Domains of weak continuity of statistical functionals with a view toward robust statistics

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Abstract

Many standard estimators such as several maximum likelihood estimators or the empirical estimator for any law-invariant convex risk measure are not (qualitatively) robust in the classical sense. However, these estimators may nevertheless satisfy a weak [13, 14] or a local [22] robustness property on relevant sets of distributions. One aim of our paper is to identify sets of local robustness, and to explain the benefit of the knowledge of such sets. For instance, we will be able to demonstrate that many maximum likelihood estimators are robust on their natural parametric domains. A second aim consists in extending the general theory of robust estimation to our local framework. In particular we provide a corresponding Hampel-type theorem linking local robustness of a plug-in estimator with a certain continuity condition.

Keywords: (ψ_k) -weak topology; w-set; qualitative robustness; Hampel's theorem; maximum likelihood estimator; law-invariant convex risk measure; aggregation robustness; Orlicz space

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1 Introduction and problem statement

Recently, in [22] qualitative robustness of plug-in estimators was considered as a local property, i.e., on strict subsets of the natural domain of the corresponding statistical functional, and a respective Hampel-type criterion was proven. The latter says that if the statistical functional is continuous for a certain topology finer than the weak topology, then qualitative robustness holds on every set of distributions on which the relative weak topology coincides with the finer topology. Such sets of distributions were characterized in [22], but the provided characterization is rather technical and not at all useful for checking the concurrence of the topologies for any given set. The aim of the present paper is to provide more useful characterizations of such sets, and to illustrate their use in the context of qualitative robustness. Compared to [22] we will also allow for more general topologies on sets of distributions which will turn out to increase the flexibility to check qualitative robustness for statistical functionals. As applications, robustness of maximum likelihood estimators and of empirical estimators of law-invariant convex risk measures are studied in detail. In particular we will demonstrate that many maximum likelihood estimators are robust on their natural parametric domains and even on broader sets. A further field of application is quantitative risk management. In recent contributions in this field the property of robustness has been pointed out as an important requirement for risk assessment; see, for instance, [3, 6, 14]. Again the empirical estimators of well-founded statistical functionals like those associated with law-invariant convex risk measures fail to be robust but might satisfy this property on domains of interest.

To explain our intension more precisely, let E be a Polish space and \mathcal{M}_1 be the set of all Borel probability measures on E. Consider the statistical model

$$(\Omega, \mathcal{F}, \{\mathbb{P}^{\theta} : \theta \in \Theta\}) := (E^{\mathbb{N}}, \mathcal{B}(E)^{\otimes \mathbb{N}}, \{\mathbb{P}^{\mu} : \mu \in \mathcal{M}\}),$$
(1)

where $\mathcal{M} \subseteq \mathcal{M}_1$ is any set of Borel probability measures on E and

$$\mathbb{P}^{\mu} := \mu^{\otimes \mathbb{N}} \tag{2}$$

is the infinite product measure of μ . Note that the coordinate projections on $E^{\mathbb{N}}$ are i.i.d. with law μ under \mathbb{P}^{μ} . For every $\boldsymbol{x} = (x_1, x_2, \ldots) \in E^{\mathbb{N}}$ and $n \in \mathbb{N}$, we define the empirical probability measure

$$\widehat{m}_n(\boldsymbol{x}) := \widehat{m}_n(x_1, \dots, x_n) := \frac{1}{n} \sum_{i=1}^n \delta_{x_i}.$$

Assume that \mathcal{M} contains the set

$$\mathfrak{E} := \{ \widehat{m}_n(x_1, \dots, x_n) : x_1, \dots, x_n \in E, n \in \mathbb{N} \}$$

of all empirical probability measures. Let (Σ, d_{Σ}) be a complete and separable metric space and $T : \mathcal{M} \to \Sigma$ be any map (statistical functional). The empirical probability measure \widehat{m}_n induces a nonparametric estimator $\widehat{T}_n : \Omega \to \Sigma$ for $T(\mu)$ in the statistical model (1) through

$$\widehat{T}_n(\boldsymbol{x}) := T(\widehat{m}_n(x)), \quad \boldsymbol{x} = (x_1, x_2, \ldots) \in \Omega,$$
(3)

provided \widehat{T}_n is $(\mathcal{F}, \mathcal{B}(\Sigma))$ -measurable.

The following Definition 1.1 generalizes Hampel's classical notion of (qualitative) robustness for the sequence (\hat{T}_n) as introduced in [9]. Recall from Theorem 2.14 in [10] that the set of all Borel probability measures on Σ equipped with the weak topology is Polish and can be metrized by the Prohorov metric π . Moreover denote by \mathcal{O}_w the weak topology on \mathcal{M}_1 .

Definition 1.1 For a given set $M \subseteq \mathcal{M}$ and $\mu \in M$, the sequence of estimators (\widehat{T}_n) is said to be M-robust at μ if for every $\varepsilon > 0$ there exists an open neighborhood $U = U(\mu, \varepsilon; M)$ of μ for the relative weak topology $\mathcal{O}_w \cap M$ such that

$$\nu \in U \implies \pi(\mathbb{P}^{\mu} \circ \widehat{T}_n^{-1}, \mathbb{P}^{\nu} \circ \widehat{T}_n^{-1}) \leq \varepsilon \quad for \ all \ n \in \mathbb{N}.$$

The sequence (\widehat{T}_n) is said to be robust on M if it is M-robust at every $\mu \in M$.

In their pioneer work, Hampel [9] and Cuevas [4] used (mainly the first part of) Definition 1.1 with specifically $M = \mathcal{M} = \mathcal{M}_1$ and established several criteria for robustness; cf. Theorems 1–2 in [9] and Theorems 1–2 in [4]. In the present paper, our focus will be on the second part of Definition 1.1, i.e. on robustness of (\hat{T}_n) on subsets M of \mathcal{M} . In this context the following two criteria are already known for $M = \mathcal{M}$.

- (I) If $T : \mathcal{M} \to \Sigma$ is continuous for the relative weak topology $\mathcal{O}_{w} \cap \mathcal{M}$, then (\widehat{T}_{n}) is robust on \mathcal{M} .
- (II) If (\widehat{T}_n) is weakly consistent and robust on \mathcal{M} , then $T : \mathcal{M} \to \Sigma$ is continuous for the relative weak topology $\mathcal{O}_{w} \cap \mathcal{M}$.

Assertion (I) is a straightforward generalization of Theorem 2 in [4] (where the author assumed $\mathcal{M} = \mathcal{M}_1$) and assertion (II) is a special case of Theorem 1 in [4].

Recall that we assumed the set \mathfrak{E} of all empirical probability measures to be contained in \mathcal{M} . As \mathfrak{E} is dense in \mathcal{M}_1 w.r.t. the weak topology \mathcal{O}_w (cf. Theorem A.38 in [8] reformulated for probability measures), this implies that weak continuity of the map $T: \mathcal{M} \to \Sigma$ is a relatively strict requirement. For instance, in the case $E = \mathbb{R}$ the mean functional $T(\mu) := \int x \mu(dx)$ is not weakly continuous on \mathfrak{E} (indeed, letting $x_{n,1} := n$ and $x_{n,i} = 0$ for i = 2, ..., n and $n \in \mathbb{N}$, the sequence $(\widehat{m}_n(x_{n1}, ..., x_{nn}))_{n \in \mathbb{N}}$ converges to δ_0 w.r.t. \mathcal{O}_w , but $\int x \widehat{m}(x_{n1}, ..., x_{nn})(dx) = 1 \not\rightarrow 0 \int x \delta_0(dx)$). In view of (I)–(II), this simple example indicates that there are only a few relevant statistical functionals $T: \mathcal{M} \to \Sigma$ for which the corresponding sequence of estimators (\hat{T}_n) is robust on the whole domain \mathcal{M} . Nevertheless, for general statistical functionals one might ask for those subsets M of \mathcal{M} on which robustness of (\hat{T}_n) holds. The following simple example shows that this question can be reasonable.

Example 1.2 Let $E = (0, \infty)$ and \mathcal{E} be the class of all exponential distributions with mean θ (cf. Example 3.6), $\theta \in (0, \infty)$. The unique maximum likelihood estimator for the parameter θ is known to be $\widehat{T}_n(\boldsymbol{x}) = \overline{\boldsymbol{x}}_n$, where $\overline{\boldsymbol{x}}_n = \frac{1}{n} \sum_{i=1}^n x_i$ for $\boldsymbol{x} = (x_1, x_2, \ldots)$. It can be represented by $\widehat{T}_n(\boldsymbol{x}) = T(\widehat{m}_n(\boldsymbol{x}))$ for the functional $T(\mu) = \int x \, \mu(dx)$ with domain $\mathcal{M} = \{\mu \in \mathcal{M}_1 : \int |x| \, \mu(dx) < \infty\}$ and state space $\Sigma = (0, \infty)$. Since T is weakly consistent by the law of large numbers but *not* weakly continuous on \mathcal{M} , assertions (I)– (II) imply that the sequence (\widehat{T}_n) is not robust on \mathcal{M} . However, in Subsection 4.1.2 we will see that our results yield robustness of (\widehat{T}_n) on relatively large subsets of \mathcal{M} , in particular on \mathcal{E} . That means that the maximum likelihood estimator is robust at least against small deviations within the underlying parametric set of distributions \mathcal{E} . This statement could not be derived in the conventional theory of robustness. Note that robustness on \mathcal{E} is of interest if one starts from the premise that both the target distribution μ and the distribution ν underlying the observations lie in \mathcal{E} .

In fact the maximum likelihood estimator is even robust against certain deviations out of \mathcal{E} , even though not against arbitrary deviations within the whole domain \mathcal{M} . For instance, at the end of Subsection 4.1.2 we will see that the maximum likelihood estimator is also robust on the broader class Γ of all Gamma distributions (with location parameter 0). Robustness on $\Gamma \supseteq \mathcal{E}$ is of interest if one assumes that the target distribution lies in \mathcal{E} (so that the maximum likelihood principle is reasonable) but the distribution ν underlying the observations may lie in the broader class Γ .

The issue of robustness on subsets was approached in Section 3.1 of [22]. The latter paper develops further the theory of [13, 14] and provides the following criteria (i)–(ii), where \mathcal{M} is assumed to be contained in the set \mathcal{M}_1^{ψ} of all $\mu \in \mathcal{M}_1$ with $\int \psi \, d\mu < \infty$ for some given continuous function $\psi : E \to [0, \infty)$. By ψ -weak topology \mathcal{O}_{ψ} on \mathcal{M}_1^{ψ} we mean the coarsest topology for which all mappings $\mu \mapsto \int f \, d\mu, f \in \mathcal{C}_{\psi}$ are continuous, where \mathcal{C}_{ψ} refers to the space of all continuous functions $f : E \to \mathbb{R}$ for which $\sup_{x \in E} |f(x)/(1+\psi(x))| < \infty$. For $E = \mathbb{R}^d$ and $\psi(x) := ||x||^p$ with $p \ge 1$, the set \mathcal{M}_1^{ψ} is just the set of all Borel probability measures on $E = \mathbb{R}^d$ with finite p-th absolute moment and the ψ -weak topology is metrizable by the L^p -Wasserstein metric; cf. Lemma 8.3 in [1].

(i) If $T : \mathcal{M} \to \Sigma$ is continuous for the relative ψ -weak topology $\mathcal{O}_{\psi} \cap \mathcal{M}$, then (T_n) is robust on every subset $M \subseteq \mathcal{M}$ with $\mathcal{O}_{\psi} \cap M = \mathcal{O}_{w} \cap M$.

(ii) If (\widehat{T}_n) is weakly consistent on \mathcal{M} and robust on every subset $M \subseteq \mathcal{M}$ with $\mathcal{O}_{\psi} \cap M = \mathcal{O}_{w} \cap M$, then $T : \mathcal{M} \to \Sigma$ is continuous for the ψ -weak topology \mathcal{O}_{ψ} .

In general the ψ -weak topology \mathcal{O}_{ψ} is finer than the relative weak topology $\mathcal{O}_{w} \cap \mathcal{M}_{1}^{\psi}$, and the two topologies coincide for $\psi \equiv 1$. Thus the criteria (i)–(ii) generalize the criteria (I)-(II). Assertion (i) says that for ψ -weakly continuous functionals $T: \mathcal{M} \to \Sigma$ the sequence (\widehat{T}_n) is robust on every subset M of \mathcal{M} for which the relative ψ -weak topology $\mathcal{O}_{\psi} \cap M$ and the relative weak topology $\mathcal{O}_{w} \cap M$ coincide. Lemma 3.4 of [22] provides the following characterization of those subsets M of \mathcal{M}_1^{ψ} for which $\mathcal{O}_{\psi} \cap M = \mathcal{O}_{w} \cap M$ holds: the latter identity holds if and only if M is locally uniformly ψ -integrating in the sense of Definition 2.2 below. On the one hand, this characterization is the basis for the proof of the criteria (i)–(ii) and is also relevant for robustness of more general estimators than plug-in estimators as defined in (3); see [21] for an example. On the other hand, the condition "locally uniformly ψ -integrating" is rather technical and not at all useful for checking the identity $\mathcal{O}_{\psi} \cap M = \mathcal{O}_{w} \cap M$ for any given set M. The aim of the present paper is to provide more useful characterizations of those subsets M of \mathcal{M}_1^{ψ} for which the identity $\mathcal{O}_{\psi} \cap M = \mathcal{O}_{w} \cap M$ holds, and to illustrate their use. For the sake of brevity we will refer to any $M \subseteq \mathcal{M}_1^{\psi}$ satisfying the condition $\mathcal{O}_{\psi} \cap M = \mathcal{O}_{w} \cap M$ as w-set in \mathcal{M}_1^{ψ} .

Theorem 2.3 below gives three further equivalent conditions for a set to be a w-set in \mathcal{M}_1^{ψ} . Based on this theorem, we will obtain in Section 3 several specific examples for wsets in \mathcal{M}_1^{ψ} for various choices of ψ . Among others, we will investigate popular parametric families of distribution such as normal, Pareto, Gumbel, or Gamma distributions, and also consider sets of distributions derived from Fréchet classes of univariate marginal distributions via aggregation operators like the sum. The latter sets of distributions are of particular interest in the context of risk assessment. The results of Section 3 together with assertion (i) above (and the results of Section 4) in particular justify Example 1.2. In Section 4 we will provide examples for ψ -weakly continuous functionals T; we will study statistical functionals underlying the maximum likelihood method and law-invariant convex risk measures. In Section 4 we will also discuss the property of robustness of the corresponding plug-in estimators (\hat{T}_n) on subsets of T's domain. Sections 5 and 6 contain longer proofs of our results.

Finally, note that we will in fact work with a slightly more general topology than \mathcal{O}_{ψ} , namely with the so-called (ψ_k) -weak topology $\mathcal{O}_{(\psi_k)}$ to be introduced at the beginning of Section 2. This generalization does not have priority, but the respective theory covers some more examples than the theory for the ψ -weak topology \mathcal{O}_{ψ} . In particular, we need to establish a corresponding extension of the criteria (i)–(ii), which can be found in Theorem 2.6.

2 Concurrence of weak and (ψ_k) -weak topologies and applications in robust statistics

As before let E be a Polish space and use the notation introduced in Section 1. Let (ψ_k) be a sequence of gauge functions, i.e., a sequence of continuous functions $\psi_k : E \to [0,\infty)$. Let \mathcal{C}_{ψ_k} be the space of all continuous functions $f : E \to \mathbb{R}$ for which $\sup_{x \in E} |f(x)/(1+\psi_k(x))| < \infty$. Let $\mathcal{M}_1^{(\psi_k)}$ be the set of all Borel probability measures μ on E for which $\int \psi_k d\mu < \infty$ for every $k \in \mathbb{N}$. The (ψ_k) -weak topology $\mathcal{O}_{(\psi_k)}$ on $\mathcal{M}_1^{(\psi_k)}$ is defined to be the coarsest topology for which all mappings $\mu \mapsto \int f d\mu$, $f \in \mathcal{C}_{\psi_k}$, $k \in \mathbb{N}$, are continuous. When $\psi_k = \psi$ for all $k \in \mathbb{N}$, we have $\mathcal{M}_1^{(\psi_k)} = \mathcal{M}_1^{\psi}$ and $\mathcal{O}_{(\psi_k)} = \mathcal{O}_{\psi}$.

Lemma 2.1 The set $\mathcal{M}_1^{(\psi_k)}$ equipped with the (ψ_k) -weak topology is a Polish space. In addition the (ψ_k) -weak topology is metrizable by the metric

$$d_{(\psi_k)}(\mu,\nu) := d_{\mathbf{w}}(\mu,\nu) + \sum_{k=1}^{\infty} 2^{-k} \Big(\Big| \int \psi_k \, d\mu - \int \psi_k \, d\nu \Big| \wedge 1 \Big), \tag{4}$$

where d_{w} is any metric for the weak topology. Moreover, for every $(\mu_n)_{n \in \mathbb{N}_0} \subset \mathcal{M}_1^{(\psi_k)}$ the following statements are equivalent.

- (a) $\mu_n \to \mu_0 \ (\psi_k)$ -weakly.
- (b) $\mu_n \to \mu_0$ weakly and $\int \psi_k d\mu_n \to \int \psi_k d\mu_0$ for every $k \in \mathbb{N}$.

In Theorem 2.3 below we will specify those subsets of $\mathcal{M}_1^{(\psi_k)}$ on which the relative (ψ_k) -weak topology and the relative weak topology coincide. We will use the following terminology, which extends Definition 2.12 in [14] and Definition 3.1 in [22].

Definition 2.2 A set $M \subseteq \mathcal{M}_1$ is said to be locally uniformly (ψ_k) -integrating if for every $\mu \in M$, $\varepsilon > 0$, and $k \in \mathbb{N}$ there exist a > 0 and a weakly open neighborhood U of μ such that

$$\nu \in M \cap U \implies \int \psi_k \mathbb{1}_{\{\psi_k \ge a\}} d\nu \le \varepsilon.$$

The set M is said to be uniformly (ψ_k) -integrating if for every $\varepsilon > 0$ and $k \in \mathbb{N}$ there exists some a > 0 such that

$$\sup_{\mu \in M} \int \psi_k \mathbb{1}_{\{\psi_k \ge a\}} d\mu \le \varepsilon.$$

If (ψ_k) consists of a single gauge function, say ψ , we shall speak of (locally) uniformly ψ -integrating sets instead of (locally) uniformly (ψ_k) -integrating sets.

Note that a set M is (locally) uniformly (ψ_k) -integrating if and only if it is (locally) uniformly ψ_k -integrating for every $k \in \mathbb{N}$. Of course, any uniformly (ψ_k) -integrating set M is also locally uniformly (ψ_k) -integrating, and any locally uniformly (ψ_k) -integrating set M is a subset of $\mathcal{M}_1^{(\psi_k)}$. If all ψ_k are bounded, then the set \mathcal{M}_1 coincides with $\mathcal{M}_1^{(\psi_k)}$ and is uniformly (ψ_k) -integrating.

Let us now turn to the characterization of those subsets of $\mathcal{M}_1^{(\psi_k)}$ on which the relative (ψ_k) -weak topology and the relative weak topology coincide. For ψ -weak topologies, the equivalence (a) \Leftrightarrow (b) in the following theorem is already known from Lemma 3.6 in [22].

Theorem 2.3 Let (ψ_k) be any sequence of gauge functions and $M \subseteq \mathcal{M}_1^{(\psi_k)}$ be given. Then the following conditions are equivalent:

- (a) $\mathcal{O}_{(\psi_k)} \cap M = \mathcal{O}_{w} \cap M$.
- (b) M is locally uniformly (ψ_k) -integrating.
- (c) Every weakly compact subset of M is uniformly (ψ_k) -integrating.
- (d) Every sequence in M which converges weakly in M is uniformly (ψ_k) -integrating.
- (e) For every sequence $(\mu_n) \subseteq M$ for which μ_n converges weakly to μ_0 the convergence $\int \psi_k d\mu_n \to \int \psi_k d\mu_0$ holds for all $k \in \mathbb{N}$.

Definition 2.4 Let (ψ_k) be any sequence of gauge functions and $M \subseteq \mathcal{M}_1^{(\psi_k)}$. Then M is said to be a w-set in $\mathcal{M}_1^{(\psi_k)}$ if condition (a) (and thus each of the equivalent conditions (a)-(e)) in Theorem 2.3 holds.

Remark 2.5 Let (ψ_k) and $(\tilde{\psi}_k)$ be sequences of gauge functions satisfying $\tilde{\psi}_k \leq \psi_k$ pointwise for every $k \in \mathbb{N}$. Then $\mathcal{M}_1^{(\psi_k)} \subseteq \mathcal{M}_1^{(\tilde{\psi}_k)}$, and the (ψ_k) -weak topology is finer than the $(\tilde{\psi}_k)$ -weak topology on $\mathcal{M}_1^{(\psi_k)}$. In particular, every w-set in $\mathcal{M}_1^{(\psi_k)}$ is also a w-set in $\mathcal{M}_1^{(\tilde{\psi}_k)}$. Moreover, if $\psi_k \equiv 1$ for every $k \in \mathbb{N}$, then every subset of $\mathcal{M}_1 = \mathcal{M}_1^{(\psi_k)}$ is a w-set.

We obtain the following generalization of Hampel's theorem, where by weak consistency of (\hat{T}_n) on \mathcal{M} we mean that $\lim_{n\to\infty} \mathbb{P}^{\mu}[|\hat{T}_n - T(\mu)| \ge \eta] = 0$ for all $\eta > 0$ and $\mu \in \mathcal{M}$.

Theorem 2.6 Let the statistical model $(\Omega, \mathcal{F}, \{\mathbb{P}^{\mu} : \mu \in \mathcal{M}\})$ (with $\mathcal{M} \subseteq \mathcal{M}_{1}^{(\psi_{k})}$), the functional $T : \mathcal{M} \to \Sigma$, and the sequence of estimators (\widehat{T}_{n}) be as introduced in Section 1. Then the following two assertions hold:

- (i) If for any w-set M in $\mathcal{M}_1^{(\psi_k)}$ with $M \subseteq \mathcal{M}$ the functional T is continuous at every $\mu \in M$ for the relative (ψ_k) -weak topology $\mathcal{O}_{(\psi_k)} \cap \mathcal{M}$, then (\widehat{T}_n) is robust on M. In particular, if T is continuous for the relative (ψ_k) -weak topology $\mathcal{O}_{(\psi_k)} \cap \mathcal{M}$, then (\widehat{T}_n) is robust on every w-set M in $\mathcal{M}_1^{(\psi_k)}$ with $M \subseteq \mathcal{M}$.
- (ii) If (\widehat{T}_n) is weakly consistent on \mathcal{M} and robust on every w-set M in $\mathcal{M}_1^{(\psi_k)}$ with $M \subseteq \mathcal{M}$, then T is continuous for the relative (ψ_k) -weak topology $\mathcal{O}_{(\psi_k)} \cap \mathcal{M}$.

In the case where $\psi_k = \psi$ for all $k \in \mathbb{N}$, Theorem 2.6 is already known from Theorem 3.8 in [22]; previous versions of this result were obtained in [13, 14]. The extension in terms of the general notion of (ψ_k) -weak topology is motivated by the example of maximum likelihood estimation which will be studied in Subsection 4.1. In particular, in order to establish local robustness for the maximum likelihood estimator of the scale parameter of Gumbel distributions the possibility to use nonconstant sequences of gauge functions will prove to be convenient; cf. Subsection 4.1.3 below. Likewise, it will turn out that the full generality of Theorem 2.6 is useful when investigating local robustness of certain law-invariant convex risk measures; cf. the discussion subsequent to Corollary 4.19.

In many situations the functional $T : \mathcal{M} \to \Sigma$ can be shown to be (ψ_k) -weakly continuous on the whole domain \mathcal{M} . In some cases, however, it is beneficial that in condition (i) of Theorem 2.6 we only require continuity of T at every point of \mathcal{M} . To give an example, let $E = \mathbb{R}$, $\psi_k = \psi \equiv 1$ (hence $\mathcal{M}_1^{(\psi_k)} = \mathcal{M}_1$), $\alpha \in (0, 1)$, and $T : \mathcal{M}_1 \to \mathbb{R}$ be the functional that assigns to a Borel probability measure its (lower) α -quantile. This (quantile) functional T is $(\psi$ -) weakly continuous at every point of the set \mathcal{M} of all Borel probability measures with a unique α -quantile, but not weakly continuous on \mathcal{M}_1 .

3 Examples of w-sets in $\mathcal{M}_1^{(\psi_k)}$

The following lemma provides a simple but general class of w-sets in $\mathcal{M}_{1}^{(\psi_{k})}$.

Lemma 3.1 Every set $M \subseteq \mathcal{M}_1^{(\psi_k)}$ that is relatively compact for the (ψ_k) -weak topology is a w-set in $\mathcal{M}_1^{(\psi_k)}$.

Proof It suffices to show that on M weak convergence implies (ψ_k) -weak convergence. So let us suppose that (μ_n) is a sequence in M that converges weakly to some $\mu \in M$. Then by (ψ_k) -weak compactness, every subsequence of (μ_n) has a subsequence that converges (ψ_k) -weakly toward some $\nu \in \mathcal{M}_1^{(\psi_k)}$. Since (ψ_k) -weak convergence implies weak convergence, we must have $\nu = \mu$. It hence follows that $\mu_n \to \mu$ also (ψ_k) -weakly which completes the proof. \Box

The preceding lemma has the following consequence.

Proposition 3.2 Suppose that (ψ_k) is a sequence of gauge functions such that all sets of the form $\{\psi_{k+1} \leq n\psi_k\}$ are compact for $k, n \in \mathbb{N}$. Then $\mathcal{M}_1^{(\psi_k)}$ is (ψ_k) -weakly compact and thus itself a w-set.

Proof Clearly, $\mathcal{M}_1^{(\psi_k)}$ is (ψ_k) -weakly closed, while relative compactness follows from Lemma 5.1 (d) by taking $\phi_k := \psi_{k+1}$. Lemma 3.1 finally gives that $\mathcal{M}_1^{(\psi_k)}$ is a w-set. \Box

The most interesting case is where E is non-compact. In this case Proposition 3.2 is not applicable to constant sequences of gauge functions (i.e. $\psi_k = \psi$ for all $k \in \mathbb{N}$), because then the sets $\{\psi \leq n\psi\} = E, n \in \mathbb{N}$, are not compact. However, as an immediate consequence of condition (e) in Theorem 2.3 we obtain the following alternative device.

Proposition 3.3 Let Θ be a topological space, and μ_{θ} be an element of $\mathcal{M}_{1}^{(\psi_{k})}$ for every $\theta \in \Theta$. Then the set $\mathcal{M}_{\Theta} := \{\mu_{\theta} : \theta \in \Theta\}$ is a w-set in $\mathcal{M}_{1}^{(\psi_{k})}$ if the following two conditions are satisfied:

- (a) For every sequence $(\theta_n)_{n \in \mathbb{N}_0}$ in Θ , weak convergence of μ_{θ_n} to μ_{θ_0} implies $\theta_n \to \theta_0$.
- (b) For every sequence $(\theta_n)_{n\in\mathbb{N}_0}$ in Θ , convergence of θ_n to θ_0 implies $\int \psi_k d\mu_{\theta_n} \to \int \psi_k d\mu_{\theta_0}$ for all $k \in \mathbb{N}$.

3.1 Parametric classes of distributions

In this section, we consider a few examples in which parametric classes of probability distributions belong to $\mathcal{M}_1^{(\psi_k)}$ for suitably chosen sequences (ψ_k) satisfying the hypotheses of Proposition 3.2 or Proposition 3.3. Note that in view of Remark 2.5, the assertions in the following examples can be reformulated for many other (coarser) topologies; see the second part of Example 3.6 for an illustration.

Example 3.4 Let $E = \mathbb{R}^d$ equipped with the euclidean norm $\|\cdot\|$ and \mathcal{N} be the class of all *d*-dimensional normal distributions $N(m, \Sigma)$, where $m \in \mathbb{R}^d$ and Σ is a semidefinite $d \times d$ covariance matrix. If we let $\psi_k(x) := \exp(\lambda_k \|x\|^{\alpha_k})$, where $\lambda_k \uparrow \infty$ and $\alpha_k \uparrow 2$, we have $\mathcal{N} \subseteq \mathcal{M}_1^{(\psi_k)}$, and Proposition 3.2 yields that \mathcal{N} is a w-set in $\mathcal{M}_1^{(\psi_k)}$.

Example 3.5 For fixed parameters $\alpha > 0$ and $x_{\min} > 0$, let $\mathcal{P}_{\alpha,x_{\min}}$ be the class of type-1 Pareto distributions with shape parameter $a \ge \alpha$. That is, $\mathcal{P}_{\alpha,x_{\min}}$ consists of all Borel probability measures on \mathbb{R} with Lebesgue density

$$f_a(x) = \frac{a}{x_{\min}} \left(\frac{x_{\min}}{x}\right)^{a+1} \mathbb{1}_{[x_{\min},\infty)}(x)$$

for some $a \geq \alpha$. If we let $\psi_k(x) := |x|^{p_k}$, where $p_k > 0$ and $p_k \uparrow \alpha$, we have $\mathcal{P}_{\alpha, x_m} \subseteq \mathcal{M}_1^{(\psi_k)}$, and Proposition 3.2 yields that $\mathcal{P}_{\alpha, x_{\min}}$ is a w-set in $\mathcal{M}_1^{(\psi_k)}$.

Example 3.6 Let Γ denote the class of all Gamma distributions with location parameter 0. That is, Γ is the class of all Borel probability measures on $(0, \infty)$ with Lebesgue density

$$f_{\kappa,\theta}(x) = \frac{x^{\kappa-1}e^{-x/\theta}}{\theta^{\kappa}\,\Gamma(k)}$$

for some $\theta, \kappa > 0$. When taking $\psi_k(x) := x^k$ or $\psi_k(x) := e^{\lambda_k x^{\beta_k}}$ where $\lambda_k \uparrow \infty$ and $\beta_k \uparrow 1$, we have $\Gamma \subseteq \mathcal{M}_1^{(\psi_k)}$, and Proposition 3.2 yields that Γ is a w-set in $\mathcal{M}_1^{(\psi_k)}$.

When the parameter κ is fixed and set to 1, the Lebesgue density $f_{\kappa,\theta}$ simplifies to the Lebesgue density

$$f_{\theta}(x) := f_{1,\theta}(x) = \frac{e^{-x/\theta}}{\theta}$$

of the exponential distribution to the parameter $\theta > 0$, and the corresponding class of all exponential distributions will be denoted by \mathcal{E} . Again by Proposition 3.2 we obtain that \mathcal{E} is a w-set in $\mathcal{M}_1^{(\psi_k)}$ for the sequences (ψ_k) of gauge functions mentioned above. In Subsection 4.1.2 we will consider the single gauge function $\psi(x) := x$ which is dominated by $\psi_k(x) := e^{\lambda_k x^{\beta_k}}$ (with λ_k, β_k as above) for every k. Thus $\mathcal{E} \subseteq \mathcal{M}_1^{(\psi_k)} \subseteq \mathcal{M}_1^{\psi}$, and by Remark 2.5 we obtain that \mathcal{E} is also a w-set in \mathcal{M}_1^{ψ} .

Example 3.7 Let \mathcal{G} denote the class of all Gumbel distributions, i.e., the class of all Borel probability measures G_a on \mathbb{R} with Lebesgue density

$$f_a(x) = ae^{-ax-e^{-ax}}$$

for some a > 0. By letting $\psi_k(x) := |x|^k$ or $\psi_k(x) := e^{\lambda_k |x|^{\beta_k}}$ where $\lambda_k \uparrow \infty$ and $\beta_k \uparrow 1$, we obtain that $\mathcal{G} \subseteq \mathcal{M}_1^{(\psi_k)}$, and Proposition 3.2 yields that \mathcal{G} is a w-set in $\mathcal{M}_1^{(\psi_k)}$.

In Subsection 4.1.3 we will consider the gauge functions $\psi_k(x) := |\log a_k - a_k x - e^{-a_k x}|$ for some sequence (a_k) representing $(0, \infty) \cap \mathbb{Q}$. Since for a > 0 the moment generating function of G_a is well defined on $(-\infty, 1/a)$ enclosing 0, the integrals $\int \tilde{a}|x|e^{-ax-e^{-ax}}dx$ and $\int e^{-\tilde{a}x}e^{-ax-e^{-ax}}dx$ are finite for any $a, \tilde{a} > 0$, and thus $G_a \in \mathcal{M}_1^{(\psi_k)}$ for all a > 0. We now verify conditions (a)–(b) of Proposition 3.3 to show that \mathcal{G} is also a w-set in $\mathcal{M}_1^{(\psi_k)}$ for this choice of gauge functions.

(a): Let $(G_{a_n})_{n \in \mathbb{N}}$ be any sequence in \mathcal{G} which weakly converges to some distribution $G_{a_0} \in \mathcal{G}$. Then corresponding sequence $(F_{a_n})_{n \in \mathbb{N}_0}$ of distribution functions satisfies

$$e^{-e^{-a_nx}} = F_{a_n}(x) \longrightarrow F_a(x) = e^{-e^{-ax}}$$
 for all $x \in \mathbb{R}$.

Thus necessarily $a_n \to a_0$.

(b): Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $(0,\infty)$ which converges to some $a_0 \in (0,\infty)$. Set $\underline{a} := \inf_{n\in\mathbb{N}} a_n$ and $\overline{a} := \sup_{n\in\mathbb{N}} a_n$, and note that $\underline{a} > 0$ and $\overline{a} < \infty$. For any $x \in \mathbb{R}$, the mapping $a \mapsto -ax - e^{-ax}$ is nonincreasing on $(0,\infty)$. Thus

$$\sup_{n \in \mathbb{N}} \psi_k(x) a_n e^{-a_n x - e^{-a_n x}} \le \psi_k(x) \overline{a} e^{-\underline{a}x - e^{-\underline{a}x}} \quad \text{for every } x \in \mathbb{R}.$$

Hence by $G_{\overline{a}} \in \mathcal{M}_1^{(\psi_k)}$, we may apply the dominated convergence theorem to conclude that for every $k \in \mathbb{N}$,

$$\int \psi_k(x) \operatorname{G}_{a_n}(dx) = \int \psi_k(x) a_n e^{-a_n x - e^{-a_n x}} dx$$
$$\longrightarrow \int \psi_k(x) a_0 e^{-a_0 x - e^{-a_0 x}} dx = \int \psi_k(x) \operatorname{G}_{a_0}(dx)$$

As we have shown that conditions (a)–(b) of Proposition 3.3 are satisfies, the proposition ensures that \mathcal{G} is indeed a w-set in $\mathcal{M}_1^{(\psi_k)}$.

3.2 Images of Fréchet classes

In this section we consider certain classes M that consist of the image measures of the probability measures in a given Fréchet class w.r.t. fixed one-dimensional marginal distributions. The motivation for studying this examples is the notion of aggregation robustness recently introduced by Embreches et al. [6]; see also Subsection 4.2 below.

Let $E = \mathbb{R}^r$ be endowed with the Euclidean norm $\|\cdot\|$. For given Borel probability measures μ_1, \ldots, μ_d on \mathbb{R} denote by $\mathcal{M}(d; \mu_1, \ldots, \mu_d)$ the set of all Borel probability measures on \mathbb{R}^d whose one-dimensional marginal distributions are μ_1, \ldots, μ_d . The set $\mathcal{M}(d; \mu_1, \ldots, \mu_d)$ is sometimes called *Fréchet class* associated with μ_1, \ldots, μ_d . For any Borel measurable map $A_d : \mathbb{R}^d \to \mathbb{R}^r$ let

$$M = M(\mu_1, \dots, \mu_d; A_d) := \left\{ \boldsymbol{\mu} \circ A_d^{-1} : \, \boldsymbol{\mu} \in \boldsymbol{\mathcal{M}}(d; \mu_1, \dots, \mu_d) \right\} \subseteq \mathcal{M}_1 \tag{5}$$

be the class of the images under A_d of all probability measures from $\mathcal{M}(d; \mu_1, \ldots, \mu_d)$. In the following Example 3.8 we give a few examples of maps A_d we have in mind. All these maps are Lipschitz continuous and thus satisfy condition (a) of Proposition 3.9 ahead. This proposition will provide a general criterion for determining whether $M(\mu_1, \ldots, \mu_d; A_d)$ is a w-set.

Example 3.8 A simple but relevant example for a map $A_d : \mathbb{R}^d \to \mathbb{R}^d$ is the identity map:

(i) $A_d(\boldsymbol{x}) := \boldsymbol{x}$.

In this case the set $M(\mu_1, \ldots, \mu_d; A_d)$ defined by (5) is nothing but the Fréchet class $\mathcal{M}(d; \mu_1, \ldots, \mu_d)$ itself. Examples for maps $A_d : \mathbb{R}^d \to \mathbb{R}$ that are relevant in risk management and insurance are given by

- (ii) $A_d(x_1, \ldots, x_d) := \sum_{i=1}^d x_i,$
- (iii) $A_d(x_1, \ldots, x_d) := \max\{x_1, \ldots, x_d\},\$

- (iv) $A_d(x_1, ..., x_d) := \sum_{i=1}^d (x_i t_i)^+$ for thresholds $t_1, ..., t_d > 0$,
- (v) $A_d(x_1, ..., x_d) := (\sum_{i=1}^d x_i t)^+$ for a threshold t > 0;

see, for instance, [15, p. 248]. For an application of (ii) see Subsection 4.2 below. \diamond

Let us now turn over to our main criterion to check whether the set M defined in (5) is a w-set in $\mathcal{M}_1^{(\tilde{\psi}_k)}$. Here we use the notation $\tilde{\psi}_k(\cdot) := \psi_k(\|\cdot\|)$. Note that $\tilde{\psi}_k = \psi_k$, $k \in \mathbb{N}$, when r = 1.

Proposition 3.9 Let $A_d : \mathbb{R}^d \to \mathbb{R}^r$ be any Borel measurable map. Let (ψ_k) be any sequence of gauge functions on \mathbb{R} that are all convex and even, and consider the sequence of gauge functions $(\widetilde{\psi}_k)$ on \mathbb{R}^r with $\widetilde{\psi}_k$ as above. Further assume that the following assertions hold:

- (a) There exist constants b, c > 0 such that $||A_d(\boldsymbol{x})|| \leq b + c \sum_{i=1}^d |x_i|$ for all $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$.
- (b) For every $k \in \mathbb{N}$ there exist $\ell_k \in \mathbb{N}$ and $c_k > 0$ such that $\psi_k((d+1)cx) \leq c_k \psi_{\ell_k}(x)$ for all $x \in \mathbb{R}$, where the constant c is given by (a).
- (c) $\mu_1, \ldots, \mu_d \in \mathcal{M}_1^{(\psi_k)}$.

Then M defined by (5) is uniformly $(\widetilde{\psi}_k)$ -integrating, and thus it is a w-set in $\mathcal{M}_1^{(\widetilde{\psi}_k)}$.

4 Examples for (ψ_k) -weakly continuous functionals T

In this section, we will discuss continuity of some relevant statistical functionals w.r.t. the (ψ_k) -weak topology for suitable sequences of gauge functions (ψ_k) . In Section 4.1 we will consider statistical functionals which underly the maximum likelihood estimation principle. In Section 4.2 we will consider statistical functionals associated with so-called risk measures. The latter play an important role in quantitative risk management; see, for instance, [15, 19].

4.1 Maximum likelihood functionals

Let $\Theta \subseteq \mathbb{R}^d$ and μ_{θ} be a Borel probability measure on E for every $\theta \in \Theta$. Let

$$(\Omega, \mathcal{F}) := (E^{\mathbb{N}}, \mathcal{B}(E)^{\otimes \mathbb{N}}) \quad \text{and} \quad \mathbb{P}^{\theta} := \mu_{\theta}^{\otimes \mathbb{N}} \text{ for every } \theta \in \Theta.$$

Then $(\Omega, \mathcal{F}, \{\mathbb{P}^{\theta} : \theta \in \Theta\})$ is a parametric statistical product model. We will assume that our parametric statistical model is dominated. That is, we assume that there is some

 σ -finite measure λ on $(E, \mathcal{B}(E))$ (called the dominating measure) such that for every $\theta \in \Theta$ the law μ_{θ} is absolutely continuous w.r.t. λ with Radon–Nikodym derivative

$$f_{\theta} := \frac{d\mu_{\theta}}{d\lambda}.$$

In particular, when X_i denotes the *i*-th coordinate projection on $\Omega = E^{\mathbb{N}}$, the law $\mathbb{P}^{\theta} \circ (X_1, \ldots, X_n)^{-1} = \mu_{\theta}^{\otimes n}$ of a sample of size *n* is absolutely continuous w.r.t. $\lambda^{\otimes n}$ with Radon–Nikodym derivative $d\mu_{\theta}^{\otimes n}/d\lambda^{\otimes n}$ satisfying

$$\frac{d\mu_{\theta}^{\otimes n}}{d\lambda^{\otimes n}}(x_1,\ldots,x_n) = \prod_{i=1}^n f_{\theta}(x_i) =: L_n(x_1,\ldots,x_n;\theta) \text{ for all } x_1,\ldots,x_n \in E$$

for every $\theta \in \Theta$ and $n \in \mathbb{N}$.

By definition, a maximum likelihood estimator \widehat{T}_n for the parameter θ based on a sample of size n satisfies

$$\widehat{T}_n(\boldsymbol{x}) = \widehat{T}_n(x_1, \dots, x_n) \in \underset{\theta \in \Theta}{\operatorname{arg\,max}} L_n(x_1, \dots, x_n; \theta) \quad \text{for all } \boldsymbol{x} = (x_1, x_2, \dots) \in \Omega \quad (6)$$

for every $\theta \in \Theta$. Let us assume that $f_{\theta} > 0$ on E for every $\theta \in \Theta$. Then condition (6) is equivalent to

$$\widehat{T}_n(\boldsymbol{x}) = \widehat{T}_n(x_1, \dots, x_n) \in \operatorname*{arg\,max}_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^n \log f_\theta(x_i) \quad \text{ for all } \boldsymbol{x} = (x_1, x_2, \dots) \in \Omega.$$

In particular, if $\mathcal{M} \subseteq \mathcal{M}_1$ contains the set \mathfrak{E} of all empirical probability measures and $T: \mathcal{M} \to \Theta$ is a functional satisfying

$$T(\mu) \in \underset{\theta \in \Theta}{\operatorname{arg\,max}} \int \log f_{\theta} \, d\mu \quad \text{for all } \mu \in \mathcal{M},$$
(7)

then we can define a maximum likelihood estimator by

$$\widehat{T}_n(\boldsymbol{x}) = \widehat{T}_n(x_1, \dots, x_n) = T(\widehat{m}_n(x_1, \dots, x_n)) \quad \text{for all } \boldsymbol{x} = (x_1, x_2, \dots) \in \Omega.$$
(8)

Inspired by the representation (8) of the maximum likelihood estimator we introduce the following terminology. For any subset $\mathcal{M} \subseteq \mathcal{M}_1$ containing \mathfrak{E} , a map $T : \mathcal{M} \to \Theta$ is called *maximum likelihood functional* associated with the statistical model $(\Omega, \mathcal{F}, \{\mathbb{P}^{\theta} : \theta \in \Theta\})$ if for every $\mu \in \mathcal{M}$ the integrals $\int |\log f_{\theta}| d\mu, \theta \in \Theta$, are finite and (7) holds.

Remark 4.1 Although it is not crucial for our purposes, note that a maximum likelihood functional T evaluated at μ_{θ} takes the parameter θ as its value, i.e., θ provides a maximizer of the mapping $\vartheta \mapsto \int \log f_{\vartheta} d\mu_{\theta}$ on Θ . Moreover, θ is the unique maximizer

as soon as $\mu_{\theta} \neq \mu_{\vartheta}$ for all $\vartheta \in \Theta \setminus \{\theta\}$. Indeed, since we assumed $f_{\theta} > 0$ for every $\theta \in \Theta$, the strict concavity of the logarithm and Jensen's inequality yield

$$\int \log f_{\vartheta} \, d\mu_{\theta} - \int \log f_{\theta} \, d\mu_{\theta} = \int \log \left(\frac{f_{\vartheta}}{f_{\theta}}\right) d\mu_{\theta} < \log \left(\int \frac{f_{\vartheta}}{f_{\theta}} \, d\mu_{\theta}\right) = \log(1) = 0$$

for every $\vartheta \in \Theta$ for which $f_{\vartheta} \neq f_{\theta}$; in the third step we have used the fact that $f_{\vartheta}/f_{\theta} = d\mu_{\vartheta}/d\mu_{\theta}$.

In view of (8) the minimal domain of a maximum likelihood functional $T : \mathcal{M} \to \Theta$ is $\mathcal{M} = \mathfrak{E}$, where as before \mathfrak{E} refers to the set of all empirical probability measures on E. In order to apply Theorem 2.6 to maximum likelihood estimators, the domain \mathcal{M} has to be chosen so large that it contains both \mathfrak{E} and the set

$$\mathcal{M}_{\Theta} := \{ \mu_{\theta} : \theta \in \Theta \}.$$

Indeed, if $\mathfrak{E} \cup \mathcal{M}_{\Theta} \subseteq \mathcal{M} \subseteq \mathcal{M}_{1}^{(\psi_{k})}$ and a maximum likelihood functional $T : \mathcal{M} \to \Theta$ is (ψ_{k}) -weakly continuous for some sequence of gauge functions (ψ_{k}) , then Theorem 2.6 shows that the corresponding sequence of maximum likelihood estimators (\widehat{T}_{n}) is robust on every w-set M in $\mathcal{M}_{1}^{(\psi_{k})}$ with $M \subseteq \mathcal{M}$ (in particular with $M \subseteq \mathcal{M}_{\Theta}$). Recall that robustness of (\widehat{T}_{n}) on M means that for every $\mu \in M$ and $\varepsilon > 0$ there exists an open neighborhood $U = U(\mu, \varepsilon; M)$ of μ for the relative weak topology $\mathcal{O}_{w} \cap M$ such that $\pi(\mathbb{P}^{\mu} \circ \widehat{T}_{n}^{-1}, \mathbb{P}^{\nu} \circ \widehat{T}_{n}^{-1}) \leq \varepsilon$ for all $\nu \in U$ and $n \in \mathbb{N}$.

Remark 4.2 In many specific situations the set \mathcal{M}_{Θ} itself can be shown to be a w-set in $\mathcal{M}_1^{(\psi_k)}$; see Examples 3.4 through 3.7. In this case the sequence (\hat{T}_n) of maxium likelihood estimators is robust on \mathcal{M}_{Θ} .

Note that robustness on \mathcal{M}_{Θ} is of interest if one starts from the premise that both the target distribution μ and the distribution ν underlying the observations lie in the parametric class of distributions \mathcal{M}_{Θ} . On the other hand, robustness on $M \supseteq \mathcal{M}_{\Theta}$ is of interest if one assumes that the target distribution lies in \mathcal{M}_{Θ} (so that the maximum likelihood principle is reasonable) but the distribution ν underlying the observations may lie in a broader class M (nested between \mathcal{M}_{Θ} and \mathcal{M}).

4.1.1 (ψ_k) -weak continuity of maximum likelihood functionals

Here we are going to investigate a fixed maximum likelihood functional $T : \mathcal{M} \to \Theta$ for continuity w.r.t. a suitable (ψ_k) -weak topology. To this end, let Θ be an open convex subset of \mathbb{R}^d equipped with the Euclidean norm $\|\cdot\|$. We will assume throughout that the following three conditions hold:

$$f_{\theta}(x) > 0$$
 for every $\theta \in \Theta$ and $x \in E$. (9)

$$x \mapsto \log f_{\theta}(x)$$
 is continuous for every $\theta \in \Theta$. (10)

$$\theta \mapsto \log f_{\theta}(x)$$
 is concave for every $x \in E$. (11)

In general we cannot expect that the maximum likelihood functional T is weakly continuous. On the other hand, it should often be possible to find a sequence of gauge functions (ψ_k) for which T is (ψ_k) -weakly continuous. At this abstract level the most obvious candidates for the sequence (ψ_k) is built upon a sequence $(\log f_{\theta_k}(\cdot))$ with (θ_k) a sequence in Θ representing $\Theta \cap \mathbb{Q}^d$. For instance, we may and do choose

$$\psi_k(x) := |\log f_{\theta_k}(x)|, \quad x \in E.$$
(12)

For every $\mu \in \mathcal{M}_1$ with $\int |\log f_{\theta}| d\mu < \infty$ for all $\theta \in \Theta$ we can now define a map $\mathcal{L}_{\mu} : \Theta \to \mathbb{R}$ by

$$\mathcal{L}_{\mu}(\theta) := \int \log f_{\theta} \, d\mu.$$

Note that \mathcal{L}_{μ} is concave by (11) and thus continuous. Moreover, $T(\mu)$ is a maximum point of \mathcal{L}_{μ} by the characteristic property (7) of the functional T.

Theorem 4.3 Let $\mathcal{M} \subseteq \mathcal{M}_1$ and $T : \mathcal{M} \to \Theta$ be a maximum likelihood functional. Assume that (9)–(11) hold, and let the function ψ_k be defined by (12) for every $k \in \mathbb{N}$. Let $\mu \in \mathcal{M}$ be such that the map \mathcal{L}_{μ} has a unique maximum point. Then $T(\mu_n) \to T(\mu)$ for any sequence $(\mu_n) \subset \mathcal{M}$ which satisfies $\mathcal{L}_{\mu_n}(\theta_k) \to \mathcal{L}_{\mu}(\theta_k)$ for every $k \in \mathbb{N}$. In particular, T is continuous at μ w.r.t. the (ψ_k) -weak topology.

Remark 4.4 Let the gauge functions ψ_k be defined as in (12). Furthermore, let $k_0 \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ there exist constants $B_k \geq 0$ and $C_k > 0$ with $\psi_k \leq B_k + C_k \psi_{k_0}$. So $\overline{\psi}_k := B_k + C_k \psi_{k_0}$ defines a new sequence $(\overline{\psi}_k)$ of gauge functions such that $\mathcal{M}_1^{\psi_{k_0}} = \mathcal{M}_1^{(\overline{\psi}_k)} = \mathcal{M}_1^{(\psi_k)}$. Then on the one hand by the genuine definition of (ψ_k) -weak topology, the $(\overline{\psi}_k)$ -weak topology is finer than the (ψ_k) -weak topology. On the other hand by Lemma 2.1 the $(\overline{\psi}_k)$ -weak topology is coarser than the ψ_{k_0} -weak topology. In particular the (ψ_k) -topology coincides with the ψ_{k_0} -weak topology so that we may replace in Theorem 4.3 the (ψ_k) -weak topology with the ψ_{k_0} -weak topology. \diamond

We round out Section 4.1 with two specific examples illustrating Theorem 4.3.

4.1.2 Example exponential distribution

Let specifically $E = (0, \infty)$, $\Theta = (0, \infty)$ and $\mathcal{M}_{\Theta} := \{ \operatorname{Exp}_{\theta} : \theta \in (0, \infty) \}$ be the class of all exponential distributions $\operatorname{Exp}_{\theta}$ with parameter θ . In this case we have

$$\log f_{\theta}(x) = -\log \theta - x/\theta \quad \text{for all } \theta \in (0, \infty) \text{ and } x \in (0, \infty).$$
(13)

If we choose the sequence of gauge functions (ψ_k) by (12), then we obtain

$$\psi_k(x) = |\log f_{\theta_k}(x)| = |-\log \theta_k - x/\theta_k| \quad \text{for all } x \in (0,\infty) \text{ and } k \in \mathbb{N}$$
(14)

for some sequence (θ_k) in $\Theta = (0, \infty)$ representing $(0, \infty) \cap \mathbb{Q}$. It can easily be seen from (13) that conditions (9)–(11) hold. Since $|\log f_{\theta}| \leq B_{\theta} + C_{\theta}\psi_{k_0}$ for $B_{\theta} := |\log \theta|, C_{\theta} := 1/\theta$ and those k_0 for which $\theta_{k_0} = 1$, we may observe $\mathcal{M}_1^{\psi_{k_0}} = \bigcap_{\theta>0} \mathcal{M}_1^{|\log f_{\theta}|} = \mathcal{M}_1^{(\psi_k)}$. Moreover, by Remark 4.4, the (ψ_k) -weak topology coincides with the ψ -weak topology on $\mathcal{M}_1^{(\psi_k)} = \mathcal{M}_1^{\psi}$, where $\psi := \psi_{k_0}$, i.e.

$$\psi(x) := x, \quad x \in (0, \infty). \tag{15}$$

For $\mu \in \mathcal{M}_1^{\psi}$ the map $\mathcal{L}_{\mu} : (0, \infty) \to \mathbb{R}$ defined by

$$\mathcal{L}_{\mu}(\theta) := \int \log f_{\theta} \, d\mu = \int (-\log \theta - x/\theta) \, \mu(dx)$$

has a unique maximum point, namely $\hat{\theta} = \int x \,\mu(dx)$. In particular, there exists exactly one maximum likelihood functional on \mathcal{M}_1^{ψ} . That is, there exists exactly one functional $T : \mathcal{M}_1^{\psi} \to (0, \infty)$ satisfying (7). In the present setting, (7) and Remark 4.1 imply that $T(\mu) = \arg \max_{\theta \in (0,\infty)} \int (-\log \theta - x/\theta) \,\mu(dx)$ for all $\mu \in \mathcal{M}_1^{\psi}$. Now, combining Theorem 4.3 with Remark 4.4 we obtain immediately the following result.

Proposition 4.5 The unique maximum likelihood functional $T : \mathcal{M}_1^{\psi} \to (0, \infty)$ is ψ -weakly continuous.

In view of part (i) of Theorem 2.6, Proposition 4.5 and the second part of Example 3.6 together imply the following corollary, where robustness is understood as in Definition 1.1.

Corollary 4.6 The sequence of maximum likelihood estimators (\widehat{T}_n) is robust on \mathcal{M}_{Θ} .

The preceding corollary shows that the maximum likelihood estimator for the parameter of the exponential distribution is robust on its natural parametric domain, i.e., on the class \mathcal{M}_{Θ} of all exponential distributions. To see that it is even robust on, for instance, the broader class of all Gamma distributions (with location parameter 0), let the sequence of gauge function (ψ_k) no longer be given by (14) but rather by

$$\psi_k(x) = |x|^k$$
 for all $x \in \mathbb{R}$ and $k \in \mathbb{N}$.

Then we clearly have $\mathcal{M}_{\Theta} \subset \mathcal{M}_{1}^{(\psi_{k})} \subset \mathcal{M}_{1}^{\psi}$ for the single gauge function ψ defined in (15). In particular, by Proposition 4.5 the restriction of the unique maximum likelihood functional T to $\mathcal{M}_{1}^{(\psi_{k})}$ is clearly (ψ_{k}) -weakly continuous. Together with part (i) of Theorem 2.6 and the first part of Example 3.6 this implies the following corollary.

Corollary 4.7 The sequence of maximum likelihood estimators (\widehat{T}_n) is robust on the class Γ of all Gamma distributions (with location parameter 0) introduced in Example 3.6.

4.1.3 Example Gumbel distribution

Let specifically $E = \mathbb{R}$, $\Theta = (0, \infty)$ and $\mathcal{M}_{\Theta} := \{G_a : a \in (0, \infty)\}$ be the class of all Gumbel distributions $G_a = G_{0,a}$ with location parameter 0 and scale parameter 1/a for a > 0; cf. Example 3.7. In this case we have

$$\log f_a(x) = \log a - ax - e^{-ax}$$
 for all $a \in (0, \infty)$ and $x \in \mathbb{R}$.

It is easily seen that conditions (9)–(11) are satisfied. Let the sequence of gauge function (ψ_k) be given by (12), i.e.

$$\psi_k(x) := |\log f_{a_k}(x)| = |\log a_k - a_k x - e^{-a_k x}|, \quad x \in \mathbb{R}, \ k \in \mathbb{N}$$

for some sequence (a_k) in $\Theta = (0, \infty)$ representing $(0, \infty) \cap \mathbb{Q}$. For this choice of gauge functions we can observe the following.

Lemma 4.8 $\bigcap_{a>0} \mathcal{M}_1^{|\log f_a|} = \mathcal{M}_1^{(\psi_k)}.$

Proof Let $0 < a < \overline{a}$. Then by the de l'Hospital rule we may observe

$$\lim_{x \to \infty} \frac{\log f_a(x)}{\log f_{\overline{a}}(x)} = \lim_{x \to \infty} \frac{-a + ae^{-ax}}{-\overline{a} + \overline{a}e^{-\overline{a}x}} = \frac{a}{\overline{a}},$$

and

$$\lim_{x \to -\infty} axe^{ax} = 0 = \lim_{x \to -\infty} \overline{a}xe^{ax}.$$

In addition

$$\lim_{x \to -\infty} \frac{\log f_a(x)}{\log f_{\overline{a}}(x)} = \lim_{x \to -\infty} \frac{\log a \ e^{ax} - axe^{ax} - 1}{\log \overline{a} \ e^{ax} - \overline{a}xe^{ax} - e^{(a-\overline{a})x} - 0} = 0,$$

where in the last step the assumption $a < \overline{a}$ has been invoked. Then

$$\lim_{x \to -\infty} \frac{|\log f_a(x)|}{|\log f_{\overline{a}}(x)|} = 0 \quad \text{and} \quad \lim_{x \to \infty} \frac{|\log f_a(x)|}{|\log f_{\overline{a}}(x)|} = \frac{a}{\overline{a}}$$

so that for some $\delta > 0$

$$|\log f_a(x)| \le 2 \frac{a}{\overline{a}} |\log f_{\overline{a}}(x)| \quad \text{if } |x| > \delta.$$
(16)

Now let $\mu \in \mathcal{M}_1^{|\log f_{\overline{a}}(x)|}$. Firstly by (16),

$$\int \mathbb{1}_{\mathbb{R}\setminus[-\delta,\delta]}(x) |\log f_a(x)| \ \mu(dx) < \infty.$$

Secondly in view of (10)

$$\int \mathbb{1}_{[-\delta,\delta]}(x) |\log f_a(x)| \ \mu(dx) \le \sup_{x \in [-\delta,\delta]} |\log f_a(x)| \ \mu([-\delta,\delta]) < \infty.$$

Hence $\mu \in \mathcal{M}_1^{|\log f_a|}$. Since for any a > 0 there is some $k \in \mathbb{N}$ such that $a < a_k$, we may conclude $\bigcap_{a>0} \mathcal{M}_1^{|\log f_a|} = \mathcal{M}_1^{(\psi_k)}$.

Using δ_0 to denote the Dirac measure at 0, Proposition 4.10 below shows that there is exactly one maximum likelihood functional on

$$\mathcal{M} := \mathcal{M}_1^{(\psi_k)} \setminus \{\delta_0\}$$

and that this functional is (ψ_k) -weakly continuous. The proof of Proposition 4.10 relies on Theorem 4.3 and the following lemma.

Lemma 4.9 For every $\mu \in \mathcal{M}$, the map $\mathcal{L}_{\mu} : (0, \infty) \to \mathbb{R}$ defined by

$$\mathcal{L}_{\mu}(a) := \int \log f_a \, d\mu = \int (\log a - ax - e^{-ax}) \, \mu(dx)$$

has a unique maximum point.

Proof The function $a \mapsto \log f_a(x)$ is strictly concave for every $x \in \mathbb{R}$, and so $a \mapsto \mathcal{L}_{\mu}(a)$ is also strictly concave for every $\mu \in \mathcal{M}$. We will show below that for $\mu \neq \delta_0$,

$$\limsup_{a \downarrow 0} \mathcal{L}_{\mu}(a) = -\infty \quad \text{and} \quad \limsup_{a \uparrow \infty} \mathcal{L}_{\mu}(a) = -\infty, \tag{17}$$

and so \mathcal{L}_{μ} has indeed a unique maximum point. To show (17), we first note that $\log f_a(x) \leq \log a$ holds for all $x \in \mathbb{R}$ and $a \in (0, \infty)$. It follows that $\limsup_{a \downarrow 0} \mathcal{L}_{\mu}(a) \leq \limsup_{a \downarrow 0} \log a = -\infty$. To prove the second identity in (17), note that

$$\frac{1}{a} \int (ax + e^{-ax}) \,\mu(dx) \ge \int_{[0,\infty)} x \,\mu(dx) + \int_{(-\infty,0)} \left(x + e^{-ax}/a\right) \,\mu(dx).$$

Since $e^{-ax}/a \uparrow \infty$ as $a \uparrow \infty$ for every x < 0, the rightmost integral tends to $+\infty$ as soon as $\mu((-\infty, 0)) > 0$. Altogether, we obtain $\liminf_{a\uparrow\infty} \frac{1}{a} \int (ax + e^{-ax}) \mu(dx) > 0$ for $\mu \neq \delta_0$, which clearly implies $\limsup_{a\uparrow\infty} \frac{1}{a} \mathcal{L}_{\mu}(a) < 0$ and in turn (17). \Box

Lemma 4.9 says that there exists exactly one maximum likelihood functional on \mathcal{M} , i.e., exactly one functional $T : \mathcal{M} \to (0, \infty)$ satisfying (7). In the present setting, (7) and Remark 4.1 imply that $T(\mu) = \arg \max_{a \in (0,\infty)} \int (\log a - ax - e^{-ax}) \mu(dx)$ for all $\mu \in \mathcal{M}$.

Proposition 4.10 The unique maximum likelihood functional $T : \mathcal{M} \to (0, \infty)$ is (ψ_k) -weakly continuous.

Proof As already mentioned above, conditions (9)–(11) hold. Moreover, by Lemma 4.9 the map \mathcal{L}_{μ} possesses a unique maximum point for every $\mu \in \mathcal{M}$. Thus the claim follows by an application of Theorem 4.3.

From the second part of Example 3.7 we know that \mathcal{M}_{Θ} is a w-set in $\mathcal{M}^{(\psi_k)}$. Together with part (i) of Theorem 2.6 and Proposition 4.10, this yields the following corollary, where robustness is understood as in Definition 1.1.

Corollary 4.11 The sequence of maximum likelihood estimators (\widehat{T}_n) is robust on \mathcal{M}_{Θ} .

4.2 Risk functionals

4.2.1 Risk measures

In this section we let specifically $E = \mathbb{R}$. Let $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous nondecreasing convex function such that $0 = \Psi(0)$ and $\lim_{x\uparrow\infty} \Psi(x) = \infty$. Such a function is sometimes referred to as a finite Young function; see, e.g., [2]. Fix any atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote by $L^0 = L^0(\Omega, \mathcal{F}, \mathbb{P})$ the set of all \mathbb{P} -a.s. finite random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. The Orlicz heart on $(\Omega, \mathcal{F}, \mathbb{P})$ associated with Ψ is defined by

$$H^{\Psi} = H^{\Psi}(\Omega, \mathcal{F}, \mathbb{P}) := \left\{ X \in L^0 : \mathbb{E}[\Psi(c|X|)] < \infty \text{ for all } c > 0 \right\}.$$

It is the largest vector subspace contained in the Orlicz class $Y^{\Psi} = Y^{\Psi}(\Omega, \mathcal{F}, \mathbb{P}) := \{X \in L^0 : \mathbb{E}[\Psi(|X|)] < \infty\}$. The Orlicz class in turn is a convex subset of the Orlicz space $L^{\Psi} = L^{\Psi}(\Omega, \mathcal{F}, \mathbb{P}) := \{X \in L^0 : \mathbb{E}[\Psi(c|X|)] < \infty$ for some $c > 0\}$. In general we have $L^{\infty} \subseteq H^{\Psi} \subseteq Y^{\Psi} \subseteq L^{\Psi} \subseteq L^1$, and these inclusions may all be strict. In fact, it is known form Theorem 2.1.17 (b) in [5] that the identity $H^{\Psi} = L^{\Psi}$ holds if and only if Ψ satisfies the so-called Δ_2 -condition:

There are
$$C, x_0 > 0$$
 such that $\Psi(2x) \le C\Psi(x)$ for all $x \ge x_0$. (18)

This condition is clearly satisfied when specifically $\Psi(x) = x^p/p$ for some $p \in [1, \infty)$. In this case, L^{Ψ} coincides with the usual L^p -space $L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 4.12 Let Ψ be a finite Young function. A law-invariant convex risk measure on H^{Ψ} will be a map $\rho: H^{\Psi} \to \mathbb{R}$ satisfying the following three conditions:

- Monotonicity: $\rho(X) \ge \rho(Y)$ for $X, Y \in H^{\Psi}$ with $X \le Y \mathbb{P}$ -a.s.
- Convexity: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda \rho(X) + (1-\lambda)\rho(Y)$ for $X, Y \in H^{\Psi}$ and $\lambda \in [0,1]$.
- Law-invariance: $\rho(X) = \rho(Y)$ for $X, Y \in H^{\Psi}$ with $\mathbb{P} \circ X^{-1} = \mathbb{P} \circ Y^{-1}$.

In a financial context, one typically requires that a law-invariant convex risk measure ρ is also *monetary* in the sense that it also satisfies the following additional property:

• Cash additivity: $\rho(X+m) = \rho(X) + m$ for $X \in H^{\Psi}$ and $m \in \mathbb{R}$;

see, e.g., [8]. Here, however, cash additivity will not be needed and so we will work with our more general class of not necessarily monetary law-invariant convex risk measures. As argued in [2], Orlicz hearts are natural domains for law-invariant convex risk measures. **Example 4.13** Let $g : [0,1] \to [0,1]$ be concave, nonincreasing, and continuous with g(0) = 0 and g(1) = 1. Let Ψ be a (finite) Young function with the conjugate Ψ^* defined by $\Psi^*(y) := \sup_{x\geq 0} (xy - \Psi(x))$. It was shown in Proposition 2.22 in [14] that if the right-sided derivative g' of g fulfills the condition $\int_0^1 \Psi^*(g'(t)) dt < \infty$, then

$$\rho_g(X) := \int_{-\infty}^0 g(F_X(x)) \, dx - \int_0^\infty \left(1 - g(F_X(x))\right) \, dx$$

defines a monetary law-invariant convex risk measure $\rho_g : H^{\Psi} \to \mathbb{R}$, where F_X stands for the distribution function of X. It is called *distortion risk measure* associated with g. For the specific distortion function $g(t) = (t/\alpha) \wedge 1$ the associated distortion risk measure ρ_g reads as

$$\rho_g(X) = -\frac{1}{\alpha} \int_0^\alpha F_X^{\leftarrow}(\beta) \ d\beta,$$

where F_X^{\leftarrow} denotes the left-continuous quantile function of the distribution F_X . This distortion risk measure is also called *Average Value at Risk* at level $\alpha \in (0, 1)$, and it is denoted by AV@R_{α}.

Example 4.14 Let for finite Young function Ψ the map $\rho_{\Psi}: H^{\Psi} \to \mathbb{R}$ be defined by

$$\rho_{\Psi}(X) := \inf \left\{ m \in \mathbb{R} : \mathbb{E} \left[\Psi \left((-X - m)^+ \right) \right] \le x_0 \right\}$$

for some $x_0 > 0$. This is a monetary law-invariant convex risk measure known as the *utility-based shortfall risk measure* with loss function $\ell_{\Psi} : \mathbb{R} \to \mathbb{R}$ defined by $\ell_{\Psi}(x) := \Psi(x^+)$; cf. e.g. [8] and, for the extension to Orlicz hearts, [14].

Example 4.15 Let specifically $\Psi(x) = x^p/p$ for some $p \in [1, \infty)$. Then $H^{\Psi} = L^p$ and $(\mathcal{M}_1^{(\psi_k)}, \mathcal{O}_{(\psi_k)}) = (\mathcal{M}_1^{\psi}, \mathcal{O}_{\psi}))$ for $\psi(x) = |x|^p/p$. The map $\rho_p : L^p \to \mathbb{R}$ defined by

$$\rho_p(X) := \mathbb{E}[(X^-)^p]$$

obviously defines a law-invariant convex risk measure in the sense of Definition 4.12. Here, $X^- := -\min\{0, X\}$ denotes the negative part of X.

4.2.2 (ψ_k) -weak continuity of the associated risk functionals

Denote by $\mathcal{M}(H^{\Psi})$ the set of the distributions of all random variables from H^{Ψ} and note that

$$\mathcal{M}(H^{\Psi}) = \mathcal{M}_1^{(\psi_k)} \quad \text{for } \psi_k := \Psi(k|\cdot|), \ k \in \mathbb{N}.$$
(19)

The inclusion \subseteq is obvious and the inclusion \supseteq holds because $(\Omega, \mathcal{F}, \mathbb{P})$ is assumed to be atomless. Note that if Ψ satisfies the Δ_2 -condition (18), then $\mathcal{M}(H^{\Psi}) = \mathcal{M}_1^{\psi}$ and $\mathcal{O}_{(\psi_k)} = \mathcal{O}_{\psi}$ for

$$\psi := \Psi(|\cdot|).$$

The law-invariance of ρ is equivalent to the existence of a map $\mathcal{R}_{\rho} : \mathcal{M}_{1}^{(\psi_{k})} \to \mathbb{R}$ such that

$$\rho(X) = \mathcal{R}_{\rho}(\mathbb{P} \circ X^{-1}) \quad \text{for all } X \in H^{\Psi}.$$
(20)

This map \mathcal{R}_{ρ} will be called the *risk functional* associated with ρ .

Theorem 4.16 Let $\mathcal{R}_{\rho} : \mathcal{M}_{1}^{(\psi_{k})} \to \mathbb{R}$ be the risk functional associated with a lawinvariant convex risk measure $\rho : H^{\Psi} \to \mathbb{R}$. Then \mathcal{R}_{ρ} is continuous w.r.t. the (ψ_{k}) -weak topology.

Remark 4.17 As a consequence of Theorem 4.16, the risk functional $\mathcal{R}_{\rho} : \mathcal{M}_{1}^{(\psi_{k})} \to \mathbb{R}$ is weakly continuous on every w-set in $\mathcal{M}_{1}^{(\psi_{k})} = \mathcal{M}(H^{\Psi})$. At the beginning of Section 3 we discussed how to check when a subset of $\mathcal{M}_{1}^{(\psi_{k})}$ is a w-set.

If Ψ satisfies the Δ_2 -condition (18), then the (ψ_k) -weak topology can be replaced by the ψ -weak topology in Theorem 4.16. On the other hand, if Ψ does not satisfy the Δ_2 -condition (18), then we can always find a law-invariant convex risk measure on H^{Ψ} which fails to be continuous w.r.t. the ψ -weak topology:

Example 4.18 Let Ψ be a finite Young function which does not satisfy Δ_2 -condition (18), and let ρ_{Ψ} denote the shortfall risk measure as in Example 4.14. In [14, proof of Theorem 2.8], there was constructed as sequence (X_n) in L^{∞} which converges to δ_0 w.r.t. the ψ -weak topology such that $\sup_n \inf\{m \in \mathbb{R} : \mathbb{E}[\Psi(8(-X_n - m)^+)] \leq x_0\} = \infty$. Hence $Y_n := 8X_n$ defines a sequence (Y_n) in L^{∞} whose laws converge weakly to δ_0 , while $\rho_{\Psi}(Y_n) \to \infty$. In particular ρ_{Ψ} is not continuous w.r.t. the ψ -weak topology.

As an immediate consequence of Theorem 4.16 and Example 4.18 we get the following corollary. The corollary extends Theorem 2.8 of [14], where considerations have been restricted to *monetary* law-invariant convex risk measures.

Corollary 4.19 Let $\mathcal{R}_{\rho} : \mathcal{M}_{1}^{(\psi_{k})} \to \mathbb{R}$ be the risk functional associated with a lawinvariant convex risk measure $\rho : H^{\Psi} \to \mathbb{R}$. Then the following conditions are equivalent:

- (a) For every law-invariant convex risk measure ρ on H^{Ψ} , the map $\mathcal{R}_{\rho} : \mathcal{M}(H^{\Psi}) \to \mathbb{R}$ is continuous for the ψ -weak topology.
- (b) Ψ satisfies the Δ_2 -condition (18).

Let us emphasize as a further implication of Corollary 4.19 that, if Ψ does *not* satisfy the Δ_2 -condition (18), then we cannot apply Theorem 2.6 for the ψ -weak topology but only for the (ψ_k)-weak topology.

4.2.3 Robustness on parametric classes of distributions

Consider the statistical model (1)–(3) with specifically

$$E := \mathbb{R}, \qquad \mathcal{M} := \mathcal{M}(H^{\Psi}), \qquad T := \mathcal{R}_{\rho}, \tag{21}$$

where $\rho: H^{\Psi} \to \mathbb{R}$ is any law-invariant convex risk measure. By Theorem 4.16 we know that $\mathcal{R}_{\rho}: \mathcal{M}(H^{\Psi}) \to \mathbb{R}$ is (ψ_k) -weakly continuous for the sequence (ψ_k) introduced in (19), and it is clear that the set \mathfrak{E} of all empirical probability measures on \mathbb{R} is contained in $\mathcal{M}(H^{\Psi})$. Thus Theorem 2.6 yields that the sequence of estimators (\widehat{T}_n) is robust on every w-set M in $\mathcal{M}(H^{\Psi})$. In particular, the sequence (\widehat{T}_n) is robust on many parametric families $\mathcal{M}_{\Theta} := {\mu_{\theta} : \theta \in \Theta}$ of univariate distributions. This is illustrated by the following examples, which rely on Examples 3.4–3.7.

Example 4.20 The Average Value at Risk AV@R_{α} introduced in Example 4.13 is defined on L^1 , i.e., on H^{Ψ} with $\Psi(x) = x$. Since this Ψ satisfies the Δ_2 -condition (18), we have $\mathcal{O}_{(\psi_k)} = \mathcal{O}_{\psi}$ for $\psi(x) = |x|$. Thus the sequence (\widehat{T}_n) is robust on each of the sets $\mathcal{N}, \mathcal{P}_{\alpha, x_{\min}}$ (with $\alpha > 1$), Γ , and \mathcal{G} introduced in Examples 3.4–3.7.

Example 4.21 Let ρ_{Ψ} be the utility-based shortfall risk measure on H^{Ψ} as introduced in Example 4.14, and let \mathcal{N} , $\mathcal{P}_{\alpha,x_{\min}}$ (with $\alpha > q \ge 1$), Γ , and \mathcal{G} denote the parametric families of distributions from Examples 3.4–3.7. Then the sequence (\hat{T}_n) is robust on

- (a) \mathcal{N} if there exists $\lambda > 0$ such that $\Psi(x) = O(e^{\lambda x^2})$ as $x \uparrow \infty$;
- (b) $\mathcal{P}_{\alpha,x_{\min}}$ if there exists $q \in [1,\alpha)$ such that $\Psi(x) = O(x^q)$ as $x \uparrow \infty$;
- (c) Γ and \mathcal{G} if there is some $\beta \in (0,1)$ such that $\Psi(x) = O(e^{x^{\beta}})$ as $x \uparrow \infty$.

Example 4.22 The risk measure ρ_p introduced in Example 4.15 is defined on L^p , i.e., on H^{Ψ} with $\Psi(x) = x^p/p$. Since this Ψ satisfies the Δ_2 -condition (18), we have $\mathcal{O}_{(\psi_k)} = \mathcal{O}_{\psi}$ for $\psi(x) = |x|^p/p$. Thus the sequence (\widehat{T}_n) is robust on each of the sets $\mathcal{N}, \mathcal{P}_{\alpha, x_{\min}}$ (with $\alpha > p$), Γ , and \mathcal{G} introduced in Examples 3.4–3.7.

4.2.4 Aggregation robustness

For $\mu_1, \ldots, \mu_d \in \mathcal{M}_1$ and $A_d(x_1, \ldots, x_d) := \sum_{i=1}^d x_i$ we let $\mathfrak{S}(\mu_1, \ldots, \mu_d) := \{ \boldsymbol{\mu} \circ A_d^{-1} : \boldsymbol{\mu} \in \mathcal{M}(d; \mu_1, \ldots, \mu_d) \},$

where $\mathcal{M}(d; \mu_1, \ldots, \mu_d)$ denotes the Fréchet class w.r.t. μ_1, \ldots, μ_d ; cf. Subsection 3.2. As before we consider a law-invariant convex risk measure ρ on H^{Ψ} as well as the associated risk functional $\mathcal{R}_{\rho} : \mathcal{M}(H^{\Psi}) \to \mathbb{R}$. If $\mu_1, \ldots, \mu_d \in \mathcal{M}(H^{\Psi})$ are regarded as distributions of single positions Y_1, \ldots, Y_d of a financial portfolio, then the set $\mathfrak{S}(\mu_1, \ldots, \mu_d)$ may be seen as the set of all possible distributions of the portfolio sum $S_d := \sum_{i=1}^d Y_i$. It is argued by Embrechts et al. [6] that it is often relatively easy to model the marginal distributions μ_1, \ldots, μ_d , while it can be difficult to obtain accurate information on the dependence structure of Y_1, \ldots, Y_d . This situation roughly corresponds to the setting where the marginal distributions μ_1, \ldots, μ_d are known but the law of $A_d(Y_1, \ldots, Y_d) = \sum_{i=1}^d Y_i$ can vary within $\mathfrak{S}(\mu_1, \ldots, \mu_d)$. Motivated by this issue, Embrechts et al. [6] raise the question of robustness of the empirical estimator for $\rho(S_d) = \mathcal{R}_{\rho}(\boldsymbol{\mu} \circ A_d^{-1})$ for known marginal distributions μ_1, \ldots, μ_d . More precisely, the statistical model (1)–(3) is specialized to

$$E := \mathbb{R}, \qquad \mathcal{M} := \mathfrak{S}(\mu_1, \dots, \mu_d), \qquad T := \mathcal{R}_{\rho}|_{\mathfrak{S}(\mu_1, \dots, \mu_d)}, \tag{22}$$

where the observations (i.e. the coordinates on $\Omega = \mathbb{R}^{\mathbb{N}}$) should be seen as i.i.d. copies of S_d . Theorem 2.6 and Proposition 3.9 imply that the sequence (\widehat{T}_n) is robust on $\mathfrak{S}(\mu_1, \ldots, \mu_d)$, because the risk functional \mathcal{R}_{ρ} is always (ψ_k) -weakly continuous on its domain $\mathcal{M}(H^{\Psi})$, according to Theorem 4.16. The crucial point is that the set $\mathfrak{S}(\mu_1, \ldots, \mu_d)$ is a w-set in $\mathcal{M}(H^{\Psi}) = \mathcal{M}_1^{(\psi_k)}$ by Proposition 3.9, and so the risk functional \mathcal{R}_{ρ} is weakly continuous on $\mathfrak{S}(\mu_1, \ldots, \mu_d)$. Embrechts et al. [6] referred to the weak continuity of the functional \mathcal{R}_{ρ} on $\mathfrak{S}(\mu_1, \ldots, \mu_d)$ as aggregation robustness of \mathcal{R}_{ρ} . Maybe it is even more appropriate to use the terminology aggregation robustness for the sequence of estimators (\widehat{T}_n) in the statistical model given by (1)–(3) and (22).

The above considerations are not restricted to the particular aggregation function $A_d(x_1, \ldots, x_d) := \sum_{i=1}^d x_i$. The latter can be replaced by any other function $A_d : \mathbb{R}^d \to \mathbb{R}$ satisfying condition (a) of Proposition 3.9. Recall that the set $M(\mu_1, \ldots, \mu_d; A_d)$ was defined in (5) and that $M(\mu_1, \ldots, \mu_d; A_d) = \mathfrak{S}(\mu_1, \ldots, \mu_d)$ when $A_d(x_1, \ldots, x_d) = \sum_{i=1}^d x_i$.

Theorem 4.23 Let $A_d : \mathbb{R}^d \to \mathbb{R}$ be any Borel-measurable map satisfying condition (a) of Proposition 3.9, and let $\mathcal{R}_{\rho} : \mathcal{M}(H^{\Psi}) \to \mathbb{R}$ be the risk functional associated with any law-invariant convex risk measure ρ on H^{Ψ} . Moreover fix $\mu_1, \ldots, \mu_d \in \mathcal{M}(H^{\Psi})$. Then $\mathcal{R}_{\rho}|_{\mathcal{M}(\mu_1,\ldots,\mu_d;A_d)}$ is weakly continuous. In particular, the sequence of estimators (\widehat{T}_n) in the statistical model given by (1)–(3) and (22) is robust. **Proof** Recall that $\psi_k = \Psi(k|\cdot|)$ for $k \in \mathbb{N}$, and let ℓ be any fixed integer exceeding the real number c. By monotonicity of Ψ we have that $\psi_k((d+1)c|x|) \leq \psi_{k(d+1)c}(x)$ for all $x \in \mathbb{R}$. Then the first statement of Theorem 4.23 follows immediately from Proposition 3.9 along with Theorem 4.16. The second statement can then be derived with the help of Theorem 2.6.

5 Proofs of results from Section 2

5.1 Proof of Lemma 2.1

By construction, a base for the (ψ_k) -weak topology is given by sets of the form $U_{k_1} \cap \cdots \cap U_{k_n} \cap \mathcal{M}_1^{(\psi_k)}$, where $n \in \mathbb{N}, k_1, \ldots, k_n \in \mathbb{N}$, and each U_{k_i} belongs to a base for the ψ_{k_i} -weak topology on $\mathcal{M}_1^{\psi_{k_i}}$. Since the ψ_k -weak topology on $\mathcal{M}_1^{\psi_k}$ is metrizable by a separable metric by [8, Corollary A.45] and hence admits a countable base, it follows that the (ψ_k) -weak topology also has a countable base. Then it is known that a subset of $\mathcal{M}_1^{(\psi_k)}$ is closed w.r.t. the (ψ_k) -weak topology if and only if together with any sequence it contains all its accumulation points; cf. Theorem 1.6.14 in [7]. Hence under the equivalence of (a) and (b) the (ψ_k) -weak topology is obviously metrizable by $d_{(\psi_k)}$ as defined in the display of Lemma 2.1.

As a metrizable topology with countable base the (ψ_k) -weak topology is separable. Moreover, by [8, Corollary A.45] the ψ_k -weak topology is completely and separably metrizable by say d_k for every $k \in \mathbb{N}$. Then the equivalence of (a) and (b) implies that the metric d on $\mathcal{M}_1^{(\psi_k)}$ defined by $d(\mu, \nu) := \sum_{k=1}^{\infty} (d_k(\mu, \nu) \wedge 1) 2^{-k}$ metrizes $\mathcal{O}_{(\psi_k)}$. This metric is separable by separability of $\mathcal{O}_{(\psi_k)}$. Now, every d-Cauchy sequence (μ_n) is a d_k -Cauchy sequence for any $k \in \mathbb{N}$. Then by completeness of the metrics d_k $(k \in \mathbb{N})$, we may find for any $k \in \mathbb{N}$ some $\nu_k \in \mathcal{M}_1^{\psi_k}$ such that $d_k(\mu_n, \nu_k) \to 0$ as $n \to \infty$. Since each ψ_k -weak topology is finer than the weak topology, we obtain $\mu_n \to \nu_k$ as $n \to \infty$ for each $k \in \mathbb{N}$. Hence by Hausdorff property of the weak topology, all the ν_k coincide, and thus by definition of the metric d we have $d(\mu_n, \mu) \to 0$ for some $\mu \in \mathcal{M}_1^{(\psi_k)}$. Thus we have shown that d is a complete metric. In particular, $\mathcal{M}_1^{(\psi_k)}$ equipped with $\mathcal{O}_{(\psi_k)}$ is a Polish space. So it is left to show the equivalence of (a) and (b).

The implication (a) \Rightarrow (b) is obvious. Conversely, let statement (b) be satisfied. We have to show that for every $f \in C_{\psi_k}$, $k \in \mathbb{N}$, and $\varepsilon > 0$ there exists some $n_0 \in \mathbb{N}$ such that

$$\left| \int f \, d\mu_n - \int f \, d\mu_0 \right| \le \varepsilon \qquad \text{for all } n \ge n_0.$$
(23)

The left hand side of (23) is bounded above by

$$\left|\int f\mathbb{1}_{\{|f|\leq a\}} d\mu_n - \int f\mathbb{1}_{\{|f|\leq a\}} d\mu_0\right| + \left|\int f\mathbb{1}_{\{|f|>a\}} d\mu_n - \int f\mathbb{1}_{\{|f|>a\}} d\mu_0\right|$$
(24)

for every a > 0. For notational simplicity we set $\tilde{\psi}_k := 1 + \psi_k$. Then the second summand in (24) is bounded above by

$$C_{f,k} \int \widetilde{\psi}_k \mathbb{1}_{\{\widetilde{\psi}_k > a\}} d\mu_n + C_{f,k} \int \widetilde{\psi}_k \mathbb{1}_{\{\widetilde{\psi}_k > a\}} d\mu_0$$
(25)

for some suitable constant $C_{f,k} > 0$ satisfying $|f(x)| \leq C_{f,k} \widetilde{\psi}_k(c)$ for all $x \in E$. Now we can choose a > 0 so large that the second summand in (25) is at most $\varepsilon/5$. The first summand in is bounded above by

$$C_{f,k} \left| \int \widetilde{\psi}_k \mathbb{1}_{\{\widetilde{\psi}_k > a\}} d\mu_n - \int \widetilde{\psi}_k \mathbb{1}_{\{\widetilde{\psi}_k > a\}} d\mu_0 \right| + C_{f,k} \int \widetilde{\psi}_k \mathbb{1}_{\{\widetilde{\psi}_k > a\}} d\mu_0$$
(26)

As see above, the second summand in (26) is at most $\varepsilon/5$. The first summand in (26) is bounded above by

$$C_{f,k} \left| \int \widetilde{\psi}_k \, d\mu_n - \int \widetilde{\psi}_k \, d\mu_0 \right| + C_{f,k} \left| \int \widetilde{\psi}_k \mathbb{1}_{\{\widetilde{\psi}_k \le a\}} \, d\mu_n - \int \widetilde{\psi}_k \mathbb{1}_{\{\widetilde{\psi}_k \le a\}} \, d\mu_0 \right|. \tag{27}$$

The first summand in (27) converges to 0 as $n \to \infty$ by assumption. Thus we can find $n_0 \in \mathbb{N}$ such that it is bounded above by $\varepsilon/5$ for every $n \ge n_0$. Since $\mu_0 \circ \widetilde{\psi}_k^{-1}$ as a probability measure on the real line has at most countably many atom, we may and do assume that a > 0 is chosen such that $\mu_0[\{\widetilde{\psi}_k = a\}] = 0$. Since $\mu_n \to \mu_0$ weakly by assumption, it follows by the portmanteau theorem that the second summand in (27) converges to 0 as $n \to \infty$. By possibly increasing n_0 we obtain that the second summand in (27) is at most $\varepsilon/5$ for all $n \ge n_0$. So far we have shown that the second summand in (24) is bounded above by $4\varepsilon/5$ for all $n \ge n_0$. Using the same arguments as for second summand in (24) is bounded above by $\varepsilon/5$ for all $n \ge n_0$. That is, we indeed arrive at (23).

5.2 Proof of Theorem 2.3

First we shall provide the following characterization of relative compact subsets for the (ψ_k) -weak topology, which will be needed in the proof of Lemma 5.2. For the ψ -weak topology this characterization is already known from Corollary A.47 in [8].

Lemma 5.1 Let (ψ_k) be any sequence of gauge functions and $M \subseteq \mathcal{M}_1^{(\psi_k)}$ be given. Then the following conditions are equivalent:

- (a) M is relatively compact for the (ψ_k) -weak topology.
- (b) For every $k \in \mathbb{N}$, M is relatively compact for the ψ_k -weak topology.

(c) For every $k \in \mathbb{N}$ and $\varepsilon > 0$ there exists a compact set $K_k \subseteq E$ such that

$$\sup_{\mu \in M} \int_{K_k^{\mathbf{c}}} \psi_k \, d\mu \le \varepsilon.$$

(d) For every $k \in \mathbb{N}$ there exists a measurable function $\phi_k : E \to \mathbb{R}_+$ such that each set $\{\phi_k \leq n\psi_k\}, n \in \mathbb{N}$, is compact in E and such that

$$\sup_{\mu \in M} \int \phi_k \, d\mu < \infty.$$

Proof (b) \Leftrightarrow (c) \Leftrightarrow (d): These implications follow immediately from Corollary A.47 in [8].

(a) \Leftrightarrow (b): Since the (ψ_k) -weak and the ψ_k -weak topologies are metrizable, for any of these topologies the relatively compact subsets are exactly the relatively sequentially compact ones. Then the implication (a) \Rightarrow (b) is obvious. To prove the implication (b) \Rightarrow (a), let M be relatively compact for the ψ_k -weak topology for each $k \in \mathbb{N}$. In particular, every ψ_k -weak closure M_k of M in $\mathcal{M}_1^{\psi_k}$ is ψ_k -weakly compact. Then by Tychonoff's theorem the set $\times_{k=1}^{\infty} M_k$ is a compact subset of $\times_{k=1}^{\infty} \mathcal{M}_1^{\psi_k}$ for the product topology generated by the ψ_k -weak topologies. Notice that the product topology is metrizable by the metric

$$d_{\text{prod}}(\boldsymbol{\mu}, \boldsymbol{\nu}) := \sum_{k=1}^{\infty} 2^{-k} \big(d_{\psi_k}(\boldsymbol{\mu}(k), \boldsymbol{\nu}(k)) \wedge 1 \big).$$

Let (μ_n) be any sequence in M. We will construct a (ψ_k) -weakly converging subsequence. For every $n \in \mathbb{N}$ we obtain an element $\boldsymbol{\mu}_n \in \times_{k=1}^{\infty} M_k$ by setting $\boldsymbol{\mu}_n(k) := \mu_n, k \in \mathbb{N}$. By compactness of $\times_{n=1}^{\infty} M_k$ we may extract a subsequence $(\boldsymbol{\mu}_{n(j)})$ from $(\boldsymbol{\mu}_n)$ that converges to some $\boldsymbol{\mu} \in \times_{k=1}^{\infty} \mathcal{M}_1^{\psi_k}$, i.e., $d_{\text{prod}}(\boldsymbol{\mu}_{n(j)}, \boldsymbol{\mu}) \to 0$. In particular, $d_{\psi_k}(\boldsymbol{\mu}_{n(j)}(k), \boldsymbol{\mu}(k)) \to 0$ for every $k \in \mathbb{N}$, i.e., $(\boldsymbol{\mu}_{n(j)}(k))$ converges ψ_k -weakly to $\boldsymbol{\mu}(k)$ for every $k \in \mathbb{N}$. Now, if we can show that $\boldsymbol{\mu}(k) = \boldsymbol{\mu}(1) =: \mu$ holds for every $k \in \mathbb{N}$, then it follows that $\mu_{n(j)} \to \mu$ ψ_k -weakly for every $k \in \mathbb{N}$ and thus $\mu_{n(j)} \to \mu$ (ψ_k) -weakly. In the rest of the proof we show that $\boldsymbol{\mu}(k) = \boldsymbol{\mu}(1)$ holds for every $k \in \mathbb{N}$.

For fixed $k \in \mathbb{N}$, the set M is a subset of $\mathcal{M}_1^{\psi_1 + \psi_k}$. Since M is also a relatively ψ_i compact subset of $\mathcal{M}_1^{\psi_i}$ we may find by (c) for every $\varepsilon > 0$ some compact subset $K_i \subseteq E$ such that

$$\sup_{\mu \in M} \int_{K_i^{\mathsf{c}}} \psi_i \ d\mu \le \varepsilon/2$$

for $i \in \{1, k\}$. Then $K := K_1 \cup K_k$ is a compact subset of E such that

$$\sup_{\mu \in M} \int_{K^{\mathsf{c}}} (\psi_1 + \psi_k) \, d\mu \le \sup_{\mu \in M} \int_{K_1^{\mathsf{c}}} \psi_1 \, d\mu + \sup_{\mu \in M} \int_{K_k^{\mathsf{c}}} \psi_k \, d\mu \le \varepsilon.$$

Hence in view of Corollary A.47 in [8] the set M is also a relatively compact subset of $\mathcal{M}_1^{\psi_{1,k}}$ for the $\psi_{1,k}$ -weak topology with $\psi_{1,k} := \psi_1 + \psi_k$. Therefore we may select a subsequence $(\mu_{n(j(\iota))})$ of $(\mu_{n(j)})$ which converges $\psi_{1,k}$ -weakly to some $\mu \in \mathcal{M}_1^{\psi_{1,k}}$. In particular, for every $f \in C_{\psi_1} \cup C_{\psi_k}$ we clearly have $f \in C_{\psi_{1,k}}$ and thus $\int f d\mu_{n(j(\iota))} \rightarrow \int f d\mu$. This means that $(\mu_{n(j(\iota))})$ converges to μ w.r.t. both the ψ_1 -weak topology and the ψ_k -weak topology. This implies $\mu(1) = \mu = \mu(k)$, and the proof is complete. \Box

In the case where each set $\{\psi \leq n\}, n \in \mathbb{N}$, is relatively compact in E, a set $M \subseteq \mathcal{M}_1^{\psi}$ is relatively compact for the ψ -weak topology if and only if it is uniformly ψ -integrating; cf. Lemma 3.4 in [22]. The following Lemma shows that the same is true for general gauge functions (and for sequences (ψ_k) of general gauge functions) when M is assumed to be relatively compact for the weak topology.

Lemma 5.2 Let (ψ_k) be any sequence of gauge functions and $M \subseteq \mathcal{M}_1$ be given. Then the following conditions are equivalent:

- (a) M is uniformly (ψ_k) -integrating and relatively compact for the weak topology.
- (b) M is relatively compact for the (ψ_k) -weak topology.

Proof (a) \Rightarrow (b): Let $k \in \mathbb{N}$ and $\varepsilon > 0$ be given. Since M is assumed to be uniformly (ψ_k) -integrating, there exists $a_k > 0$ such that $\sup_{\mu \in M} \int \psi_k \mathbb{1}_{\{\psi_k > a_k\}} d\mu \leq \varepsilon/2$. Since M is assumed to be weakly relatively compact, we moreover obtain by Prohorov's theorem a compact set $C_k \subseteq E$ such that $\sup_{\mu \in M} \mu[C_k^c] \leq \varepsilon/(2a_k)$. The set $K_k := C_k \cap \{\psi_k \leq a_k\}$ is a compact subset of E and satisfies $K_k^c = \{\psi_k > a_k\} \cup (C_k^c \cap \{\psi_k \leq a_k\})$. Hence,

$$\sup_{\mu \in M} \int_{K_k^{\mathbf{c}}} \psi_k \, d\mu \, \leq \, \sup_{\mu \in M} \int \psi_k \mathbb{1}_{\{\psi_k > a_k\}} \, d\mu + \sup_{\mu \in M} \int_{C_k^{\mathbf{c}}} \psi_k \mathbb{1}_{\{\psi_k \le a_k\}} \, d\mu \, \leq \, \varepsilon.$$

It follows by the implication (c) \Rightarrow (a) of Lemma 5.1 that M is relatively compact for the (ψ_k) -weak topology.

(b) \Rightarrow (a): By the implication (a) \Rightarrow (b) of Lemma 5.1 the set M is ψ_k -weakly relatively compact for each $k \in \mathbb{N}$. Hence M is uniformly ψ_k -integrating for each $k \in \mathbb{N}$ due to Lemma A.2 in [14]. Moreover, relative compactness of M for the weak topology follows from the fact that the weak topology is coarser than the (ψ_k) -weak topology. \Box

Proof of Theorem 2.3: (b) \Rightarrow (c): Let $M_0 \subseteq M$ be weakly compact, and fix $\varepsilon > 0$ and $k \in \mathbb{N}$. By assumption there exists for every $\mu \in M_0$ some weakly open neighborhood U_{μ} of μ and some $a_{\mu} > 0$ such that $\int \psi_k \mathbb{1}_{\{\psi_k \ge a_{\mu}\}} d\nu < \varepsilon$ for all $\nu \in U_{\mu} \cap M$. By weak compactness of M_0 we can extract a finite cover of M_0 consisting of such neighborhoods $U_{\mu_1}, \ldots, U_{\mu_m}$ (with $\mu_1, \ldots, \mu_m \in M$), and it follows that $\sup_{\nu \in M_0} \int \psi_k \mathbb{1}_{\{\psi_k \ge a\}} d\nu \le \varepsilon$ if we take $a := \max_{i=1,\ldots,m} a_{\mu_i}$. (c) \Rightarrow (b): Let us suppose by way of contradiction that there exist $\mu \in M$, $k \in \mathbb{N}$, $\varepsilon > 0$, and a sequence (ν_n) in M such that $\nu_n \to \mu$ weakly but $\int \psi_k \mathbb{1}_{\{\psi_k \ge n\}} d\nu_n \ge \varepsilon$ for all n. Then $\{\nu_1, \nu_2, \ldots\} \cup \{\mu\}$ is weakly compact and not uniformly (ψ_k) -integrating. This gives a contradiction.

(c) \Rightarrow (a): Since both topologies are metrizable, it suffices to show that they coincide on any given weakly compact set $M_0 \subseteq M$. By (c) and Lemma 5.2, M_0 is compact for the (ψ_k) -weak topology, and so the two topologies coincide on M_0 by Lemma 3.1.

(a) \Rightarrow (c): Every weakly compact subset of M is also (ψ_k) -weakly compact due to (a), and hence uniformly (ψ_k) -integrating by Lemma 5.2.

(c) \Leftrightarrow (d): The implication (c) \Rightarrow (d) is obvious. Conversely suppose by way of contradiction that (d) holds but that there is a weakly compact $M_0 \subseteq M$ that is not uniformly (ψ_k) -integrating. Then there exist $k \in \mathbb{N}$, $\varepsilon > 0$, and a sequence (μ_n) in M such that $\int \psi_k \mathbb{1}_{\{\psi_k \ge n\}} d\mu_n \ge \varepsilon$ for all n. By selecting a weakly convergent subsequence we arrive at a contradiction to (d).

(a) \Leftrightarrow (e): This equivalence is obvious since both topologies are metrizable. \Box

5.3 Proof of Theorem 2.6

5.3.1 Proof of part (i)

The proof of this part is organized as follows. Below we will show that conditions (a)–(b) of Lemma 5.3 and conditions (c)–(d) of Lemma 5.4 are satisfied for every w-set $M \subseteq \mathcal{M}$ $(\subseteq \mathcal{M}_1^{(\psi_k)})$. Then, if for any w-set $M \subseteq \mathcal{M}$ the functional T is continuous at every $\mu \in M$ for the relative (ψ_k) -weak topology $\mathcal{O}_{(\psi_k)} \cap \mathcal{M}$, the (ψ_k) -robustness of the sequence (\widehat{T}_n) on M is a consequence of the two lemmas and the fact that $\mathcal{O}_{w} \cap M = \mathcal{O}_{(\psi_k)} \cap M$ for every w-set M.

Lemma 5.3 Let $M \subseteq \mathcal{M}$ and assume that the following two conditions hold:

- (a) $T: \mathcal{M} \to \Sigma$ is $(d_{(\psi_k)}, d_{\Sigma})$ -continuous at every $\mu \in M$.
- (b) For every $\mu \in M$, $\varepsilon > 0$, and $\eta > 0$ there are some $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\nu \in M$$
, $d_{(\psi_k)}(\mu, \nu) \leq \delta \implies \mathbb{P}^{\nu} \left[d_{(\psi_k)}(\widehat{m}_n, \nu) \geq \eta \right] \leq \varepsilon \text{ for all } n \geq n_0.$

Then for every $\mu \in M$ and $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and an open neighborhood $U_{(\psi_k)} = U_{(\psi_k)}(\mu, \varepsilon; M)$ of μ for the relative (ψ_k) -weak topology $\mathcal{O}_{(\psi_k)} \cap M$ such that

$$\nu \in U_{(\psi_k)} \implies \pi(\mathbb{P}^{\mu} \circ \widehat{T}_n^{-1}, \mathbb{P}^{\nu} \circ \widehat{T}_n^{-1}) \le \varepsilon \quad \text{for all } n \ge n_0.$$

Proof Note that the proof of Theorem 2.1 in [22] still works when in assumption (a) of this theorem one only requires that the sequence (V_n) is asymptotically $(d_{\Upsilon}, d_{\Sigma})$ continuous at every point of Θ_0 (and not on all of Θ); take into account that in the

proof the asymptotic continuity of (V_n) is used only subsequent to (41). Further note that in [22] the assumption that the metric space (Υ, d_{Υ}) be complete and separable is superfluous (and nowhere used). Then the claim follows by (the generalization of) Theorem 2.1 in [22] with $(\Upsilon, d_{\Upsilon}) := (\mathcal{M}, d_{(\psi_k)}), U(\mu) := \mu, V_n := T$ for all $n \in \mathbb{N}$, and $\widehat{U}_n(x_1, x_2, \ldots) := \widehat{m}_n(x_1, \ldots, x_n).$

For every $n \in \mathbb{N}$ we equip the *n*-fold product space E^n with the product topology. Note that the corresponding Borel σ -field coincides with the *n*-fold product $\mathcal{B}(E)^{\otimes n}$ of the Borel σ -algebra $\mathcal{B}(E)$ on E, and let π_n be any metric that metrizes the weak topology on the set of all probability measures on $(E^n, \mathcal{B}(E)^{\otimes n})$. Let X_i be the *i*-th coordinate projection on $\Omega = E^{\mathbb{N}}$ and let $\hat{t}_n : E^n \to \Sigma$ be the estimator \hat{T}_n regarded as a map on E^n ; recall (3) and note that $\hat{m}_n(x)$ depends only on the first *n* coordinates of $x = (x_1, x_2, \ldots) \in E^{\mathbb{N}}$.

Lemma 5.4 Let $M \subseteq \mathcal{M}$ and assume that the following two conditions hold:

- (c) $E^n \ni (x_1, \ldots, x_n) \mapsto \hat{t}_n(x_1, \ldots, x_n)$ is continuous for every $n \in \mathbb{N}$.
- (d) $M \ni \mu \mapsto \mathbb{P}^{\mu} \circ (X_1, \ldots, X_n)^{-1}$ is (d_w, π_n) -continuous for every $n \in \mathbb{N}$.

Then for every $\mu \in M$, $n \in \mathbb{N}$, and $\varepsilon > 0$ there exist and an open neighborhood $U_{(\psi_k)} = U_{(\psi_k)}(\mu, \varepsilon; M)$ of μ for the relative (ψ_k) -weak topology $\mathcal{O}_{(\psi_k)} \cap M$ such that

$$\nu \in U_{(\psi_k)} \implies \pi(\mathbb{P}^{\mu} \circ \widehat{T}_n^{-1}, \mathbb{P}^{\nu} \circ \widehat{T}_n^{-1}) \leq \varepsilon.$$

Proof The lemma is a direct consequence of Theorem 2.5 and Example 2.6 in [22]. \Box

As already discussed at the beginning of the proof, it remains to show that conditions (a)–(d) are satisfied, where for (b) we have to assume that $M \subseteq \mathcal{M}$ is a w-set in $\mathcal{M}_1^{(\psi_k)}$. (a): Condition (a) holds by assumption.

(a): Condition (a) holds by assumption. (b): To verify condition (b) for any fixed w-set $M \subseteq \mathcal{M} (\subseteq \mathcal{M}_1^{(\psi_k)})$, we assume without

(b): To verify condition (b) for any fixed w-set $M \subseteq \mathcal{M} (\subseteq \mathcal{M}_1^{(n)})$, we assume without loss of generality that the metric d_w in (4) is given by the Prohorov metric d_P , i.e.,

$$d_{(\psi_k)}(\mu,\nu) = d_{\mathcal{P}}(\mu,\nu) + \sum_{k=1}^{\infty} 2^{-k} \Big(\Big| \int \psi_k \, d\mu_1 - \int \psi_k \, d\mu_2 \Big| \wedge 1 \Big).$$

Let $\mu \in M$, $\varepsilon > 0$, and $\eta > 0$ be fixed. Choose $k_{\eta} \in \mathbb{N}$ so large such that $\sum_{k=k_{\varepsilon}+1}^{\infty} 2^{-k} < \eta/3$. Then, for every $\nu \in \mathcal{M}$,

$$\mathbb{P}^{\nu}\left[d_{(\psi_{k})}(\widehat{m}_{n},\nu) \geq \eta\right]$$

$$\leq \mathbb{P}^{\nu}\left[d_{(\psi_{k})}(\widehat{m}_{n},\nu) \geq \eta/3\right] + \sum_{k=1}^{k_{\varepsilon}} \mathbb{P}^{\nu}\left[2^{-k} \left|\int \psi_{k} \, d\widehat{m}_{n} - \int \psi_{k} \, d\nu\right| \geq \eta/(3k_{\varepsilon})\right].$$

Lemma 4 in [17] shows that $\lim_{n\to\infty} \sup_{\nu\in\mathcal{M}_1} \mathbb{P}^{\nu}[d_{\mathbb{P}}(\widehat{m}_n,\nu) \geq \eta] = 0$. So we can find some $n_{\mathbb{P}} \in \mathbb{N}$ such that

$$\sup_{\nu \in \mathcal{M}_1} \mathbb{P}^{\mu}[d_{\mathcal{P}}(\widehat{m}_n, \nu) \ge \eta] \le \varepsilon/2 \quad \text{for all } n \ge n_{\mathcal{P}}.$$
(28)

So it remains to show that for every $k = 1, ..., k_{\varepsilon}$ there exist $\delta_k > 0$ and $n_k \in \mathbb{N}$ such that

$$\nu \in M, \ d_{(\psi_k)}(\mu, \nu) \le \delta_k \implies \mathbb{P}^{\nu} \left[\left| \int \psi_k \, d\widehat{m}_n - \int \psi_k \, d\nu \right| \ge \frac{2^k \eta}{3k_{\varepsilon}} \right] \le \frac{\varepsilon}{2k_{\varepsilon}} \text{ for all } n \ge n_k.$$
(29)

By choosing $\delta := \min\{\delta_1, \ldots, \delta_{k_{\varepsilon}}\}$ and $n_0 := \max\{n_{\mathbf{P}}, n_1, \ldots, n_{k_{\varepsilon}}\}$ we then obtain (b).

To prove (29), we take into account that M is a w-set in $\mathcal{M}_1^{(\psi_k)}$. By Theorem 2.3 this means that M is locally uniformly (ψ_k) -integrating. Thus for every $k \in \mathbb{N}$ we can find some $\delta_k > 0$ and $a_k > 0$ such that $\int \psi_k \mathbb{1}_{\{\psi_k \ge a_k\}} d\nu < \min\{\frac{2^k \eta}{3k_{\varepsilon}}; \frac{2^k \eta \varepsilon}{9k_{\varepsilon} 2}\}$ for all $\nu \in M$ with $d_{\mathrm{P}}(\mu_1, \mu_2) \le \delta$. For every $\nu \in M$ with $d_{\mathrm{P}}(\mu, \nu) \le \delta_k$ we then obtain

$$\begin{aligned} \mathbb{P}^{\nu} \Big[\Big| \int \psi_k \, d\widehat{m}_n - \int \psi_k \, d\nu \Big| &\geq \frac{2^k \eta}{3k_{\varepsilon}} \Big] \\ &\leq \mathbb{P}^{\nu} \Big[\int \psi_k \mathbb{1}_{\{\psi_k \geq a_k\}} \, d\widehat{m}_n \geq \frac{2^k \eta}{9k_{\varepsilon}} \Big] \\ &\quad + \mathbb{P}^{\nu} \Big[\Big| \int \psi_k \mathbb{1}_{\{\psi_k < a_k\}} \, d\widehat{m}_n - \int \psi \mathbb{1}_{\{\psi < a_k\}} \, d\nu \Big| \geq \frac{2^k \eta}{9k_{\varepsilon}} \Big] \\ &\quad + \mathbb{P}^{\nu} \Big[\int \psi_k \mathbb{1}_{\{\psi_k \geq a_k\}} \, d\nu \geq \frac{2^k \eta}{9k_{\varepsilon}} \Big] \\ &=: S_1(k, n, a_k) + S_2(k, n, a_k) + S_3(k, a_k), \end{aligned}$$

where $S_3(k, a_k) = 0$ and $S_1(k, n, a_k) \leq (9k_{\varepsilon}/(2^k\eta) \int \psi \mathbb{1}_{\{\psi_k \geq a_k\}} d\nu \leq \varepsilon/2$ for all $n \in \mathbb{N}$ (by Markov's inequality). Further, by Chebychev's inequality we can find some $n_k \in \mathbb{N}$ such that $S_2(k, n, a) \leq \varepsilon/2$ for all $n \geq n_k$ (and all $\nu \in \mathcal{M}_1$). This proves (29) with $d_{(\psi_k)}$ replaced by $d_{\mathbb{P}}$. Since $d_{\mathbb{P}} \leq d_{(\psi_k)}$, we arrive at (29).

(c): The mapping $(x_1, \ldots, x_n) \mapsto \hat{t}_n(x_1, \ldots, x_n) = T(\frac{1}{n} \sum_{i=1}^n \delta_{x_i})$ is continuous, because the statistical functional T is $(d_{(\psi_k)}, d_{\Sigma})$ -continuous by assumption and the mapping $(x_1, \ldots, x_n) \mapsto \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$ is easily seen to be $(d_{E^n}, d_{(\psi_k)})$ -continuous, where d_{E_n} is any metric which metrizes the product topology on E^n .

(d): The (d_w, π_n) -continuity of the mapping $M \ni \mu \mapsto \mathbb{P}^{\mu} \circ (X_1, \ldots, X_n)^{-1} = \mu^{\otimes n}$ for every $n \in \mathbb{N}$ is obvious. too.

5.3.2 Proof of part (ii)

Now assume that (\widehat{T}_n) is (ψ_k) -robust and weakly consistent. The (ψ_k) -robustness means that (\widehat{T}_n) is robust on every w-set $M \subseteq \mathcal{M} (\subseteq \mathcal{M}_1^{(\psi_k)})$. By the classical Hampel theorem

(Theorem 1 in [4]) we can conclude that $T|_M$ is weakly continuous for every w-set $M \subseteq \mathcal{M}(\subseteq \mathcal{M}_1^{(\psi_k)})$. In the remainder we will show that this implies (ψ_k) -weak continuity of T. Let $\mu, \mu_1, \mu_2, \ldots \in \mathcal{M}$ such that $\mu_n \to \mu$ (ψ_k) -weakly. Since $T|_M$ is (d_w, d_Σ) -continuous for every w-set $M \subseteq \mathcal{M}(\subseteq \mathcal{M}_1^{(\psi_k)})$, it suffice to show that the set $M := \{\mu, \mu_1, \mu_2, \ldots\}$ is a w-set in $M \subseteq \mathcal{M}(\subseteq \mathcal{M}_1^{(\psi_k)})$. By assumption, the set M is compact for the (ψ_k) -weak topology since this topology is metrizable. Thus by Lemma 3.1 the set M is also a w-set in $M \subseteq \mathcal{M}$. This completes the proof.

6 Remaining proofs

6.1 Proof of Proposition 3.9

For every $i = 1, \ldots, d$ we define $\mu'_i := \mu_i \circ f_{d,c}^{-1}$ for $f_{d,c}(x) := (d+1)cx$. By assumption (b) $\{\mu'_1, \ldots, \mu'_d\}$ is a finite subset of $\mathcal{M}_1^{(\psi_k)}$, and thus uniformly (ψ_k) -integrating. In view of de la Vallée-Poussin theorem for sets of measures (analogue Theorem II.T22 in [16]) one can thus find for every $k \in \mathbb{N}$ a convex and increasing function $h_k : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{x\to\infty} h_k(x)/x = \infty$ and

$$\max_{i=1,\dots,d} \int h_k(\psi_k) \, d\mu'_i < \infty. \tag{30}$$

Since ψ_k is convex and nonnegative, it is also nondecreasing on $[0, \infty)$. In addition ψ_k is assumed to be even, so that the composition $h_k \circ \psi_k = h_k \circ \psi_k(|\cdot|)$ is convex. Together with assumption (a) and (30) this yields

$$\begin{split} \int h_k(\widetilde{\psi}_k) \, d\boldsymbol{\mu} \circ A_d^{-1} &= \int h_k \circ \widetilde{\psi}_k(A_d(\boldsymbol{x})) \, \boldsymbol{\mu}(d\boldsymbol{x}) \\ &= \int h_k \circ \psi_k(\|A_d(\boldsymbol{x})\|) \, \boldsymbol{\mu}(d\boldsymbol{x}) \\ &\leq \int h_k \circ \psi_k \Big(b + c \sum_{i=1}^d |x_i| \Big) \, \boldsymbol{\mu}(d\boldsymbol{x}) \\ &\leq \int \frac{1}{d+1} \sum_{i=0}^d h_k \circ \psi_k \big((d+1)cx_i \big) \, \boldsymbol{\mu}(d\boldsymbol{x}) \\ &\leq h_k \circ \psi_k \big((d+1)b \big) \, \lor \, \max_{j=1,\dots,d} \int h_k \circ \psi_k \big((d+1)cx \big) \, \mu_j(dx) \\ &= h_k \circ \psi_k \big((d+1)b \big) \, \lor \, \max_{j=1,\dots,d} \int h_k \circ \psi_k(x) \, \mu_j'(dx) \, < \, \infty \end{split}$$

for all $k \in \mathbb{N}$ and $\boldsymbol{\mu} \in \mathcal{M}(d; N)$, where we used the convention $x_0 := b/c$. This implies

$$\sup_{\nu \in M} \int h_k(\widetilde{\psi}_k) \, d\nu \, < \, \infty$$

for all $k \in \mathbb{N}$, and by another application of the de la Vallée-Poussin theorem for sets of measures we can conclude that M is uniformly $(\tilde{\psi}_k)$ -integrable. \Box

6.2 Proof of Theorem 4.3

Let $(\mu_n)_{n\in\mathbb{N}}$ be any sequence in \mathcal{M} such that $\mathcal{L}_{\mu_n}(\theta_k) \to \mathcal{L}_{\mu}(\theta_k)$ holds for every $k \in \mathbb{N}$. In particular the sequence $(\mathcal{L}_{\mu_n})_{n\in\mathbb{N}}$ converges pointwise on a dense subset of Θ to \mathcal{L}_{μ} . Together with the concavity of \mathcal{L}_{μ} and \mathcal{L}_{μ_n} , $n \in \mathbb{N}$, this implies that $(\mathcal{L}_{\mu_n})_{n\in\mathbb{N}}$ converges even pointwise to \mathcal{L}_{μ} ; cf. Corollary 7.18 in [18].

Further, by assumption $\arg \max_{\theta \in \Theta} \mathcal{L}_{\mu}(\theta) = \{T(\mu)\}$ and $T(\mu_n) \in \arg \max_{\theta \in \Theta} \mathcal{L}_{\mu_n}(\theta)$ for every $n \in \mathbb{N}$. Since $(\mathcal{L}_{\mu_n})_{n \in \mathbb{N}}$ is a sequence of concave maps which converges pointwise to the concave map \mathcal{L}_{μ} , we may draw on well-known results concerning stability of convex minimization (e. g. Theorem 5.3.25(f) in [12]) to conclude $T(\mu_n) \to T(\mu)$. So the first part of Theorem 4.3 is shown. The remaining part follows immediately from the first part, because convergence $\mu_n \to \mu$ w.r.t. the (ψ_k) -weak topology implies $\mathcal{L}_{\mu_n}(\theta_k) \to \mathcal{L}_{\mu}(\theta_k)$ for every $k \in \mathbb{N}$. Now, the proof is complete. \Box

6.3 Proof of Theorem 4.16

It is known from Theorem 2.1.11 in [5] that the Orlicz heart H^{Ψ} is a Banach space when endowed with the Luxemburg norm

$$||X||_{\Psi} := \inf \left\{ \lambda > 0 : \mathbb{E}[\Psi(|X|/\lambda)] \le 1 \right\}.$$

Moreover, we may observe that $||X||_{\Psi} \leq ||X||_{\Psi}$ whenever $|X| \leq |Y|$ P-a.s. This means that H^{Ψ} equipped with $|| \cdot ||_{\Psi}$ and the P-a.s. order is a Banach lattice. It follows by Proposition 3.1 in [20] that

$$\rho \text{ is continuous w.r.t. } \|\cdot\|_{\Psi}.$$
(31)

The missing link between (31) and Theorem 4.16 is provided by the following representation result which is interesting in its own right. Recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless so that it supports a random variable which is uniformly distributed on the open unit interval.

Theorem 6.1 A sequence (μ_n) in $\mathcal{M}_1^{(\psi_k)}$ converges w.r.t. the (ψ_k) -weak topology to some $\mu_0 \in \mathcal{M}_1^{(\psi_k)}$ if and only if $||F_{\mu_n}^{\leftarrow}(U) - F_{\mu_0}^{\leftarrow}(U)||_{\Psi} \to 0$, where U is an arbitrary random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ that is uniformly distributed on (0, 1).

Proof We let $X_n := F_{\mu_n}^{\leftarrow}(U)$ and prove first that $||X_n - X_0||_{\Psi} \to 0$ implies that $\mu_n \to \mu_0$ in the (ψ_k) -weak topology. By Proposition 2.1.10 in [5], $||X_n - X_0||_{\Psi} \to 0$ yields $\mathbb{E}[\psi_{2k}(X_n - X_0)] \to 0$ for all $k \in \mathbb{N}$ and $X_n \to X_0$ in probability. Convexity and monotonicity of Ψ imply that $0 \leq \psi_k(X_n) \leq \frac{1}{2}\psi_{2k}(X_n - X_0) + \frac{1}{2}\psi_{2k}(X_0)$. Hence,

 $\psi_k(X_n)$ is uniformly integrable, and we obtain by Vitali's theorem in the form of [11, Proposition 3.12 (iii) \Rightarrow (ii)] that

$$\int \psi_k(x) \,\mu_n(dx) = \mathbb{E}[\psi_k(X_n)] \longrightarrow \mathbb{E}[\psi_k(X_0)] = \int \psi_k(x) \,\mu_0(dx)$$

Moreover, since $X_n \to X_0 \mathbb{P}$ -a.s., the corresponding laws (μ_n) converge weakly. It follows that (μ_n) converges to μ w.r.t. the (ψ_k) -weak topology.

Conversely, assume that $\mu_n \to \mu_0$ in the (ψ_k) -weak topology. Then $\mu_n \to \mu_0$ weakly, and the continuity of Ψ and the fact that $\Psi(0) = 0$ yield that

$$\psi_k(X_n) \longrightarrow \psi_k(X_0) \qquad \mathbb{P}\text{-a.s. for all } k \ge 0,$$
(32)

$$\psi_k(X_n - X_0) \longrightarrow 0 \qquad \mathbb{P}\text{-a.s. for all } k \ge 0.$$
 (33)

Moreover, the convergence $\mu_n \to \mu_0$ w.r.t. the (ψ_k) -weak topology implies that

$$\mathbb{E}[\psi_k(X_n)] = \int \psi_k(x) \,\mu_n(dx) \longrightarrow \int \psi_k(x) \,\mu_0(dx) = \mathbb{E}[\psi_k(X_0)]. \tag{34}$$

In particular each expectation $\mathbb{E}[\psi_k(X_n)]$ is finite so that we have $X_n \in H^{\Psi}$. Now, (32), (34), and Vitali's theorem in the form of [11, Proposition 3.12 (ii) \Rightarrow (iii)] imply that the sequence $(\psi_k(X_n))_{n\in\mathbb{N}_0}$ is uniformly integrable for every k. Since Ψ is nondecreasing and convex we obtain $\psi_k(X_n-X_0) \leq \frac{1}{2}\psi_{2k}(X_0) + \frac{1}{2}\psi_{2k}(X_0)$. Since the sequence $(\psi_{2k}(X_n))_{n\in\mathbb{N}_0}$ is uniformly integrable, we may thus conclude that the sequence $(\psi_k(X_n - X_0))_{n\in\mathbb{N}}$ is uniformly integrable. Therefore, (33) and another application of Vitali's theorem, this time in the form of [11, Proposition 3.12 (iii) \Rightarrow (ii)], yield $\mathbb{E}[\psi_k(X_n - X_0)] \rightarrow 0$ for every k > 0, which implies $||X_n - X_0||_{\Psi} \rightarrow 0$ according to Proposition 2.1.10 in [5]. \Box

Proof of Theorem 4.16: Since $\mathbb{P} \circ (F_{\nu}^{\leftarrow}(U))^{-1} = \nu$ for every $\nu \in \mathcal{M}_1$, the asserted (ψ_k) -weak continuity of the risk functional \mathcal{R}_{ρ} is an immediate consequence of (31) and Theorem 6.1.

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