Limiting laws for extreme eigenvalues of large-dimensional spiked Fisher matrices with a divergent number of spikes

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Abstract

Consider the $p \times p$ matrix that is the product of a population covariance matrix and the inverse of another population covariance matrix. Suppose that their difference has a divergent rank with respect to p, when two samples of sizes n and T from the two populations are available, we construct its corresponding sample version. In the regime of high dimension where both n and T are proportional to p, we investigate the limiting laws for extreme (spiked) eigenvalues of the sample (spiked) Fisher matrix when the number of spikes is divergent and these spikes are unbounded.

Keywords: Extreme eigenvalue, Fisher matrix, Phase transition phenomenon, Random matrix theory, Spiked population model.

1 Introduction

In the last few decades, as the remarkable development in storage devices and computing capability, the demand for processing complex-structured data increases dramatically. One of the features as well as the challenges of these data sets is their high dimensions. The difficulty is that the classical limit theory for multivariate statistical analysis fails to ensure reliable inference for high-dimensional data analysis. Classical limit theorems require "small *p* large *n*" to keep their validity, which conflicts with the situation "large *p* large *n*" in high-dimensional settings in the sense that $p/n \rightarrow c > 0$ as the asymptotic properties are rather different. To attack the relevant issues, random matrix theory (RMT) serves as a powerful tool in addressing statistical problems in high dimensions. The first research of random matrices in multivariate statistics was about the Wishart matrices in [18]. Abundant research has been established for various topics in this field during the past half century, especially in recent years. In the area of RMT in statistics, we refer to monographs [2] and [19] for systematical study and [12] for a comprehensive review.

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A relevant topic in multivariate statistics is about testing the equality of two covariance matrices:

$$H_0: \Sigma_1 = \Sigma_2 \quad \text{vs.} \quad H_1: \Sigma_1 = \Sigma_2 + \Delta, \tag{1.1}$$

where Σ_1 and Σ_2 are two covariance matrices corresponding to two *p*-variate populations, and Δ is a non-negative definite matrix with rank *q*. Let \mathbf{S}_1 and \mathbf{S}_2 be the sample covariance matrices from these two populations, respectively. When \mathbf{S}_2 is invertible, the random matrix $\mathbf{F} = \mathbf{S}_2^{-1} \mathbf{S}_1$ is called a Fisher matrix.

The difference between the null hypothesis and the alternative hypothesis relies on those extreme eigenvalues of **F**. Under the null hypothesis, $\Sigma_1 = \Sigma_2$, [16] established the well-known Wacheter distribution as the limiting spectral distribution (LSD) of **F**. Some extensions were built later (see examples in [13], [14] and [15]). Furthermore, [1] pointed out the fact that the largest eigenvalue of **F** converges to the upper bound of the support of the LSD of **F**. Under the alternative hypothesis, **F** is called a *spiked Fisher matrix* (see [17]), because $\Sigma_2^{-1}\Sigma_1$ has a spiked structure similar to that of a *spiked population model* proposed by [10]. More specifically, the matrix $\Sigma_2^{-1}\Sigma_1$ is assumed to have the spectrum

$$\operatorname{spec}(\Sigma_2^{-1}\Sigma_1) = \{\lambda_1, \dots, \lambda_q, 1, \dots, 1\},$$
 (1.2)

where $\lambda_1 \ge ... \ge \lambda_q > 1$. When the rank q of Δ is finite, [6] showed the phase transition phenomenon of the extreme eigenvalues of **F** under Gaussian population assumption. That is, for $1 \le i \le q$, the *i*-th largest eigenvalue of **F** will depart from the upper bound of the support of LSD of **F** if and only if λ_1 exceeds certein phase transition point. [17] extended it to the cases without Gaussian assumption and established central limit theorems for the outlier eigenvalues of **F**.

We in this paper consider, as a reasonable extension in theory and applications, the case of divergent q with respect to the dimension p. We will investigate the convergence in probability and central limit theorems for spiked eigenvalues of spiked Fisher matrices. We formulate our problem as follows.

Assume that

$$\mathbf{Y} = (\mathbf{y}_1, \dots, \mathbf{y}_T) = (y_{ij})_{1 \le i \le p, 1 \le j \le T} \in \mathbb{R}^{p \times T} \quad \text{and} \quad \mathbf{Z} = (\mathbf{z}_1, \dots, \mathbf{z}_n) = (z_{ij})_{1 \le i \le p, 1 \le j \le n} \in \mathbb{R}^{p \times n}$$
(1.3)

are two independent arrays of independent real-valued random variables with zero mean and unit variance. We consider two samples $\{\Sigma_1^{1/2} \mathbf{y}_i\}_{1 \le i \le T}$ and $\{\Sigma_2^{1/2} \mathbf{z}_i\}_{1 \le i \le n}$, then their corresponding sample covariance matrices can respectively be written as

$$\mathbf{S}_{1} = \frac{1}{T} \sum_{i=1}^{T} \Sigma_{1}^{\frac{1}{2}} \mathbf{y}_{i} \mathbf{y}_{i}^{\mathsf{T}} \Sigma_{1}^{\frac{1}{2}} = \frac{1}{T} \Sigma_{1}^{\frac{1}{2}} \mathbf{Y} \mathbf{Y}^{\mathsf{T}} \Sigma_{1}^{\frac{1}{2}} \quad \text{and} \quad \mathbf{S}_{2} = \frac{1}{n} \sum_{i=1}^{n} \Sigma_{2}^{\frac{1}{2}} \mathbf{z}_{i} \mathbf{z}_{i}^{\mathsf{T}} \Sigma_{2}^{\frac{1}{2}} = \frac{1}{n} \Sigma_{2}^{\frac{1}{2}} \mathbf{Z} \mathbf{Z}^{\mathsf{T}} \Sigma_{2}^{\frac{1}{2}}.$$

Also, define the Fisher matrix $\mathbf{F} := \mathbf{S}_2^{-1} \mathbf{S}_1$, as the sample version of matrix $\Sigma_2^{-1} \Sigma_1$. We aim to investigate the limiting properties of the eigenvalues of **F**. As the eigenvalues of **F** remain

invariant under the linear transformation

$$(\mathbf{S}_{1}, \mathbf{S}_{2}) \to \left(\Sigma_{2}^{-\frac{1}{2}} \mathbf{S}_{1} \Sigma_{2}^{-\frac{1}{2}}, \Sigma_{2}^{-\frac{1}{2}} \mathbf{S}_{2} \Sigma_{2}^{-\frac{1}{2}}\right),$$
(1.4)

thus we can assume $\Sigma_2 = \mathbf{I}_p$ throughout this paper without loss of generality. Under the assumption (1.2), eigenvalues of Σ_1 are $\lambda_1 \ge \ldots \ge \lambda_q > \lambda_{q+1} = \ldots = \lambda_p = 1$. Recalling (1.1) that Σ_1 is a rank q pertubation of $\Sigma_2 = \mathbf{I}_p$, we simply assume

$$\Sigma_1 = \begin{pmatrix} \Sigma_{11} & 0\\ 0 & \mathbf{I}_{p-q} \end{pmatrix}.$$
 (1.5)

For the sake of brevity and readability, we write the eigenvalues of **F** in descending order $\widehat{\lambda}_1 \ge \ldots \ge \widehat{\lambda}_p$, simplifying the double subscripts as single ones. It should be noted that $\widehat{\lambda}_i$ is related to the sample size *n*.

We then describe the related work and our contributions in this paper. When the number of the spiked eigenvalues q is fixed, and all the spiked eigenvalue $\lambda_i, i = 1, \ldots, q$, are bounded, there are some results on the limiting properties of the eigenvalues of **F** in the literature. Such as, the almost surely convergence (strong consistency) and central limit theorem (CLT) of spiked eigenvalues ([17]) and asymptotically Tracy-Widom distribution for the largest non-spiked eigenvalues $q = q(p) \rightarrow \infty$ as $p \rightarrow \infty$, and spiked eigenvalues $\lambda_i, 1 \le i \le q$ diverge as $p \rightarrow \infty$. To the best of our knowledge, there is no relevant result in the literature. A relevant work is [5] who studied spiked population models, where the asymptotics for spiked eigenvalues, including convergence in probability (weak consistency) and CLT, as well as Tracy-Widom law for the largest nonspiked eigenvalue were built under a quite general framework. Unlike the case of fixed q and bounded spikes λ_i , $1 \le i \le q$, normalizations for $\hat{\lambda}_i, 1 \le i \le q$ are needed for the divergent q case. Consider the normalized eigenvalues $\hat{\lambda}_i/\lambda_i$ in consistency and $(\hat{\lambda}_i - \theta_i)/\theta_i$ in CLT, where θ_i is a centered parameter defined later.

The basic approach behind the proofs of the asymptotics for spiked eigenvalues is the analysis of an equation for the determinant of a $q \times q$ random matrix (indexed by n). When q is bounded, [17] derived the almost sure entrywise convergence of the $q \times q$ matrix (and hence the convergence with respect to matrix norms) and then solving the equation to lead to the almost sure limits of spiked eigenvalues. This argument does not work in the divergent q case where the convergence of a $q \times q$ matrix with respect to some norm could not be directly implied by the entrywise convergence. Instead, we use the CLT for random sequilinear forms in [3] to derive the convergence rate of each entry, and then use Chebyshev's inequality to put all entries together to derive the convergence rate of the matrix in ℓ_{∞} norm. In this way, we achieve the convergence in probability as well as the CLT of spiked eigenvalues (after proper normalizations). This approach is similar to that used in [5], so some technical assumptions are also imposed similarly.

The remaining parts of the paper are organized as follows. Section 2 establishes the main results, including the convergence in probability of $\hat{\lambda}_i/\lambda_i$ and central limit theorems

of $(\widehat{\lambda}_i - \theta_i)/\theta_i$, for those spiked eigenvalues of spiked Fisher matrix **F**. Here, θ_i , $1 \le i \le q$, is a sequence of centering parameters defined in this section. In the Section 3, we show the proofs of our main results in Section 2. Some important technical lemmas and their proofs are displayed in the Section 4.

2 Main results

2.1 Notations and assumptions

Considering the linear transformation (1.4), we assume that $\Sigma_2 = \mathbf{I}_p$ without loss of generality, and then Σ_1 has the structure as shown in (1.5). Further, we decompose the Σ_{11} in (1.5) as

$$\Sigma_{11} = \mathbf{U}^{\mathsf{T}} \boldsymbol{\Lambda}_1 \mathbf{U}.$$

Here, $\mathbf{U} \equiv (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_q)^{\top}$ is a $q \times q$ orthogonal matrix and

$$\Lambda_1 = \operatorname{diag}(\underbrace{\lambda_1, \ldots, \lambda_{N_1}}_{n_1}, \ldots, \underbrace{\lambda_{N_{\ell-1}+1}, \ldots, \lambda_q}_{n_\ell}),$$

where $\lambda_1 = \ldots = \lambda_{N_1} > \ldots > \lambda_{N_{\ell-1}+1} = \ldots = \lambda_q$ and $N_i := \sum_{j=1}^i n_j$ for $1 \le i \le \ell$. In this case, Σ_1 can be decomposed as

$$\Sigma_1 = \begin{pmatrix} \mathbf{U}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{p-q} \end{pmatrix} \begin{pmatrix} \Lambda_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{p-q} \end{pmatrix} \begin{pmatrix} \mathbf{U} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{p-q} \end{pmatrix} = : \begin{pmatrix} \mathbf{U}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{p-q} \end{pmatrix} \Lambda \begin{pmatrix} \mathbf{U} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{p-q} \end{pmatrix}.$$

We give decompositions of the sample covariance matrices \mathbf{S}_1 and \mathbf{S}_2 as follows. We fisrt decompose the matrices \mathbf{Y} and \mathbf{Z} defined in (1.3) as $\mathbf{Y} = (\mathbf{Y}_1^{\mathsf{T}}, \mathbf{Y}_2^{\mathsf{T}})^{\mathsf{T}}$ and $\mathbf{Z} = (\mathbf{Z}_1^{\mathsf{T}}, \mathbf{Z}_2^{\mathsf{T}})^{\mathsf{T}}$, where $\mathbf{Y}_1, \mathbf{Z}_1 \in \mathbb{R}^{q \times n}$ and $\mathbf{Y}_2, \mathbf{Z}_2 \in \mathbb{R}^{(p-q) \times n}$. Let $\mathbf{X} := \Sigma_1^{1/2} \mathbf{Y}$. Then we can similarly write $\mathbf{X} = (\mathbf{X}_1^{\mathsf{T}}, \mathbf{X}_2^{\mathsf{T}})^{\mathsf{T}}$, where $\mathbf{X}_1 = \Sigma_{11}^{1/2} \mathbf{Y}_1 = \mathbf{U}^{\mathsf{T}} \Lambda_1^{1/2} \mathbf{U} \mathbf{Y}_1 \in \mathbb{R}^{q \times T}$ and $\mathbf{X}_2 = \mathbf{Y}_2 \in \mathbb{R}^{(p-q) \times T}$. It follows that

$$\mathbf{S}_{1} = \begin{pmatrix} \frac{1}{T} \mathbf{X}_{1} \mathbf{X}_{1}^{\top} & \frac{1}{T} \mathbf{X}_{1} \mathbf{X}_{2}^{\top} \\ \frac{1}{T} \mathbf{X}_{2} \mathbf{X}_{1}^{\top} & \frac{1}{T} \mathbf{X}_{2} \mathbf{X}_{2}^{\top} \end{pmatrix} \quad \text{and} \quad \mathbf{S}_{2} = \begin{pmatrix} \frac{1}{n} \mathbf{Z}_{1} \mathbf{Z}_{1}^{\top} & \frac{1}{n} \mathbf{Z}_{1} \mathbf{Z}_{2}^{\top} \\ \frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{1}^{\top} & \frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \end{pmatrix}.$$
(2.1)

For $\lambda \in \mathbb{R} \setminus \{0\}$, we introduce

$$\mathbf{F}_{0} = \left(\frac{1}{n}\mathbf{Z}_{2}\mathbf{Z}_{2}^{\mathsf{T}}\right)^{-1} \left(\frac{1}{T}\mathbf{X}_{2}\mathbf{X}_{2}^{\mathsf{T}}\right), \quad \mathbf{M}(\lambda) = \mathbf{I}_{p-q} - \frac{\mathbf{F}_{0}}{\lambda},$$
$$\widetilde{m}_{\theta}(z) = \frac{1}{p-q} \operatorname{tr} \left(z\mathbf{I}_{p-q} - \frac{\mathbf{F}_{0}}{\theta}\right)^{-1}, \ \theta \in \mathbb{R}, \ z \in \mathbb{C}^{+}.$$
(2.2)

Let $\mu_1 \ge ... \ge \mu_{p-q}$ be the eigenvalues of the Fisher matrix \mathbf{F}_0 . Then the empirical spectral distribution (ESD) of \mathbf{F}_0 can be defined as

$$F_n(x) = \frac{1}{p-q} \sum_{j=1}^{p-q} \mathbf{1}_{\{\mu_j \le x\}}, \ x \in \mathbb{R}.$$

By the result in [17], under the assumption of $p/n \to y \in (0, 1)$ and $p/T \to c > 0$, almost surely, the empirical spectral distribution F_n weakly converges to the limiting spectral distribution $F_{c,y}$, whose Stieltjes transform $S(z) = \int_{-\infty}^{\infty} (x-z)^{-1} dF_{c,y}(x)$ satisfies, for $z \notin [a, b]$

$$S(z) = \frac{1-c}{zc} - \frac{c[z(1-y)+1-c] + 2zy - c\sqrt{[z(1-y)+1-c]^2 - 4z}}{2zc(c+zy)},$$
 (2.3)

where $a = (1 - \sqrt{c + y - cy})^2 (1 - y)^{-2}$ and $b = (1 + \sqrt{c + y - cy})^2 (1 - y)^{-2}$.

In the following, for any complex matrix A, we use $s_i(A)$ to denote the *i*-th largest singular value, and ||A|| to denote the largest singular value throughout the paper. Write $a_n = O_{a.s.}(b_n)$ if it almost surely holds that $a_n = O(b_n)$. Throughout this paper C is a constant that may vary from place to place.

The following assumptions are required.

Assumption 2.1. $y_p := p/n \rightarrow y \in (0, 1), \ \widetilde{y}_p := (p-q)/n; \ c_p := p/T \rightarrow c > 0, \ \widetilde{c}_p := (p-q)/T; \ q = q(n) \rightarrow \infty \text{ as } n \rightarrow \infty \text{ but } q = o(n^{\frac{1}{6}}).$

Assumption 2.2. For any $1 \le i \le q$, λ_i satisfies $q^2/\lambda_i \to 0$ and either of the two following conditions:

(a).
$$\lambda_i^{-1} \sum_{j=1}^q \lambda_j = o(q^{-\frac{1}{2}} n^{\frac{1}{4}}) \text{ and } \lambda_i^{-2} \sum_{j=1}^q \lambda_j = o(q^{-1}); (b). \ \lambda_i \sum_{j=1}^q \lambda_j^{-1} = o(q^{-\frac{1}{2}} n^{\frac{1}{4}}).$$

Assumption 2.3. Random vectors in $\{\mathbf{y}_i : 1 \le i \le T\} \bigcup \{\mathbf{z}_i : 1 \le i \le n\}$ are independent identically distributed, $Ez_{ij} = 0$, $E|z_{ij}|^2 = 1 \ \forall 1 \le i \le p$, $1 \le j \le n$ and $\sup_{1 \le i \le p} E|z_{ij}|^4 < \infty$.

Assumption 2.4. There exists a constant C > 1 such that $\lambda_{N_i}/\lambda_{N_{i+1}} \ge C$ for any $1 \le i \le \ell - 1$.

Assumption 2.5. Suppose that $\{\lambda_i\}_{1 \le i \le q}$ are of bounded multiplicities, i.e., $\sup_{1 \le i \le q} n_i < \infty$.

2.2 Weak consistency

The weak consistency of $\widehat{\lambda}_i$ is stated below. Due to the fact that λ_i may go to infinity with *n*, consider the limit in probability for the ratio $\widehat{\lambda}_i/\lambda_i$, $1 \le i \le q$.

Theorem 2.6. Assume that Assumptions 2.1, 2.2 and 2.3 hold. Then for all $1 \le i \le q$,

$$\frac{\lambda_i}{\lambda_i} = \frac{1}{1-y} + O\left(y_p - y\right) + \kappa q \cdot O_p\left(\frac{1}{\sqrt{n}} + \lambda_i^{-1}\right),$$

where $\kappa := \min\{\kappa_1, \kappa_2\}$ with $\kappa_1 := q + \lambda_i^{-1} \sum_{j=1}^q \lambda_j$ and $\kappa_2 := q + \lambda_i \sum_{j=1}^q \lambda_j^{-1}$.

Remark 2.7. Note that the limit of the ratio $\hat{\lambda}_i/\lambda_i$ is 1/(1 - y) > 1, for all $1 \le i \le q$. This is different from the relevant limit for spiked population model with divergent q, which is 1 (see Theorem 2.1 in [5]). Roughly speaking, when we take $y \to 0$ with $1/(1 - y) \to 1$, asymptotically, a spiked Fisher matrix behaves similarly as the sample covariance matrix in a spiked population model.

Remark 2.8. In the case of fixed q and bounded spikes λ_i , $1 \le i \le q$, Theorem 3.1 in [17] shows that almost surely the spiked eigenvalue $\widehat{\lambda}_i$ converges to the limit $\lambda_i(\lambda_i + c - 1)(\lambda_i - \lambda_i y - 1)^{-1}$. Simply taking $\lambda_i \to \infty$, the limit of $\lambda_i(\lambda_i + c - 1)(\lambda_i - \lambda_i y - 1)^{-1}\lambda_i^{-1}$ equals to 1/(1 - y). Thus, Theorem 2.6 indicates that for the divergent q case, the result coincides with the result for the fixed q case in [17].

Remark 2.9. In Theorem 2.6 we only consider unbounded spikes, but actually it can be readily extended to handle the case with both bounded and unbounded spikes. Consider the model

$$\Sigma_1 = \begin{pmatrix} \mathbf{U}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{p-q} \end{pmatrix} \Lambda \begin{pmatrix} \mathbf{U} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{p-q} \end{pmatrix},$$

where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_q, \lambda_{q+1}, \dots, \lambda_{q+q_0}, 1, \dots, 1), q = o(n^{1/6})$ and q_0 is bounded. Assume that spikes $\lambda_1 \ge \dots \ge \lambda_q$ are unbounded as in Theorem 2.6 and $\lambda_{q+1} \ge \dots \ge \lambda_{q+q_0}$ are bounded. For $q + 1 \le i \le q + q_0$, by Theorem A.10 in [2], we have

$$\widehat{\lambda}_{i} = s_{i}\left(\mathbf{S}_{2}^{-1}\mathbf{S}_{1}\right) \leq s_{i}\left(\mathbf{S}_{1}\right) s_{1}\left(\mathbf{S}_{2}^{-1}\right) \leq s_{i}\left(\boldsymbol{\Sigma}_{1}\right) s_{1}\left(\frac{1}{T}\mathbf{Y}\mathbf{Y}^{\top}\right) s_{1}\left(\mathbf{S}_{2}^{-1}\right) < \infty$$

almost surely. So it holds that

$$\det\left(\frac{\widehat{\lambda}_i}{n}\mathbf{Z}_1\mathbf{Z}_1^{\top} - \frac{1}{T}\mathbf{X}_1\mathbf{X}_1^{\top}\right) \neq 0.$$

Similar to the decomposition in (3.3), we have

$$\det\left\{ \left(\frac{\widehat{\lambda}_i}{n} \mathbf{Z}_2 \mathbf{Z}_2^\top - \frac{1}{T} \mathbf{X}_2 \mathbf{X}_2^\top \right) - \left(\frac{\widehat{\lambda}_i}{n} \mathbf{Z}_2 \mathbf{Z}_1^\top - \frac{1}{T} \mathbf{X}_2 \mathbf{X}_1^\top \right) \left(\frac{\widehat{\lambda}_i}{n} \mathbf{Z}_1 \mathbf{Z}_1^\top - \frac{1}{T} \mathbf{X}_1 \mathbf{X}_1^\top \right)^{-1} \left(\frac{\widehat{\lambda}_i}{n} \mathbf{Z}_1 \mathbf{Z}_2^\top - \frac{1}{T} \mathbf{X}_1 \mathbf{X}_2^\top \right) \right\} = 0$$
(2.4)

In the same manner as used in the proof of Theorem 2.6, it can be checked that

$$\left\| \left(\frac{\widehat{\lambda}_i}{n} \mathbf{Z}_2 \mathbf{Z}_1^\top - \frac{1}{T} \mathbf{X}_2 \mathbf{X}_1^\top \right) \left(\frac{\widehat{\lambda}_i}{n} \mathbf{Z}_1 \mathbf{Z}_1^\top - \frac{1}{T} \mathbf{X}_1 \mathbf{X}_1^\top \right)^{-1} \left(\frac{\widehat{\lambda}_i}{n} \mathbf{Z}_1 \mathbf{Z}_2^\top - \frac{1}{T} \mathbf{X}_1 \mathbf{X}_2^\top \right) \right\|_{\infty} = \mathbf{o}_p(1).$$

Then the solution of equation (2.4) is close to that of the equation

$$\det\left(\frac{\widehat{\lambda}_i}{n}\mathbf{Z}_2\mathbf{Z}_2^{\top} - \frac{1}{T}\mathbf{X}_2\mathbf{X}_2^{\top}\right) = 0.$$
(2.5)

Note that the solution of (2.5) is an eigenvlaue of the spiked Fisher matrix $(\mathbf{Z}_2 \mathbf{Z}_2^\top / n)^{-1} (\mathbf{X}_2 \mathbf{X}_2^\top / T)$ which has been well studied by [17]. Thus, the weak consistency for all outliers $\hat{\lambda}_i$, $1 \le i \le q + q_0$, could be achieved by combining Theorem 3.1 in [17] and Theorem 2.6. Such a kind of extension could also be considered for the CLT in Theorem 2.10.

2.3 Central limit theorem

As λ_i , $1 \le i \le q$, goes to infinity, the consistency of $\widehat{\lambda}_i / \lambda_i$ in Theorem 2.6 does not mean that $(1-y)\widehat{\lambda}_i$ is a good estimator of λ_i . In this section, we establish the CLT for $\widehat{\lambda}_i$ to provide further properties.

We first introduce a centered parameter for $\widehat{\lambda}_i$. Let $\theta_i \in \mathbb{R}$, $1 \le i \le q$, satisfy

$$1 - \frac{1}{n} \mathbb{E}\left[\operatorname{tr}\left\{\mathbf{M}^{-1}(\theta_{i})\right\}\right] = \frac{\lambda_{i}}{\theta_{i}} \left(1 + \frac{1}{T} \mathbb{E}\left[\operatorname{tr}\left\{\mathbf{M}^{-1}(\theta_{i}) \frac{\mathbf{F}_{0}}{\theta_{i}}\right\}\right]\right),$$
(2.6)

and define δ_i , for $1 \le i \le q$, as

$$\delta_i = \frac{\widehat{\lambda}_i - \theta_i}{\theta_i}.$$
(2.7)

By Lemma 4.1, when $n \to \infty$, we can easily see that

$$\frac{1}{p-q} \mathbb{E}\left[\operatorname{tr}\left\{\mathbf{M}^{-1}(\theta_{i})\right\}\right] = \mathbb{E}\left\{\widetilde{m}_{\theta_{i}}(1)\right\} \to 1 \quad \text{and} \quad \frac{1}{p-q} \mathbb{E}\left[\operatorname{tr}\left\{\mathbf{M}^{-1}(\theta_{i})\frac{\mathbf{F}_{0}}{\theta_{i}}\right\}\right] \to 0.$$

It follows by (2.6) that

$$\frac{\lambda_i}{\theta_i} = \left(1 - \frac{p - q}{n}\right) + o(1) \to 1 - y$$

Since the equation in Definition 2.6 for θ_i is hard to calculate, an alternative definition for θ_i is proposed as follows. Recall the definition of $\widetilde{m}_{\theta}(z)$ in (2.2):

$$\widetilde{m}_{\theta}(z) = \frac{1}{p-q} \operatorname{tr} \left(z \mathbf{I}_{p-q} - \frac{\mathbf{F}_0}{\theta} \right)^{-1}, \ \theta \in \mathbb{R}, \ z \in \mathbb{C}^+.$$

Denoting $f_{\theta}(x) = \theta/(\theta - x)$ for any fixed $\theta \in \mathbb{R}$, we have

$$\widetilde{m}_{\theta}(1) = \frac{1}{p-q} \operatorname{tr} \left(\mathbf{I}_{p-q} - \frac{\mathbf{F}_0}{\theta} \right)^{-1} = \int_{-\infty}^{\infty} \frac{\theta}{\theta - x} dF_n(x) =: F_n(f_{\theta}),$$

where F_n denotes the ESD of the matrix \mathbf{F}_0 . By the CLT for linear spectral statistics (LSS) of Fisher matrices (see Theorem 3.10 in [19]), for any fixed θ ,

$$p\{F_n(f_\theta) - F_{\widetilde{c}_p, \widetilde{y}_p}(f_\theta)\}$$

converges weakly to a Gaussian variable. It follows that

$$\widetilde{m}_{\theta}(1) = F_{\widetilde{c}_{p},\widetilde{y}_{p}}(f_{\theta}) + O_{p}(n^{-1}) = -\theta \widetilde{S}(\theta) + O_{p}(n^{-1})$$

$$= \frac{\widetilde{c}_{p} - 1}{\widetilde{c}_{p}} + \frac{\widetilde{c}_{p}\{\theta(1 - \widetilde{y}_{p}) + 1 - \widetilde{c}_{p}\} + 2\theta \widetilde{y}_{p} - \widetilde{c}_{p}\sqrt{\{\theta(1 - \widetilde{y}_{p}) + 1 - \widetilde{c}_{p}\}^{2} - 4\theta}}{2\widetilde{c}_{p}(\widetilde{c}_{p} + \theta \widetilde{y}_{p})} + O_{p}(n^{-1}),$$

where $\widetilde{\mathcal{S}}(\cdot)$ denotes the stieltjes transform of $F_{\widetilde{c}_p,\widetilde{y}_p}$. This leads to

$$E\left\{\widetilde{m}_{\theta}\left(1\right)\right\} = -\theta\widetilde{S}(\theta) + O(n^{-1}).$$
(2.8)

The definition of θ_i in (2.6) can be rewritten as

$$1 - \widetilde{y}_p \mathbb{E}\left\{\widetilde{m}_{\theta_i}(1)\right\} = \frac{\lambda_i}{\theta_i} \left[1 - \widetilde{c}_p + \widetilde{c}_p \mathbb{E}\left\{\widetilde{m}_{\theta_i}(1)\right\}\right].$$

According to (2.8), it is equivalent to

$$1 + \widetilde{y}_p \theta_i \widetilde{\mathcal{S}}(\theta_i) + \mathcal{O}(n^{-1}) = \frac{\lambda_i}{\theta_i} \left\{ 1 - \widetilde{c}_p - \widetilde{c}_p \theta_i \widetilde{\mathcal{S}}(\theta_i) + \mathcal{O}(n^{-1}) \right\}.$$
 (2.9)

Thus, we give another definition of θ_i by the following equation

$$1 + \widetilde{y}_p \theta_i \widetilde{\mathcal{S}}(\theta_i) = \frac{\lambda_i}{\theta_i} \left\{ 1 - \widetilde{c}_p - \widetilde{c}_p \theta_i \widetilde{\mathcal{S}}(\theta_i) \right\}.$$
(2.10)

It is notable that the θ_i defined by (2.10) is also applicable to the CLT of δ_i in the later section. Comparing two equations (2.9) and (2.10), we can derive that the difference between two δ_i 's respectively derived from these two equations is at most $O(n^{-1})$, which is smaller than the scale $n^{-1/2}$ of δ_i . Even Taylor's expansion on the stieltjes transformantion $\widetilde{S}(\cdot)$ can be simply used to the equation (2.10) and then get the explicit forms of θ_i , although some errors would appear. In the remaining parts of this paper, we use θ_i defined by (2.6) in all results and their proofs.

Consider the case where all the spiked eigenvalues are simple, that is, $n_i = 1$ for all $1 \le i \le \ell$, which means that $\Lambda_1 = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_q)$.

Theorem 2.10. Under Assumptions 2.1, 2.2, 2.3, 2.4 and that $n_i = 1$, $1 \le i \le l$, i.e., l = q, it holds that, for all $1 \le i \le q$,

$$\sqrt{p}\frac{\delta_i}{\sigma_i} \xrightarrow{d} \mathcal{N}(0,1)$$

with $\sigma_i^2 := (y+c)v_i - c - y(1-3y)(1-y)^{-1}$, where $v_i = E|\mathbf{u}_i^\top \mathbf{Z}_1 \mathbf{e}_1|^4$, $\mathbf{e}_1 = (1, 0, \dots, 0)^\top \in \mathbb{R}^q$ and $\mathbf{u}_i \in \mathbb{R}^q$ is the *i*-th column of the matrix \mathbf{U}^\top .

Remark 2.11. When the value of the variance σ_i^2 at the population level is unknown, for statistical inference, estimating σ_i^2 is in need. A natural estimation way would be to estimate the eigenvector \mathbf{u}_i first. For the spiked population model, [5] shows that when a leading eigenvalue of the sample covariance matrix is divergent, the corresponding sample eigenvector is a good estimator for its population counterpart in terms of their inner product. However, the situation becomes much more difficult when it comes to the spiked Fisher matrix. Recalling the assumed structure $\Sigma_2^{-1/2} \Sigma_1 \Sigma_2^{-1/2} = \mathbf{I}_p + \Delta$, we suppose that $\mathbf{v}_i := (\mathbf{u}_i^{\top}, 0, \dots, 0)^{\top} \in \mathbb{R}^p$ is the eigenvector of $\Sigma_2^{-1/2} \Sigma_1 \Sigma_2^{-1/2} = \mathbf{I}_p + \Delta$ corresponding to λ_i and $\widehat{\mathbf{v}}_i$ is that of $\mathbf{S}_1 = \Sigma_1^{1/2} \mathbf{Y} \mathbf{Y}^{\top} \Sigma_1^{1/2}$. Then $\Sigma_2^{1/2} \widehat{\mathbf{v}}_i$ is the eigenvector of $(\mathbf{I}_p + \Delta)^{1/2} \mathbf{Y} \mathbf{Y}^{\top} (\mathbf{I}_p + \Delta)^{1/2}$ corresponding to the *i*-th largest eigenvalue. If Σ_2 is known or can be consistently estimated, $\Sigma_2^{1/2} \widehat{\mathbf{v}}_i$ is a good estimator of \mathbf{v}_i , by Theorem 4.1 in [5]. But actually Σ_2 cannot be easily recovered based on \mathbf{S}_2 because of the delocalization of those eigenvectors for non-outliers (see [4]). Thus, how to construct a consistent estimation of

 Σ_2 becomes a challenging issue. As a special case, when entries of **Y** and **Z** are Gaussian, the parameter v_i equals to 3, which is independent of the value of \mathbf{u}_i . In practice, the bootstrap approximation would be an alternative way to achieve a reliable estimation of σ_i^2 . For estimation of the variance of the largest sample eigenvalue in a spiked population model, spiked population model, [11] shows that the bootstrap approximation works when the largest eigenvalue is quite large. This deserves a further study.

To check the practical applicability of Theorem 2.10, a simulation is conducted. Set p = 200, T = 600, n = 1000, $q = \lceil 2 \log p \rceil$, $\lambda_i = (3/2)^{q+1-i} (\log p/3)^3$ for $1 \le i \le q$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Let $\Sigma_1 = \text{diag}(\lambda_1, \ldots, \lambda_q, 1, \ldots, 1)$ and $\Sigma_2 = \mathbf{I}_p$. Draw a sample $\{\mathbf{x}_i\}_{1\le i\le T}$ of size T from $\mathcal{N}(0, \Sigma_1)$ and a sample $\{\mathbf{z}_i\}_{1\le i\le n}$ of size n from $\mathcal{N}(0, \Sigma_2)$. Compute the largest q eigenvalues $\widehat{\lambda}_i$, $1 \le i \le q$, of the Fisher matrix $\mathbf{F} = \mathbf{S}_2^{-1}\mathbf{S}_1$ and then δ_i accordingly, where $\mathbf{S}_1 = \sum_{i=1}^T \mathbf{x}_i \mathbf{x}_i^\top / T$ and $\mathbf{S}_2 = \sum_{i=1}^n \mathbf{z}_i \mathbf{z}_i^\top / n$. We draw qq plots of $\sqrt{p}\delta_1/\sigma_1$ and $\sqrt{p}\delta_q/\sigma_q$ from 1000 independent replications in Figure 1. It suggests that both of $\sqrt{p}\delta_1/\sigma_1$ and $\sqrt{p}\delta_q/\sigma_q$ are well approximated by the standard normal distribution.

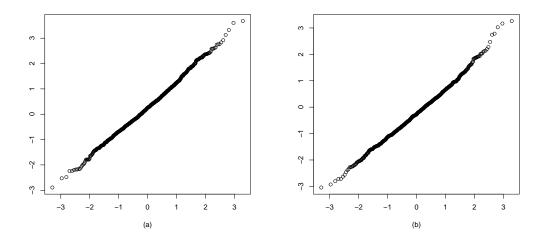


Figure 1: (a) The qq plot of the normalized largest spiked eigenvalue $\sqrt{p}\delta_1/\sigma_1$ from 1000 independent replications. (b) The qq plot of the normalized smallest spiked eigenvalue $\sqrt{p}\delta_q/\sigma_q$ from 1000 independent replications.

Next, consider the case where some spiked eigenvalues are possibly multiple:

$$\Lambda_1 = \operatorname{diag}(\underbrace{\lambda_1, \ldots, \lambda_{N_1}}_{n_1}, \ldots, \underbrace{\lambda_{N_{\ell-1}+1}, \ldots, \lambda_q}_{n_\ell}),$$

where $\lambda_1 = \ldots = \lambda_{N_1} > \ldots > \lambda_{N_{\ell-1}+1} = \ldots = \lambda_q$, $N_i := \sum_{j=1}^i n_j$ for $1 \le i \le \ell$ and there exists a constant $C < \infty$ such that $1 \le n_i \le C$ for all $1 \le i \le \ell$. According to

the multiplicities of spiked eigenvalues, we divide the index set $\{1, \ldots, q\}$ into ℓ subsets, $J_i = \{N_{i-1} + 1, \ldots, N_i\}, 1 \le i \le \ell$. Here we denote $N_0 = 0$. For any $1 \le i \le \ell$, and $1 \le h, k, h_1, h_2, h_2 \le n_i$, define

$$\mathcal{M}_{N_{i},h,k} := \mathbf{E} \left(\mathbf{u}_{N_{i-1}+h}^{\top} \mathbf{Z}_{1} \mathbf{e}_{1} \mathbf{u}_{N_{i-1}+k}^{\top} \mathbf{Z}_{1} \mathbf{e}_{1} \right),$$
$$\mathcal{M}_{N_{i},h_{1},k_{1},h_{2},k_{2}} := \mathbf{E} \left(\mathbf{u}_{N_{i-1}+h_{1}}^{\top} \mathbf{Z}_{1} \mathbf{e}_{1} \mathbf{u}_{N_{i-1}+k_{1}}^{\top} \mathbf{Z}_{1} \mathbf{e}_{1} \mathbf{u}_{N_{i-1}+h_{2}}^{\top} \mathbf{Z}_{1} \mathbf{e}_{1} \mathbf{u}_{N_{i-1}+k_{2}}^{\top} \mathbf{Z}_{1} \mathbf{e}_{1} \right).$$

Theorem 2.12. Suppose that Assumptions 2.1, 2.2, 2.3, 2.4 and 2.5 hold. Define $\phi_i(\widehat{\lambda}_j) = (\widehat{\lambda}_j - \theta_j)/\theta_j$, for $1 \le i \le \ell$ and $j \in J_i$. Then $\sqrt{p}\{\phi_i(\widehat{\lambda}_j), j \in J_i\}$ converges weakly to the distribution of the eigenvalues of the $n_i \times n_i$ random matrix $\mathfrak{R}^{(i)}$, where $\mathfrak{R}^{(i)} = (R_{hk}^{(i)})_{1 \le h,k \le n_i}$ is a symmetric matrix with independent Gaussian entries of mean zero and covariance structure

$$\cos\left(R_{h_1,k_1}^{(i)}, R_{h_2,k_2}^{(i)}\right) = (1-y)^{-2} \omega \left(\mathcal{M}_{N_i,h_1,k_1,h_2,k_2} - \mathcal{M}_{N_i,h_1,k_1} \mathcal{M}_{N_i,h_2,k_2}\right) + (1-y)^{-2} \left(\beta - \omega\right) \left(\mathcal{M}_{N_i,h_1,k_2} \mathcal{M}_{N_i,h_2,k_1} + \mathcal{M}_{N_i,h_1,h_2} \mathcal{M}_{N_i,k_1,k_2}\right),$$

where $\omega = (y + c)(1 - y)^2$ and $\beta = y(1 - y) + c(1 - y)^2$.

3 Proofs of the theorems

We begin with a summary of the proofs. Roughly, the proof of Theorem 2.6 proceeds in three steps. First, we prove that the spiked eigenvalue $\hat{\lambda}_i$, $1 \le i \le q$, solves the equation (3.5) whose left-hand side is the determinant of a $q \times q$ matrix which can be decomposed into four terms, namely $U\Xi_A U^{\top}$, $U\Xi_B U^{\top}$, $U\Xi_C U^{\top}$ and $U\Xi_D U^{\top}$ defined below. Second, we derive the limit of each entry of these four matrices and their convergence rates in ℓ_{∞} norm, where the CLT for random sequilinear forms in [3] and Chebyshev's inequality are repeatedly used. Third, using eigenvalue perturbation theorems on (3.5), we estimate the fluctuation of the scaled eigenvalue $\hat{\lambda}_i/\lambda_i$ and reach the result. As for the proof of Theorem 2.10, we also work on the equation (3.5) in three main steps. First, we rewrite the matrix in (3.5) as the sum of $U\Theta_{1n}U^{\top}$, $U\delta_i\Theta_{2n}U^{\top}$ and $U\Theta_{3n}U^{\top}$. See equation (3.30) below. Second, we prove the CLT for each diagonal entry of $U\Theta_{1n}U^{\top}$ (Lemma 4.2) and estimate the ℓ_{∞} norm of $U\Theta_{1n}U^{\top}$ (Lemma 4.3), $U\Theta_{2n}U^{\top}$ (Lemma 4.4) and $U\Theta_{3n}U^{\top}$. Third, we expand the determinant in (3.30) by Leibniz formula and then achieve the CLT for δ_i . In this section, we will cite the lemmas given in the next section without the proofs whose details are postponed to the next section.

Proof of Theorem 2.6. We first show that for $1 \le i \le q$, $\widehat{\lambda}_i$ converges to infinity at the same order with λ_i almost surely, i.e., there exists some constant C > 1 such that $C^{-1} < \widehat{\lambda}_i / \lambda_i < C$ almost surely.

For any $1 \le i \le q$, by Theorem A.10 in [2], we have that

$$\widehat{\lambda}_i = s_i(\mathbf{S}_2^{-1}\mathbf{S}_1) \le s_i(\mathbf{S}_1)s_1(\mathbf{S}_2^{-1}) = s_i(\mathbf{S}_1)s_p^{-1}(\mathbf{S}_2) \text{ and } s_i(\mathbf{S}_1) \le s_i(\mathbf{S}_2^{-1}\mathbf{S}_1)s_1(\mathbf{S}_2).$$

Noting a basic fact that $s_1(\mathbf{S}_2) \to (1 + \sqrt{y})^2$ and $s_p(\mathbf{S}_2) \to (1 - \sqrt{y})^2 > 0$ almost surely, we have $0 < C_1 < \widehat{\lambda}_i / s_i(\mathbf{S}_1) \le C_2 < +\infty$ almost surely for some constants C_1 and C_2 .

Again, by Theorem A.10 in [2] and Weyl's inequality, we have

$$s_i(\mathbf{S}_1) \le s_i(\Sigma_1) s_1\left(\frac{1}{T}\mathbf{Y}\mathbf{Y}^{\top}\right) = \lambda_i s_1\left(\frac{1}{T}\mathbf{Y}\mathbf{Y}^{\top}\right)$$

and

$$s_{i}(\mathbf{S}_{1}) = s_{i}\left(\frac{1}{T}\mathbf{Y}^{\mathsf{T}}\boldsymbol{\Sigma}_{1}\mathbf{Y}\right) = s_{i}\left(\frac{1}{T}\mathbf{Y}_{1}^{\mathsf{T}}\boldsymbol{\Sigma}_{11}\mathbf{Y}_{1} + \frac{1}{T}\mathbf{Y}_{2}^{\mathsf{T}}\mathbf{Y}_{2}\right) \ge s_{i}\left(\frac{1}{T}\mathbf{Y}_{1}^{\mathsf{T}}\boldsymbol{\Sigma}_{11}\mathbf{Y}_{1}\right)$$
$$\ge s_{i}\left(\boldsymbol{\Sigma}_{11}\right)s_{q}\left(\frac{1}{T}\mathbf{Y}_{1}\mathbf{Y}_{1}^{\mathsf{T}}\right) = \lambda_{i}s_{q}\left(\frac{1}{T}\mathbf{Y}_{1}\mathbf{Y}_{1}^{\mathsf{T}}\right).$$

Due to the fact that

$$s_1\left(\frac{1}{T}\mathbf{Y}\mathbf{Y}^{\mathsf{T}}\right) \to (1+\sqrt{c})^2 \quad \text{and} \quad s_q\left(\frac{1}{T}\mathbf{Y}_1\mathbf{Y}_1^{\mathsf{T}}\right) \to 1$$

almost surely, we have $0 < C_3 < s_i(\mathbf{S}_1)/\lambda_i < C_4 < +\infty$ almost surely for some constants C_3 and C_4 .

Thus, we conclude that $C^{-1} < \hat{\lambda}_i / \lambda_i < C$ almost surely for some constant *C*.

For any $1 \le i \le q$, by the definition of $\widehat{\lambda}_i$, it solves the equation det $(\widehat{\lambda}_i \mathbf{I} - \mathbf{S}_2^{-1} \mathbf{S}_1) = 0$, or equivalently,

$$\det\left(\widehat{\lambda}_i \mathbf{S}_2 - \mathbf{S}_1\right) = 0. \tag{3.1}$$

By the decomposition of S_1 and S_2 in (2.1), the equation (3.1) can be rewritten as

$$\det \begin{pmatrix} \frac{\lambda_i}{n} \mathbf{Z}_1 \mathbf{Z}_1^{\mathsf{T}} - \frac{1}{T} \mathbf{X}_1 \mathbf{X}_1^{\mathsf{T}} & \frac{\lambda_i}{n} \mathbf{Z}_1 \mathbf{Z}_2^{\mathsf{T}} - \frac{1}{T} \mathbf{X}_1 \mathbf{X}_2^{\mathsf{T}} \\ \frac{\lambda_i}{n} \mathbf{Z}_2 \mathbf{Z}_1^{\mathsf{T}} - \frac{1}{T} \mathbf{X}_2 \mathbf{X}_1^{\mathsf{T}} & \frac{\lambda_i}{n} \mathbf{Z}_2 \mathbf{Z}_2^{\mathsf{T}} - \frac{1}{T} \mathbf{X}_2 \mathbf{X}_2^{\mathsf{T}} \end{pmatrix} = 0.$$
(3.2)

By the formula of the determinant of partitioned matrices, we know that $det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = det(D) det(A - BD^{-1}C)$ when D is nonsingular. As for $1 \le i \le q$, $\widehat{\lambda}_i$ is an outlier eigenvalue of $\mathbf{S}_2^{-1}\mathbf{S}_1$ because $\widehat{\lambda}_i$ goes to infinity at the same order with λ_i , which means

$$\det\left(\frac{\widehat{\lambda}_i}{n}\mathbf{Z}_2\mathbf{Z}_2^{\top} - \frac{1}{T}\mathbf{X}_2\mathbf{X}_2^{\top}\right) \neq 0,$$

then it follows by (3.2) that

$$\det\left\{ \left(\frac{\widehat{\lambda}_{i}}{n} \mathbf{Z}_{1} \mathbf{Z}_{1}^{\mathsf{T}} - \frac{1}{T} \mathbf{X}_{1} \mathbf{X}_{1}^{\mathsf{T}} \right) - \left(\frac{\widehat{\lambda}_{i}}{n} \mathbf{Z}_{1} \mathbf{Z}_{2}^{\mathsf{T}} - \frac{1}{T} \mathbf{X}_{1} \mathbf{X}_{2}^{\mathsf{T}} \right) \left(\frac{\widehat{\lambda}_{i}}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\mathsf{T}} - \frac{1}{T} \mathbf{X}_{2} \mathbf{X}_{2}^{\mathsf{T}} \right)^{-1} \left(\frac{\widehat{\lambda}_{i}}{n} \mathbf{Z}_{2} \mathbf{Z}_{1}^{\mathsf{T}} - \frac{1}{T} \mathbf{X}_{2} \mathbf{X}_{1}^{\mathsf{T}} \right) \right\} = 0$$

$$(3.3)$$

For $\lambda \in \mathbb{R}$, defining

$$\mathbf{A}(\lambda) = \mathbf{Z}_2^{\mathsf{T}} \mathbf{M}^{-1}(\lambda) \left(\frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_2^{\mathsf{T}}\right)^{-1} \frac{1}{n} \mathbf{Z}_2,$$

$$\mathbf{B}(\lambda) = \mathbf{X}_{2}^{\mathsf{T}} \mathbf{M}^{-1}(\lambda) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\mathsf{T}}\right)^{-1} \frac{1}{\lambda T} \mathbf{X}_{2},$$
$$\mathbf{C}(\lambda) = \mathbf{Z}_{2}^{\mathsf{T}} \mathbf{M}^{-1}(\lambda) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\mathsf{T}}\right)^{-1} \frac{1}{\lambda T} \mathbf{X}_{2},$$
$$\mathbf{D}(\lambda) = \mathbf{X}_{2}^{\mathsf{T}} \mathbf{M}^{-1}(\lambda) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\mathsf{T}}\right)^{-1} \frac{1}{\lambda n} \mathbf{Z}_{2},$$

it holds that $\mathbf{A}(\lambda) = \mathbf{A}(\lambda)^{\mathsf{T}}$, $\mathbf{B}(\lambda) = \mathbf{B}(\lambda)^{\mathsf{T}}$ and $T\mathbf{C}(\lambda) = n\mathbf{D}(\lambda)^{\mathsf{T}}$. Then some elementary calculations lead to

$$\det\left[\widehat{\lambda}_{i}\frac{\mathbf{Z}_{1}\left\{\mathbf{I}_{n}-\mathbf{A}(\widehat{\lambda}_{i})\right\}\mathbf{Z}_{1}^{\top}}{n}-\frac{\mathbf{X}_{1}\left\{\mathbf{I}_{T}+\mathbf{B}(\widehat{\lambda}_{i})\right\}\mathbf{X}_{1}^{\top}}{T}+\widehat{\lambda}_{i}\frac{\mathbf{Z}_{1}\mathbf{C}(\widehat{\lambda}_{i})\mathbf{X}_{1}^{\top}}{n}+\widehat{\lambda}_{i}\frac{\mathbf{X}_{1}\mathbf{D}(\widehat{\lambda}_{i})\mathbf{Z}_{1}^{\top}}{T}\right]=0.$$
(3.4)

To ease the notation, we define

$$\Xi_A := \widehat{\lambda}_i \frac{\mathbf{Z}_1 \left\{ \mathbf{I}_n - \mathbf{A}(\widehat{\lambda}_i) \right\} \mathbf{Z}_1^{\top}}{n},$$

$$\Xi_B := \frac{\mathbf{X}_1 \left\{ \mathbf{I}_T + \mathbf{B}(\widehat{\lambda}_i) \right\} \mathbf{X}_1^{\top}}{T},$$

$$\Xi_C := \widehat{\lambda}_i \frac{\mathbf{Z}_1 \mathbf{C}(\widehat{\lambda}_i) \mathbf{X}_1^{\top}}{n},$$

$$\Xi_D := \widehat{\lambda}_i \frac{\mathbf{X}_1 \mathbf{D}(\widehat{\lambda}_i) \mathbf{Z}_1^{\top}}{T}.$$

Multiplying the matrix in (3.4) by U on the left side hand and by U^{\top} on the right side, we have

$$\det\left\{\mathbf{U}\left(\Xi_{A}-\Xi_{B}+\Xi_{C}+\Xi_{D}\right)\mathbf{U}^{\mathsf{T}}\right\}=0.$$
(3.5)

Next, we analyze these four terms in (3.5) in the following.

For the term $\mathbf{U}\Xi_A\mathbf{U}^{\mathsf{T}}$, we first consider the decomposition

$$\frac{1}{n}\mathbf{Z}_{1}\left\{\mathbf{I}_{n}-\mathbf{A}(\widehat{\lambda}_{i})\right\}\mathbf{Z}_{1}^{\top}=\frac{1}{n}\mathbf{Z}_{1}\left\{\mathbf{I}_{n}-\mathbf{A}(\lambda_{i})\right\}\mathbf{Z}_{1}^{\top}+\frac{1}{n}\mathbf{Z}_{1}\left\{\mathbf{A}(\widehat{\lambda}_{i})-\mathbf{A}(\lambda_{i})\right\}\mathbf{Z}_{1}^{\top}.$$

By Lemma 4.1 below, we have $\widetilde{m}_{\lambda_i}(1) - 1 = O_{a.s.}(\lambda_i^{-1})$, which implies

$$\frac{1}{n}\operatorname{tr}\left\{\mathbf{I}_{n}-\mathbf{A}(\lambda_{i})\right\}=1-\frac{p-q}{n}\widetilde{m}_{\lambda_{i}}(1)=1-y_{p}+\frac{q}{n}+\mathsf{O}_{a.s.}(\lambda_{i}^{-1}).$$

Note that $E(\mathbf{Z}_1\mathbf{Z}_1^\top/n) = \mathbf{I}_q$ and that $(\mathbf{X}_1, \mathbf{Z}_1)$ is independent of $(\mathbf{X}_2, \mathbf{Z}_2)$. Under Assumption 2.3, by using Theorem 7.2 of [3], we have that, for all $1 \le j \le q$,

$$\mathbf{e}_{j}^{\mathsf{T}}\left[\frac{1}{n}\mathbf{Z}_{1}\left\{\mathbf{I}_{n}-\mathbf{A}(\lambda_{i})\right\}\mathbf{Z}_{1}^{\mathsf{T}}\right]\mathbf{e}_{j}-\left\{1-\frac{p-q}{n}\widetilde{m}_{\lambda_{i}}(1)\right\}=O_{p}\left(\frac{1}{\sqrt{n}}\right)$$
(3.6)

and

$$\mathbf{E}\left(\mathbf{e}_{j}^{\mathsf{T}}\left[\frac{1}{n}\mathbf{Z}_{1}\left\{\mathbf{I}_{n}-\mathbf{A}(\lambda_{i})\right\}\mathbf{Z}_{1}^{\mathsf{T}}\right]\mathbf{e}_{j}-\left\{1-\frac{p-q}{n}\widetilde{m}_{\lambda_{i}}(1)\right\}\right)^{2}=\mathbf{O}\left(\frac{1}{n}\right)$$
(3.7)

for all $1 \le j \le q$. For those off-diagonal elements, we have that, for any $1 \le j_1 \ne j_2 \le q$,

$$\mathbf{e}_{j_1}^{\mathsf{T}} \left[\frac{1}{n} \mathbf{Z}_1 \left\{ \mathbf{I}_n - \mathbf{A}(\lambda_i) \right\} \mathbf{Z}_1^{\mathsf{T}} \right] \mathbf{e}_{j_2} = \mathbf{O}_p \left(\frac{1}{\sqrt{n}} \right)$$
(3.8)

and

$$\mathbf{E}\left(\mathbf{e}_{j_{1}}^{\top}\left[\frac{1}{n}\mathbf{Z}_{1}\left\{\mathbf{I}_{n}-\mathbf{A}(\lambda_{i})\right\}\mathbf{Z}_{1}^{\top}\right]\mathbf{e}_{j_{2}}\right)^{2}=\mathbf{O}\left(\frac{1}{n}\right),$$
(3.9)

which is implied by Theorem 7.1 and Corollary 7.1 in [3]. Also we can write

$$\mathbf{A}(\lambda_{i}) - \mathbf{A}\left(\widehat{\lambda}_{i}\right) = \mathbf{Z}_{2}^{\top} \left\{ \mathbf{M}^{-1}(\lambda_{i}) - \mathbf{M}^{-1}\left(\widehat{\lambda}_{i}\right) \right\} \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{n} \mathbf{Z}_{2}$$
$$= \mathbf{Z}_{2}^{\top} \mathbf{M}^{-1}(\lambda_{i}) \left\{ \mathbf{M}\left(\widehat{\lambda}_{i}\right) - \mathbf{M}(\lambda_{i}) \right\} \mathbf{M}^{-1}\left(\widehat{\lambda}_{i}\right) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{n} \mathbf{Z}_{2}$$
$$= \left(\lambda_{i}^{-1} - \widehat{\lambda}_{i}^{-1} \right) \mathbf{Z}_{2}^{\top} \mathbf{M}^{-1}(\lambda_{i}) \mathbf{F}_{0} \mathbf{M}^{-1}\left(\widehat{\lambda}_{i}\right) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{n} \mathbf{Z}_{2}.$$

It can be bounded by

$$\begin{aligned} \left\| \mathbf{A} \left(\lambda_{i} \right) - \mathbf{A} \left(\widehat{\lambda}_{i} \right) \right\| &= \left\| \left(\lambda_{i}^{-1} - \widehat{\lambda}_{i}^{-1} \right) \mathbf{Z}_{2}^{\mathsf{T}} \mathbf{M}^{-1} \left(\lambda_{i} \right) \mathbf{F}_{0} \mathbf{M}^{-1} \left(\widehat{\lambda}_{i} \right) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\mathsf{T}} \right)^{-1} \frac{1}{n} \mathbf{Z}_{2} \right\| \\ &\leq \left| \lambda_{i}^{-1} - \widehat{\lambda}_{i}^{-1} \right| \left\| \frac{1}{\sqrt{n}} \mathbf{Z}_{2}^{\mathsf{T}} \right\| \left\| \mathbf{M}^{-1} \left(\lambda_{i} \right) \right\| \left\| \mathbf{F}_{0} \right\| \left\| \mathbf{M}^{-1} \left(\widehat{\lambda}_{i} \right) \right\| \left\| \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\mathsf{T}} \right)^{-1} \right\| \left\| \frac{1}{\sqrt{n}} \mathbf{Z}_{2} \right\| = \mathbf{O}(\lambda_{i}^{-1}) \end{aligned}$$

almost surely. It follows that, for any $1 \le j_1, j_2 \le q$,

$$\mathbf{e}_{j_1}^{\mathsf{T}} \left[\frac{1}{n} \mathbf{Z}_1 \left\{ \mathbf{A}(\widehat{\lambda}_i) - \mathbf{A}(\lambda_i) \right\} \mathbf{Z}_1^{\mathsf{T}} \right] \mathbf{e}_{j_2} = \mathbf{O}_{a.s.}(\lambda_i^{-1}).$$
(3.10)

Combining (3.6), (3.8)) and (3.10), we can get that, for any $1 \le j \le q$,

$$\mathbf{e}_{j}^{\top}\left[\frac{1}{n}\mathbf{Z}_{1}\left\{\mathbf{I}_{n}-\mathbf{A}(\widehat{\lambda}_{i})\right\}\mathbf{Z}_{1}^{\top}\right]\mathbf{e}_{j}-\left(1-\frac{p-q}{n}\right)=\mathbf{O}_{p}\left(\frac{1}{\sqrt{n}}\right)+\mathbf{O}_{a.s.}(\lambda_{i}^{-1})$$

and that, for any $1 \le j_1 \ne j_2 \le q$,

$$\mathbf{e}_{j_1}^{\top} \left[\frac{1}{n} \mathbf{Z}_1 \left\{ \mathbf{I}_n - \mathbf{A}(\widehat{\lambda}_i) \right\} \mathbf{Z}_1^{\top} \right] \mathbf{e}_{j_2} = \mathbf{O}_p \left(\frac{1}{\sqrt{n}} \right) + \mathbf{O}_{a.s.}(\lambda_i^{-1}).$$

Replacing Z_1 by UZ_1 , it is easy to check that all the above conclusions still hold:

$$\mathbf{e}_{j}^{\mathsf{T}}\left[\frac{1}{n}\mathbf{U}\mathbf{Z}_{1}\left\{\mathbf{I}_{n}-\mathbf{A}(\widehat{\lambda}_{i})\right\}\mathbf{Z}_{1}^{\mathsf{T}}\mathbf{U}^{\mathsf{T}}\right]\mathbf{e}_{j}=1-\frac{p-q}{n}+O_{p}\left(\frac{1}{\sqrt{n}}\right)+O_{a.s.}(\lambda_{i}^{-1})$$
(3.11)

for all $1 \le j \le q$, and

$$\mathbf{e}_{j_1}^{\mathsf{T}} \left[\frac{1}{n} \mathbf{U} \mathbf{Z}_1 \left\{ \mathbf{I}_n - \mathbf{A}(\widehat{\lambda}_i) \right\} \mathbf{Z}_1^{\mathsf{T}} \mathbf{U}^{\mathsf{T}} \right] \mathbf{e}_{j_2} = \mathbf{O}_p \left(\frac{1}{\sqrt{n}} \right) + \mathbf{O}_{a.s.}(\lambda_i^{-1})$$
(3.12)

for all $1 \le j_1 \ne j_2 \le q$. By the definition of Ξ_A in (3.4), together with (3.11) and (3.12), we can see that, for all $1 \le j \le q$,

$$\mathbf{e}_{j}^{\mathsf{T}}\mathbf{U}\Xi_{A}\mathbf{U}^{\mathsf{T}}\mathbf{e}_{j} = \widehat{\lambda}_{i}\left(1 - \frac{p-q}{n}\right) + \lambda_{i} \cdot O_{p}\left(\frac{1}{\sqrt{n}}\right) + O_{a.s.}(1)$$
(3.13)

and that, for all $1 \le j_1 \ne j_2 \le q$,

$$\mathbf{e}_{j_1}^{\top} \mathbf{U} \Xi_A \mathbf{U}^{\top} \mathbf{e}_{j_2} = \lambda_i \cdot \mathcal{O}_p \left(\frac{1}{\sqrt{n}} \right) + \mathcal{O}_{a.s.}(1).$$
(3.14)

For the term $\mathbf{U}\Xi_{B}\mathbf{U}^{\mathsf{T}}$, by the definition of \mathbf{X}_{1} , we can derive that

$$\mathbf{U}\Xi_{B}\mathbf{U}^{\mathsf{T}} = \frac{1}{T}\Lambda_{1}^{\frac{1}{2}}\mathbf{U}\mathbf{Y}_{1}\left\{\mathbf{I}_{T} + \mathbf{B}(\widehat{\lambda}_{i})\right\}\mathbf{Y}_{1}^{\mathsf{T}}\mathbf{U}^{\mathsf{T}}\Lambda_{1}^{\frac{1}{2}} = \frac{1}{T}\Lambda_{1}^{\frac{1}{2}}\mathbf{U}\mathbf{Y}_{1}\left\{\mathbf{I}_{T} + \mathbf{B}(\lambda_{i})\right\}\mathbf{Y}_{1}^{\mathsf{T}}\mathbf{U}^{\mathsf{T}}\Lambda_{1}^{\frac{1}{2}} + \frac{1}{T}\Lambda_{1}^{\frac{1}{2}}\mathbf{U}\mathbf{Y}_{1}\left\{\mathbf{B}(\widehat{\lambda}_{i}) - \mathbf{B}(\lambda_{i})\right\}\mathbf{Y}_{1}^{\mathsf{T}}\mathbf{U}^{\mathsf{T}}\Lambda_{1}^{\frac{1}{2}},$$

where

$$\frac{1}{T} \operatorname{tr} \left\{ \mathbf{I}_{T} + \mathbf{B} \left(\lambda_{i} \right) \right\} = \frac{1}{T} \operatorname{tr} \left\{ \mathbf{I}_{T} + \mathbf{X}_{2}^{\top} \mathbf{M}^{-1} \left(\lambda_{i} \right) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{\lambda_{i} T} \mathbf{X}_{2} \right\}$$
$$= 1 + \frac{1}{T} \operatorname{tr} \left\{ \mathbf{M}^{-1} \left(\lambda_{i} \right) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{\lambda_{i} T} \mathbf{X}_{2} \mathbf{X}_{2}^{\top} \right\}$$
$$= 1 + \frac{1}{T} \operatorname{tr} \left\{ \left(\mathbf{I}_{p-q} - \frac{\mathbf{F}_{0}}{\lambda_{i}} \right)^{-1} \frac{\mathbf{F}_{0}}{\lambda_{i}} \right\}$$
$$= 1 + \frac{p-q}{T} \left\{ \widetilde{m}_{\lambda_{i}}(1) - 1 \right\}$$

and

$$\begin{split} \mathbf{B}(\widehat{\lambda}_{i}) - \mathbf{B}(\lambda_{i}) &= \mathbf{X}_{2}^{\top} \left\{ \widehat{\lambda}_{i}^{-1} \mathbf{M}^{-1} \left(\widehat{\lambda}_{i} \right) - \lambda_{i}^{-1} \mathbf{M}^{-1} (\lambda_{i}) \right\} \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{T} \mathbf{X}_{2} \\ &= \widehat{\lambda}_{i}^{-1} \lambda_{i}^{-1} \mathbf{X}_{2}^{\top} \mathbf{M}^{-1} \left(\widehat{\lambda}_{i} \right) \left\{ \lambda_{i} \mathbf{M} (\lambda_{i}) - \widehat{\lambda}_{i} \mathbf{M} \left(\widehat{\lambda}_{i} \right) \right\} \mathbf{M}^{-1} (\lambda_{i}) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{T} \mathbf{X}_{2} \\ &= \left(\widehat{\lambda}_{i}^{-1} - \lambda_{i}^{-1} \right) \mathbf{X}_{2}^{\top} \mathbf{M}^{-1} \left(\widehat{\lambda}_{i} \right) \mathbf{M}^{-1} (\lambda_{i}) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{T} \mathbf{X}_{2}. \end{split}$$

The same arguments for deriving (3.13) and (3.14) lead to that, for all $1 \le j \le q$,

$$\mathbf{e}_{j}^{\mathsf{T}}\mathbf{U}\Xi_{B}\mathbf{U}^{\mathsf{T}}\mathbf{e}_{j} = \lambda_{j} + \lambda_{j} \cdot \mathbf{O}_{p}\left(\frac{1}{\sqrt{n}}\right) + \lambda_{j} \cdot \mathbf{O}_{a.s.}(\lambda_{i}^{-1})$$
(3.15)

and that, for $1 \leq j_1, j_2 \leq q$,

$$\mathbf{e}_{j_1}^{\mathsf{T}} \mathbf{U} \Xi_B \mathbf{U}^{\mathsf{T}} \mathbf{e}_{j_2} = \lambda_{j_1}^{\frac{1}{2}} \lambda_{j_2}^{\frac{1}{2}} \cdot \mathbf{O}_p \left(\frac{1}{\sqrt{n}} \right) + \lambda_{j_1}^{\frac{1}{2}} \lambda_{j_2}^{\frac{1}{2}} \cdot \mathbf{O}_{a.s.}(\lambda_i^{-1})$$
(3.16)

for all $1 \le j_1 \ne j_2 \le q$.

For the term $\mathbf{U}(\boldsymbol{\Xi}_{C} + \boldsymbol{\Xi}_{D}) \mathbf{U}^{\mathsf{T}}$, by using the fact that $\mathbf{Y}_{1} = \mathbf{U}^{\mathsf{T}} \boldsymbol{\Lambda}_{1}^{-\frac{1}{2}} \mathbf{U} \mathbf{X}_{1}$, we have that

$$\mathbf{U}\left(\Xi_C + \Xi_D\right)\mathbf{U}^{\mathsf{T}}$$

$$= \mathbf{U} \left\{ \widehat{\lambda}_{i} \frac{\mathbf{Z}_{1} \mathbf{C}(\widehat{\lambda}_{i}) \mathbf{X}_{1}^{\top}}{n} + \widehat{\lambda}_{i} \frac{\mathbf{X}_{1} \mathbf{D}(\widehat{\lambda}_{i}) \mathbf{Z}_{1}^{\top}}{T} \right\} \mathbf{U}^{\top}$$

$$= \mathbf{U} \left\{ \lambda_{i} \frac{\mathbf{Z}_{1} \mathbf{C}(\lambda_{i}) \mathbf{X}_{1}^{\top}}{n} + \lambda_{i} \frac{\mathbf{X}_{1} \mathbf{D}(\lambda_{i}) \mathbf{Z}_{1}^{\top}}{T} \right\} \mathbf{U}^{\top} + \mathbf{U} \mathbf{Z}_{1} \left\{ \widehat{\lambda}_{i} \mathbf{C}(\widehat{\lambda}_{i}) - \lambda_{i} \mathbf{C}(\lambda_{i}) \right\} \frac{1}{n} \mathbf{Y}_{1}^{\top} \mathbf{U}^{\top} \boldsymbol{\Lambda}_{1}^{\frac{1}{2}}$$

$$+ \boldsymbol{\Lambda}_{1}^{\frac{1}{2}} \mathbf{U} \mathbf{Y}_{1} \left\{ \widehat{\lambda}_{i} \mathbf{D}(\widehat{\lambda}_{i}) - \lambda_{i} \mathbf{D}(\lambda_{i}) \right\} \frac{1}{T} \mathbf{Z}_{1}^{\top} \mathbf{U}^{\top},$$

and that

$$\begin{aligned} \mathbf{U} & \left\{ \lambda_{i} \frac{\mathbf{Z}_{1} \mathbf{C}(\lambda_{i}) \mathbf{X}_{1}^{\top}}{n} + \lambda_{i} \frac{\mathbf{X}_{1} \mathbf{D}(\lambda_{i}) \mathbf{Z}_{1}^{\top}}{T} \right\} \mathbf{U}^{\top} \\ = & \mathbf{U} \begin{pmatrix} \mathbf{Z}_{1} & \mathbf{X}_{1} \end{pmatrix} \begin{pmatrix} O & \frac{\lambda_{i} \mathbf{C}(\lambda_{i})}{n} \\ \frac{\lambda_{i} \mathbf{D}(\lambda_{i})}{T} & O \end{pmatrix} \begin{pmatrix} \mathbf{Z}_{1}^{\top} \\ \mathbf{X}_{1}^{\top} \end{pmatrix} \mathbf{U}^{\top} \\ = & \left(\mathbf{U} \mathbf{Z}_{1} & \Lambda_{1}^{\frac{1}{2}} \mathbf{U} \mathbf{Y}_{1} \right) \begin{pmatrix} O & \frac{\lambda_{i} \mathbf{C}(\lambda_{i})}{n} \\ \frac{\lambda_{i} \mathbf{D}(\lambda_{i})}{T} & O \end{pmatrix} \begin{pmatrix} \mathbf{Z}_{1}^{\top} \mathbf{U}^{\top} \\ \mathbf{X}_{1}^{\top} \end{pmatrix} \mathbf{U}^{\top} \end{aligned}$$

Then we have that, for all $1 \le j_1, j_2 \le q$,

$$\begin{aligned} \mathbf{e}_{j_{1}}^{\mathsf{T}} \mathbf{U} \left\{ \lambda_{i} \frac{\mathbf{Z}_{1} \mathbf{C}(\lambda_{i}) \mathbf{X}_{1}^{\mathsf{T}}}{n} + \lambda_{i} \frac{\mathbf{X}_{1} \mathbf{D}(\lambda_{i}) \mathbf{Z}_{1}^{\mathsf{T}}}{T} \right\} \mathbf{U}^{\mathsf{T}} \mathbf{e}_{j_{2}} \\ = \mathbf{e}_{j_{1}}^{\mathsf{T}} \left(\mathbf{U} \mathbf{Z}_{1} \quad \Lambda_{1}^{\frac{1}{2}} \mathbf{U} \mathbf{Y}_{1} \right) \begin{pmatrix} O & \frac{\lambda_{i} \mathbf{C}(\lambda_{i})}{n} \\ \frac{\lambda_{i} \mathbf{D}(\lambda_{i})}{T} & O \end{pmatrix} \begin{pmatrix} \mathbf{Z}_{1}^{\mathsf{T}} \mathbf{U}^{\mathsf{T}} \\ \mathbf{Y}_{1}^{\mathsf{T}} \mathbf{U}^{\mathsf{T}} \Lambda_{1}^{\frac{1}{2}} \end{pmatrix} \mathbf{e}_{j_{2}} \\ = \mathbf{e}_{j_{1}}^{\mathsf{T}} \left(\mathbf{U} \mathbf{Z}_{1} \quad \lambda_{j_{1}}^{\frac{1}{2}} \mathbf{U} \mathbf{Y}_{1} \right) \begin{pmatrix} O & \frac{\lambda_{i} \mathbf{C}(\lambda_{i})}{n} \\ \frac{\lambda_{i} \mathbf{D}(\lambda_{i})}{T} & O \end{pmatrix} \begin{pmatrix} \mathbf{Z}_{1}^{\mathsf{T}} \mathbf{U}^{\mathsf{T}} \\ \lambda_{j_{2}}^{\frac{1}{2}} \mathbf{Y}_{1}^{\mathsf{T}} \mathbf{U}^{\mathsf{T}} \end{pmatrix} \mathbf{e}_{j_{2}} \\ = \frac{\lambda_{j_{1}}^{\frac{1}{2}} + \lambda_{j_{2}}^{\frac{1}{2}}}{2} \cdot \mathbf{e}_{j_{1}}^{\mathsf{T}} \left(\mathbf{U} \mathbf{Z}_{1} \quad \mathbf{U} \mathbf{Y}_{1} \right) \begin{pmatrix} O & \frac{\lambda_{i} \mathbf{C}(\lambda_{i})}{n} \\ \frac{\lambda_{i} \mathbf{D}(\lambda_{i})}{T} & O \end{pmatrix} \begin{pmatrix} \mathbf{Z}_{1}^{\mathsf{T}} \mathbf{U}^{\mathsf{T}} \\ \mathbf{Y}_{1}^{\mathsf{T}} \mathbf{U}^{\mathsf{T}} \end{pmatrix} \mathbf{e}_{j_{2}} \\ + \frac{\lambda_{j_{1}}^{\frac{1}{2}} - \lambda_{j_{2}}^{\frac{1}{2}}}{2\mathbf{i}} \cdot \mathbf{e}_{j_{1}}^{\mathsf{T}} \left(\mathbf{U} \mathbf{Z}_{1} \quad \mathbf{U} \mathbf{Y}_{1} \right) \begin{pmatrix} O & \frac{\lambda_{i} \mathbf{C}(\lambda_{i})}{n} \\ -\frac{\lambda_{i} \mathbf{D}(\lambda_{i})}{n} & O \end{pmatrix} \begin{pmatrix} \mathbf{Z}_{1}^{\mathsf{T}} \mathbf{U}^{\mathsf{T}} \\ \mathbf{Y}_{1}^{\mathsf{T}} \mathbf{U}^{\mathsf{T}} \end{pmatrix} \mathbf{e}_{j_{2}} \\ + \frac{\lambda_{j_{1}}^{\frac{1}{2}} - \lambda_{j_{2}}^{\frac{1}{2}}}{2\mathbf{i}} \cdot \mathbf{e}_{j_{1}}^{\mathsf{T}} \left(\mathbf{U} \mathbf{Z}_{1} \quad \mathbf{U} \mathbf{Y}_{1} \right) \begin{pmatrix} O & \frac{\lambda_{i} \mathbf{C}(\lambda_{i})}{n} \\ -\frac{\lambda_{i} \mathbf{D}(\lambda_{i})}{n} & O \end{pmatrix} \begin{pmatrix} \mathbf{Z}_{1}^{\mathsf{T}} \mathbf{U}^{\mathsf{T}} \\ \mathbf{Y}_{1}^{\mathsf{T}} \mathbf{U}^{\mathsf{T}} \end{pmatrix} \mathbf{e}_{j_{2}} \\ = \frac{\lambda_{j_{1}}^{\frac{1}{2}} - \lambda_{j_{2}}^{\frac{1}{2}}}{2\mathbf{i}} \cdot \mathbf{e}_{j_{1}}^{\mathsf{T}} \left(\mathbf{U} \mathbf{Z}_{1} \quad \mathbf{U} \mathbf{Y}_{1} \right) \begin{pmatrix} O & \frac{\lambda_{i} \mathbf{C}(\lambda_{i})}{n} \\ -\frac{\lambda_{i} \mathbf{D}(\lambda_{i})}{n} & O \end{pmatrix} \begin{pmatrix} \mathbf{Z}_{1}^{\mathsf{T}} \mathbf{U}^{\mathsf{T}} \\ \mathbf{Y}_{1}^{\mathsf{T}} \mathbf{U}^{\mathsf{T}} \end{pmatrix} \mathbf{e}_{j_{2}} \\ = \frac{\lambda_{j_{1}}^{\frac{1}{2}} - \lambda_{j_{2}}^{\frac{1}{2}}}{2\mathbf{i}} \cdot \mathbf{O}_{j} \begin{pmatrix} O & \frac{\lambda_{i} \mathbf{C}(\lambda_{i})}{n} \\ -\frac{\lambda_{i} \mathbf{D}(\lambda_{i})}{n} & O \end{pmatrix} \begin{pmatrix} \mathbf{U} \mathbf{U} \\ \mathbf{U} \\$$

where $\mathbf{i} := \sqrt{-1}$ is the imaginary unit and the penultimate equality is implied by Theorem 7.1 in [3]. Due to the fact that

$$\begin{split} \widehat{\lambda}_{i} \mathbf{C}(\widehat{\lambda}_{i}) &- \lambda_{i} \mathbf{C} \left(\lambda_{i}\right) = \mathbf{Z}_{2}^{\top} \left\{ \mathbf{M}^{-1}(\widehat{\lambda}_{i}) - \mathbf{M}^{-1}(\lambda_{i}) \right\} \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{T} \mathbf{X}_{2} \\ &= \mathbf{Z}_{2}^{\top} \mathbf{M}^{-1}(\widehat{\lambda}_{i}) \left\{ \mathbf{M}(\lambda_{i}) - \mathbf{M}(\widehat{\lambda}_{i}) \right\} \mathbf{M}^{-1}(\lambda_{i}) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{T} \mathbf{X}_{2} \\ &= \left(\widehat{\lambda}_{i}^{-1} - \lambda_{i}^{-1} \right) \mathbf{Z}_{2}^{\top} \mathbf{M}^{-1}(\widehat{\lambda}_{i}) \mathbf{F}_{0} \mathbf{M}^{-1}(\lambda_{i}) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{T} \mathbf{X}_{2}, \\ \widehat{\lambda}_{i} \mathbf{D}(\widehat{\lambda}_{i}) - \lambda_{i} \mathbf{D} \left(\lambda_{i}\right) = \mathbf{X}_{2}^{\top} \left\{ \mathbf{M}^{-1}(\widehat{\lambda}_{i}) - \mathbf{M}^{-1}(\lambda_{i}) \right\} \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{n} \mathbf{Z}_{2} \end{split}$$

$$= \mathbf{X}_{2}^{\top} \mathbf{M}^{-1}(\widehat{\lambda}_{i}) \left\{ \mathbf{M}(\lambda_{i}) - \mathbf{M}(\widehat{\lambda}_{i}) \right\} \mathbf{M}^{-1}(\lambda_{i}) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{n} \mathbf{Z}_{2}$$
$$= \left(\widehat{\lambda}_{i}^{-1} - \lambda_{i}^{-1} \right) \mathbf{X}_{2}^{\top} \mathbf{M}^{-1}(\widehat{\lambda}_{i}) \mathbf{F}_{0} \mathbf{M}^{-1}(\lambda_{i}) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{n} \mathbf{Z}_{2},$$

we can get

$$\mathbf{e}_{j_1}^{\top} \mathbf{U} \mathbf{Z}_1 \left\{ \widehat{\lambda}_i \mathbf{C}(\widehat{\lambda}_i) - \lambda_i \mathbf{C}(\lambda_i) \right\} \frac{1}{n} \mathbf{Y}_1^{\top} \mathbf{U}^{\top} \mathbf{\Lambda}_1^{\frac{1}{2}} \mathbf{e}_{j_2} = \lambda_{j_2}^{\frac{1}{2}} \cdot \mathbf{O}_{a.s.} \left(\lambda_i^{-1} \right)$$

and

$$\mathbf{e}_{j_1}^{\top} \boldsymbol{\Lambda}_1^{\frac{1}{2}} \mathbf{U} \mathbf{Y}_1 \left\{ \widehat{\lambda}_i \mathbf{D}(\widehat{\lambda}_i) - \lambda_i \mathbf{D}(\lambda_i) \right\} \frac{1}{T} \mathbf{Z}_1^{\top} \mathbf{U}^{\top} \mathbf{e}_{j_2} = \lambda_{j_1}^{\frac{1}{2}} \cdot \mathbf{O}_{a.s.} \left(\lambda_i^{-1} \right)$$

for any $1 \le j_1, j_2 \le q$. By using the similar arguments for proving (3.13) and (3.14), it holds that

$$\mathbf{e}_{j_1}^{\mathsf{T}} \mathbf{U} \left(\Xi_C + \Xi_D \right) \mathbf{U}^{\mathsf{T}} \mathbf{e}_{j_2} = \left(\lambda_{j_1}^{\frac{1}{2}} + \lambda_{j_2}^{\frac{1}{2}} \right) \cdot \mathbf{O}_p \left(\frac{1}{\sqrt{n}} \right) + \left(\lambda_{j_1}^{\frac{1}{2}} + \lambda_{j_2}^{\frac{1}{2}} \right) \cdot \mathbf{O}_{a.s.} \left(\lambda_i^{-1} \right)$$
(3.17)

for any $1 \le j_1, j_2 \le q$.

Combining (3.14)-(3.17) and the determinant (3.5), we can compute the limit of $\hat{\lambda}_i/\lambda_i$ for each $1 \le i \le q$. We use a new notation to denote the matrix in the determinant (3.5). Define

$$\Xi := \mathbf{U} \left(\Xi_A - \Xi_B + \Xi_C + \Xi_D \right) \mathbf{U}^{\mathsf{T}}, \quad \widetilde{\Xi} := \operatorname{diag} \left(\xi_{11}, \dots, \xi_{qq} \right),$$

where $\xi_{jj} = \widehat{\lambda}_i \{1 - (p - q)/n\} - \lambda_j$. Then by (3.14)-(3.17), we have that

$$\mathbf{e}_{j_{1}}^{\top} \left(\Xi - \widetilde{\Xi} \right) \mathbf{e}_{j_{2}} = \lambda_{i} \cdot \mathbf{O}_{p} \left(\frac{1}{\sqrt{n}} \right) + \mathbf{O}_{p}(1) + \lambda_{j_{1}}^{\frac{1}{2}} \lambda_{j_{2}}^{\frac{1}{2}} \cdot \mathbf{O}_{p} \left(\frac{1}{\sqrt{n}} \right) + \lambda_{j_{1}}^{\frac{1}{2}} \lambda_{j_{2}}^{\frac{1}{2}} \cdot \mathbf{O}_{a.s.}(\lambda_{i}^{-1}) \\ + \left(\lambda_{j_{1}}^{\frac{1}{2}} + \lambda_{j_{2}}^{\frac{1}{2}} \right) \cdot \mathbf{O}_{p} \left(\frac{1}{\sqrt{n}} \right) + \left(\lambda_{j_{1}}^{\frac{1}{2}} + \lambda_{j_{2}}^{\frac{1}{2}} \right) \cdot \mathbf{O}_{a.s.}(\lambda_{i}^{-1}) \\ = \left(\lambda_{i} + \lambda_{j_{1}}^{\frac{1}{2}} \lambda_{j_{2}}^{\frac{1}{2}} \right) \left\{ \mathbf{O}_{p} \left(\frac{1}{\sqrt{n}} \right) + \mathbf{O}_{a.s.}(\lambda_{i}^{-1}) \right\}$$

for any $1 \le j_1, j_2 \le q$, which follows that

$$\mathbf{e}_{j_1}^{\top}\lambda_i^{-1}\left(\Xi - \widetilde{\Xi}\right)\mathbf{e}_{j_2} = \left(1 + \lambda_i^{-1}\lambda_{j_1}^{\frac{1}{2}}\lambda_{j_2}^{\frac{1}{2}}\right)\left\{\mathbf{O}_p\left(\frac{1}{\sqrt{n}}\right) + \mathbf{O}_{a.s.}\left(\lambda_i^{-1}\right)\right\}.$$
(3.18)

According to (3.7) and (3.9) for Ξ_A (similar results also hold for Ξ_B , Ξ_C and Ξ_D), it can be easily checked that the variance of the term in (3.18) has the order

$$\left(1+\lambda_{i}^{-1}\lambda_{j_{1}}^{\frac{1}{2}}\lambda_{j_{2}}^{\frac{1}{2}}\right)^{2}\left(n^{-\frac{1}{2}}+\lambda_{i}^{-1}\right)^{2}.$$

By Chebyshev's inequality, we have that, for any $\epsilon > 0$,

$$\Pr\left\{\max_{1\leq j_1, j_2\leq q} \left| \mathbf{e}_{j_1}^{\mathsf{T}} \lambda_i^{-1} \left(\Xi - \widetilde{\Xi} \right) \mathbf{e}_{j_2} \right| \geq \epsilon \left(n^{-\frac{1}{2}} + \lambda_i^{-1} \right) \right\}$$

$$\leq \sum_{1 \leq j_1, j_2 \leq q} \Pr\left\{ \left| \mathbf{e}_{j_1}^{\top} \lambda_i^{-1} \left(\Xi - \widetilde{\Xi} \right) \mathbf{e}_{j_2} \right| \geq \epsilon \left(n^{-\frac{1}{2}} + \lambda_i^{-1} \right) \right\}$$

$$\leq \sum_{1 \leq j_1, j_2 \leq q} \frac{\operatorname{E}\left\{ \mathbf{e}_{j_1}^{\top} \lambda_i^{-1} \left(\Xi - \widetilde{\Xi} \right) \mathbf{e}_{j_2} \right\}^2}{\epsilon^2 \left(n^{-\frac{1}{2}} + \lambda_i^{-1} \right)^2}$$

$$= \sum_{1 \leq j_1, j_2 \leq q} \left(1 + \lambda_i^{-1} \lambda_{j_1}^{\frac{1}{2}} \lambda_{j_2}^{\frac{1}{2}} \right)^2 \cdot \operatorname{O}\left(\epsilon^{-2} \right) = \left(q + \lambda_i^{-1} \sum_{j=1}^q \lambda_j \right)^2 \cdot \operatorname{O}\left(\epsilon^{-2} \right) = \kappa_1^2 \cdot \operatorname{O}\left(\epsilon^{-2} \right),$$

which means

$$\left\|\lambda_{i}^{-1}\left(\Xi-\widetilde{\Xi}\right)\right\|_{\infty}=\max_{1\leq j_{1},j_{2}\leq q}\left|\mathbf{e}_{j_{1}}^{\top}\lambda_{i}^{-1}\left(\Xi-\widetilde{\Xi}\right)\mathbf{e}_{j_{2}}\right|=\kappa_{1}\cdot\mathbf{O}_{p}\left(\frac{1}{\sqrt{n}}+\lambda_{i}^{-1}\right)$$

and then

$$\||\lambda_i^{-1}(\Xi - \widetilde{\Xi})|\|_{\infty} \le q \|\lambda_i^{-1}(\Xi - \widetilde{\Xi})\|_{\infty} = \kappa_1 q \cdot \mathcal{O}_p\left(\frac{1}{\sqrt{n}} + \lambda_i^{-1}\right).$$

Note that the determinant equation det $(\widetilde{\Xi}) = 0$ is equivalent to det $(\lambda_i^{-1}\widetilde{\Xi}) = 0$, that is,

$$\det\left\{\frac{\widehat{\lambda}_i}{\lambda_i}\left(1-\frac{p-q}{n}\right)\mathbf{I}_q-\lambda_i^{-1}\Lambda_1\right\}=0.$$

At the same time, the equation det $(\Xi) = 0$ is equivalent to det $(\lambda_i^{-1}\Xi) = 0$, that is,

$$\det\left\{\frac{\widehat{\lambda}_i}{\lambda_i}\left(1-\frac{p-q}{n}\right)\mathbf{I}_q-\lambda_i^{-1}\Lambda_1+\lambda_i^{-1}\left(\Xi-\widetilde{\Xi}\right)\right\}=0.$$

By eigenvalue perturbation theorems (see Theorem 6.3.2 in Chapter 6, [9]), we have

$$\left|\frac{\widehat{\lambda}_i}{\lambda_i}\left(1-\frac{p-q}{n}\right)-1\right| \leq |||\lambda_i^{-1}(\Xi-\widetilde{\Xi})|||_{\infty} = \kappa_1 q \cdot \mathcal{O}_p\left(\frac{1}{\sqrt{n}}+\lambda_i^{-1}\right),$$

that is

$$\widehat{\frac{\lambda_i}{\lambda_i}} = \frac{1}{1-y} + O\left(y_p - y\right) + \kappa_1 q \cdot O_p\left(\frac{1}{\sqrt{n}} + \lambda_i^{-1}\right).$$
(3.19)

Instead, we can compare determinant equations

$$\det\left(\Lambda_1^{-\frac{1}{2}}\widetilde{\Xi}\Lambda_1^{-\frac{1}{2}}\right) = 0 \quad \text{and} \quad \det\left(\Lambda_1^{-\frac{1}{2}}\Xi\Lambda_1^{-\frac{1}{2}}\right) = 0$$

and then repeat all the derivations above to achieve an upper bound of $\||\Lambda_1^{-1/2}(\Xi - \widetilde{\Xi})\Lambda_1^{-1/2}\||_{\infty}$. In this case, we can get

$$\frac{\widehat{\lambda}_i}{\lambda_i} = \frac{1}{1-y} + O\left(y_p - y\right) + \kappa_2 q \cdot O_p\left(\frac{1}{\sqrt{n}} + \lambda_i^{-1}\right).$$
(3.20)

Thus, (3.19) and (3.20) lead to

$$\widehat{\frac{\lambda_i}{\lambda_i}} = \frac{1}{1-y} + \mathcal{O}\left(y_p - y\right) + \kappa q \cdot \mathcal{O}_p\left(\frac{1}{\sqrt{n}} + \lambda_i^{-1}\right),$$

where $\kappa q(n^{-1/2} + \lambda_i^{-1}) = o(1)$ under Assumption 2.2. The proof is finished. \Box

Proof of Theorem 2.10. We begin with the equation on $\widehat{\lambda}_i$ in (3.4). Recall that we have expressed (3.4) as

$$\det\left(\Xi_A - \Xi_B + \Xi_C + \Xi_D\right) = 0. \tag{3.21}$$

For the first term Ξ_A , we can write

$$\begin{split} \Xi_{A} = & \widehat{\frac{\lambda_{i}}{n}} \mathbf{Z}_{1} \left[\left\{ \mathbf{I}_{n} - \mathbf{A}(\widehat{\lambda}_{i}) \right\} - \left\{ \mathbf{I}_{n} - \mathbf{A}(\theta_{i}) \right\} \right] \mathbf{Z}_{1}^{\top} + \frac{\widehat{\lambda}_{i}}{n} \left(\mathbf{Z}_{1} \left\{ \mathbf{I}_{n} - \mathbf{A}(\theta_{i}) \right\} \mathbf{Z}_{1}^{\top} - \mathbf{E} \left[\mathbf{Z}_{1} \left\{ \mathbf{I}_{n} - \mathbf{A}(\theta_{i}) \right\} \mathbf{Z}_{1}^{\top} \right] \right) \\ & + \frac{\widehat{\lambda}_{i}}{n} \mathbf{E} \left[\mathbf{Z}_{1} \left\{ \mathbf{I}_{n} - \mathbf{A}(\theta_{i}) \right\} \mathbf{Z}_{1}^{\top} \right]. \end{split}$$

Using the fact

$$\left\{\mathbf{I}_n - \mathbf{A}(\widehat{\lambda}_i)\right\} - \left\{\mathbf{I}_n - \mathbf{A}(\theta_i)\right\} = -\delta_i \mathbf{A}(\theta_i) + \delta_i \mathbf{Z}_2^{\mathsf{T}} \mathbf{M}^{-1}(\widehat{\lambda}_i) \mathbf{M}^{-1}(\theta_i) \left(\frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_2^{\mathsf{T}}\right)^{-1} \frac{1}{n} \mathbf{Z}_2$$

and $\widehat{\lambda}_i = \theta_i (1 + \delta_i)$ by (2.7), we can get

$$\begin{aligned} \Xi_{A} &= \theta_{i}\delta_{i}(1+\delta_{i})\frac{1}{n} \left(\mathbf{Z}_{1} \left\{ \mathbf{I}_{n} - \mathbf{A}(\theta_{i}) \right\} \mathbf{Z}_{1}^{\top} - \mathbf{E} \left[\mathbf{Z}_{1} \left\{ \mathbf{I}_{n} - \mathbf{A}(\theta_{i}) \right\} \mathbf{Z}_{1}^{\top} \right] \right) \\ &+ \theta_{i}\delta_{i}(1+\delta_{i})\frac{1}{n} \mathbf{Z}_{1} \mathbf{Z}_{2}^{\top} \mathbf{M}^{-1}(\widehat{\lambda}_{i}) \mathbf{M}^{-1}(\theta_{i}) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{1}^{\top} \\ &- \theta_{i}\delta_{i}(1+\delta_{i})\frac{1}{n} \mathbf{Z}_{1} \mathbf{Z}_{1}^{\top} + \theta_{i}\delta_{i}(1+\delta_{i})\frac{1}{n} \mathbf{E} \left[\mathbf{Z}_{1} \left\{ \mathbf{I}_{n} - \mathbf{A}(\theta_{i}) \right\} \mathbf{Z}_{1}^{\top} \right] \\ &+ \theta_{i}(1+\delta_{i})\frac{1}{n} \left(\mathbf{Z}_{1} \left\{ \mathbf{I}_{n} - \mathbf{A}(\theta_{i}) \right\} \mathbf{Z}_{1}^{\top} - \mathbf{E} \left[\mathbf{Z}_{1} \left\{ \mathbf{I}_{n} - \mathbf{A}(\theta_{i}) \right\} \mathbf{Z}_{1}^{\top} \right] \right) \\ &+ \theta_{i}(1+\delta_{i})\frac{1}{n} \mathbf{E} \left[\mathbf{Z}_{1} \left\{ \mathbf{I}_{n} - \mathbf{A}(\theta_{i}) \right\} \mathbf{Z}_{1}^{\top} \right] \\ &= \theta_{i}(1+\delta_{i})^{2}\frac{1}{n} \left(\mathbf{Z}_{1} \left\{ \mathbf{I}_{n} - \mathbf{A}(\theta_{i}) \right\} \mathbf{Z}_{1}^{\top} - \mathbf{E} \left[\mathbf{Z}_{1} \left\{ \mathbf{I}_{n} - \mathbf{A}(\theta_{i}) \right\} \mathbf{Z}_{1}^{\top} \right] \right) \\ &+ \theta_{i}\delta_{i}(1+\delta_{i})\frac{1}{n} \mathbf{Z}_{1} \mathbf{Z}_{2}^{\top} \mathbf{M}^{-1}(\widehat{\lambda}_{i}) \mathbf{M}^{-1}(\theta_{i}) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{1}^{\top} \\ &- \theta_{i}\delta_{i}(1+\delta_{i})\frac{1}{n} \mathbf{Z}_{1} \mathbf{Z}_{1}^{\top} + \theta_{i}(1+\delta_{i})^{2}\frac{1}{n} \mathbf{E} \left[\mathbf{Z}_{1} \left\{ \mathbf{I}_{n} - \mathbf{A}(\theta_{i}) \right\} \mathbf{Z}_{1}^{\top} \right] \\ &=: \theta_{i}(1+\delta_{i})^{2} \Xi_{A1} + \theta_{i}\delta_{i}(1+\delta_{i}) \Xi_{A2} - \theta_{i}\delta_{i}(1+\delta_{i}) \Xi_{A3} + \theta_{i}(1+\delta_{i})^{2} \Xi_{A4}. \end{aligned}$$
(3.22)

For the second term Ξ_B , we can similarly write

$$\Xi_{B} = \frac{1}{T} \mathbf{X}_{1} \left\{ \mathbf{B}(\widehat{\lambda}_{i}) - \mathbf{B}(\theta_{i}) \right\} \mathbf{X}_{1}^{\top} + \frac{1}{T} \left(\mathbf{X}_{1} \left\{ \mathbf{I}_{T} + \mathbf{B}(\theta_{i}) \right\} \mathbf{X}_{1}^{\top} - \mathbf{E} \left[\mathbf{X}_{1} \left\{ \mathbf{I}_{T} + \mathbf{B}(\theta_{i}) \right\} \mathbf{X}_{1}^{\top} \right] \right) \\ + \frac{1}{T} \mathbf{E} \left[\mathbf{X}_{1} \left\{ \mathbf{I}_{T} + \mathbf{B}(\theta_{i}) \right\} \mathbf{X}_{1}^{\top} \right] \\ = \frac{1}{T} \left(\mathbf{X}_{1} \left\{ \mathbf{I}_{T} + \mathbf{B}(\theta_{i}) \right\} \mathbf{X}_{1}^{\top} - \mathbf{E} \left[\mathbf{X}_{1} \left\{ \mathbf{I}_{T} + \mathbf{B}(\theta_{i}) \right\} \mathbf{X}_{1}^{\top} \right] \right) \\ - \frac{\delta_{i}}{\widehat{\lambda}_{i}} \cdot \frac{1}{T} \mathbf{X}_{1} \mathbf{X}_{2}^{\top} \mathbf{M}^{-1}(\widehat{\lambda}_{i}) \mathbf{M}^{-1}(\theta_{i}) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{T} \mathbf{X}_{2} \mathbf{X}_{1}^{\top} + \frac{1}{T} \mathbf{E} \left[\mathbf{X}_{1} \left\{ \mathbf{I}_{T} + \mathbf{B}(\theta_{i}) \right\} \mathbf{X}_{1}^{\top} \right]$$

$$=:\Xi_{B1} - \frac{\delta_i}{\widehat{\lambda}_i} \Xi_{B2} + \Xi_{B3}, \tag{3.23}$$

where the second equality above uses the fact

$$\mathbf{B}(\widehat{\lambda}_i) - \mathbf{B}(\theta_i) = -\frac{\delta_i}{\widehat{\lambda}_i} \mathbf{X}_2^{\mathsf{T}} \mathbf{M}^{-1}(\widehat{\lambda}_i) \mathbf{M}^{-1}(\theta_i) \left(\frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_2^{\mathsf{T}}\right)^{-1} \frac{1}{T} \mathbf{X}_2.$$

For the term Ξ_C , we have

$$\Xi_{C} = \frac{\widehat{\lambda}_{i}}{n} \mathbf{Z}_{1} \left\{ \mathbf{C}(\widehat{\lambda}_{i}) - \mathbf{C}(\theta_{i}) \right\} \mathbf{X}_{1}^{\top} + \frac{\widehat{\lambda}_{i} - \theta_{i}}{n} \mathbf{Z}_{1} \mathbf{C}(\theta_{i}) \mathbf{X}_{1}^{\top} + \frac{\theta_{i}}{n} \left[\mathbf{Z}_{1} \mathbf{C}(\theta_{i}) \mathbf{X}_{1}^{\top} - \mathbf{E} \left\{ \mathbf{Z}_{1} \mathbf{C}(\theta_{i}) \mathbf{X}_{1}^{\top} \right\} \right].$$

Using the fact

$$\mathbf{C}(\widehat{\lambda}_i) - \mathbf{C}(\theta_i) = -\frac{\delta_i}{\widehat{\lambda}_i} \mathbf{Z}_2^{\mathsf{T}} \mathbf{M}^{-1}(\widehat{\lambda}_i) \mathbf{M}^{-1}(\theta_i) \left(\frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_2^{\mathsf{T}}\right)^{-1} \frac{1}{T} \mathbf{X}_2,$$

we have the decomposition

$$\Xi_{C} = \theta_{i} \frac{1}{n} \left[\mathbf{Z}_{1} \mathbf{C}(\theta_{i}) \mathbf{X}_{1}^{\top} - \mathbf{E} \left\{ \mathbf{Z}_{1} \mathbf{C}(\theta_{i}) \mathbf{X}_{1}^{\top} \right\} \right] - \delta_{i} \frac{1}{n} \mathbf{Z}_{1} \mathbf{Z}_{2}^{\top} \mathbf{M}^{-1}(\widehat{\lambda}_{i}) \mathbf{M}^{-1}(\theta_{i}) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{T} \mathbf{X}_{2} \mathbf{X}_{1}^{\top} + \theta_{i} \delta_{i} \frac{1}{n} \mathbf{Z}_{1} \mathbf{C}(\theta_{i}) \mathbf{X}_{1}^{\top} =: \theta_{i} \Xi_{C1} - \delta_{i} \Xi_{C2} + \theta_{i} \delta_{i} \Xi_{C3}.$$
(3.24)

Similarly, we can write the last term Ξ_D as

$$\Xi_{D} = \theta_{i} \frac{1}{T} \left[\mathbf{X}_{1} \mathbf{D}(\theta_{i}) \mathbf{Z}_{1}^{\top} - \mathbf{E} \left\{ \mathbf{X}_{1} \mathbf{D}(\theta_{i}) \mathbf{Z}_{1}^{\top} \right\} \right] - \delta_{i} \frac{1}{T} \mathbf{X}_{1} \mathbf{X}_{2}^{\top} \mathbf{M}^{-1}(\widehat{\lambda}_{i}) \mathbf{M}^{-1}(\theta_{i}) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{1}^{\top} + \theta_{i} \delta_{i} \frac{1}{T} \mathbf{X}_{1} \mathbf{D}(\theta_{i}) \mathbf{Z}_{1}^{\top} =: \theta_{i} \Xi_{D1} - \delta_{i} \Xi_{D2} + \theta_{i} \delta_{i} \Xi_{D3}.$$
(3.25)

Putting (3.22)-(3.25) into (3.21), we have

$$\det(\theta_i \Theta_{1n} + \theta_i \delta_i \Theta_{2n} + \theta_i \Theta_{3n}) = 0, \qquad (3.26)$$

where

$$\Theta_{1n} := (1+\delta_i)^2 \Xi_{A1} - \theta_i^{-1} \Xi_{B1} + \Xi_{C1} + \Xi_{D1}, \qquad (3.27)$$

$$\Theta_{2n} := (1+\delta_i)\Xi_{A2} - (1+\delta_i)\Xi_{A3} + \frac{1}{\theta_i\widehat{\lambda}_i}\Xi_{B2} - \theta_i^{-1}\Xi_{C2} + \Xi_{C3} - \theta_i^{-1}\Xi_{D2} + \Xi_{D3}, \quad (3.28)$$

$$\Theta_{3n} := (1+\delta_i)^2 \Xi_{A4} - \theta_i^{-1} \Xi_{B3}.$$
(3.29)

Multiplying both sides of the matrix in (3.26) by $\theta_i^{-1/2}$ **U** from the left hand side and $\theta_i^{-1/2}$ **U**^T from the right hand side, we get

$$\det\left\{\mathbf{U}(\Theta_{1n} + \delta_i \Theta_{2n} + \Theta_{3n})\mathbf{U}^{\mathsf{T}}\right\} = 0.$$
(3.30)

Recall that \mathbf{e}_i is the *q*-dimensional vector whose *i*-th element is 1 and others are 0. By Lemma 4.2 below, we have

$$\sqrt{p}\widehat{S}_i := \sqrt{p}\mathbf{e}_i^\top \mathbf{U}\Theta_{1n}\mathbf{U}^\top \mathbf{e}_i \xrightarrow{d} \mathcal{N}(0, \widetilde{\sigma}_i^2), \qquad (3.31)$$

where $\tilde{\sigma}_i^2 = (y+c)(1-y)^2 v_i - y(1-y)(1-3y) - c(1-y)^2$. It follows by Lemma 4.3 below that

$$\|\mathbf{U}\Theta_{1n}\mathbf{U}^{\mathsf{T}}\|_{\infty} = \mathcal{O}_p\left(\frac{q}{\sqrt{n}} + \frac{\sum_j \lambda_j}{\sqrt{n}\lambda_i}\right).$$
(3.32)

By Lemma 4.4 below, we also have

$$\max_{1 \le j \le q} \left| \mathbf{e}_{j}^{\mathsf{T}} \mathbf{U} \Theta_{2n} \mathbf{U}^{\mathsf{T}} \mathbf{e}_{j} - (y-1) \right| = O_{p} \left(\frac{\sqrt{q} \delta_{i}}{\lambda_{i}} + \frac{\sqrt{q}}{\sqrt{n}} + \frac{\sqrt{\sum_{j} \lambda_{j}^{2}}}{\lambda_{i}^{2}} + \frac{\sqrt{\sum_{j} \lambda_{j}}}{\lambda_{i}} \right), \quad (3.33)$$

$$\max_{1 \le j_1 \ne j_2 \le q} \left| \mathbf{e}_{j_1}^{\mathsf{T}} \mathbf{U} \Theta_{2n} \mathbf{U}^{\mathsf{T}} \mathbf{e}_{j_2} \right| = \mathcal{O}_p \left(\frac{q \delta_i}{\lambda_i} + \frac{q}{\sqrt{n}} + \frac{\sum_j \lambda_j}{\lambda_i^2} + \frac{\sqrt{q \sum_j \lambda_j}}{\lambda_i} \right).$$
(3.34)

For the term $\mathbf{U}\Theta_{3n}\mathbf{U}^{\top}$ in (3.30), by considering its (j_1, j_2) entry for all $1 \le j_1, j_2 \le q$, we can easily get that

$$(1+\delta_i)^2 \mathbf{U} \Xi_{A4} \mathbf{U}^{\mathsf{T}} = (1+\delta_i)^2 \left[1 - \frac{p-q}{n} \mathbf{E} \left\{ \widetilde{m}_{\theta_i}(1) \right\} \right] \mathbf{I}_q, \qquad (3.35)$$

$$\mathbf{U}\Xi_{B3}\mathbf{U}^{\mathsf{T}} = \left(1 + \frac{p-q}{T}\left[-1 + \mathbf{E}\left\{\widetilde{m}_{\theta_i}(1)\right\}\right]\right)\mathbf{\Lambda}_1.$$
(3.36)

By the definition of θ_i in (2.6), we know

$$1 - \frac{p-q}{n} \mathbb{E}\left\{\widetilde{m}_{\theta_i}(1)\right\} = \frac{\lambda_i}{\theta_i} \left(1 + \frac{p-q}{T} \left[-1 + \mathbb{E}\left\{\widetilde{m}_{\theta_i}(1)\right\}\right]\right),$$

which, together with the results in Lemma 4.1 below and Theorem 2.6, yields that

$$(1+\delta_{i})^{2} \left[1 - \frac{p-q}{n} \mathbb{E}\left\{\widetilde{m}_{\theta_{i}}(1)\right\}\right] - \frac{\lambda_{i}}{\theta_{i}} \left(1 + \frac{p-q}{T} \left[-1 + \mathbb{E}\left\{\widetilde{m}_{\theta_{i}}(1)\right\}\right]\right)$$
$$= 2\delta_{i} \left[1 - \frac{p-q}{n} \mathbb{E}\left\{\widetilde{m}_{\theta_{i}}(1)\right\}\right] + \delta_{i}^{2} \left[1 - \frac{p-q}{n} \mathbb{E}\left\{\widetilde{m}_{\theta_{i}}(1)\right\}\right] = 2\delta_{i} \left\{1 - \frac{p-q}{n} + o(1)\right\}. \quad (3.37)$$

Combining (3.35)-(3.37) and the definition of Θ_{3n} in (3.29), we can get that, for $1 \le j \le q$,

$$\mathbf{e}_{j}^{\mathsf{T}}\mathbf{U}\Theta_{3n}\mathbf{U}^{\mathsf{T}}\mathbf{e}_{j}^{\mathsf{T}} = \left\{ (1+\delta_{i})^{2} - \frac{\lambda_{j}}{\lambda_{i}} \right\} \left[1 - \frac{p-q}{n} \mathbb{E}\left\{ \widetilde{m}_{\theta_{i}}(1) \right\} \right],$$

which converges to zero if and only if $\lambda_j = \lambda_i$ because $(1 + \delta_i)^2 - \lambda_j/\lambda_i > C > 0$ for some constant *C* if $\lambda_j \neq \lambda_i$ under Assumption 2.4. When $\lambda_j = \lambda_i$, we have

$$\mathbf{e}_{j}^{\mathsf{T}}\mathbf{U}\Theta_{3n}\mathbf{U}^{\mathsf{T}}\mathbf{e}_{j}^{\mathsf{T}} = 2\delta_{i}\left\{1 - \frac{p-q}{n} + \mathrm{o}(1)\right\}.$$
(3.38)

Note that all off-diagonal entries of the matrix $\mathbf{U}\Theta_{3n}\mathbf{U}^{\mathsf{T}}$ is zero, i.e.

$$\mathbf{e}_{j_1}^{\mathsf{T}} \mathbf{U} \Theta_{3n} \mathbf{U}^{\mathsf{T}} \mathbf{e}_{j_2} = 0, \forall 1 \le j_1 \ne j_2 \le q.$$
(3.39)

Inserting (3.31), (3.32), (3.33), (3.34), (3.38) and (3.39) into (3.30), we can solve the determinant equation (3.30) and get the limiting distribution of $\delta_i (1 \le i \le q)$ immediately. Since diagonal elements of $\mathbf{U}\Theta_{3n}\mathbf{U}^{\mathsf{T}}$ are at least constant order, when $\mathbf{e}_j^{\mathsf{T}}\mathbf{U}\Theta_{3n}\mathbf{U}^{\mathsf{T}}\mathbf{e}_j^{\mathsf{T}}$ goes to infinity for some *j*'s, we can divide these rows by $\mathbf{e}_j^{\mathsf{T}}\mathbf{U}\Theta_{3n}\mathbf{U}^{\mathsf{T}}\mathbf{e}_j^{\mathsf{T}}$. In this way, we can get

$$\det \begin{pmatrix} O_p(1) & \dots & O_p(*) & \dots & O_p(*) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ O_p(*) & \dots & \widehat{S}_i + (1 - y + O_p(1))\delta_i & \dots & O_p(*) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ O_p(*) & \dots & O_p(*) & \dots & O_p(1) \end{pmatrix} = 0$$

where $\sqrt{p}\widehat{S}_i \xrightarrow{d} \mathcal{N}(0, \widetilde{\sigma}_i^2)$ and

$$* = \frac{q}{\sqrt{n}} + \frac{\sum_{j} \lambda_{j}}{\sqrt{n}\lambda_{i}} + \frac{q\delta_{i}^{2}}{\lambda_{i}} + \frac{\delta_{i} \sum_{j} \lambda_{j}}{\lambda_{i}^{2}} + \frac{\delta_{i} \sqrt{q \sum_{j} \lambda_{j}}}{\lambda_{i}}.$$

By Leibniz formula for determinants, we can get that $\widehat{S}_i + \{1 - y + o_p(1)\}\delta_i + qO_p(*^2) = 0$, that is

$$\widehat{S}_i + \left\{1 - y + o_p(1)\right\}\delta_i + O_p\left(\frac{q^3}{n} + \frac{q(\sum_j \lambda_j)^2}{n\lambda_i^2} + \frac{q^3\delta_i^4}{\lambda_i^2} + \frac{q\delta_i^2(\sum_j \lambda_j)^2}{\lambda_i^4} + \frac{q^2\delta_i^2\sum_j \lambda_j}{\lambda_i^2}\right) = 0.$$

Under Assumptions 2.1 and 2.2(a), we have $q = o(n^{\frac{1}{6}})$ and $\lambda_i^{-1} \sum_j \lambda_j = o(q^{-\frac{1}{2}}n^{\frac{1}{4}})$, then it follows that

$$\frac{q^{3}}{n} = o(n^{-\frac{1}{2}}), \quad \frac{q^{2} \sum_{j} \lambda_{j}}{n \lambda_{i}^{2}} = o(n^{-\frac{1}{2}}), \quad \frac{q^{3} \delta_{i}^{4}}{\lambda_{i}^{2}} = o_{p}(\delta_{i}^{2} n^{\frac{1}{2}})$$
$$\frac{q \delta_{i}^{2} (\sum_{j} \lambda_{j})^{2}}{\lambda_{i}^{4}} = o_{p}(\delta_{i}^{2} n^{\frac{1}{2}}), \quad \frac{q^{2} \delta_{i}^{2} \sum_{j} \lambda_{j}}{\lambda_{i}^{2}} = o_{p}(\delta_{i}^{2} n^{\frac{1}{2}}).$$

It leads to

$$\widehat{S}_i + \left\{ 1 - y + o_p(1) \right\} \delta_i + o_p(\delta_i^2 n^{\frac{1}{2}}) + o(n^{-\frac{1}{2}}) = 0.$$

By multiplying \sqrt{p} on both sides, we further obtain that

$$\sqrt{p}\widehat{S}_i + \left\{1 - y + o_p(1)\right\} \cdot \sqrt{p}\delta_i + o_p(1) \cdot p\delta_i^2 + o(1) = 0.$$

Recalling that $\sqrt{p}\widehat{S}_i \xrightarrow{d} \mathcal{N}(0, \widetilde{\sigma}_i^2)$, we can reach to $\sqrt{p}\delta_i \xrightarrow{d} \mathcal{N}(0, \sigma_i^2)$, where

$$\sigma_i^2 = \frac{\widetilde{\sigma}_i^2}{(1-y)^2} = (y+c)v_i - c - \frac{y(1-3y)}{1-y}.$$

Instead, we can consider the determinant

$$\det\left\{\widetilde{\mathbf{\Lambda}}^{-\frac{1}{2}}\mathbf{U}(\theta_i\Theta_{1n}+\theta_i\delta_i\Theta_{2n}+\theta_i\Theta_{3n})\mathbf{U}^{\mathsf{T}}\widetilde{\mathbf{\Lambda}}^{-\frac{1}{2}}\right\}=0,$$
(3.40)

where $\widetilde{\Lambda} = \text{diag}(\theta_1, \dots, \theta_q) \in \mathbb{R}^{q \times q}$. Repeating all the derivations above, we can get

$$\|\widetilde{\mathbf{\Lambda}}^{-\frac{1}{2}}\mathbf{U}\theta_i \Theta_{1n} \mathbf{U}^{\mathsf{T}} \widetilde{\mathbf{\Lambda}}^{-\frac{1}{2}}\|_{\infty} = \mathcal{O}_p \left(\frac{q}{\sqrt{n}} + \frac{\lambda_i \sum_j \lambda_j^{-1}}{\sqrt{n}} \right), \tag{3.41}$$

$$\max_{1 \le j \le q} \left| \mathbf{e}_{j}^{\mathsf{T}} \widetilde{\mathbf{\Lambda}}^{-\frac{1}{2}} \theta_{i} \mathbf{U} \Theta_{2n} \mathbf{U}^{\mathsf{T}} \widetilde{\mathbf{\Lambda}}^{-\frac{1}{2}} \mathbf{e}_{j} - (y-1) \frac{\theta_{i}}{\theta_{j}} \right| = \mathbf{O}_{p} \left(\delta_{i} \sqrt{\sum_{j} \lambda_{j}^{-2}} + \frac{\lambda_{i} \sqrt{\sum_{j} \lambda_{j}^{-2}}}{\sqrt{n}} + \frac{\sqrt{q}}{\lambda_{i}} + \sqrt{\sum_{j} \lambda_{j}^{-1}} \right)$$
(3.42)

$$\max_{1 \le j_1 \ne j_2 \le q} \left| \mathbf{e}_{j_1}^{\top} \widetilde{\mathbf{\Lambda}}^{-\frac{1}{2}} \theta_i \mathbf{U} \Theta_{2n} \mathbf{U}^{\top} \widetilde{\mathbf{\Lambda}}^{-\frac{1}{2}} \mathbf{e}_{j_2} \right| = \mathcal{O}_p \left(\delta_i \sum_j \lambda_j^{-1} + \frac{\lambda_i \sum_j \lambda_j^{-1}}{\sqrt{n}} + \frac{q}{\lambda_i} + \sqrt{q \sum_j \lambda_j^{-1}} \right),$$
(3.43)

$$\mathbf{e}_{j}^{\mathsf{T}}\widetilde{\mathbf{\Lambda}}^{-\frac{1}{2}}\theta_{i}\mathbf{U}\Theta_{3n}\mathbf{U}^{\mathsf{T}}\widetilde{\mathbf{\Lambda}}^{-\frac{1}{2}}\mathbf{e}_{j}^{\mathsf{T}} = \left\{ (1+\delta_{i})^{2} - \frac{\lambda_{j}}{\lambda_{i}} \right\} \left[1 - \frac{p-q}{n}\mathbf{E}\left\{ \widetilde{m}_{\theta_{i}}(1) \right\} \right], \tag{3.44}$$

$$\mathbf{e}_{j_1}^{\top}\widetilde{\mathbf{\Lambda}}^{-\frac{1}{2}}\theta_i \mathbf{U}\Theta_{3n}\mathbf{U}^{\top}\widetilde{\mathbf{\Lambda}}^{-\frac{1}{2}}\mathbf{e}_{j_2} = 0, \ \forall 1 \le j_1 \ne j_2 \le q.$$
(3.45)

Inserting (3.41)-(3.45) into (3.40), we can similarly prove $\sqrt{p}\delta_i \xrightarrow{d} \mathcal{N}(0, \sigma_i^2)$ under Assumption 2.2(b). Thus the proof is completed. \Box

Proof of Theorem 2.12. The proof of Theorem 2.12 is similar to that of Theorem 2.10, the only difference is that we take the $J_i \times J_i$ block as a typical object to analyse, some useful lemmas can also be obtained from Lemmas 4.2-4.4 below. Similar arguments for deriving the proof of Theorem 4.1 in [17] can be used. Thus, we omit the details. \Box

4 Some Technical Lemmas

Lemma 4.1. Suppose that Assumptions 2.1 and 2.3 hold. For any $\theta \to \infty$, we have $\widetilde{m}_{\theta}(1) - 1 = O_{a.s.}(\theta^{-1})$.

Proof of lemma 4.1. By the definition of $\widetilde{m}_{\theta}(z)$ in (2.2),

$$\widetilde{m}_{\theta}(1) = \frac{1}{p-q} \operatorname{tr} \left(\mathbf{I}_{p-q} - \frac{\mathbf{F}_0}{\theta} \right)^{-1} = 1 + \frac{1}{p-q} \operatorname{tr} \left\{ \frac{\mathbf{F}_0}{\theta} \left(\mathbf{I}_{p-q} - \frac{\mathbf{F}_0}{\theta} \right)^{-1} \right\},$$

we have

$$\widetilde{m}_{\theta}(1) - 1 = \frac{1}{p-q} \operatorname{tr}\left\{\frac{\mathbf{F}_{0}}{\theta} \left(\mathbf{I}_{p-q} - \frac{\mathbf{F}_{0}}{\theta}\right)^{-1}\right\} = \theta^{-1} \left(\frac{1}{p-q} \sum_{1 \le j \le p-q} \frac{\mu_{j}}{1 - \mu_{j}/\theta}\right).$$

Since all the eigenvalues of \mathbf{F}_0 , namely $\mu_1 \ge \ldots \ge \mu_{p-q}$, are almost surely bounded, we can get that $\widetilde{m}_{\theta}(1) - 1 = \mathcal{O}_{a.s.}(\theta^{-1})$. \Box

Recall that \mathbf{e}_i is the *q*-dimensional vector whose *i*-th element is 1 and others are 0, $\mathbf{U}^{\top} = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_q)$, where $\mathbf{u}_i \in \mathbb{R}^q$ is the *i*-th column of the matrix \mathbf{U}^{\top} . Then we get the following lemma. **Lemma 4.2.** For any fixed $1 \le i \le q$, denote $\mathbf{G}_{ni} = \sqrt{p} \mathbf{U} \Theta_{1n} \mathbf{U}^{\top}$. Under the assumptions of Theorem 2.10, we have

$$\mathbf{e}_i^{\mathsf{T}} \mathbf{G}_{ni} \mathbf{e}_i \xrightarrow{d} \mathcal{N}(0, \widetilde{\sigma}_i^2),$$

where $\tilde{\sigma}_i^2 = (y+c)(1-y)^2 v_i - y(1-y)(1-3y) - c(1-y)^2$ and $v_i = \mathbb{E}|\mathbf{u}_i^\top \mathbf{Z}_1 \mathbf{e}_1|^4$ for $1 \le i \le q$.

Proof of Lemma 4.2. From the definition of Θ_{1n} in (3.27) and the fact that $\mathbf{Y}_1 = \Sigma_1^{-\frac{1}{2}} \mathbf{X}_1 = \mathbf{U}^{\mathsf{T}} \mathbf{\Lambda}^{-\frac{1}{2}} \mathbf{U} \mathbf{X}_1$, we have the decomposition

$$\mathbf{e}_{i}^{\mathsf{T}}\mathbf{G}_{ni}\mathbf{e}_{i} = \mathbf{u}_{i}^{\mathsf{T}} \Big[\frac{(1+\delta_{i})^{2} \sqrt{p}}{n} \mathbf{Z}_{1} \left\{ \mathbf{I}_{n} - \mathbf{A}(\theta_{i}) \right\} \mathbf{Z}_{1}^{\mathsf{T}} - \frac{\lambda_{i}}{\theta_{i}} \frac{\sqrt{p}}{T} \mathbf{Y}_{1} \left\{ \mathbf{I}_{T} + \mathbf{B}(\theta_{i}) \right\} \mathbf{Y}_{1}^{\mathsf{T}} \\ + \frac{\sqrt{p\lambda_{i}}}{n} \mathbf{Z}_{1} \mathbf{C}(\theta_{i}) \mathbf{Y}_{1}^{\mathsf{T}} + \frac{\sqrt{p\lambda_{i}}}{T} \mathbf{Y}_{1} \mathbf{D}(\theta_{i}) \mathbf{Z}_{1}^{\mathsf{T}} \Big] \mathbf{u}_{i} - \mathbf{E}(\cdot), \qquad (4.1)$$

where $E[\cdot]$ is the expectation of all the preceding terms after the equal sign.

By Theorem 2.6, δ_i converges in probability to 0, thus we only need to consider the limit of

$$\mathbf{e}_{i}^{\top} \widetilde{\mathbf{G}}_{ni} \mathbf{e}_{i} := \mathbf{u}_{i}^{\top} \Big[\frac{\sqrt{p}}{n} \mathbf{Z}_{1} \{ \mathbf{I}_{n} - \mathbf{A}(\theta_{i}) \} \mathbf{Z}_{1}^{\top} - \frac{\lambda_{i}}{\theta_{i}} \frac{\sqrt{p}}{T} \mathbf{Y}_{1} \{ \mathbf{I}_{T} + \mathbf{B}(\theta_{i}) \} \mathbf{Y}_{1}^{\top} + \frac{\sqrt{p\lambda_{i}}}{n} \mathbf{Z}_{1} \mathbf{C}(\theta_{i}) \mathbf{Y}_{1}^{\top} \\ + \frac{\sqrt{p\lambda_{i}}}{T} \mathbf{Y}_{1} \mathbf{D}(\theta_{i}) \mathbf{Z}_{1}^{\top} \Big] \mathbf{u}_{i} - \mathbf{E}[\cdot].$$

For the first two terms, Theorem 7.2 in [3] implies that, for any $1 \le i \le q$,

$$\frac{1}{\sqrt{n}} \left[\mathbf{u}_i^\top \mathbf{Z}_1 \left\{ \mathbf{I}_n - \mathbf{A}(\theta_i) \right\} \mathbf{Z}_1^\top \mathbf{u}_i - \operatorname{tr} \left\{ \mathbf{I}_n - \mathbf{A}(\theta_i) \right\} \right] \xrightarrow{d} \mathcal{N}(0, \widetilde{\sigma}_{i\mathbf{A}}^2),$$

$$\frac{1}{\sqrt{T}} \left[\mathbf{u}_i^\top \mathbf{Y}_1 \left\{ \mathbf{I}_T + \mathbf{B}(\theta_i) \right\} \mathbf{Y}_1^\top \mathbf{u}_i - \operatorname{tr} \left\{ \mathbf{I}_T + \mathbf{B}(\theta_i) \right\} \right] \xrightarrow{d} \mathcal{N}(0, \widetilde{\sigma}_{i\mathbf{B}}^2),$$

with $\widetilde{\sigma}_{iA}^2 = \omega_{\mathbf{I}_n - \mathbf{A}(\theta_i)}(v_i - 3) + 2\beta_{\mathbf{I}_n - \mathbf{A}(\theta_i)}$ and $\widetilde{\sigma}_{iB}^2 = \omega_{\mathbf{I}_T + \mathbf{B}(\theta_i)}(v_i - 3) + 2\beta_{\mathbf{I}_T + \mathbf{B}(\theta_i)}$, where

$$\begin{aligned} \nu_i &= \mathbf{E} |\mathbf{u}_i^{\mathsf{T}} \mathbf{Z}_1 \mathbf{e}_1|^4 = \mathbf{E} |\mathbf{u}_i^{\mathsf{T}} \mathbf{Y}_1 \mathbf{e}_1|^4, \\ \omega_{\mathbf{I}_n - \mathbf{A}(\theta_i)} &= \lim_{n \to \infty} \frac{1}{n} \sum_{1 \le k \le n} \left[\{ \mathbf{I}_n - \mathbf{A}(\theta_i) \} (k, k) \right]^2, \\ \beta_{\mathbf{I}_n - \mathbf{A}(\theta_i)} &= \lim_{n \to \infty} \frac{1}{n} \operatorname{tr} \{ \mathbf{I}_n - \mathbf{A}(\theta_i) \}^2, \end{aligned}$$

 $\omega_{\mathbf{I}_T + \mathbf{B}(\theta_i)}$ and $\beta_{\mathbf{I}_T + \mathbf{B}(\theta_i)}$ are similarly defined. Here the fact that $\mathbf{E}|\mathbf{u}_i^\top \mathbf{Z}_1 \mathbf{e}_1|^4 = \mathbf{E}|\mathbf{u}_i^\top \mathbf{Y}_1 \mathbf{e}_1|^4$ is implied by Assumption 2.3. Based on the facts that

$$E\left[\mathbf{u}_{i}^{\top}\mathbf{Z}_{1}\left\{\mathbf{I}_{n}-\mathbf{A}(\theta_{i})\right\}\mathbf{Z}_{1}^{\top}\mathbf{u}_{i}\right] = E\left(\operatorname{tr}\left[\mathbf{Z}_{1}^{\top}\mathbf{u}_{i}\mathbf{u}_{i}^{\top}\mathbf{Z}_{1}\left\{\mathbf{I}_{n}-\mathbf{A}(\theta_{i})\right\}\right]\right)$$
$$=\operatorname{tr}\left[E\left(\mathbf{Z}_{1}^{\top}\mathbf{u}_{i}\mathbf{u}_{i}^{\top}\mathbf{Z}_{1}\right)E\left\{\mathbf{I}_{n}-\mathbf{A}(\theta_{i})\right\}\right] = E\left[\operatorname{tr}\left\{\mathbf{I}_{n}-\mathbf{A}(\theta_{i})\right\}\right] = n - (p-q)E\left\{\widetilde{m}_{\theta_{i}}(1)\right\},$$

and that $\widetilde{m}_{\theta_i}(1) - \mathbb{E} \{ \widetilde{m}_{\theta_i}(1) \} = \mathcal{O}_p(n^{-1})$, we can get that

$$\frac{1}{\sqrt{n}} \left(\mathbf{E} \left[\mathbf{u}_i^\top \mathbf{Z}_1 \left\{ \mathbf{I}_n - \mathbf{A}(\theta_i) \right\} \mathbf{Z}_1^\top \mathbf{u}_i^\top \right] - \operatorname{tr} \left\{ \mathbf{I}_n - \mathbf{A}(\theta_i) \right\} \right) = \mathbf{o}_p(1).$$

Then it follows that

$$\frac{1}{\sqrt{n}} \left(\mathbf{u}_i^\top \mathbf{Z}_1 \left\{ \mathbf{I}_n - \mathbf{A}(\theta_i) \right\} \mathbf{Z}_1^\top \mathbf{u}_i - \mathbf{E} \left[\mathbf{u}_i^\top \mathbf{Z}_1 \left\{ \mathbf{I}_n - \mathbf{A}(\theta_i) \right\} \mathbf{Z}_1^\top \mathbf{u}_i^\top \right] \right) \xrightarrow{d} \mathcal{N}(0, \widetilde{\sigma}_{i\mathbf{A}}^2),$$

and similarly,

$$\frac{1}{\sqrt{T}} \left(\mathbf{u}_i^{\mathsf{T}} \mathbf{Y}_1 \left\{ \mathbf{I}_T + \mathbf{B}(\theta_i) \right\} \mathbf{Y}_1^{\mathsf{T}} \mathbf{u}_i - \mathbf{E} \left[\mathbf{u}_i^{\mathsf{T}} \mathbf{Y}_1 \left\{ \mathbf{I}_T + \mathbf{B}(\theta_i) \right\} \mathbf{Y}_1^{\mathsf{T}} \mathbf{u}_i \right] \right) \xrightarrow{d} \mathcal{N}(0, \widetilde{\sigma}_{i\mathbf{B}}^2).$$

For other two terms, by the same approach in the proof of Theorem 2.6, we have that

$$\mathbf{u}_i^{\top} \left\{ \frac{\sqrt{p\lambda_i}}{n} \mathbf{Z}_1 \mathbf{C}(\theta_i) \mathbf{Y}_1^{\top} + \frac{\sqrt{p\lambda_i}}{T} \mathbf{Y}_1 \mathbf{D}(\theta_i) \mathbf{Z}_1^{\top} \right\} \mathbf{u}_i = \mathbf{O}_p \left(\frac{1}{\sqrt{\lambda_i}} \right).$$

By all these arguments above, we can derive that $\mathbf{e}_i^{\mathsf{T}} \mathbf{G}_{ni} \mathbf{e}_i \xrightarrow{d} \mathcal{N}(0, \widetilde{\sigma}_i^2)$ with $\widetilde{\sigma}_i^2 = y \sigma_{i\mathbf{A}}^2 + c(1-y)^2 \sigma_{i\mathbf{B}}^2$.

We compute $\omega_{\mathbf{I}_n - \mathbf{A}(\theta_i)}$, $\beta_{\mathbf{I}_n - \mathbf{A}(\theta_i)}$, $\omega_{\mathbf{I}_T + \mathbf{B}(\theta_i)}$ and $\beta_{\mathbf{I}_T + \mathbf{B}(\theta_i)}$ in the following. By the derivations in the proof of Lemma 6 in [17],

$$\{\mathbf{I}_n - \mathbf{A}(\theta_i)\}(k, k) = 1 - \left\{\mathbf{Z}_2^{\top} \mathbf{M}(\theta_i)^{-1} \left(\frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_2^{\top}\right)^{-1} \frac{1}{n} \mathbf{Z}_2\right\}(k, k)$$
$$= 1 - \frac{\theta_i}{n} \left\{\mathbf{Z}_2^{\top} \left(\theta_i \cdot \frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_2^{\top} - \frac{1}{T} \mathbf{X}_2 \mathbf{X}_2^{\top}\right)^{-1} \mathbf{Z}_2\right\}(k, k)$$
$$= \frac{1}{1 + \frac{\theta_i}{n} \left\{\eta_k^{\top} \left(\theta_i \frac{1}{n} \mathbf{Z}_{2k} \mathbf{Z}_{2k}^{\top} - \frac{1}{T} \mathbf{X}_2 \mathbf{X}_2^{\top}\right)^{-1} \eta_k\right\}},$$

where η_k is the *k*-th column of \mathbb{Z}_2 and \mathbb{Z}_{2k} is defined by removing the *k*-th column of \mathbb{Z}_2 . Note that

$$\left(\frac{1}{n}\mathbf{Z}_{2k}\mathbf{Z}_{2k}^{\top} - \frac{1}{\theta_{i}T}\mathbf{X}_{2}\mathbf{X}_{2}^{\top}\right)^{-1} - \left(\frac{1}{n}\mathbf{Z}_{2k}\mathbf{Z}_{2k}^{\top}\right)^{-1}$$

$$= \left(\frac{1}{n}\mathbf{Z}_{2i}\mathbf{Z}_{2i}^{\top} - \frac{1}{\theta_{i}T}\mathbf{X}_{2}\mathbf{X}_{2}^{\top}\right)^{-1} \left\{\frac{1}{n}\mathbf{Z}_{2k}\mathbf{Z}_{2k}^{\top} - \left(\frac{1}{n}\mathbf{Z}_{2k}\mathbf{Z}_{2k}^{\top} - \frac{1}{\theta_{i}T}\mathbf{X}_{2}\mathbf{X}_{2}^{\top}\right)\right\} \left(\frac{1}{n}\mathbf{Z}_{2k}\mathbf{Z}_{2k}^{\top}\right)^{-1}$$

$$= \theta_{i}^{-1}\left(\frac{1}{n}\mathbf{Z}_{2i}\mathbf{Z}_{2i}^{\top} - \frac{1}{\theta_{i}T}\mathbf{X}_{2}\mathbf{X}_{2}^{\top}\right)^{-1} \left(\frac{1}{T}\mathbf{X}_{2}\mathbf{X}_{2}^{\top}\right) \left(\frac{1}{n}\mathbf{Z}_{2k}\mathbf{Z}_{2k}^{\top}\right)^{-1}$$

$$(4.2)$$

and

$$\frac{1}{p-q-1} \operatorname{tr}\left(\frac{1}{n} \mathbf{Z}_{2k} \mathbf{Z}_{2k}^{\top}\right)^{-1} = \mathcal{S}_{\mathrm{MP}}(0) + \mathcal{O}_{p}\left(p^{-1}\right) = \frac{1}{1-y} + \mathcal{O}_{p}\left(p^{-1}\right), \quad (4.3)$$

where S_{MP} denotes the Stieltjes transform of the Marcenko-Pastur law. Then we have that

$$\frac{1}{p-q}\theta_{i} \mathbf{E}\left\{ \operatorname{tr}\left(\theta_{i}\frac{1}{n}\mathbf{Z}_{2k}\mathbf{Z}_{2k}^{\top}-\frac{1}{T}\mathbf{X}_{2}\mathbf{X}_{2}^{\top}\right)^{-1}\right\} = \mathbf{E}\left\{\frac{1}{p-q}\operatorname{tr}\left(\frac{1}{n}\mathbf{Z}_{2k}\mathbf{Z}_{2k}^{\top}\right)^{-1} + \mathbf{O}_{a.s.}\left(\theta_{i}^{-1}\right)\right\}$$
$$= \mathbf{E}\left\{\frac{1}{1-y} + \mathbf{O}_{a.s.}\left(\theta_{i}^{-1}\right) + \mathbf{O}_{p}\left(p^{-1}\right)\right\} \to \frac{1}{1-y}.$$

By Lemma A.2. in [17], it holds that

$$\{\mathbf{I}_n - \mathbf{A}(\theta_i)\}(k, k) \to \frac{1}{1 + y(1 - y)^{-1}} = 1 - y,$$

which implies

$$\omega_{\mathbf{I}_n - \mathbf{A}(\theta_i)} = \lim_{n \to \infty} \frac{1}{n} \sum_{1 \le k \le n} \left[\{ \mathbf{I}_n - \mathbf{A}(\theta_i) \} (k, k) \right]^2 = (1 - y)^2.$$

By the similar argument, we can obtain that

$$\omega_{\mathbf{I}_T + \mathbf{B}(\theta_i)} = \lim_{T \to \infty} \frac{1}{T} \sum_{1 \le k \le T} \left[\{ \mathbf{I}_T + \mathbf{B}(\theta_i) \} (k, k) \right]^2 = 1.$$

Now we come to the calculation of $\beta_{\mathbf{I}_n - \mathbf{A}(\theta_i)}$ and $\beta_{\mathbf{I}_T + \mathbf{B}(\theta_i)}$. Since $\theta_i \to +\infty$ as *n* goes to infinity, we have

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \frac{\theta_i}{\theta_i - x} dF_n(x) = 1, \lim_{n \to \infty} \int_{-\infty}^{\infty} \frac{\theta_i^2}{(\theta_i - x)^2} dF_n(x) = 1,$$
$$\lim_{T \to \infty} \int_{-\infty}^{\infty} \frac{x}{\theta_i - x} dF_n(x) = 0, \lim_{T \to \infty} \int_{-\infty}^{\infty} \frac{x^2}{(\theta_i - x)^2} dF_n(x) = 0.$$

Then these calculations lead to

$$\begin{split} \beta_{\mathbf{I}_n - \mathbf{A}(\theta_i)} &= \lim_{n \to \infty} \frac{1}{n} \operatorname{tr} \{\mathbf{I}_n - \mathbf{A}(\theta_i)\}^2 = \lim_{n \to \infty} \frac{1}{n} \operatorname{tr} \{\mathbf{I}_n - 2\mathbf{A}(\theta_i) + \mathbf{A}^2(\theta_i)\} \\ &= 1 - 2\lim_{n \to \infty} \left(\frac{p - q}{n} \int_{-\infty}^{\infty} \frac{\theta_i}{\theta_i - x} dF_n(x) \right) + \lim_{n \to \infty} \left\{ \frac{p - q}{n} \int_{-\infty}^{\infty} \frac{\theta_i^2}{(\theta_i - x)^2} dF_n(x) \right\} \\ &= 1 - 2y + y = 1 - y, \\ \beta_{\mathbf{I}_T + \mathbf{B}(\theta_i)} &= \lim_{T \to \infty} \frac{1}{T} \operatorname{tr} \{\mathbf{I}_T + \mathbf{B}(\theta_i)\}^2 = \lim_{T \to \infty} \frac{1}{T} \operatorname{tr} \{\mathbf{I}_T + 2\mathbf{B}(\theta_i) + \mathbf{B}^2(\theta_i)\} \\ &= 1 + 2\lim_{T \to \infty} \left\{ \frac{p - q}{T} \int_{-\infty}^{\infty} \frac{x}{\theta_i - x} dF_n(x) \right\} + \lim_{T \to \infty} \left\{ \frac{p - q}{T} \int_{-\infty}^{\infty} \frac{x^2}{(\theta_i - x)^2} dF_n(x) \right\} \\ &= 1 + 0 + 0 = 1. \end{split}$$

Thus, we can write

$$\begin{split} \widetilde{\sigma}_{i}^{2} &= y \widetilde{\sigma}_{i\mathbf{A}}^{2} + c(1-y)^{2} \widetilde{\sigma}_{i\mathbf{B}}^{2} \\ &= y \{ \omega_{\mathbf{I}_{n} - \mathbf{A}(\theta_{i})}(v_{i} - 3) + 2\beta_{\mathbf{I}_{n} - \mathbf{A}(\theta_{i})} \} + c(1-y)^{2} \{ \omega_{\mathbf{I}_{T} + \mathbf{B}(\theta_{i})}(v_{i} - 3) + 2\beta_{\mathbf{I}_{T} + \mathbf{B}(\theta_{i})} \} \\ &= (y + c) (1 - y)^{2} v_{i} - y (1 - y) (1 - 3y) - c (1 - y)^{2} \,. \end{split}$$

Thus the proof is completed. \Box

Lemma 4.3. Under the assumptions of Theorem 2.10,

$$\|\mathbf{U}\Theta_{1n}\mathbf{U}^{\top}\|_{\infty} = \mathbf{O}_p\left(\frac{q}{\sqrt{n}} + \frac{\sum_j \lambda_j}{\sqrt{n}\lambda_i}\right).$$

Proof of Lemma 4.3. By the definition of Θ_{1n} in (3.27) again, we know

$$\Theta_{1n} = (1 + \delta_i)^2 \frac{1}{n} \mathbf{Z}_1 \{ \mathbf{I}_n - \mathbf{A}(\theta_i) \} \mathbf{Z}_1^\top - \theta_i^{-1} \frac{1}{T} \mathbf{X}_1 \{ \mathbf{I}_T + \mathbf{B}(\theta_i) \} \mathbf{X}_1^\top + \frac{1}{n} \mathbf{Z}_1 \mathbf{C}(\theta_i) \mathbf{X}_1^\top + \frac{1}{T} \mathbf{X}_1 \mathbf{D}(\theta_i) \mathbf{Z}_1^\top - \mathbf{E}(\cdot),$$
(4.4)

where $E(\cdot)$ is the expectation of all the preceding terms.

Denote

$$\eta_{n1} = \frac{1}{n} \mathbf{Z}_{1} \left\{ \mathbf{I}_{n} - \mathbf{A}(\theta_{i}) \right\} \mathbf{Z}_{1}^{\top} - \mathbf{E} \left[\frac{1}{n} \mathbf{Z}_{1} \left\{ \mathbf{I}_{n} - \mathbf{A}(\theta_{i}) \right\} \mathbf{Z}_{1}^{\top} \right],$$

$$\eta_{n2} = \frac{1}{T} \mathbf{Y}_{1} \left\{ \mathbf{I}_{T} + \mathbf{B}(\theta_{i}) \right\} \mathbf{Y}_{1}^{\top} - \mathbf{E} \left[\frac{1}{T} \mathbf{Y}_{1} \left\{ \mathbf{I}_{T} + \mathbf{B}(\theta_{i}) \right\} \mathbf{Y}_{1}^{\top} \right],$$

$$\eta_{n3} = \sqrt{\theta_{i}} \frac{1}{n} \mathbf{Z}_{1} \mathbf{C}(\theta_{i}) \mathbf{Y}_{1}^{\top} - \mathbf{E} \left\{ \sqrt{\theta_{i}} \frac{1}{n} \mathbf{Z}_{1} \mathbf{C}(\theta_{i}) \mathbf{Y}_{1}^{\top} \right\},$$

$$\eta_{n4} = \sqrt{\theta_{i}} \frac{1}{T} \mathbf{Y}_{1} \mathbf{D}(\theta_{i}) \mathbf{Z}_{1}^{\top} - \mathbf{E} \left\{ \sqrt{\theta_{i}} \frac{1}{T} \mathbf{Y}_{1} \mathbf{D}(\theta_{i}) \mathbf{Z}_{1}^{\top} \right\}.$$

By the fact $\mathbf{X}_1 = \mathbf{U}^{\top} \mathbf{\Lambda}_1^{\frac{1}{2}} \mathbf{U} \mathbf{Y}_1$, we can write

$$\mathbf{U}\Theta_{1n}\mathbf{U}^{\top} := \sum_{i=1}^{4} \mathbf{V}_{ni},\tag{4.5}$$

where

$$\begin{aligned} \mathbf{V}_{n1} &= (1+\delta_{i})^{2} \mathbf{U} \left(\frac{1}{n} \mathbf{Z}_{1} \left\{ \mathbf{I}_{n} - \mathbf{A}(\theta_{i}) \right\} \mathbf{Z}_{1}^{\top} - \mathbf{E} \left[\frac{1}{n} \mathbf{Z}_{1} \left\{ \mathbf{I}_{n} - \mathbf{A}(\theta_{i}) \right\} \mathbf{Z}_{1}^{\top} \right] \right) \mathbf{U}^{\top} \\ &= (1+\delta_{i})^{2} \mathbf{U} \eta_{n1} \mathbf{U}^{\top}, \end{aligned}$$
(4.6)
$$\mathbf{V}_{n2} &= -\theta_{i}^{-1} \mathbf{U} \left(\frac{1}{T} \mathbf{X}_{1} \left\{ \mathbf{I}_{T} + \mathbf{B}(\theta_{i}) \right\} \mathbf{X}_{1}^{\top} - \mathbf{E} \left[\frac{1}{T} \mathbf{X}_{1} \left\{ \mathbf{I}_{T} + \mathbf{B}(\theta_{i}) \right\} \mathbf{X}_{1}^{\top} \right] \right) \mathbf{U}^{\top} \\ &= -\theta_{i}^{-1} \mathbf{A}^{\frac{1}{2}} \mathbf{U} \left(\frac{1}{T} \mathbf{Y}_{1} \left\{ \mathbf{I}_{T} + \mathbf{B}(\theta_{i}) \right\} \mathbf{Y}_{1}^{\top} - \mathbf{E} \left[\frac{1}{T} \mathbf{Y}_{1} \left\{ \mathbf{I}_{T} + \mathbf{B}(\theta_{i}) \right\} \mathbf{Y}_{1}^{\top} \right] \right) \mathbf{U}^{\top} \mathbf{A}^{\frac{1}{2}} \\ &= -\theta_{i}^{-1} \mathbf{A}^{\frac{1}{2}} \mathbf{U} \eta_{n2} \mathbf{U}^{\top} \mathbf{A}^{\frac{1}{2}}, \end{aligned}$$
(4.7)
$$\mathbf{V}_{n3} = \mathbf{U} \left[\frac{1}{n} \mathbf{Z}_{1} \mathbf{C}(\theta_{i}) \mathbf{X}_{1}^{\top} - \mathbf{E} \left\{ \frac{1}{n} \mathbf{Z}_{1} \mathbf{C}(\theta_{i}) \mathbf{X}_{1}^{\top} \right\} \right] \mathbf{U}^{\top} \mathbf{A}^{\frac{1}{2}} \\ &= \mathbf{U} \left[\frac{1}{n} \mathbf{Z}_{1} \mathbf{C}(\theta_{i}) \mathbf{Y}_{1}^{\top} - \mathbf{E} \left\{ \frac{1}{n} \mathbf{Z}_{1} \mathbf{C}(\theta_{i}) \mathbf{Y}_{1}^{\top} \right\} \right] \mathbf{U}^{\top} \mathbf{A}^{\frac{1}{2}} \\ &= \theta_{i}^{-\frac{1}{2}} \mathbf{U} \eta_{n3} \mathbf{U}^{\top} \mathbf{A}^{\frac{1}{2}} \end{aligned}$$
(4.8)
$$\mathbf{V}_{A} = \mathbf{U} \left[\frac{1}{n} \mathbf{X}_{1} \mathbf{D}(\theta_{i}) \mathbf{Z}^{\top} - \mathbf{E} \left\{ \frac{1}{n} \mathbf{X}_{1} \mathbf{D}(\theta_{i}) \mathbf{Z}^{\top} \right\} \right] \mathbf{U}^{\top} - \mathbf{E} \left\{ \frac{1}{n} \mathbf{X}_{1} \mathbf{D}(\theta_{i}) \mathbf{Z}^{\top} \right\} \end{aligned}$$

$$\mathbf{V}_{n4} = \mathbf{U} \left[\frac{1}{T} \mathbf{X}_1 \mathbf{D}(\theta_i) \mathbf{Z}_1^\top - \mathbf{E} \left\{ \frac{1}{T} \mathbf{X}_1 \mathbf{D}(\theta_i) \mathbf{Z}_1^\top \right\} \right] \mathbf{U}^\top = \theta_i^{-\frac{1}{2}} \mathbf{\Lambda}^{\frac{1}{2}} \mathbf{U} \eta_{n4} \mathbf{U}^\top.$$
(4.9)

Similarly as the arguments in the proof of Lemma 4.2, it holds that, for $1 \le j_1, j_2 \le q$,

$$\mathbf{e}_{j_1}^{\mathsf{T}}\eta_{n1}\mathbf{e}_{j_2} = \mathbf{O}_p\left(\frac{1}{\sqrt{n}}\right), \quad \mathbf{e}_{j_1}^{\mathsf{T}}\eta_{n2}\mathbf{e}_{j_2} = \mathbf{O}_p\left(\frac{1}{\sqrt{n}}\right),$$

$$\mathbf{e}_{j_1}^{\mathsf{T}} \eta_{n3} \mathbf{e}_{j_2} = \mathcal{O}_p \left(\frac{1}{\sqrt{n\lambda_i}} \right), \quad \mathbf{e}_{j_1}^{\mathsf{T}} \eta_{n4} \mathbf{e}_{j_2} = \mathcal{O}_p \left(\frac{1}{\sqrt{n\lambda_i}} \right).$$

Noting that U is an orthogonal matrix, we have that

$$\mathbf{e}_{j_1}^{\top} \mathbf{V}_{n1} \mathbf{e}_{j_2} = \mathbf{e}_{j_1}^{\top} (1+\delta_i)^2 \mathbf{U} \eta_{n1} \mathbf{U}^{\top} \mathbf{e}_{j_2} = \mathbf{O}_p \left(\frac{1}{\sqrt{n}}\right),$$

$$\mathbf{e}_{j_1}^{\top} \mathbf{V}_{n2} \mathbf{e}_{j_2} = -\mathbf{e}_{j_1}^{\top} \theta_i^{-1} \mathbf{\Lambda}^{\frac{1}{2}} \mathbf{U} \eta_{n2} \mathbf{U}^{\top} \mathbf{\Lambda}^{\frac{1}{2}} \mathbf{e}_{j_2} = \lambda_i^{-1} \lambda_{j_1}^{\frac{1}{2}} \lambda_{j_2}^{\frac{1}{2}} \cdot \mathbf{O}_p \left(\frac{1}{\sqrt{n}}\right),$$

$$\mathbf{e}_{j_1}^{\top} \mathbf{V}_{n3} \mathbf{e}_{j_2} = \mathbf{e}_{j_1}^{\top} \theta_i^{-\frac{1}{2}} \mathbf{U} \eta_{n3} \mathbf{U}^{\top} \mathbf{\Lambda}^{\frac{1}{2}} \mathbf{e}_{j_2} = \lambda_i^{-1} \lambda_{j_2}^{\frac{1}{2}} \cdot \mathbf{O}_p \left(\frac{1}{\sqrt{n}}\right),$$

$$\mathbf{e}_{j_1}^{\top} \mathbf{V}_{n4} \mathbf{e}_{j_2} = \mathbf{e}_{j_1}^{\top} \theta_i^{-\frac{1}{2}} \mathbf{\Lambda}^{\frac{1}{2}} \mathbf{U} \eta_{n4} \mathbf{U}^{\top} \mathbf{e}_{j_2} = \lambda_i^{-1} \lambda_{j_1}^{\frac{1}{2}} \cdot \mathbf{O}_p \left(\frac{1}{\sqrt{n}}\right).$$

Then by Chebyshev's inequality, we can deduce that

$$\|\mathbf{V}_{n1}\|_{\infty} = \mathbf{O}_p\left(\frac{q}{\sqrt{n}}\right), \quad \|\mathbf{V}_{n2}\|_{\infty} = \mathbf{O}_p\left(\frac{\sum_j \lambda_j}{\sqrt{n}\lambda_i}\right), \quad \|\mathbf{V}_{n3} + \mathbf{V}_{n4}\|_{\infty} = \mathbf{O}_p\left(\frac{\sqrt{q\sum_j \lambda_j}}{\sqrt{n}\lambda_i}\right),$$

where $\sqrt{q \sum_j \lambda_j} = o(\sum_j \lambda_j)$. Thus we complete the proof by (4.5). \Box

Lemma 4.4. Under the assumptions of Theorem 2.10,

$$\max_{1 \le j \le q} \left| \mathbf{e}_{j}^{\mathsf{T}} \mathbf{U} \Theta_{2n} \mathbf{U}^{\mathsf{T}} \mathbf{e}_{j} - (y-1) \right| = \mathbf{O}_{p} \left(\frac{\sqrt{q} \delta_{i}}{\lambda_{i}} + \frac{\sqrt{q}}{\sqrt{n}} + \frac{\sqrt{\sum_{j} \lambda_{j}^{2}}}{\lambda_{i}^{2}} + \frac{\sqrt{\sum_{j} \lambda_{j}}}{\lambda_{i}} \right), \quad (4.10)$$

$$\max_{1 \le j_1 \ne j_2 \le q} |\mathbf{e}_{j_1}^\top \mathbf{U} \Theta_{2n} \mathbf{U}^\top \mathbf{e}_{j_2}| = \mathcal{O}_p \left(\frac{q\delta_i}{\lambda_i} + \frac{q}{\sqrt{n}} + \frac{\sum_j \lambda_j}{\lambda_i^2} + \frac{\sqrt{q\sum_j \lambda_j}}{\lambda_i} \right).$$
(4.11)

Proof of Lemma 4.4. Recall the definition of Θ_{2n} in (3.28):

$$\begin{split} \Theta_{2n} &= (1+\delta_i) \frac{1}{n} \mathbf{Z}_1 \mathbf{Z}_2^{\mathsf{T}} \mathbf{M}^{-1}(\widehat{\lambda}_i) \mathbf{M}^{-1}(\theta_i) \left(\frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_2^{\mathsf{T}} \right)^{-1} \frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_1^{\mathsf{T}} - (1+\delta_i) \frac{1}{n} \mathbf{Z}_1 \mathbf{Z}_1^{\mathsf{T}} \\ &+ \frac{1}{\widehat{\lambda}_i \theta_i} \frac{1}{T} \mathbf{X}_1 \mathbf{X}_2^{\mathsf{T}} \mathbf{M}^{-1}(\widehat{\lambda}_i) \mathbf{M}^{-1}(\theta_i) \left(\frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_2^{\mathsf{T}} \right)^{-1} \frac{1}{T} \mathbf{X}_2 \mathbf{X}_1^{\mathsf{T}} \\ &- \frac{(1+\delta_i)}{\widehat{\lambda}_i} \frac{1}{n} \mathbf{Z}_1 \mathbf{Z}_2^{\mathsf{T}} \mathbf{M}^{-1}(\widehat{\lambda}_i) \mathbf{M}^{-1}(\theta_i) \left(\frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_2^{\mathsf{T}} \right)^{-1} \frac{1}{T} \mathbf{X}_2 \mathbf{X}_1^{\mathsf{T}} + \frac{1}{n} \mathbf{Z}_1 \mathbf{C}(\theta_i) \mathbf{X}_1^{\mathsf{T}} \\ &- \frac{(1+\delta_i)}{\widehat{\lambda}_i} \frac{1}{T} \mathbf{X}_1 \mathbf{X}_2^{\mathsf{T}} \mathbf{M}^{-1}(\widehat{\lambda}_i) \mathbf{M}^{-1}(\theta_i) \left(\frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_2^{\mathsf{T}} \right)^{-1} \frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_1^{\mathsf{T}} + \frac{1}{T} \mathbf{X}_1 \mathbf{D}(\theta_i) \mathbf{Z}_1^{\mathsf{T}}. \end{split}$$

Noting that

$$\mathbf{M}^{-1}(\widehat{\lambda}_i) - \mathbf{M}^{-1}(\theta_i) = \mathbf{M}^{-1}(\widehat{\lambda}_i) \left\{ \mathbf{M}(\theta_i) - \mathbf{M}(\widehat{\lambda}_i) \right\} \mathbf{M}^{-1}(\theta_i) = -\frac{\delta_i}{\widehat{\lambda}_i} \mathbf{M}^{-1}(\widehat{\lambda}_i) \mathbf{F}_0 \mathbf{M}^{-1}(\theta_i),$$

we decompose the first term in Θ_{2n} as

$$\frac{1}{n} \mathbf{Z}_1 \mathbf{Z}_2^{\mathsf{T}} \mathbf{M}^{-1}(\widehat{\lambda}_i) \mathbf{M}^{-1}(\theta_i) \left(\frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_2^{\mathsf{T}}\right)^{-1} \frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_1^{\mathsf{T}}$$

$$= \frac{1}{n} \mathbf{Z}_1 \mathbf{Z}_2^{\mathsf{T}} \left\{ \mathbf{M}^{-1}(\widehat{\lambda}_i) - \mathbf{M}^{-1}(\theta_i) \right\} \mathbf{M}^{-1}(\theta_i) \left(\frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_2^{\mathsf{T}} \right)^{-1} \frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_1^{\mathsf{T}} + \frac{1}{n} \mathbf{Z}_1 \mathbf{Z}_2^{\mathsf{T}} \mathbf{M}^{-2}(\theta_i) \left(\frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_2^{\mathsf{T}} \right)^{-1} \frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_1^{\mathsf{T}} \\ = -\frac{\delta_i}{\widehat{\lambda}_i} \frac{1}{n} \mathbf{Z}_1 \mathbf{Z}_2^{\mathsf{T}} \mathbf{M}^{-1}(\widehat{\lambda}_i) \mathbf{F}_0 \mathbf{M}^{-2}(\theta_i) \left(\frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_2^{\mathsf{T}} \right)^{-1} \frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_1^{\mathsf{T}} + \frac{1}{n} \mathbf{Z}_1 \mathbf{Z}_2^{\mathsf{T}} \mathbf{M}^{-2}(\theta_i) \left(\frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_2^{\mathsf{T}} \right)^{-1} \frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_1^{\mathsf{T}}$$

On one hand, similar to the arguments in the proof of Theorem 2.6, we can derive that

$$\max_{1 \le j \le q} \left| \mathbf{e}_{j}^{\top} \frac{\delta_{i}}{\widehat{\lambda}_{i}} \frac{1}{n} \mathbf{Z}_{1} \mathbf{Z}_{2}^{\top} \mathbf{M}^{-1}(\widehat{\lambda}_{i}) \mathbf{F}_{0} \mathbf{M}^{-2}(\theta_{i}) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{1}^{\top} \mathbf{e}_{j} \right| = \mathbf{O}_{p} \left(\frac{\sqrt{q} \delta_{i}}{\lambda_{i}} \right),$$
$$\max_{1 \le j_{1} \ne j_{2} \le q} \left| \mathbf{e}_{j_{1}}^{\top} \frac{\delta_{i}}{\widehat{\lambda}_{i}} \frac{1}{n} \mathbf{Z}_{1} \mathbf{Z}_{2}^{\top} \mathbf{M}^{-1}(\widehat{\lambda}_{i}) \mathbf{F}_{0} \mathbf{M}^{-2}(\theta_{i}) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\top} \right)^{-1} \frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{1}^{\top} \mathbf{e}_{j_{2}} \right| = \mathbf{O}_{p} \left(\frac{q \delta_{i}}{\lambda_{i}} \right).$$

On the other hand, similar to the proof of Lemma 4.2, we can get that

$$\frac{1}{n} \left(\mathbf{e}_j^{\mathsf{T}} \mathbf{Z}_1 \mathbf{Z}_2^{\mathsf{T}} \mathbf{M}^{-2}(\theta_i) \left(\frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_2^{\mathsf{T}} \right)^{-1} \frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_1^{\mathsf{T}} \mathbf{e}_j - \mathbf{E} \left[\operatorname{tr} \left\{ \mathbf{M}^{-2}(\theta_i) \right\} \right] \right) = \mathbf{O}_p \left(\frac{1}{\sqrt{n}} \right),$$
$$\frac{1}{n} \mathbf{e}_{j_1}^{\mathsf{T}} \mathbf{Z}_1 \mathbf{Z}_2^{\mathsf{T}} \mathbf{M}^{-2}(\theta_i) \left(\frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_2^{\mathsf{T}} \right)^{-1} \frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_1^{\mathsf{T}} \mathbf{e}_{j_2} = \mathbf{O}_p \left(\frac{1}{\sqrt{n}} \right),$$

where $\frac{1}{n} \mathbb{E} \left\{ \operatorname{tr} \mathbf{M}^{-2}(\theta_i) \right\} \to y$. It follows that

$$\max_{1 \le j \le q} \left| \frac{1}{n} \mathbf{e}_{j}^{\mathsf{T}} (1+\delta_{i}) \mathbf{Z}_{1} \mathbf{Z}_{2}^{\mathsf{T}} \mathbf{M}^{-2}(\theta_{i}) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\mathsf{T}} \right)^{-1} \frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{1}^{\mathsf{T}} \mathbf{e}_{j} - \frac{1}{n} \mathbb{E} \left[\operatorname{tr} \left\{ \mathbf{M}^{-2}(\theta_{i}) \right\} \right] \right| = O_{p} \left(\frac{\sqrt{q}}{\sqrt{n}} \right)$$
$$\max_{1 \le j_{1} \ne j_{2} \le q} \left| \frac{1}{n} \mathbf{e}_{j_{1}}^{\mathsf{T}} (1+\delta_{i}) \mathbf{Z}_{1} \mathbf{Z}_{2}^{\mathsf{T}} \mathbf{M}^{-2}(\theta_{i}) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\mathsf{T}} \right)^{-1} \frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{1}^{\mathsf{T}} \mathbf{e}_{j_{2}} \right| = O_{p} \left(\frac{q}{\sqrt{n}} \right).$$

Similarly, we can get the following for other terms:

$$\begin{split} \max_{1 \leq j \leq q} \left| \mathbf{e}_{j}^{\mathsf{T}} (1+\delta_{i}) \frac{1}{n} \mathbf{Z}_{1} \mathbf{Z}_{1}^{\mathsf{T}} \mathbf{e}_{j} - 1 \right| &= \mathcal{O}_{p} \left(\frac{\sqrt{q}}{\sqrt{n}} \right), \ \max_{1 \leq j_{1} \neq j_{2} \leq q} \left| \mathbf{e}_{j_{1}}^{\mathsf{T}} (1+\delta_{i}) \frac{1}{n} \mathbf{Z}_{1} \mathbf{Z}_{1}^{\mathsf{T}} \mathbf{e}_{j_{2}} \right| &= \mathcal{O}_{p} \left(\frac{q}{\sqrt{n}} \right), \\ \max_{1 \leq j_{1} \neq j_{2} \leq q} \left| \mathbf{e}_{j}^{\mathsf{T}} \frac{1}{\widehat{\lambda_{i} \theta_{i}}} \frac{1}{T} \mathbf{X}_{1} \mathbf{X}_{2}^{\mathsf{T}} \mathbf{M}^{-1} (\widehat{\lambda_{i}}) \mathbf{M}^{-1} (\theta_{i}) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\mathsf{T}} \right)^{-1} \frac{1}{T} \mathbf{X}_{2} \mathbf{X}_{1}^{\mathsf{T}} \mathbf{e}_{j} \right| &= \mathcal{O}_{p} \left(\frac{\sqrt{\sum_{j} \lambda_{j}^{2}}}{\lambda_{i}^{2}} \right), \\ \max_{1 \leq j_{1} \neq j_{2} \leq q} \left| \mathbf{e}_{j_{1}}^{\mathsf{T}} \frac{1}{\widehat{\lambda_{i} \theta_{i}}} \frac{1}{T} \mathbf{X}_{1} \mathbf{X}_{2}^{\mathsf{T}} \mathbf{M}^{-1} (\widehat{\lambda_{i}}) \mathbf{M}^{-1} (\theta_{i}) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\mathsf{T}} \right)^{-1} \frac{1}{T} \mathbf{X}_{2} \mathbf{X}_{1}^{\mathsf{T}} \mathbf{e}_{j} \right| &= \mathcal{O}_{p} \left(\frac{\sqrt{\sum_{j} \lambda_{j}}}{\lambda_{i}^{2}} \right), \\ \max_{1 \leq j_{1} \neq j_{2} \leq q} \left| \mathbf{e}_{j}^{\mathsf{T}} \frac{(1+\delta_{i})}{\widehat{\lambda_{i}}} \frac{1}{n} \mathbf{Z}_{1} \mathbf{Z}_{2}^{\mathsf{T}} \mathbf{M}^{-1} (\widehat{\lambda_{i}}) \mathbf{M}^{-1} (\theta_{i}) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\mathsf{T}} \right)^{-1} \frac{1}{T} \mathbf{X}_{2} \mathbf{X}_{1}^{\mathsf{T}} \mathbf{e}_{j} \right| &= \mathcal{O}_{p} \left(\frac{\sqrt{\sum_{j} \lambda_{j}}}{\lambda_{i}^{2}} \right), \\ \max_{1 \leq j_{1} \neq j_{2} \leq q} \left| \mathbf{e}_{j}^{\mathsf{T}} \frac{(1+\delta_{i})}{\widehat{\lambda_{i}}} \frac{1}{n} \mathbf{Z}_{1} \mathbf{Z}_{2}^{\mathsf{T}} \mathbf{M}^{-1} (\widehat{\lambda_{i}}) \mathbf{M}^{-1} (\theta_{i}) \left(\frac{1}{n} \mathbf{Z}_{2} \mathbf{Z}_{2}^{\mathsf{T}} \right)^{-1} \frac{1}{T} \mathbf{X}_{2} \mathbf{X}_{1}^{\mathsf{T}} \mathbf{e}_{j} \right| &= \mathcal{O}_{p} \left(\frac{\sqrt{\sum_{j} \lambda_{j}}}{\lambda_{i}} \right), \\ \max_{1 \leq j_{1} \neq j_{2} \leq q} \left| \mathbf{e}_{j}^{\mathsf{T}} \frac{1}{n} \mathbf{Z}_{1} \mathbf{C} (\theta_{i}) \mathbf{X}_{1}^{\mathsf{T}} \mathbf{e}_{j} \right| &= \mathcal{O}_{p} \left(\frac{\sqrt{\sum_{j} \lambda_{j}}}{\sqrt{n} \lambda_{i}} \right), \\ \max_{1 \leq j_{1} \leq j_{2} \leq q} \left| \mathbf{e}_{j}^{\mathsf{T}} \frac{1}{n} \mathbf{Z}_{1} \mathbf{C} (\theta_{i}) \mathbf{X}_{1}^{\mathsf{T}} \mathbf{e}_{j} \right| &= \mathcal{O}_{p} \left(\frac{\sqrt{\sum_{j} \lambda_{j}}}{\sqrt{n} \lambda_{i}} \right), \\ \max_{1 \leq j_{1} \leq j_{2} \leq q} \left| \mathbf{e}_{j}^{\mathsf{T}} \frac{1}{n} \mathbf{Z}_{1} \mathbf{Z}_{1}^{\mathsf{T}} \mathbf{e}_{j} \right| &= \mathcal{O}_{p} \left(\frac{\sqrt{\sum_{j} \lambda_{j}}}{\sqrt{n} \lambda_{i}} \right), \\ \max_{1 \leq j_{1} \leq j_{2} \leq q} \left| \mathbf{e}_{j}^{\mathsf{T}} \frac{1}{n} \mathbf{Z}_{1} \mathbf{Z}_{1}^{\mathsf{T}} \mathbf{Z}_{1}^{\mathsf{T}} \mathbf{U}_{1}^{\mathsf{T}} \mathbf{U}_{1}^{\mathsf{T}} \mathbf{U}_{1}^{\mathsf{T}} \mathbf{U}_{1}^{\mathsf{T}} \mathbf{U}_{1}^{\mathsf{T}} \mathbf{U}_{1}^{\mathsf{T}} \mathbf{U}_{1}^{\mathsf{T}} \mathbf{U}$$

$$\max_{1 \le j_1 \ne j_2 \le q} \left| \mathbf{e}_{j_1}^{\top} \frac{(1+\delta_i)}{\widehat{\lambda}_i} \frac{1}{T} \mathbf{X}_1 \mathbf{X}_2^{\top} \mathbf{M}^{-1}(\widehat{\lambda}_i) \mathbf{M}^{-1}(\theta_i) \left(\frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_2^{\top} \right)^{-1} \frac{1}{n} \mathbf{Z}_2 \mathbf{Z}_1^{\top} \mathbf{e}_{j_2} \right| = \mathbf{O}_p \left(\frac{\sqrt{2j} \, \lambda_j}{\lambda_i} \right),$$
$$\max_{1 \le j \le q} \left| \mathbf{e}_j^{\top} \frac{1}{T} \mathbf{X}_1 \mathbf{D}(\theta_i) \mathbf{Z}_1^{\top} \mathbf{e}_j \right| = \mathbf{O}_p \left(\frac{\sqrt{2j} \, \lambda_j}{\sqrt{n} \lambda_i} \right), \quad \max_{1 \le j_1 \ne j_2 \le q} \left| \mathbf{e}_{j_1}^{\top} \frac{1}{T} \mathbf{X}_1 \mathbf{D}(\theta_i) \mathbf{Z}_1^{\top} \mathbf{e}_{j_2} \right| = \mathbf{O}_p \left(\frac{\sqrt{2j} \, \lambda_j}{\sqrt{n} \lambda_i} \right),$$

Thus, all these inequalities lead to

$$\max_{1 \le j \le q} \left| \mathbf{e}_{j}^{\mathsf{T}} \Theta_{2n} \mathbf{e}_{j} - (y - 1) \right| = \mathbf{O}_{p} \left(\frac{\sqrt{q} \delta_{i}}{\lambda_{i}} + \frac{\sqrt{q}}{\sqrt{n}} + \frac{\sqrt{\sum_{j} \lambda_{j}^{2}}}{\lambda_{i}^{2}} + \frac{\sqrt{\sum_{j} \lambda_{j}}}{\lambda_{i}} \right)$$
$$\max_{1 \le j_{1} \ne j_{2} \le q} \left| \mathbf{e}_{j_{1}}^{\mathsf{T}} \Theta_{2n} \mathbf{e}_{j_{2}} \right| = \mathbf{O}_{p} \left(\frac{q \delta_{i}}{\lambda_{i}} + \frac{q}{\sqrt{n}} + \frac{\sum_{j} \lambda_{j}}{\lambda_{i}^{2}} + \frac{\sqrt{q} \sum_{j} \lambda_{j}}{\lambda_{i}} \right).$$

The proof is completed. \Box

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