applicable to any other module.

The following section shows an extension of the Maude strategy language adding new combinators based on the subject module in which strategies are to be applied. Using that extended language the norm-via-munorm strategy will be defined directly and more succinctly.

#### 4. Theory-dependent extensions of the strategy language

When the Maude strategy language was designed, the objective was not to offer a vast repertory of operators to concisely express a wide range of tasks, like in the case of Stratego [7], but to be compact and expressive enough. Thanks to reflection, the language can be extended to better suit a specific purpose or to incorporate a new feature. In this section, we apply this principle and describe a general schema to construct strategy language extensions by some module transformations without losing any of the advantages of the strategy language, like the interaction at the object level and with the strategy-aware model checker. In particular, the language is extended with the socalled *congruence operators* from ELAN [4] and Stratego, and the *generic traversals* from Stratego. Both operator families depend on the signature of the subject module where strategies are applied, so a module transformation is required to implement them.

The procedure we will follow is directly applicable to other extensions, and it is supported by a collection of helper modules that save work to the extension developer and avoid writing boilerplate code for each extension. It consists of the following steps illustrated in Figure 2:

- 1. Extending the universal theory of the META-LEVEL module with the metarepresentation of the new strategy combinators, probably depending on the module M where strategies are to be applied. This is similar to what we did in Section 3.
- 2. Since the builtin metaSrewrite function does not support the new combinators and in order to execute them, extended expressions are translated to the standard language by extending the skeleton of a transform function. Moreover, the translation allows using the strategy-aware model checker on extended strategies and all other strategy-related machinery of the interpreter for free.
- 3. In order to write extended strategies at the object level, an extensible grammar SLANG-GRAMMAR of the strategy language can be added productions for the new operators. Strategies are parsed as terms with the builtin metaParse function, and transformed to their metarepresentations by an extensible stratParse function.
- 4. A parameterized interactive Maude interface is provided to operate with extended strategies completely at the object level. It admits extended strategies in its **srewrite** command and in the strategy definitions of strategy modules. A umaudemcbased program for model checking LTL, CTL\*, and  $\mu$ -calculus with these strategies and modules is available too (see Section 2.4).

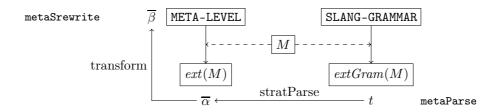


Figure 2: Typical structure of a strategy language extension

As a result, extended strategy expressions can be used almost anywhere an original expression could have been used, although not directly in the commands of the Maude interpreter. Since strategies are translated to the standard strategy language, some extensions are not easily implemented using this approach, as we discuss at the end of the section.

As said, we will exemplify this procedure with two families of operators that are not available in the Maude language. Congruence operators  $f(\alpha_1, \ldots, \alpha_n)$  are strategy combinators that reproduce the data constructors of the target module  $f(t_1, \ldots, t_n)$  with their arguments replaced by strategies, which are applied to the corresponding arguments of the subject term's top symbol if they coincide. In other words, they are semantically equivalent to a **matchrew** construct of the form

# $f(\alpha_1, \ldots, \alpha_n) \equiv \text{matchrew } f(x_1, \ldots, x_n)$ by $x_1$ using $\alpha_1, \ldots, x_n$ using $\alpha_n$ .

On the other hand, generic traversals are operators that allow applying a strategy along the structure of any term without explicitly mentioning it:  $\mathbf{gt-all}(\alpha)$  applies  $\alpha$  to all arguments of the top symbol,  $\mathbf{gt-one}(\alpha)$  applies  $\alpha$  to the first argument from left to right in which it succeeds, and  $\mathbf{gt-some}(\alpha)$  applies  $\alpha$  to as many children as possible and at least to one, so it is equivalent to  $\mathbf{test}(\mathbf{gt-one}(\alpha))$ ;  $\mathbf{gt-all}(\mathbf{try}(\alpha))$ .<sup>7</sup>

Congruence operators. First, we extend the metalevel with the metarepresentations of congruence operators: a homonym symbol of sort Strategy is introduced for each data constructor of M taking as many Strategy arguments as the arity of the original one:

The auxiliary function repeatType builds a list with the given number of repetitions of its first argument, and removeId removes identity axiom attributes of the original operators, which are meaningless in the congruence operators. Overloaded symbols may

 $<sup>^{7}</sup>$ The original names of generic traversal operators in Stratego do not include the gt- prefix, which is used here to avoid confusion with the **one** and **all** operators of the strategy language.

produce different conflicting declarations if their attributes do not coincide, so generated operators are given a second pass to remove potential conflicts.

Extended strategies can now be expressed at the metalevel, but since the builtin **metaSrewrite** is unaware of these new operators and we do not want to implement the strategy language from scratch, we should translate them to the standard subset. This translation is defined in the extended META-LEVEL as a function transform between terms of sort Strategy. The complete recursive definition of this function would be large and repetitive, so an extendable and generic one is supplied to facilitate the task of defining extensions. This is provided by the SLANG-EXTENSION-STATIC module to be included in the transformed module, where equations are given for the standard constructors and the user only has to provide equations for the new elements.

```
fmod SLANG-EXTENSION-STATIC is
protecting META-LEVEL .
op transform : Strategy Nat -> Strategy .
var N : Nat . var S : Strategy .
eq transform(idle, N) = idle .
eq transform(top(S), N) = top(transform(S, N)) .
...
...
```

endfm

The transform operator takes a natural number as a second argument, used as an index to generate fresh variables in nested **matchrews**, since their bindings are permanent. Some helper functions like **makeVar** and **makeConstant** that already appear in the previous example are included in the module too. Note that the equations defining transform are generated by the module transformation, so they must metarepresent Strategy terms and involve two levels of reflection.

```
op generateCongOpsDefs : OpDeclSet -> EquationSet .
eq generateCongOpsDefs(none) = none .
eq generateCongOpsDefs(op Q : NeTyL -> Ty [ctor Attrs] . Ops) =
  (eq 'transform[Q[makeStratVars(size(NeTyL))], 'N:Nat] =
    'matchrew_s.t._by_[
      '_'[_'][upTerm(Q), 'makeOpVars[upTerm(NeTyL), 'N:Nat]],
        'nil.EqCondition,
        'makeUsingPairs[upTerm(NeTyL),
           wrapStratList(makeStratVars(size(NeTyL))),
           '_+_['N:Nat, upTerm(size(NeTyL))], 'N:Nat]
    ] [none] .)
  generateCongOpsDefs(Ops) .
eq generateCongOpsDefs(op Q : nil -> Ty [ctor Attrs] . Ops) =
  (eq 'transform[qid(string(Q) + ".Strategy"), 'N:Nat] =
    'match_s.t._[makeConstant(Q, Ty), 'nil.EqCondition] [none] .)
  generateCongOpsDefs(Ops) .
eq generateCongOpsDefs(Op Ops) =
     generateCongOpsDefs(Ops) [owise] .
```

The second equation generates the **matchrew** constructs described at the beginning of the section, and again, simpler **match** tests are used for constants. Variable names are generated from the index passed as the second argument of **transform**, which is increased in recursive calls to ensure that the index is not used again in a subterm. Just like when declaring them, the same congruence operator may receive multiple **transform** equations for different overloaded data constructors, so they are combined afterwards in a strategy disjunction.

```
eq combineCongOpsDefs(Eqs
   (eq 'transform[T, 'N:Nat] = T1 [none] .)
   (eq 'transform[T, 'N:Nat] = T2 [none] .)) =
   combineCongOpsDefs(Eqs
      eq 'transform[T, 'N:Nat] = '_|_[T1, T2] [none] .) .
eq combineCongOpsDefs(Eqs) = Eqs [owise] .
```

Generic traversals. Since the strategy language does not provide the means to perform generic traversals of terms, and since we have chosen to translate extended strategies to standard ones, we should implement generic traversals using module-specific strategies. Namely, we can translate the strategy  $\mathbf{gt-all}(\alpha)$  to the disjunction of  $f(\alpha, \ldots, \alpha)$  for all  $f \in \Sigma$ , and  $\mathbf{gt-one}(\alpha)$  using the disjunction for all f of

```
f(\alpha, \text{ idle}, \ldots, \text{ idle}) or else \cdots or else f(\text{idle}, \text{ idle}, \ldots, \alpha).
```

These still extended strategies are translated to the standard language as explained before. For instance, the strategies in the disjunction to which **gt-all** is translated can be built with the following equations:

```
op generateGTAll : OpDeclSet -> TermList .
eq generateGTAll(none) = empty .
eq generateGTAll(op Q : NeTyL -> Ty [ctor Attrs] . Ops) =
Q[repeatTerm('S:Strategy, size(NeTyL))],
generateGTAll(Ops) .
eq generateGTAll(op Q : nil -> Ty [ctor Attrs] . Ops) =
makeConstant(Q, 'Strategy), generateGTAll(Ops) .
eq generateGTAll(Op Ops) = generateGTAll(Ops) [owise] .
```

The final shape of the extended metalevel with congruence operators and generic traversals is given by the equation below. Generic traversals are defined directly with equations instead of using transform, because they do not explicitly mention any variable indices.

```
eq extendCongOps(M) = fmod append('META-LEVEL, getName(M)) is
(extending 'SLANG-EXTENSION-STATIC .)
sorts none .
none *** subsorts
combineCongOps(generateCongOps(getOps(M)))
(op 'gt-all : 'Strategy -> 'Strategy [none] .)
(op 'gt-one : 'Strategy -> 'Strategy [none] .)
(op 'gt-some : 'Strategy -> 'Strategy [none] .)
none *** membership axioms
combineCongOpsDefs(generateCongOpsDefs(getOps(M)))
(eq 'gt-all['S:Strategy] =
```

```
'_|_[generateGTAll(getOps(M))] [none] .)
(eq 'gt-one['S:Strategy] =
        '_|_[generateGTOne(getOps(M))] [none] .)
(eq 'gt-some['S:Strategy] =
        '_|_[generateGTSome(getOps(M))] [none] .)
endfm .
```

The META-LEVEL module is imported transitively via the already-known module SLANG-EXTENSION-STATIC. The complete specification is available in the strategy language example collection [18].

Object-level usage. Writing extended strategies like f(r1[none]{empty}, gt-all(match '0.Zero s.t. nil)) at the metalevel and executing them with metaSrewrite is possible with what we have introduced so far. However, we want to be able to write them at the object level, like f(r1, gt-all(match 0)), and to use them anywhere a standard strategy can be used, namely, as arguments of the srewrite commands and in strategy definitions within modules. This kind of extensions cannot be directly handled by the Core Maude interpreter, but they can be supported through Full Maude or by custom interactive interfaces [9, 11, §17]. The second option has been chosen.

As stated at the beginning of the section, a parser for the extended strategy language at the object level is required, for which an extensible grammar of the standard strategy language is provided as the SLANG-GRAMMAR in Figure 2. The developer of the extension should complement it with the productions for the new strategy combinators. Moreover, some module-dependent productions available in the standard strategy language (rule labels for their application, sort membership tests, ...) can be added with a function provided in the skeleton. Thus, we should extend SLANG-GRAMMAR as we did with META-LEVEL. Using the predefined parsing function metaParse, extended strategy expressions can be parsed and then translated to their representations in the extended metalevel. The skeleton of a stratParse function to convert the terms parsed by this grammar to the Strategy sort is declared and defined for the standard combinators, like the previous transform, so that its definition only has to be completed for the new ones. This closes the procedure depicted in Figure 2. In addition to the grammar of expressions, a limited grammar of strategy modules is already specified to parse those with extended strategies in their definitions, which are translated to the builtin subset in the transformed module. A parametric object-oriented module specifies an interactive interface with an adapted **srewrite** command, where extended modules can be entered. The interface uses the external objects of Maude 3 for standard input/output communication and the above procedure to parse and execute the strategies.

For example, the layered normalization strategy norm-via-munorm of Section 3 can be specified using a short recursive definition with generic traversals that resumes  $\mu$ normalization in all arguments of the term. The translation from equations (and from strat annotations to frozen annotations) in LAZY-LIST, which was automatically done by the previous transformation, has been manually done here.

```
smod LAZY-LIST-STRAT is
protecting LAZY-LIST-RLS .
strat norm-via-munorm @ LazyList .
sd norm-via-munorm := one(all) ! ; gt-all(norm-via-munorm) .
endsm
```

The strategy definition is completely generic, although it is parsed in the particular LAZY-LIST-RLS module and translated to the standard subset according to it. In fact, the transformed strategy is essentially the same obtained in Section 3.

\*\* Strategy language extensions playground \*\*
SLExt> smod LAZY-LIST-STRAT is ... endsm
Module LAZY-LIST-STRAT is now the current module.
SLExt> srew take(3, natsFrom(0)) using norm-via-munorm .
Solution 1: 0:1:2:nil
No more solutions.

Another example, where congruence operators are used, is the following module that defines two constants a and b, a binary function f, a rule swap that swaps the entries of f, and another rule next that rewrites a to b.

```
mod F00 is
   sort Foo .
   ops a b : -> Foo [ctor] .
   op f : Foo Foo -> Foo [ctor] .
   vars X Y : Foo .
   rl [swap] : f(X, Y) => f(Y, X) .
   rl [next] : a => b .
endm
```

The extended strategy f(swap, gt-all(next)) can then be executed:

```
SLExt> select FOO .
Module FOO is now the current module.
SLExt> srew f(f(a,b), f(a,a)) using f(swap, gt-all(next)) .
Solution 1: f(f(b, a), f(b, b))
No more solutions.
```

Model checking. In addition to the ability of writing strategy expressions at the object level, another feature of the strategy language we want to preserve is the possibility of model checking systems controlled by strategies. Since extended strategies are finally translated into a strategy expression in the standard strategy language of the original module, model checking with extended strategies is straightforward. In the distribution of the language extension skeleton, a simple Python script makes Maude parse and translate the strategies according to the procedure of Figure 2 before passing the problem data to the unified model-checking library umaudemc, where the internal and external model checkers are used to verify LTL, CTL\*, and  $\mu$ -calculus properties (see Section 2.4). However, this procedure could have been done entirely in Maude, if we only want to check LTL properties using the builtin model checker.

Limitations of the approach. This extension procedure can be easily generalized to allow modifications on the subject module where strategies are applied, or to be parametric also on the strategy expression to be evaluated. For example, inline strategy definitions like let  $st(t_1, \ldots, t_n) := \beta$  in  $\alpha$  can be implemented by pulling the strategy definition

in the expression to the target module. However, strategy combinators that are not expressible in the Maude strategy language could not be handled with this approach. Various executable semantics of the strategy language are available to implement other lower-level extensions [18].

# 5. Multistrategies

The strategy-controlled system model proposed in Maude is the combination of a rewrite system and a strategy expression that controls it as a whole. However, many systems are better specified compositionally. Typical examples are object- or agentoriented systems, in which each object or agent would follow its own strategy. Likewise, describing the interaction of players in games with a single sequential strategy control flow is cumbersome. Hence, we propose the following model transformation to facilitate this specification problem. Instead of a single strategy expression  $\alpha$ , the system control will be specified by a *multistrategy*: an undetermined number of strategies  $\alpha_1, \ldots, \alpha_n$  and a global strategy  $\gamma$  that describes how they are combined. Two built in  $\gamma$  are provided: a concurrent one, in which the next strategy to take a step can be any one of them, and a turn-based one, in which strategies are executed in a fixed order. For example, we could provide each agent of an agent-based system with its own strategy  $\alpha_k$  that defines its behavior autonomously. Each strategy can be understood as the program of a concurrent thread of execution, which is interrupted and resumed to allow the interleaved interaction of the agents after every rewrite. Alternatively, we could fix an order and make them be executed in turns, as if they were players in a game. A fundamental question is the amount of atomic work done by a strategy  $\alpha_i$  when it is given control, in other words, the granularity of their interleaving in the global execution. A single rule application is a reasonable atomic step, but a few more strategies are executed atomically like matchrews with a non-trivial pattern and conditions in the conditional operator, since they assume a particular structure or invariant of the term that may not be preserved if another strategy thread modifies the term in the meantime.

Multistrategies are implemented using strategies at the metalevel and an augmented execution environment. Essentially, to evaluate the strategies  $\alpha_1, \ldots, \alpha_n$  on the subject term t, they are transformed into the term {  $\overline{t} :: < 1 \% \overline{\alpha}_1 > \cdots < n \% \overline{\alpha}_n >, \overline{M}$  } that includes the metarepresentation  $\overline{t}$  of the subject term, of the strategies  $\overline{\alpha}_i$ , and of the module  $\overline{M}$  where they are evaluated. The evolution of this execution context is defined by some rules, which modify the strategy representations and execute them according to their semantics, governed by the global strategy  $\gamma$ . The rules that do not alter the subject term (but choose alternatives, expand iterations...) are called *control rules*, and those modifying the term with rules of the underlying system are called *system rules*. These rules and their two categories are directly based on a small-step operational semantics proposed for the strategy language [34]. With control(N) and system(N) being the disjunction of all control and system rules applied to the thread N, global control strategies, like turn(N, M) for executing M strategies in turns starting from the N<sup>th</sup> one and freec to execute them concurrently, can be specified as follows:

vars N M K : Nat . var T : Term . var : Strategy .
var Mod : Module .

```
sd =>>(N) := control(N) * ; system(N) .
sd turns(N, M) := =>>(N) ? turns(s(N) rem M, M) : idle .
sd freec := (matchrew C s.t. { T :: < N by C using =>>(N)) ? freec : idle .
```

The =>> strategy specifies the atomic step of a strategy thread execution as explained before.<sup>8</sup> The definitions of the  $\gamma$  strategies **turns** and **freec** apply this atomic step =>> with indices that are respectively increased cyclically or selected nondeterministically by matching. Note that the **turns** strategy stops when the current strategy is unable to continue, while **freec** halts when all threads are stuck. Custom global strategies can be easily defined for other general or specific purposes. For example, the **freec** strategy can also be bound on the number of steps:

```
sd freec(0) := idle .
sd freec(s(K)) := (matchrew C s.t. { T :: < N := C by C using =>>(N)) ? freec(K) : id
```

Further details on the transformation are discussed in Section 5.2, and the complete commented Maude code is available at [18].

Auxiliary operations and an interactive environment have been prepared to easily execute multistrategies at the object level, and to obtain meaningful counterexample traces when model checking these systems. The interactive environment is similar to that used for the language extensions in Section 4, with a command for rewriting with multistrategies **srewrite** t using  $\alpha_1, \ldots, \alpha_n$  by  $\gamma$ , where  $\gamma$  can be the words **turns** or **concurrent** for the predefined strategies, or **custom** followed with an arbitrary expression. Another command **check**  $\varphi$  **from** t using  $\alpha_1, \ldots, \alpha_n$  by  $\gamma$  checks the LTL property  $\varphi$  on the given multistrategic model. Branching-time properties can also be checked using an external umaudemc-based command line tool.

Let us illustrate the execution of multistrategies with the simple LLIST example of Section 2.2, which specifies a list of letters (a, b, c, ...) that can be appended with a put rule, and a strategy seq that does so with a list of them in order. After loading that module and the interactive interface in multistrat-iface.maude, we can execute multiple seq calls by turns or concurrently:

```
** Multistrategies playground **
```

More interesting examples have been specified using multistrategies [18], including the specification of the Lamport's bakery algorithm and the tic-tac-toe game, where relevant properties are model checked using different combinations of process or player strategies. This latter example is studied in the following section.

<sup>&</sup>lt;sup>8</sup>The concepts of control transition, system transition, and the composed relation  $\Rightarrow \Rightarrow are already present in the small-step operational semantics used to define model checking for systems controlled by the Maude strategy language [34], as both pursue the similar purpose of isolating rule applications.$ 

#### 5.1. Multistrategies playing games: the tic-tac-toe

*Tic-tac-toe* or *noughts and crosses* is a popular game in which two players, circles and crosses, take turns putting their symbols in a 3x3 grid to complete a vertical, horizontal, or diagonal sequence of cells. The first player to achieve the goal is the winner. Tic-tac-toe is a solved game that always ends in a draw if no player makes a mistake. In this section, we will specify a flawless strategy for a player, and use the model checker to prove that it actually is. But before that, the representation of the board and the rules should be specified.

```
fmod TICTACTOE is
protecting NAT .
protecting EXT-BOOL .
sorts Position Player Grid .
ops 0 X - : -> Player [ctor] .
op [_,_,_] : Nat Nat Player -> Grid [ctor] .
op empty : -> Grid [ctor] .
op __ : Grid Grid -> Grid [ctor assoc comm id: empty] .
```

The Grid sort's elements are sets of triples that map a coordinate on the game board to the player that occupies that position, O for circles, X for crosses, and - to mean an empty position. In order to decide whether a game is finished, some predicates are defined in order to detect complete rows in every possible direction.

```
ops hasHRow hasVRow hasDRow : Player Grid -> Bool .
vars I1 I2 I3 J : Nat . var G : Grid . var P : Player
eq hasHRow(P, [I1, J, P] [I2, J, P] [I3, J, P] G) = true .
eq hasHRow(P, G) = false [owise] .
```

As a commutative and associative operator, the grid matches the definition pattern if the entire row J belongs to player P for some J. Predicates for the other directions are defined similarly. Using them, winning is defined by the following disjunction:

endm

where or-else is a short-circuit version of the logical disjunction provided by the predefined module EXT-BOOL. The game module also defines a constant initial for the initial grid of empty positions [k, l, -] for all  $1 \le k, l \le 3$ .

In the system module TICTACTOE-RULES, the player movements are represented by two rules putO and putX that simply place its symbol on an empty position.

mod TICTACTOE-RULES is
 protecting TICTACTOE .
 vars I J : Nat .

```
rl [put0] : [I, J, -] => [I, J, 0] .
  rl [putX] : [I, J, -] => [I, J, X] .
endm
```

In these terms, we can define strategies for playing the game in a strategy module TICTACTOE-STRAT. The simplest and most unconscious strategy is the free application of the put rules, but stopping when the game is over. This is what the following random strategies do:

```
smod TICTACTOE-STRAT is
  protecting TICTACTOE-RULES .
  strats randomO randomX @ Grid .
  vars G R : Grid . vars I1 I2 I3 I J : Nat . var P : Player .
  sd randomO := (match G s.t. not hasWon(X, G) ; putO)
                   ? randomO : idle .
  sd randomX := (match G s.t. not hasWon(0, G) ; putX)
                   ? randomX : idle .
```

Note that **random** does not mean that the positions are chosen randomly, but that any of them can be chosen, and so the strategy search commands will explore all possible selections. Moreover, randomX and randomO are not mutually recursive, since each represents the strategy of a different player and turns are handled by the multistrategies framework. These strategies can be improved with certain easy intuitions about more clever movements. For example, if the current player already has two positions in a row and the third one is empty, the symbol should be put there to win immediately. Otherwise, if that situation occurs for the opponent, the active player should occupy the empty position to prevent the other player from winning in its next turn. This is specified by the following betterO strategy for the player O (similarly for the X player).

```
strats better0 betterX @ Grid .
sd better0 := (match G s.t. not hasWon(X, G) ;
  ((matchrew G s.t. [I, J, -] R := winningPos(O, G)
      by G using putO[I <- I, J <- J])
  or-else
  (matchrew G s.t. [I, J, -] R := winningPos(X, G)
     by G using putO[I <- I, J <- J])
  or-else
  putO)) ? betterO : idle .
```

The winningPos(P, G) function returns a set of sort Grid with all the positions where player P can complete a row. These are calculated equationally, by pattern matching again.

```
ops winningPos winningHPos winningVPos winningD1Pos
    winningD2Pos : Player Grid -> Grid .
eq winningPos(P, G) = winningHPos(P, G) winningVPos(P, G)
                              23
```

```
winningD1Pos(P, G) winningD2Pos(P, G) .
eq winningHPos(P, [I1, J, P] [I2, J, P] [I3, J, -] G) =
    [I3, J, -] winningHPos(P, G) .
eq winningHPos(P, G) = empty [owise] .
```

Functions to find vertical and diagonal rows are defined similarly. At this point, we can then ask ourselves whether this strategy is *perfect*, i.e. whether it always leads to the best possible outcome: not losing the game no matter how the other player behaves. Using the integrated model checker, we can formally verify it. Making X play with **better** and O play with **random**, i.e. trying all possible moves for O, the property  $\Box \neg Owins$  tells us whether better is optimal.

```
MStrat > select TICTACTOE - CHECK .
Module TICTACTOE - CHECK is now the current module.
MStrat > check [] ~ Owins from initial
    using betterX, randomO by turns .
initial
V 0 does putX
V 1 does putO
V 0 does putX
[1, 1, -] .....[2, 3, -] [3, 1, X] ... [3, 3, X]
V 1 does put0
∨ 0 does putX
V 1 does putO
[1, 1, -] ...... [2, 1, -] [2, 2, 0] ...... [3, 3, X]
V 0 does putX
\vee 1 does putO
X [1, 1, -] [1, 2, 0] ..... [3, 3, X]
```

And it is not, since the counterexample (where we have removed the positions that have not changed) shows an execution in which the circles win even if the crosses play the **better** strategy. In fact, the only situation where the intelligence of the strategy is actually used is the last move for X, when O has two winning positions in the middle vertical and horizontal row that X cannot block at the same time.

Hence, **better** is not a perfect strategy, and further precautions should be taken not to make mistakes. In Table 1 of [13], we can find a script of a perfect strategy for playing tic-tac-toe, which is similar to the algorithm used by the Newell and Simon's tic-tac-toe program in 1972. This script includes various rules to play, which we will call *actions* not to confuse them with rewriting rules, that are not necessarily exclusive. For the strategy to be effective, these actions must be executed in order, applying at each step the first possible one. The first two are those included in **better**, (1) *Win* that completes the row where there are already two positions of the current player, and (2) *Block* that prevents the opponent from doing so in the next turn.

```
stratsperfect0perfectX@ Grid .stratperfect-step: Player @ Grid .
```

```
sd perfect0 := (match G s.t. not hasWon(X, G) ;
     perfect-step(0)) ? perfect0 : idle
sd perfectX := (match G s.t. not hasWon(0, G) ;
     perfect-step(X)) ? perfectX : idle .
sd perfect-step(P) :=
  *** Win
  (matchrew G s.t. [I, J, -] R := winningPos(P, G)
     by G using put(P, I, J))
  or-else
  *** Block
  (matchrew G s.t. [I, J, -] R := winningPos(opponent(P), G)
     by G using put(P, I, J))
  or-else
  *** Fork
  (put(P) ; hasFork(P))
  or-else
  *** Blocking an opponent's fork
  (test(put(opponent(P)) ; hasFork(opponent(P))) ; put(P) ;
    *** The opponent cannot fork
    (not(put(opponent(P)) ; hasFork(opponent(P)))
    *** The opponent is forced to block rather that fork
    | (matchrew G s.t. [I, J, -] R := winningPos(P, G)
         by G using not(put(opponent(P), I, J) ;
                        hasFork(opponent(P))))
  )
  or-else
  *** Center
  put(P, 2, 2)
  or-else
  *** Opposite corner
  ((matchrew [I, I, Q] G s.t. I =/= 2 /\ Q = opponent(P)
      by G using put(P, sd(4, I), sd(4, I)))
  | (matchrew [I, J, Q] G s.t. I =/= 2 /\ J =/= 2
      (\ Q = opponent(P) by G using put(P, J, I)))
  or-else
  *** Empty corner
  (put(P, 1, 1) | put(P, 3, 3) |
  put(P, 1, 3) | put(P, 3, 1))
  or-else
  *** Empty side
  (match G s.t. P == 0 ? (put0[I <- 2] | put0[J <- 2])
    : (putX[I <- 2] | putX[J <- 2]))
```

The or-else combinator guarantees that actions are applied in order. So as not to define the same strategy twice for each player, as we have done with the previous shorter strategies, the perfect-step strategy takes the player as an argument, and the put strategy and the opponent function have been specified.

```
strat put : Player Nat Nat @ Grid .
sd put(X, I, J) := putX[I <- I, J <- J] .
25</pre>
```

```
sd put(0, I, J) := put0[I <- I, J <- J] .
strat put : Player @ Grid .
sd put(X) := putX .
sd put(0) := put0 .
op opponent : Player ~> Player .
eq opponent(X) = 0 .
eq opponent(0) = X .
```

After the actions already present in **better**, the strategy tries (3) Fork to obtain two winning positions for the next turn, so that winning is guaranteed unless the other player completes a row immediately. Instead of calculating these positions equationally, our strategy puts a symbol randomly and then checks whether there is a fork, with the following strategy:

```
strat hasFork : Player @ Grid .
sd hasFork(P) := match G s.t. size(winningPos(P, G)) >= 2 .
```

The next actions are (4) Block fork that prevents the opponent's fork, (5) Center that occupies the center position, (6) Opposite corner that fills the diagonally-opposite corner of an opponent's position, (7) Empty corner that puts the symbol in any corner, and finally (8) Empty side that uses a side instead. Note that the Empty side action is the only remaining possibility when the previous actions have been discarded, so it is equivalent to write simply put(P) instead of the specific strategy we have used in the perfect-step definition.

Using the check command and the same property again, we discover that perfect is actually perfect no matter which player starts.

```
MStrat> check [] ~ Owins from initial
            using perfectX, randomO by turns .
The property is satisfied.
MStrat> check [] ~ Owins from initial
            using randomO, perfectX by turns .
The property is satisfied.
```

However, the strategy does not ensure that X eventually wins. The game may end in a draw, as shown by the following counterexample, which has been drawn in Figure 3.

```
MStrat> check <> Xwins from initial
            using perfectX, randomO by turns .
| initial
V 0 does perfect-step
| [1, 1, -] ...... [2, 2, X] ...... [3, 3, -]
V 1 does put0
| [1, 1, -] ..... [3, 3, 0]
V 0 does perfect-step
| [1, 1, X] ..... [3, 1, -] ... [3, 3, 0]
V 1 does put0
| [1, 1, X] ..... [3, 1, 0] [3, 2, -] [3, 3, 0]
V 0 does perfect-step
| [1, 1, X] [1, 2, -] ..... [3, 1, 0] [3, 2, X] [3, 3, 0]
V 1 does put0
```

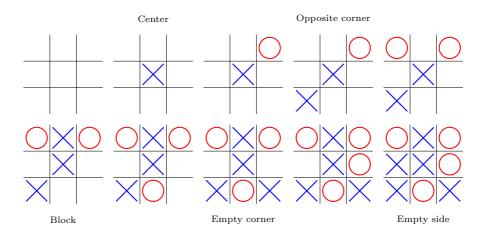


Figure 3: Game where perfectX does not win against randomO.

In particular, both players can play the perfect strategy, and then no one wins.

```
MStrat> check [] (~ Owins /\ ~ Xwins) from initial
            using perfectX, perfect0 by turns .
The property is satisfied.
```

Finally, we wonder whether the perfect strategy we have adopted from [13] is concise or it can be simplified. Repeating the check commands with variations of the strategy, we can see that the last three actions, *Opposite corner*, *Empty corner*, and *Empty side*, can be replaced by the unrestricted put(P), and so this distinction is superfluous. However, this simplification cannot be extended to the *Center* action without losing perfection. Other combinations of rules can be safely removed too.

Not only the check command is useful in this example, but the srewrite command allows obtaining, for instance, all final configurations of the game using certain strategies.

#### 5.2. A deeper look into the implementation

Unlike the previous metalevel transformations, multistrategies are not handled by an equational static manipulation of the original module and its strategies. Instead, the term to be rewritten and the multiple strategies that act on it are operated at the metalevel during their execution. The execution context for the multistrategies is not strictly parametric on the subject system, but contains it as data while being rewritten in the system module MULTISTRAT. The already-seen context {  $\overline{t} :: < 1 \% \overline{\alpha}_1 > \cdots < n \% \overline{\alpha}_n >, \overline{M}$  } is specified as:

The key fact that lets us follow the execution of the multiple strategies on the subject term is that contexts are univocally associated to terms, and their transitions to their transformations, as we will see.

```
op getTerm : MSContext -> Term .
eq getTerm({ T :: TS, M }) = T .
```

The multiple strategy threads are run on these contexts by several rules that reimplement at some extent the execution of strategies. As said before, some rules only manipulate and decompose strategies, while others may modify the term being rewritten. For example, the following are control transitions for the iteration and disjunction combinators:

rl [ms-reduct] : < N rl [ms-reduct] : < N crl [ms-choose] : < N if S1 =/= fail /\ S2 =/= fail .

System transitions are performed by the following rule that executes the strategy S:

Among the strategies considered atomicStrategy there are not only rule applications, but also matchrew strategies with multiple patterns, as we have said, since they assume a fixed structure of the term along all its execution, which may otherwise be broken by another thread acting on the term. The successors of atomic strategies are calculated using the builtin Maude engine via metaSrewrite. However, to nondeterministically select one of the possible successors, we have to collect all of them in a set with the allSuccs function, and let the rule be instantiated with each by matching the E'; Es pattern on that set of terms. Another atomic action is the evaluation of conditions in conditional operators by the following rule:

```
crl [ms-cond] : { T :: < N => { T' :: < N if T' ; Ts := allSuccs(M, T, S1) .
```

Finally, all these rules are gathered in the strategies control and system that lead us to the overview at the beginning of this example.

```
strats control system : Nat @ MSContext .
```

```
sd system(N) := ms-cond[N <- N] or-else ms-run[N <- N] .
sd control(N) := ms-reduct[N <- N] | ... | ms-def[N <- N] .</pre>
```

The Maude-based interactive interface and the command-line program to verify branchingtime properties are programmed similarly to that for the strategy language extensions seen in Section 4.

Unlike the previous examples in Sections 3 and 4, the reflective implementation of multistrategies just explained operates with the metarepresentation of the strategies at *run time*, instead of producing or *compiling* a new module that is then executed by Maude at the object level. Hence, a sometimes noticeable performance penalty is to be expected when executing and model checking with multistrategies. For example, in the Lamport's bakery algorithm mentioned before and available in the example collection [18], generating the entire state space requires 50 seconds with multistrategies but only 200 ms using a less natural alternative strategy that is also included in this specification. However, a more efficient and complex transformation of the first style or a direct implementation could be developed if multistrategies need to scale for more complex applications.

# 6. Related work and conclusions

As we have indicated throughout the paper, the reflective capabilities of Maude have widely been used to build extensions of Maude and frameworks for specific languages and utilities. In addition to Full Maude and the Maude Formal Environment [17], other relevant examples are Real Time Maude [31] for specification and verification of realtime systems, and the mobile agents extension Mobile Maude [16]. On the other hand, the strategy language was introduced to control rewriting at the object level without the conceptual difficulties of reflective computations and the intricate shape of metalevel programs. However, some tasks still require resorting to the metalevel, like writing these interactive interfaces or generating strategies depending on the specification or some input data. While Maude has a singular support for reflection and strategies, other strategy languages can also benefit from manipulating and programatically generating strategies as proposed in this paper. The pioneer strategy language ELAN does not originally come with reflective support, but a reflective extension was proposed [23] where these transformations can be implemented. However, its universal theory does not apparently represent the strategy language combinators themselves, so manipulating strategies could not be straightforward. Partially based on the experience of ELAN, the Porgy graphrewriting language [21] is reflective since its rewrite rules are graphs themselves that can also be rewritten, but this does not cover its strategy language. Since the Stratego [7] toolset is designed for program transformation, what has been done in this paper could be naturally achieved there. Moreover, some reflective transformations used as examples in this paper would not be necessary in Stratego, since its strategy language is more flexible and supports all the operators we have implemented in Section 4. In a broader sense, programmable strategies are only general programs whose atomic actions are rule applications, so what is proposed here does not differ much of what could be done in other reflective programming languages.

With the examples provided in this paper, we aim to show that manipulating, transforming, and generating strategies is accessible and has useful applications. The reflective representation of the object-level strategy language provides the means to easily do this within Maude, while having strategies executed by the efficient builtin engine. The first example in Section 3 shows that strategies can be readily generated to solve specific problems related to program evaluation; the second one in Section 4 allows extending the strategy language with new operators and experiment with them; and the multistrategies of Section 5 can be useful to specify, simulate, and verify systems with distributed control like agent-based or object-oriented systems and games. Another interesting example involving strategy generation is a framework for simulating and verifying membrane systems [36].

Declaration of competing interest. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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## References

- Baader, F., Nipkow, T., 1998. Term Rewriting and All That. Cambridge University Press. doi:10.1017/CB09781139172752.
- [2] Balland, E., Brauner, P., Kopetz, R., Moreau, P., Reilles, A., 2007. Tom: Piggybacking rewriting on Java, in: Baader, F. (Ed.), Term Rewriting and Applications, 18th International Conference, RTA 2007, Paris, France, June 26-28, 2007, Proceedings, Springer. pp. 36–47. doi:10.1007/978-3-540-73449-9\_5.
- [3] Barendregt, H., 2014. The Lambda Calculus: Its Syntax and Semantics. volume 131. 2 ed., North Holland.
- [4] Borovanský, P., Kirchner, C., Kirchner, H., Ringeissen, C., 2001. Rewriting with strategies in ELAN: A functional semantics. Int. J. Found. Comput. Sci. 12, 69–95. doi:10.1142/S0129054101000412.
- [5] Bourdier, T., Cirstea, H., Dougherty, D.J., Kirchner, H., 2009. Extensional and intensional strategies, in: Fernández, M. (Ed.), Proceedings Ninth International Workshop on Reduction Strategies in Rewriting and Programming, WRS 2009, Brasilia, Brazil, 28th June 2009, pp. 1–19. doi:10.4204/EPTCS.15.1.
- [6] Bradfield, J.C., Walukiewicz, I., 2018. The mu-calculus and model checking, in: Clarke, E.M., Henzinger, T.A., Veith, H., Bloem, R. (Eds.), Handbook of Model Checking. Springer, pp. 871–919. doi:10.1007/978-3-319-10575-8\_26.
- [7] Bravenboer, M., Kalleberg, K.T., Vermaas, R., Visser, E., 2008. Stratego/XT 0.17. A language and toolset for program transformation. Sci. Comput. Program. 72, 52–70. doi:10.1016/j.scico.2007.11.003.
- [8] Clarke, E.M., Emerson, E.A., 1981. Design and synthesis of synchronization skeletons using branching-time temporal logic, in: Kozen, D. (Ed.), Logics of Programs, Workshop, Yorktown Heights, New York, USA, May 1981, Springer. pp. 52–71. doi:10.1007/BFb0025774.
- [9] Clavel, M., 2003. Strategies and user interfaces in Maude at work, in: Gramlich, B., Lucas, S. (Eds.), Proceedings of the 3rd International Workshop on Reduction Strategies in Rewriting and Programming, WRS 2003, Valencia, Spain, June 8, 2003, Elsevier. pp. 570–592. doi:10.1016/S1571-0661(05)82612-X.
- [10] Clavel, M., Durán, F., Eker, S., Escobar, S., Lincoln, P., Martí-Oliet, N., Meseguer, J., Rubio, R., Talcott, C., 2020-10. Maude Manual v3.1. URL: http://maude.lcc.uma.es/maude31-manual-html/maude-manual.html.
- [11] Clavel, M., Durán, F., Eker, S., Lincoln, P., Martí-Oliet, N., Meseguer, J., Talcott, C.L., 2007a. All About Maude - A High-Performance Logical Framework, How to Specify, Program and Verify Systems in Rewriting Logic. volume 4350 of *Lecture Notes in Computer Science*. Springer. doi:10.1007/978-3-540-71999-1.

- [12] Clavel, M., Meseguer, J., Palomino, M., 2007b. Reflection in membership equational logic, manysorted equational logic, Horn logic with equality, and rewriting logic. Theor. Comput. Sci. 373, 70–91. doi:10.1016/j.tcs.2006.12.009.
- [13] Crowley, K., Siegler, R.S., 1993. Flexible strategy use in young children's tic-tac-toe. Cogn. Sci. 17, 531–561. URL: 10.1016/0364-0213(93)90003-Q.
- [14] Durán, F., Eker, S., Escobar, S., Martí-Oliet, N., Meseguer, J., Rubio, R., Talcott, C., 2020. Programming and symbolic computation in Maude. J. Log. Algebraic Methods Program. 110, 1–58. doi:10.1016/j.jlamp.2019.100497.
- [15] Durán, F., Escobar, S., Lucas, S., 2004. New evaluation commands for Maude within Full Maude, in: Martí-Oliet, N. (Ed.), Proceedings of the Fifth International Workshop on Rewriting Logic and its Applications, WRLA 2004, Barcelona, Spain, March 27-April 4, 2004, Elsevier. pp. 263–284. doi:10.1016/j.entcs.2004.06.014.
- [16] Durán, F., Riesco, A., Verdejo, A., 2007. A distributed implementation of Mobile Maude, in: Denker, G., Talcott, C. (Eds.), Proceedings of the 6th International Workshop on Rewriting Logic and its Applications, WRLA 2006, Vienna, Austria, April 1-2, 2006, Elsevier. pp. 113–131. doi:10.1016/j.entcs.2007.06.011.
- [17] Durán, F., Rocha, C., Álvarez, J.M., 2011. Towards a Maude Formal Environment, in: Agha, G., Danvy, O., Meseguer, J. (Eds.), Formal Modeling: Actors, Open Systems, Biological Systems -Essays Dedicated to Carolyn Talcott on the Occasion of Her 70th Birthday, Springer. pp. 329–351. doi:10.1007/978-3-642-24933-4\_17.
- [18] Eker, S., Martí-Oliet, N., Meseguer, J., Pita, I., Rubio, R., Verdejo, A., 2020. Strategy language for Maude. URL: http://maude.ucm.es/strategies.
- [19] Eker, S., Meseguer, J., Sridharanarayanan, A., 2004. The Maude LTL model checker, in: Gadducci, F., Montanari, U. (Eds.), Proceedings of the Fourth International Workshop on Rewriting Logic and its Applications, WRLA 2002, Pisa, Italy, September 19-21, 2002, Elsevier. pp. 162–187. doi:10.1016/S1571-0661(05)82534-4.
- [20] Emerson, E.A., Halpern, J.Y., 1986. "Sometimes" and "not never" revisited: on branching versus linear time temporal logic. J. ACM 33, 151–178. doi:10.1145/4904.4999.
- [21] Fernández, M., Kirchner, H., Pinaud, B., 2019. Strategic port graph rewriting: an interactive modelling framework. Mathematical Structures in Computer Science 29, 615–662. doi:10.1017/S0960129518000270.
- [22] Kant, G., Laarman, A., Meijer, J., van de Pol, J., Blom, S., van Dijk, T., 2015. LTSmin: High-performance language-independent model checking, in: Baier, C., Tinelli, C. (Eds.), Tools and Algorithms for the Construction and Analysis of Systems, 21st International Conference, TACAS 2015, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2015, London, UK, April 11-18, 2015, Proceedings, Springer. pp. 692–707. doi:10.1007/978-3-662-46681-0\_61.
- [23] Kirchner, H., Moreau, P., 1996. A reflective extension of ELAN, in: Meseguer, J. (Ed.), First International Workshop on Rewriting Logic and its Applications, WRLA 1996, Asilomar Conference Center, Pacific Grove, CA, USA, September 3-6, 1996, Elsevier. pp. 149–168. doi:10.1016/S1571-0661(04)00038-6.
- [24] Lescanne, P., 1990. Implementations of completion by transition rules + control: ORME, in: Kirchner, H., Wechler, W. (Eds.), Algebraic and Logic Programming, Second International Conference, Nancy, France, October 1-3, 1990, Proceedings, Springer. pp. 262–269. doi:10.1007/3-540-53162-9\_44.
- [25] Lilis, Y., Savidis, A., 2020. A survey of metaprogramming languages. ACM Comput. Surv. 52, 113:1–113:39. doi:10.1145/3354584.
- [26] Lucas, S., 2020. Context-sensitive rewriting. ACM Comput. Surv. 53. doi:10.1145/3397677.
- [27] Lucas, S., 2021. Applications and extensions of context-sensitive rewriting. J. Log. Algebraic
- Methods Program. 121. doi:https://doi.org/10.1016/j.jlamp.2021.100680.
  [28] Marin, M., Kutsia, T., 2006. Foundations of the rule-based system ρLog. J. Appl. Non Class. Logics
- 16, 151–168. doi:10.3166/jancl.16.151-168.
  [29] Meseguer, J., 1992. Conditional rewriting logic as a unified model of concurrency. Theor. Comput.
- Sci. 96, 73–155. doi:10.1016/0304-3975(92)90182-F.
  [30] Nieuwenhuis, R., Oliveras, A., Tinelli, C., 2006. Solving SAT and SAT modulo theories: From an abstract Davis–Putnam–Logemann–Loveland procedure to DPLL(T). J. ACM 53, 937–977.
- doi:10.1145/1217856.1217859.
  [31] Ölveczky, P.C., 2014. Real-Time Maude and its applications, in: Escobar, S. (Ed.), Rewriting Logic and Its Applications 10th International Workshop, WRLA 2014, Held as a Satellite

Event of ETAPS, Grenoble, France, April 5-6, 2014, Revised Selected Papers, Springer. pp. 42–79. doi:10.1007/978-3-319-12904-4\_3.

- [32] Pnueli, A., 1977. The temporal logic of programs, in: 18th Annual Symposium on Foundations of Computer Science, Providence, Rhode Island, USA, 31 October - 1 November 1977, IEEE Computer Society. pp. 46–57. doi:10.1109/SFCS.1977.32.
- [33] Rubio, R., 2020. Unified Maude model-checking tool (umaudemc). FaDoSS. URL: https://github.com/fadoss/umaudemc.
- [34] Rubio, R., Martí-Oliet, N., Pita, I., Verdejo, A., 2019a. Model checking strategy-controlled rewriting systems, in: Geuvers, H. (Ed.), 4th International Conference on Formal Structures for Computation and Deduction, FSCD 2019, June 24-30, 2019, Dortmund, Germany, Schloss Dagstuhl - Leibniz-Zentrum für Informatik. pp. 34:1–34:18. doi:10.4230/LIPIcs.FSCD.2019.31.
- [35] Rubio, R., Martí-Oliet, N., Pita, I., Verdejo, A., 2019b. Parameterized strategies specification in Maude, in: Fiadeiro, J., Țuțu, I. (Eds.), Recent Trends in Algebraic Development Techniques, Springer. pp. 27–44. doi:10.1007/978-3-030-23220-7\_2.
- [36] Rubio, R., Martí-Oliet, N., Pita, I., Verdejo, A., 2020a. Simulating and model checking membrane systems using strategies in Maude, in: 7th International Workshop on Rewriting Techniques for Program Transformation and Evaluation, WPTE 2020, Paris, France, pp. 1–10.
- [37] Rubio, R., Martí-Oliet, N., Pita, I., Verdejo, A., 2020b. Strategies, model checking and branchingtime properties in Maude, in: Escobar, S., Martí-Oliet, N. (Eds.), Rewriting Logic and Its Applications - 13th International Workshop, WRLA 2020, Virtual Event, October 20-22, 2020, Revised Selected Papers, Springer. pp. 156–175. doi:10.1007/978-3-030-63595-4\_9.