

On excitable β -skeletons

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Abstract

A β -skeleton, $\beta \geq 1$, is a planar proximity undirected graph of an Euclidean point set where nodes are connected by an edge if their lune-based neighborhood contains no other points of the given set. Parameter β determines size and shape of the nodes' neighborhoods. In an excitable β -skeleton every node takes three states — resting, excited and refractory, and updates its state in discrete time depending on states of its neighbors. We design families of β -skeletons with absolute and relative thresholds of excitability and demonstrate that several distinct classes of space-time excitation dynamics can be selected using β . The classes include spiral and target waves of excitation, branching domains of excitation and oscillating localizations.

Keywords: proximity graphs, β -skeletons, excitation, waves, localizations, space-time dynamics, pattern formation

1 Introduction

A planar graph consists of nodes which are points of Euclidean plane and edges which are straight segments connecting the points. A planar proximity graph is a planar graph where two points are connected by an edge if they are close in some sense. Usually a pair of points is assigned certain neighborhood, and points of the pair are connected by an edge if their neighborhood is empty. Delaunay triangulation [8], relative neighborhood graph [10] and Gabriel graph [16], and indeed spanning tree, are most known examples of proximity graphs. β -skeletons, proposed in [12], form a unique family of proximity graphs monotonously parameterised by parameter β .

Proximity graphs found their applications in fields of science and engineering: geographical variational analysis [9,16,20], evolutionary biology [15], spatial analysis in biology [13,6,7,11], simulation of epidemics [23]. Proximity graphs are used in physics to study percolation [5] and analysis of magnetic field [22].

Engineering applications of proximity graphs are in message routing in ad hoc wireless networks, see e.g. [14,21,19,17,24], and visualisation [18]. Road network analysis is yet another field where proximity graphs are invaluable. Road networks are well matched by relative neighborhood graphs, see e.g. study of Tsukuba central district [25,26]. Biological transport networks also bear remarkable similarity to certain proximity graphs. Foraging trails of ants [1] and protoplasmic networks of slime mold *Physarum polycephalum* [2,3] are most striking examples.

Structure of proximity graphs represents so wide range of natural systems that it is important to uncover basic mechanism of activity propagation on the graphs, which could be applied in future studies of particular natural systems. This is why we undertook computational experiments with excitable β -skeletons to check how space-time dynamics of excitation on β -skeletons depends on β . The paper is structured as follows. In Sect. 2 we introduce models of excitable β -skeletons. Phenomenology of skeletons with absolute threshold of excitation (a node excites depending on an absolute number of its excited neighbors) is provided in Sect. 3. Space-time dynamics of skeletons with relative threshold of excitation (a node excites depending on a ratio of excited neighbors) is studied in Sect. 4. Results of computational experiments are summarized in Section 5.

2 The model

Given a set $\mathbf{V}$ of planar points, for any two points p and q we define β -neighborhood $U_\beta(p, q)$ as an intersection of two discs with radius $\beta|p - q|/2$ centered at points $((1 - \frac{\beta}{2})p, \frac{\beta}{2}q)$ and $(\frac{\beta}{2}p, (1 - \frac{\beta}{2})q)$, $\beta \geq 1$ [12,10], see examples of the lunes in Fig. 1. Points p and q are connected by an edge in β -skeleton if the pair's β -neighborhood contains no other points from $\mathbf{V}$.

A β -skeleton is a graph $G_\beta(\mathbf{V}) = \langle \mathbf{V}, \mathbf{E}, \beta \rangle$, where nodes $\mathbf{V} \subset \mathbf{R}^2$, edges $\mathbf{E}$, and for $p, q \in \mathbf{V}$ edge $(pq) \in \mathbf{E}$ if $U_\beta(p, q) \cap \mathbf{V} \setminus \{p, q\} = \emptyset$. Parameterization β is monotonous: if $\beta_1 > \beta_2$ then $G_{\beta_1}(\mathbf{V}) \subset G_{\beta_2}(\mathbf{V})$ [12,10]. A β -skeleton is Gabriel graph [16] for $\beta = 1$ and the skeleton is relative neighbourhood graph for $\beta = 2$. We consider only skeletons with $1 \leq \beta \leq 2$ because β -skeletons are non-planar for $\beta < 1$ and they are disconnected for $\beta > 2$.

We study β -skeletons of planar set of n discs. Centers of the discs form set $\mathbf{V}$. Each disc has a radius 2.5 units. The discs are randomly distributed in a large disc with radius 480 units. We undertake computational experiments with n varying from 1000 to 15000 (Fig. 2). Number of nodes *per se* is not as important as density of their packing therefore we refer to any particular $\mathbf{V}$ via density of node packing $\varphi = 0.027, \dots, 0.407$.

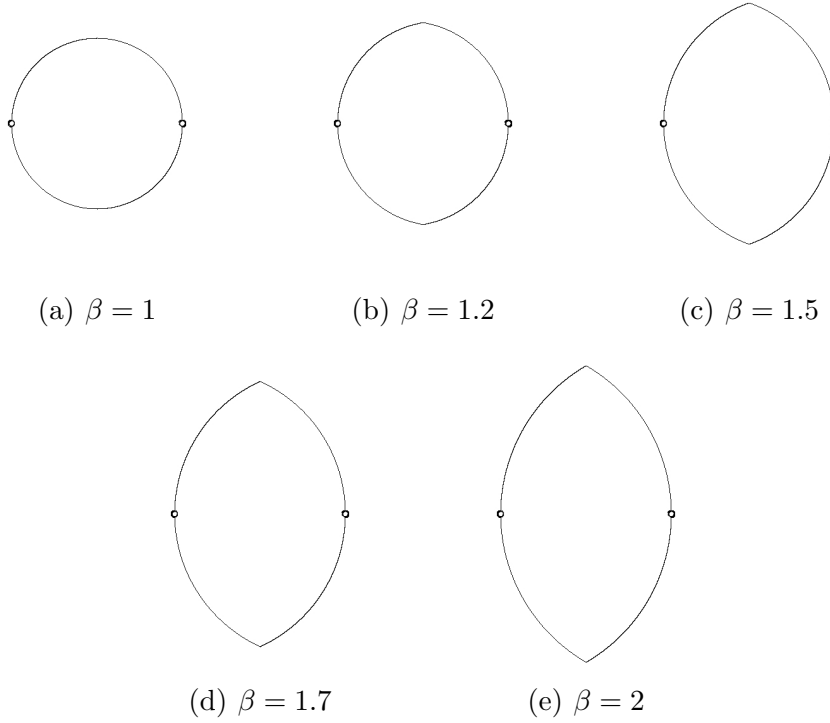


Fig. 1. Examples of the lunes between two points (small circles) for various values of β .

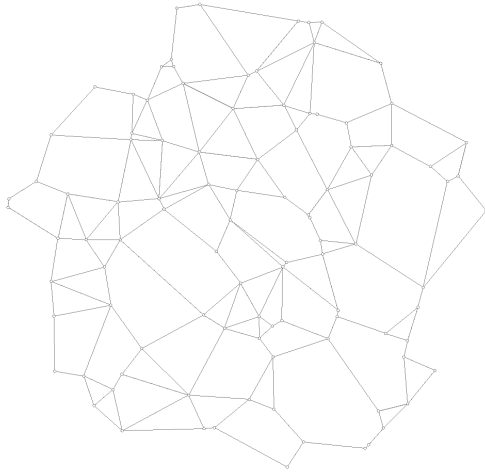
An excitable β -skeleton is defined as follows. A node $p \in \mathbf{V}$ is a finite state machine. Every node takes three states: resting ($\circ$), excited ($+$) and refractory ($-$). A node updates its states in discrete time depending on states of its neighbors. All nodes update their states simultaneously.

We assume that a resting node excites depending on a number of excited neighbors. If a node is excited at time t the node takes refractory state at time step $t + 1$, independently on states of its neighbors. Transition from refractory to resting state is also unconditional.

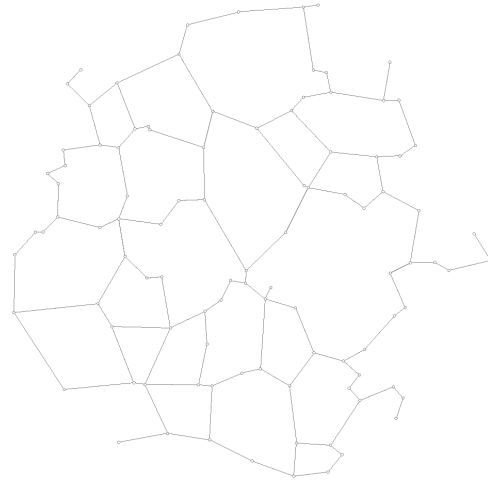
Let $\nu(p) = \{q \in \mathbf{V} : (pq) \in \mathbf{E}\}$ be a neighborhood, or set of neighbors, of node p , p^t a state of node p at time step t , $\sigma^t(p)$ a number of excited neighbors of p at step t , and $d(p)$ a degree, or a number of neighbours $|\nu(p)|$, of node p . Then the node-state transition function can be defined as follows:

$$p^{t+1} = \begin{cases} +, & \text{if } C(\nu(p)^t) \\ -, & \text{if } p^t = + \\ \circ, & \text{if } p^t = - \end{cases} \quad (1)$$

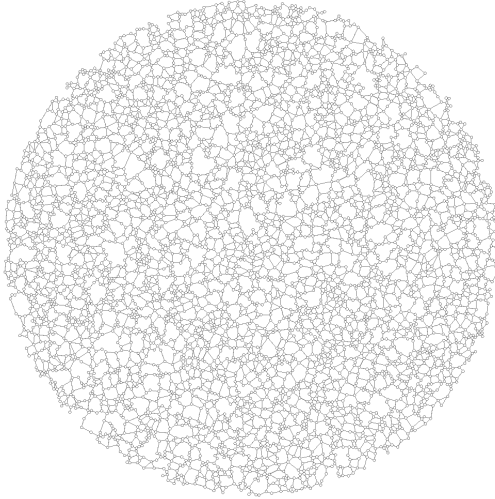
We consider two versions of the excitation condition $C(\nu(p)^t) \in \{ \text{TRUE}, \text{FALSE} \}$:



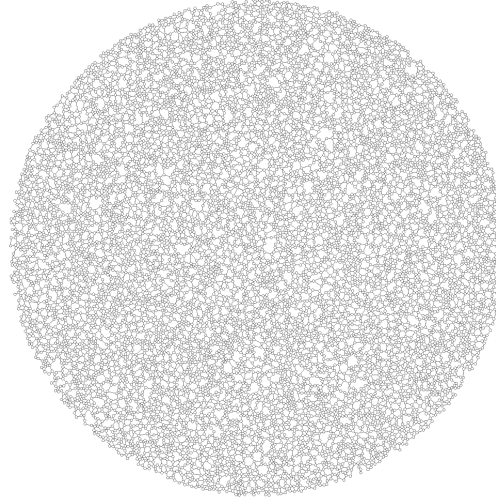
(a) $\beta = 1$



(b) $\beta = 1.5$



(c) $\beta = 1$



(d) $\beta = 1.5$

Fig. 2. Examples of β -skeletons: (a) $n = 100$, $\varphi = 0.0027$, and $\beta = 1$, (b) $n = 100$, $\varphi = 0.0027$, and $\beta = 2$, (c) $n = 5000$, $\varphi = 0.136$, and $\beta = 1.5$, (d) $n = 10000$, $\varphi = 0.271$, and $\beta = 1.5$.

- *Absolute excitability:* $C(\nu(p)^t) = \sigma^t(p) \geq \theta$, $\theta = 1, \dots$
- *Relative excitability:* $C(\nu(p)^t) = (\frac{\sigma^t(p)}{d(p)} > \epsilon)$, $\epsilon \in [0, 1]$.

Absolute excitability is the most common approach of defining rules in excitable discrete systems however it does not account for diversity of node degrees in disordered systems. Thus we also explore skeletons with relative excitability.

In the paper we illustrate space-dynamics of excitation by snapshots of skele-

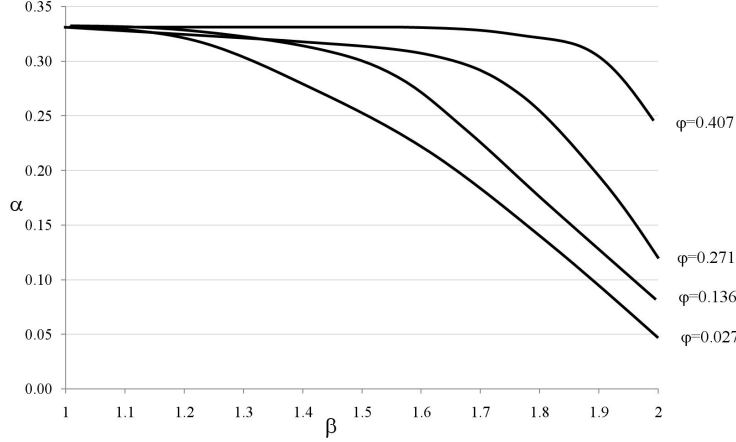


Fig. 3. Activity α versus β for various densities φ of node packing in β -skeletons with absolute excitation threshold $\theta = 1$. Skeletons are developed from initial random configuration, where a node is assigned excited state probability 0.1 and is resting otherwise.

ton configurations. We analyze integral dynamics of skeletons using activity α . The activity α is a ratio of excited nodes to a total number of nodes averaged fixed number time steps. The activity α is measured after initial transient period, when excitation patterns are given a chance to occupy the whole skeleton (usually a hundred time steps is sufficient).

3 Absolutely excitable skeletons

In β -skeletons governed by absolute excitability rule excitation persists only for threshold $\theta = 1$.

Finding 1 *Activity α is proportional to density φ and inversely proportional to β .*

Figure 3 shows exactly how activity α depends on β . Note that maximum possible α equals 0.33, for sustainable activity, because an excited node always becomes refractory, and a refractory node always becomes resting. Thus even if the whole skeleton is active, only third of its nodes are excited at any given time step. The activity decreases polynomially with increase of β , degree of the polynomial is proportional to density φ of nodes in the skeleton.

Spatial counterparts of integral activity are shown in Fig. 4. These are examples of excitation patterns emerged after a single node of the resting skeleton was excited. In β -skeletons with high density of nodes packing $\varphi = 0.271$ and 0.407 excitation of a single node of a resting skeleton leads to formation of a generator of spiral and target waves if $1 \leq \beta < 1.4$. The most noticeable

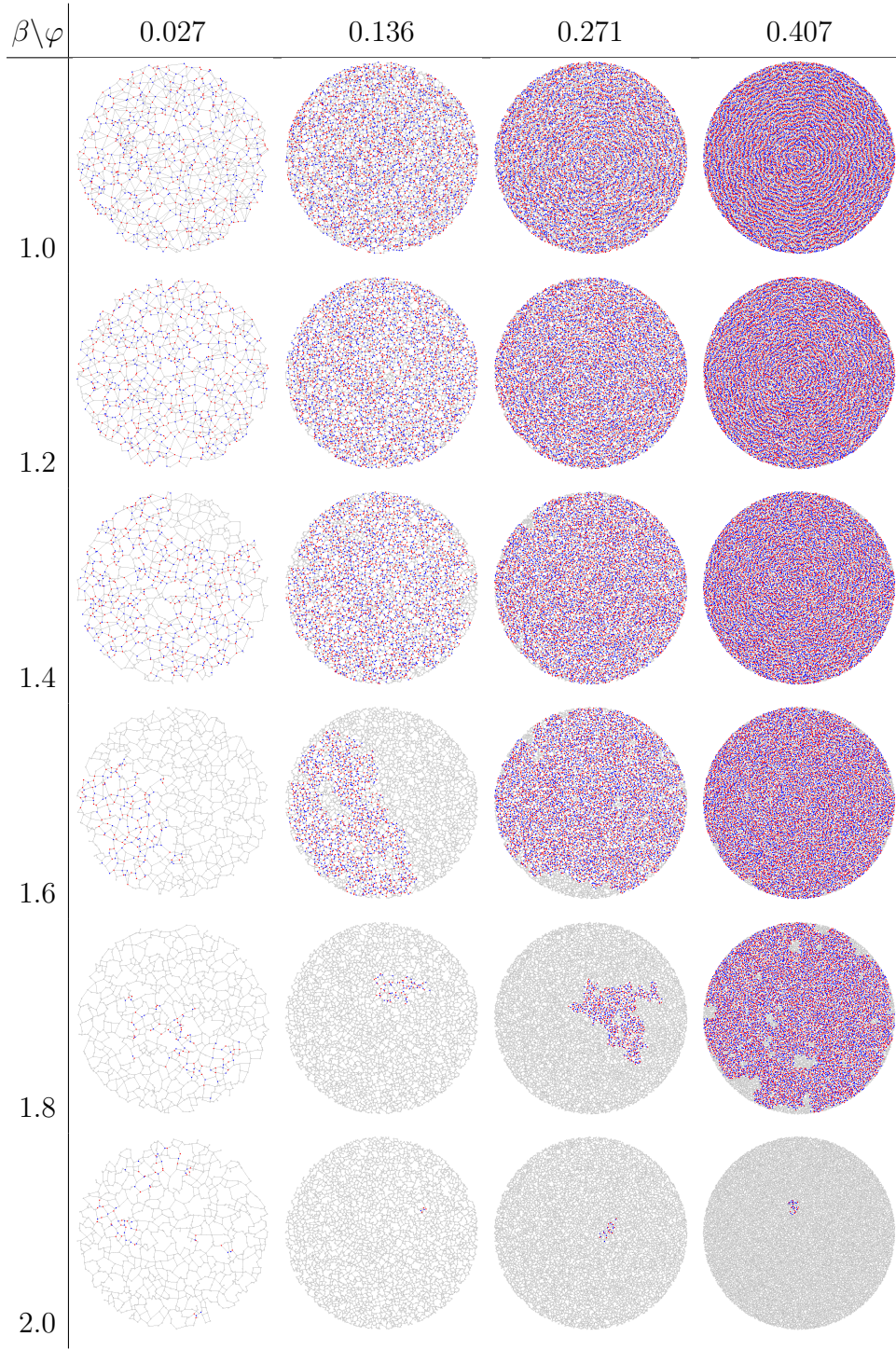


Fig. 4. Configurations of excitation on β -skeletons with node density $\varphi = 0.027, 0.136, 0.271, 0.407$ and absolute threshold of excitation $\theta = 1$. All configurations are developed from a localized initial excitation: all nodes but are resting, one node is excited. The configurations are snapped after 50 steps of development. Red (light-gray) coloured node are excited, blue (dark-gray) coloured nodes are refractory.

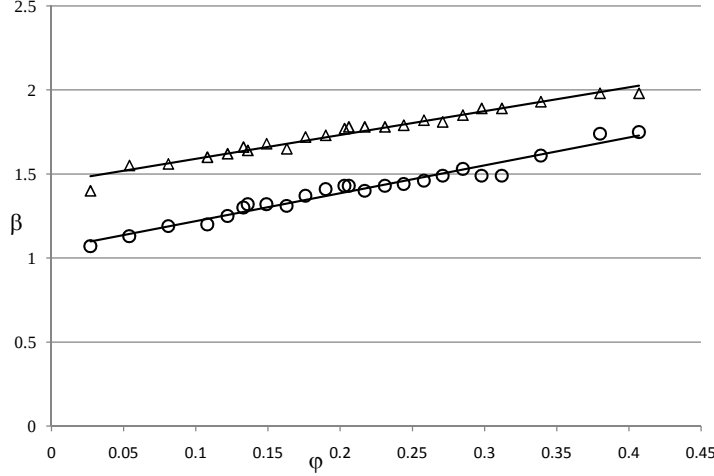


Fig. 5. Dependence of critical values β_d and β_o on φ . Data points for β_d are shown by circles, for β_o by triangles. Trend lines are $\beta_d = 1.65\varphi + 1.05$, $R^2 = 0.961$ and $\beta_o = 1.42\varphi + 1.45$, $R^2 = 0.964$.

pattern of the waves is observed for $\varphi = 0.407$ and $\beta = 1$. Increase of β leads to break up wave-fronts and only localized compact clusters of activity remain on the skeleton.

What values of φ and β cause excitation fail to span the whole graph? For what values of φ and β only tiny localized clusters of oscillating activity remain in a skeleton? We call values β critical if patterns of excitation change abruptly, e.g. excitation dynamics changes from target waves filling the whole skeleton to slowly growing, β_d , or even localized domains of activity, oscillators, β_o .

Finding 2 *Critical values β_d and β_o are linearly proportional to density φ of nodes in skeleton.*

Assuming that a skeleton fully occupied by target waves has activity level $\alpha = 0.33$ even tiniest decrease in α indicates presence of a stationary resting domain. See example in Fig. 4, $\varphi = 0.271$ and $\beta = 1.4$. Let us assume that excitation fails to span a skeleton when $\alpha \leq 0.32$ and only bounded domains of excitation are present if $\alpha \leq 0.25$. Let β_d and β_o be critical values of β thus that if for a given β -skeleton value β exceeds β_d then the skeleton is not fully occupied by excitation, and if $\beta \geq \beta_o$ then only localized oscillators are formed. Figure 5 presents critical values of β computed for 25 sample configurations of β -skeletons with φ varying from 0.027 to 0.407. What are structural correlates responsible for the transition from a wide-spread excitation to localized domains?

Finding 3 *Excitation ceases to occupy a whole skeleton when the mode of node degree distribution of the skeleton changes from 4 to 3.*

$\phi \backslash \beta$	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0.027	<u>4</u>	3	3	3	3	3	3	3	3	3	3
0.054	<u>4</u>	<u>4</u>	3	3	3	3	3	3	3	3	3
0.081	<u>4</u>	<u>4</u>	<u>4</u>	3	3	3	3	3	3	3	3
0.108	<u>4</u>	<u>4</u>	<u>4</u>	3	3	3	3	3	3	3	3
0.136	<u>4</u>	<u>4</u>	<u>4</u>	3	3	3	3	3	3	3	3
0.163	<u>4</u>	<u>4</u>	<u>4</u>	<u>4</u>	3	3	3	3	3	3	3
0.190	<u>4</u>	<u>4</u>	<u>4</u>	<u>4</u>	3	3	3	3	3	3	3
0.217	<u>4</u>	<u>4</u>	<u>4</u>	<u>4</u>	<u>4</u>	3	3	3	3	3	3
0.244	5	<u>4</u>	<u>4</u>	<u>4</u>	<u>4</u>	3	3	3	3	3	3
0.271	5	<u>4</u>	<u>4</u>	<u>4</u>	<u>4</u>	3	3	3	3	3	3
0.298	5	<u>4</u>	<u>4</u>	<u>4</u>	<u>4</u>	<u>4</u>	3	3	3	3	3
0.325	5	5	<u>4</u>	<u>4</u>	<u>4</u>	<u>4</u>	3	3	3	3	3
0.352	5	5	<u>4</u>	<u>4</u>	<u>4</u>	<u>4</u>	<u>4</u>	3	3	3	3
0.380	5	5	<u>4</u>	<u>4</u>	<u>4</u>	<u>4</u>	<u>4</u>	3	3	3	3
0.407	5	5	<u>4</u>	<u>4</u>	<u>4</u>	<u>4</u>	<u>4</u>	3	3	3	3

(a)

$\phi \backslash \beta$	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0.027	4.0	3.6	3.5	3.3	<u>3.2</u>	3.0	2.9	2.8	2.7	2.6	2.5
0.054	4.1	3.8	3.6	3.4	3.2	<u>3.1</u>	2.9	2.8	2.7	2.6	2.6
0.081	4.2	3.9	3.7	3.5	3.3	3.2	<u>3.0</u>	2.9	2.8	2.7	2.6
0.108	4.2	4.0	3.7	3.5	3.3	3.2	<u>3.1</u>	2.9	2.8	2.7	2.7
0.136	4.3	4.0	3.8	3.6	3.4	3.2	3.1	<u>3.0</u>	2.9	2.8	2.7
0.163	4.4	4.1	3.9	3.7	3.5	3.3	3.2	<u>3.0</u>	2.9	2.8	2.7
0.190	4.5	4.2	3.9	3.7	3.5	3.3	3.2	<u>3.1</u>	2.9	2.8	2.7
0.217	4.6	4.3	4.0	3.8	3.6	3.4	3.3	3.1	<u>3.0</u>	2.9	2.8
0.244	4.6	4.3	4.1	3.8	3.6	3.5	3.3	3.2	<u>3.0</u>	2.9	2.8
0.271	4.7	4.4	4.1	3.9	3.7	3.5	3.3	3.2	<u>3.1</u>	2.9	2.8
0.298	4.8	4.5	4.2	4.0	3.7	3.6	3.4	3.3	3.1	<u>3.0</u>	2.9
0.325	4.9	4.5	4.3	4.0	3.8	3.6	3.5	3.3	3.2	<u>3.0</u>	2.9
0.353	4.9	4.6	4.3	4.1	3.9	3.7	3.5	3.4	3.2	<u>3.1</u>	2.9
0.380	5.0	4.7	4.4	4.2	4.0	3.8	3.6	3.4	3.2	3.1	<u>3.0</u>
0.407	5.1	4.7	4.5	4.2	4.0	3.8	3.6	3.4	3.3	3.1	<u>3.0</u>

(b)

Fig. 6. Mode (a) and average (b) node degree of β -skeletons for $\beta \in [1, 2]$ and $\varphi = 0.027, \dots, 0.407$. In (a) entries are boldfaced to increase visibility. In (b) entries corresponding to $\beta_o(\varphi)$ are underlined.

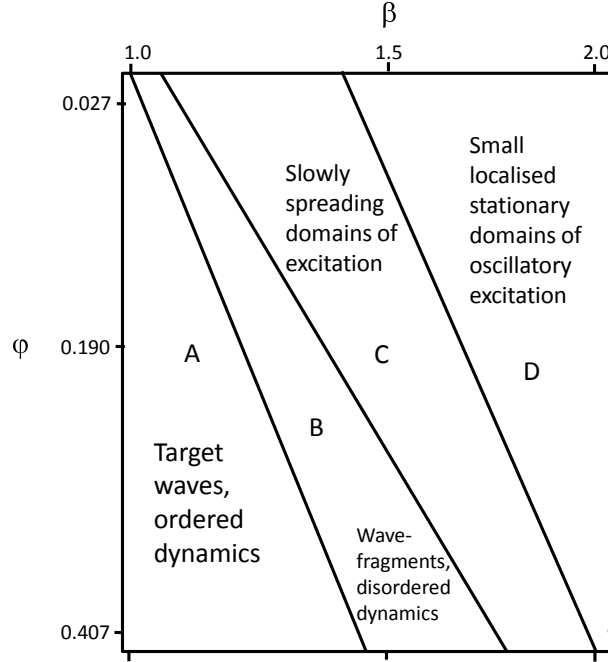


Fig. 7. Parameterisation of a space-time dynamic of β -skeleton with absolute excitation threshold by node density φ and β .

We found this by directly comparing β_d and integral characteristics of node degree distributions for a range of β and φ (Fig. 6). Mode of node degree distribution is not helpful however in detecting skeletons supporting localized oscillators. As shown in Fig. 6 skeletons with mode 3 of degree distribution can exhibit very large and very small domains of oscillating activity. An average node degree of a skeleton gives us a bit more detailed picture of φ - β -induced structural transitions and thus can be used to detect when localized oscillators are formed.

Finding 4 *A skeleton exhibits only localized oscillating domains of activity when average node degree drops below 3.*

The average node degrees corresponding to onset of localized oscillations are underlined in Fig. 6. Further comparison of average node degrees (Fig. 6), and corresponding space-time configurations of activity (see e.g. Fig. 4) allows us to state the following observation. Average node degree δ of a skeleton determines space-time dynamics of excitation as follows:

- $3.9 \leq \delta$: classical wave-like dynamics, generators of target waves, ordered dynamics (Fig. 7A),
- $3.7 \leq \delta < 3.9$: generators of wave-fragments, disordered excitation activity (Fig. 7B),
- $3.0 \leq \delta < 3.7$: formation of slowly propagating domains of excitation, some

- loci of a skeleton remain resting (Fig. 7C),
- $\delta \leq 3.0$: only small localized stationary domains of oscillating activity are formed (Fig. 7D).

4 Relatively excitable skeletons

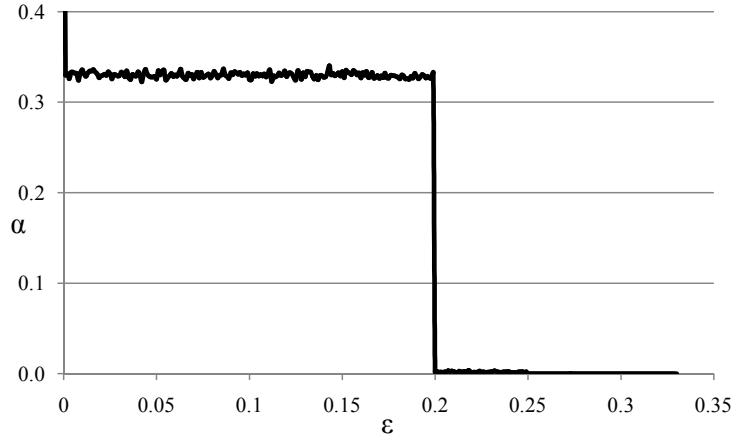
Here we consider behavior of skeletons with node density $\varphi = 0.407$. Skeletons with density $\varphi = 0.407$ (15,000 disc-nodes) show a wide representation of structural properties because they exhibit widest range of modes of node degree distributions (Fig. 6). In computational experiments we perturbed a resting skeleton with small excitation: every node is assigned an excited state with probability 0.1. We allowed the skeleton to develop its excitation patterns for 100 iterations and recorded activity α then. Examples of integral activity of skeletons for $\beta = 1, 1.5$ and 2 are shown in Fig. 8. We see that drastic changes in activity levels occur when ϵ changes from 0.199 to 2 (first drop in activity level is most clear in Fig. 8a), and when ϵ changes from 0.249 to 0.25 (second drop in activity level, see e.g. Fig. 8b).

More detailed description of how α depends on β is provided in Fig. 9. For $\epsilon \in [0, 0.2[$ activity level α is not changed with increase of β , $1 \leq \beta \leq 1.7$. However as soon as β reaches 1.8 activity level slightly decreases. The activity level drops down by one third when $\beta = 2$ (Fig. 9). For $\epsilon \in [0.25, 0.33[$ activity monotonously increases with increase of β . For $\epsilon \in [0.2, 0.249[$ activity α grows with growth of β till $\beta = 1.6$. The activity remains unchanged for $\beta = 1.7$ and then decreases with further increase of β . Let us discuss space-time dynamics of these β -skeletons for selected values of β .

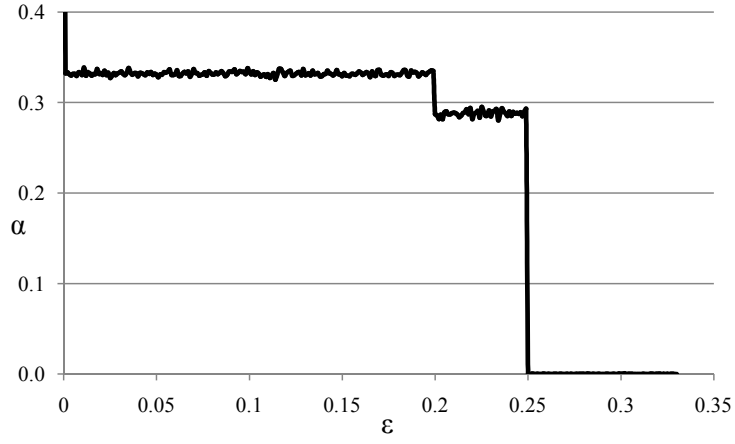
4.1 $\beta = 1$

For $\epsilon \in [0, 0.166]$ skeletons exhibit classical properties of discrete excitable media. A single excitation causes formation of a spiral wave or a generator of spiral or target waves. Successions of circular waves fill the whole skeleton (Fig. 10ab). As soon as ϵ exceeds 0.166 excitation wave-fronts break up into separate wave fragments. Thus branching domains of excitation activity are formed (Fig. 10c).

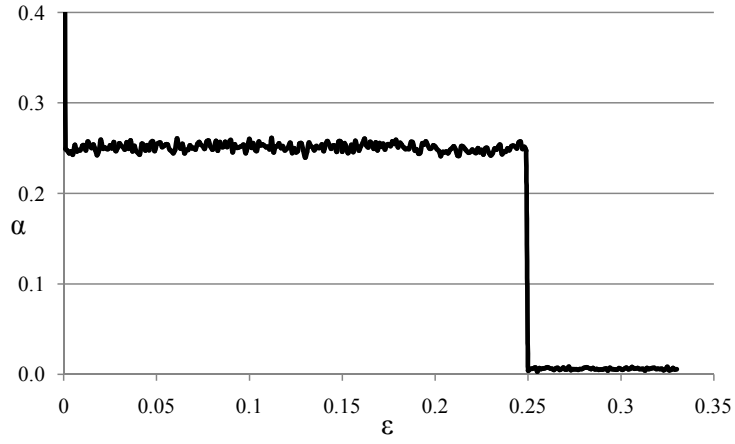
Only tiny oscillating activity domains emerge for $\epsilon \geq 0.2$ (Fig. 10d). Life-cycles of three oscillators are shown in Fig. 11. Left oscillator in Fig. 11 is a typical one. Excitation wave, consisting of one excited state and one refractory state, runs along a cycle of five nodes. The oscillator thus has period five (Fig. 11a–f). The oscillator conserves number of non-resting states.



(a) $\beta = 1$



(b) $\beta = 1.5$



(c) $\beta = 2$

Fig. 8. Activity α versus relative excitation threshold ϵ of β -skeletons for (a) $\beta = 1$, (b) $\beta = 1.5$, (c) $\beta = 2$.

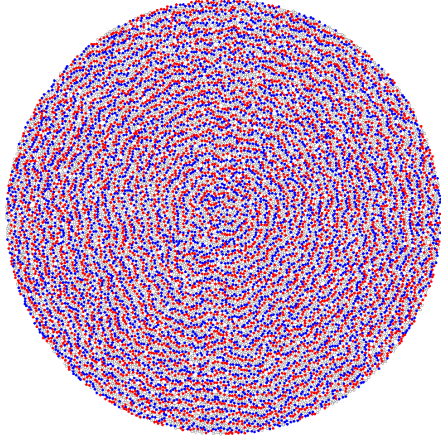
$\beta \backslash \epsilon$	$\epsilon \in [0, 0.2[$	$\epsilon \in [0.2, 0.25[$	$\epsilon \in [0.25, 0.33[$
1	0.33	0.002	0
1.1	0.33	0.01	0
1.2	0.33	0.04	0
1.3	0.33	0.15	0
1.4	0.33	0.23	0.00007
1.5	0.33	0.29	0.00010
1.6	0.33	0.31	0.00015
1.7	0.33	0.31	0.00020
1.8	0.32	0.305	0.00150
1.9	0.295	0.285	0.00300
2	0.24	0.245	0.01

Fig. 9. Dependence of activity level α of β -skeletons on β and ϵ . Initially resting skeleton is set to a random configuration of excitation by exciting each node with probability 0.1. Values of α are measured 100 iterations after initial excitation of skeletons.

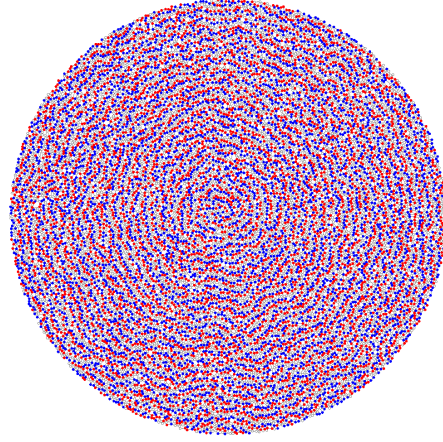
Oscillator in the top part of snapshots in Fig. 11 has period three. It changes its state between configurations with two excited and one refractory and one excited and two refractory states. The activation does not cycle but is exchanged between several nodes. First one node a excites its two neighbors. These excited nodes convey excitation to two other nodes (their neighbors), which in turn excite the original node a (Fig. 11a–d).

Oscillator shown in right part of snapshots in Fig. 11 consists of four excited and four refractory states, the number of the non-resting states is conserved. The oscillator has period three (Fig. 11a–d). This oscillator behaves like a breather, with its compressed (Fig. 11c), intermediate (Fig. 11a) and fully expanded (Fig. 11b) configuration.

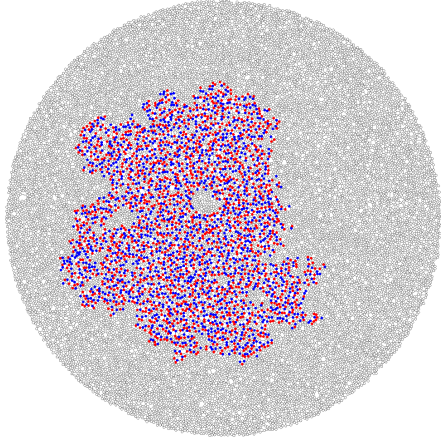
Skeleton becomes non-excitable, i.e. no excitation persists, when ϵ exceeds 0.25.



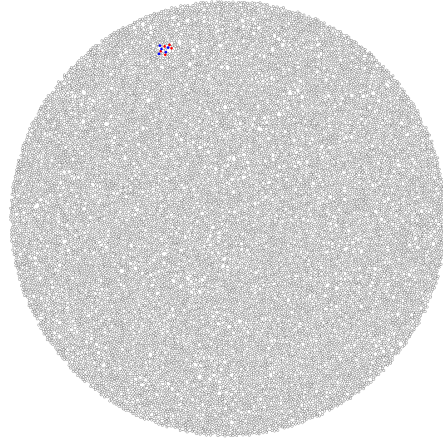
(a) $\beta = 1, \epsilon = 0.01$



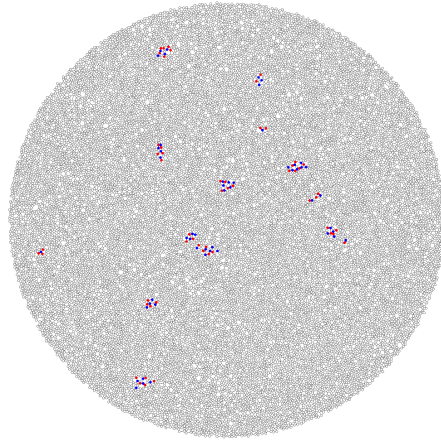
(b) $\beta = 1, \epsilon = 0.15$



(c) $\beta = 1, \epsilon = 0.167$



(d) $\beta = 1, \epsilon = 0.2$



(e) $\beta = 1, \epsilon = 0.24$

Fig. 10. Snapshots of excitation dynamics in β -skeletons ($\beta = 1$) with relative excitation threshold. (a)–(d) singular initial excitation, (e) random initial excitation, a node is excited with probability 0.1. Red (light-gray) coloured node are excited, blue (dark-gray) coloured nodes are refractory.

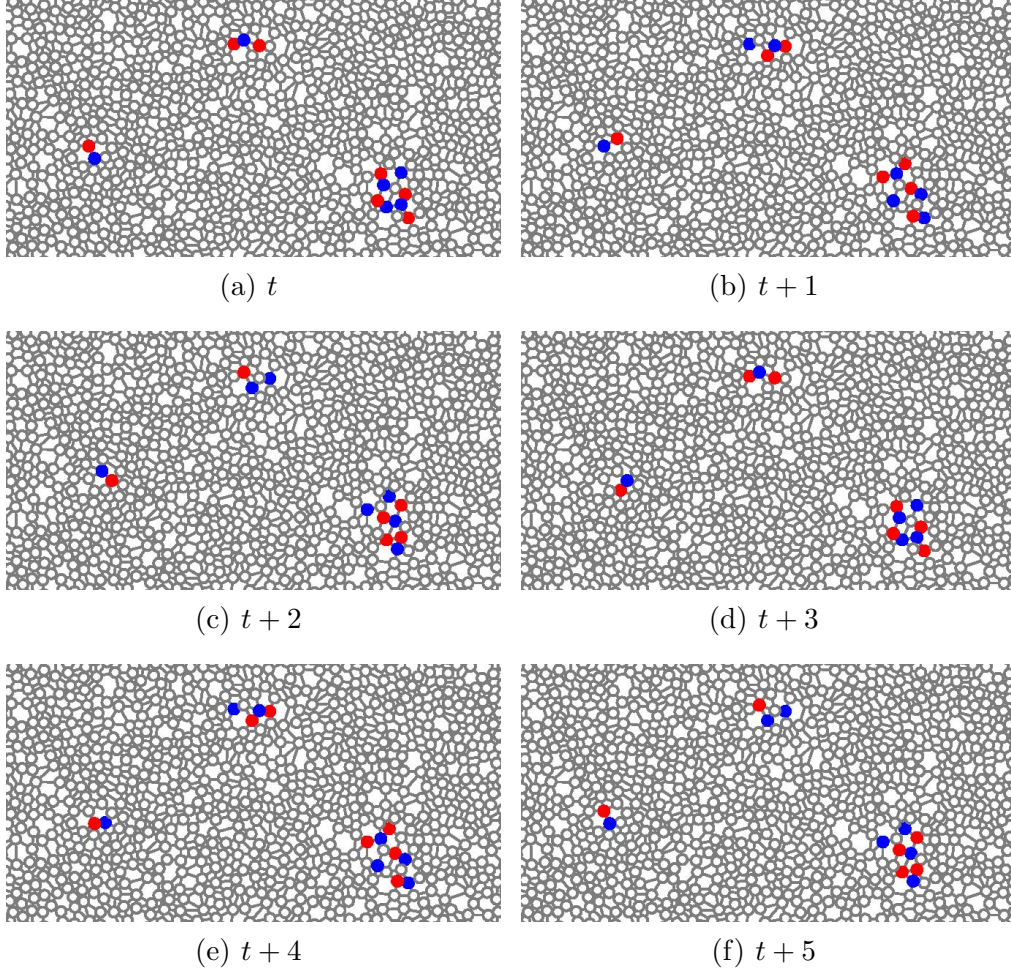


Fig. 11. Examples of oscillators in β -skeletons ($\beta = 1$) with relative excitation threshold $\epsilon = 0.24$. Red (light-gray) coloured node are excited, blue (dark-gray) coloured nodes are refractory.

4.2 $\beta = 1.5$

Propagating domains of excitation are circularly shaped however they do not show any pronounced wave-fronts, the excitation patterns are rather disordered for $\epsilon \in [0, 0.199]$ (Fig. 12ab). With ϵ exceeding 0.199 activity domains change their shapes from circular to branching, tree-like propagating domains (Fig. 12c). Size of such branching domains decreases with increase of ϵ . When $\epsilon \geq 0.25$ all domains cease propagating and just tiny clusters, oscillators, of activity are formed (Fig. 12d). A typical oscillator is a cycle of three or four nodes around which a quasi-one-dimensional excitation propagates. Sometimes the running excitation sends 'sparks' of activity to lateral nodes; these sparks extinguish in few steps of development. No excitation persists in skeletons for $\epsilon > 0.33$.

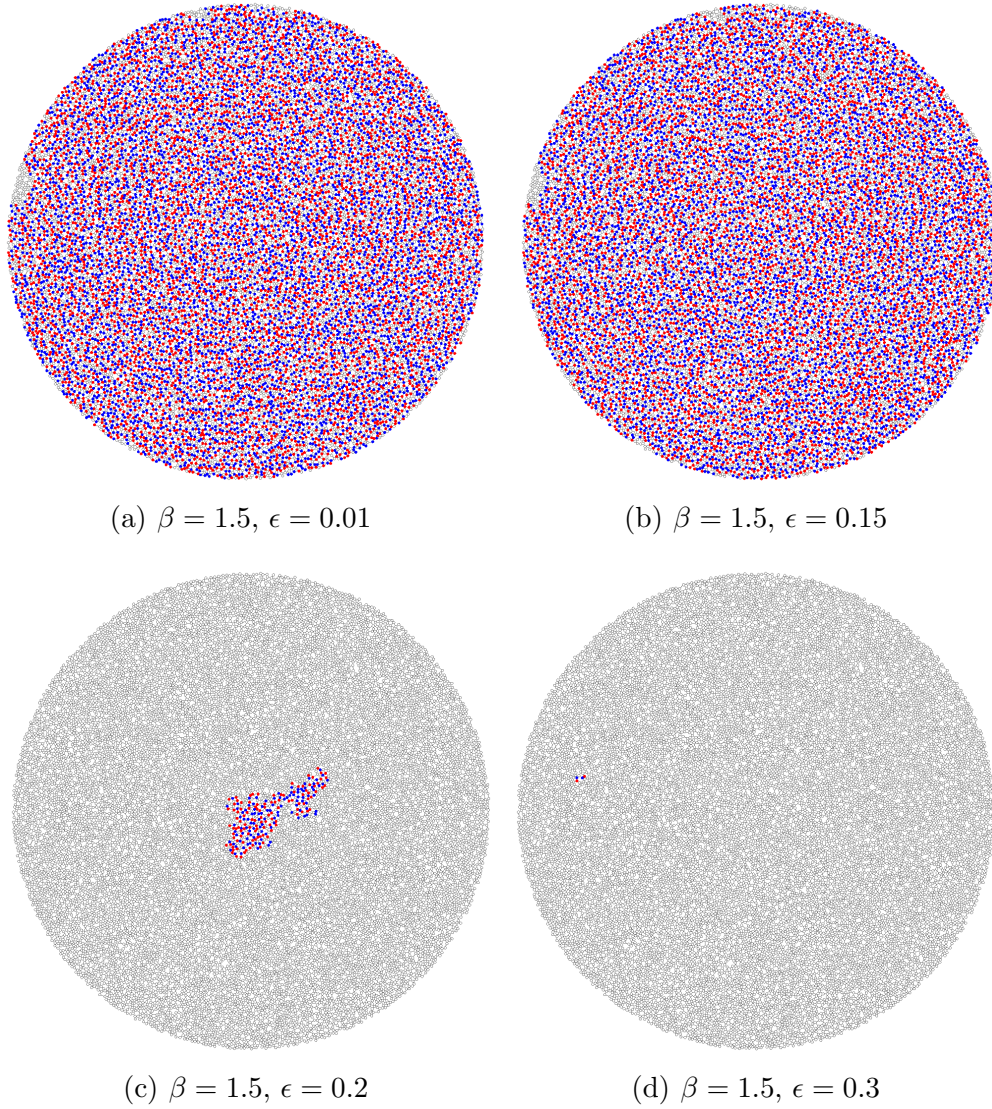


Fig. 12. Snapshots of excitation dynamics in β -skeletons ($\beta = 1.5$) with relative excitation threshold. Red (light-gray) coloured node are excited, blue (dark-gray) coloured nodes are refractory.

4.3 $\beta = 2$

Single excitation gives birth to slowly propagating irregularly-shaped domains of excitation activity (Fig. 13a). The domains originated from a single excitation never span the whole skeleton. Sizes of domains decrease with increase of ϵ (Fig. 13bc).

Only tiny clusters of excitation are formed for $\epsilon \in [0.25, 0.333[$ (Fig. 13d). A life-cycle of most typical oscillator is shown in Fig. 14. This oscillator has period seven. Its core structure is a singleton-excitation (accompanied by a

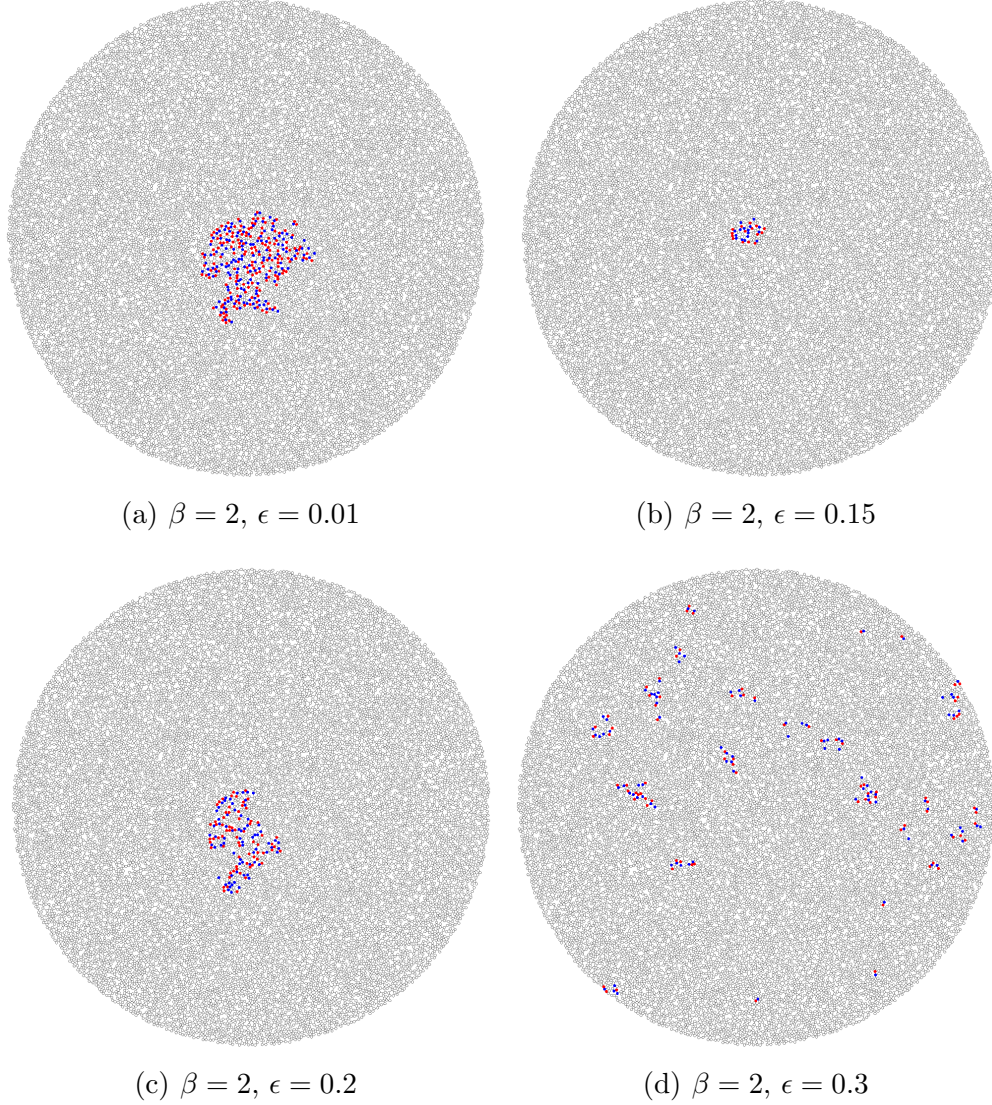


Fig. 13. Snapshots of excitation dynamics in β -skeletons ($\beta = 2$) with relative excitation threshold. Red (light-gray) coloured node are excited, blue (dark-gray) coloured nodes are refractory.

refractory tail) running around seven-node cycle, anti-clockwise. At certain moments of its life-cycle the excitation excites few nodes neighboring to the ‘core cycle’. Thus sparks of excitation are formed (Fig. 14c–f).

No excitation persists for $\epsilon \geq 0.333$.

Finding 5 *Excitable beta-skeletons with relative threshold of excitation exhibit the following classes of space-time dynamics: (1) spiral and target waves similar to classical excitation waves, (2) omnidirectional propagating patterns of disordered excitation, (3) localised domains of excitations, (4) branching domains, and, (5) tiny oscillators.*

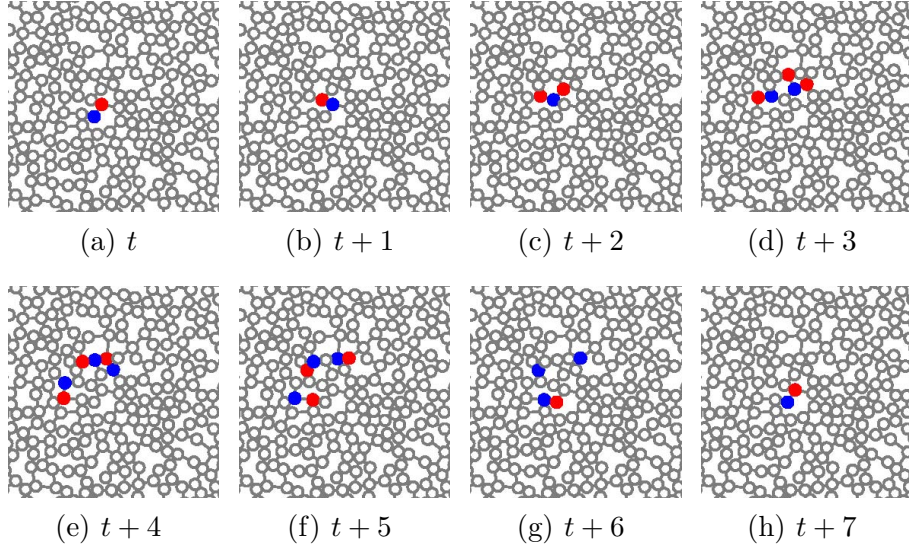


Fig. 14. Examples of oscillators in β -skeletons ($\beta = 2$) with relative excitation threshold $\epsilon = 0.29$. Red (light-gray) coloured node are excited, blue (dark-gray) coloured nodes are refractory.

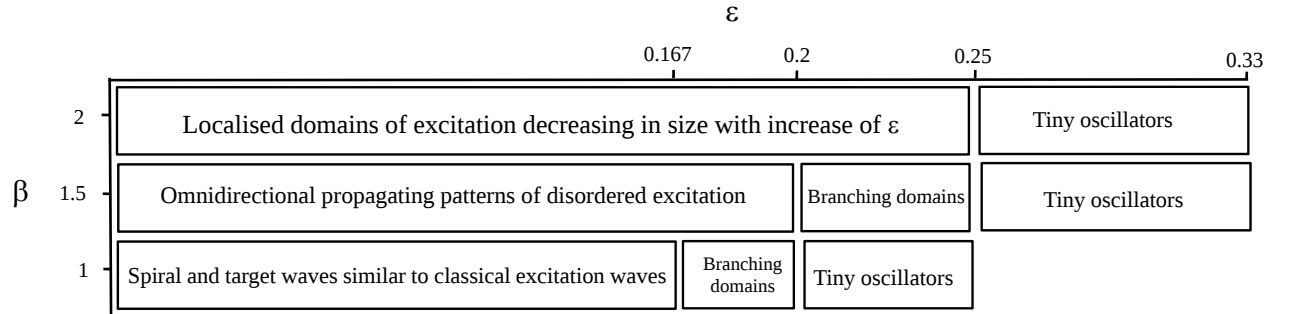


Fig. 15. Parameterisation of space-time dynamics of β -skeletons with relative excitation threshold by β and ϵ .

Position of the classes on $\beta - \epsilon$ -plane is shown in Fig. 15. There are sharp boundaries between classes, characterized by abrupt changes in morphology of excitation patterns. Critical values of relative excitability threshold are $\epsilon = 0.167, 0.2, 0.25$ and 0.33 . These thresholds reflect situations when a resting node excites if at least one of six ($\epsilon = 0.166$), one of five ($\epsilon = 0.2$), one of four ($\epsilon = 0.25$) and one of three ($\epsilon = 0.33$) neighbors is excited. Excitability threshold $\epsilon = 0.166$ is critical only for β -skeletons with β very close to 1. This is because such skeletons have mode 5 of node degree distribution and average node degree 5.1 (Fig. 6, $\varphi = 0.407$), i.e. nodes with six neighbors are second (after nodes with five neighbors) dominating nodes on the graph. For similar reasons $\epsilon = 0.2$ is not critical threshold value for β -skeletons with β close to 2 because over half of the nodes in such skeletons have three neighbors, and

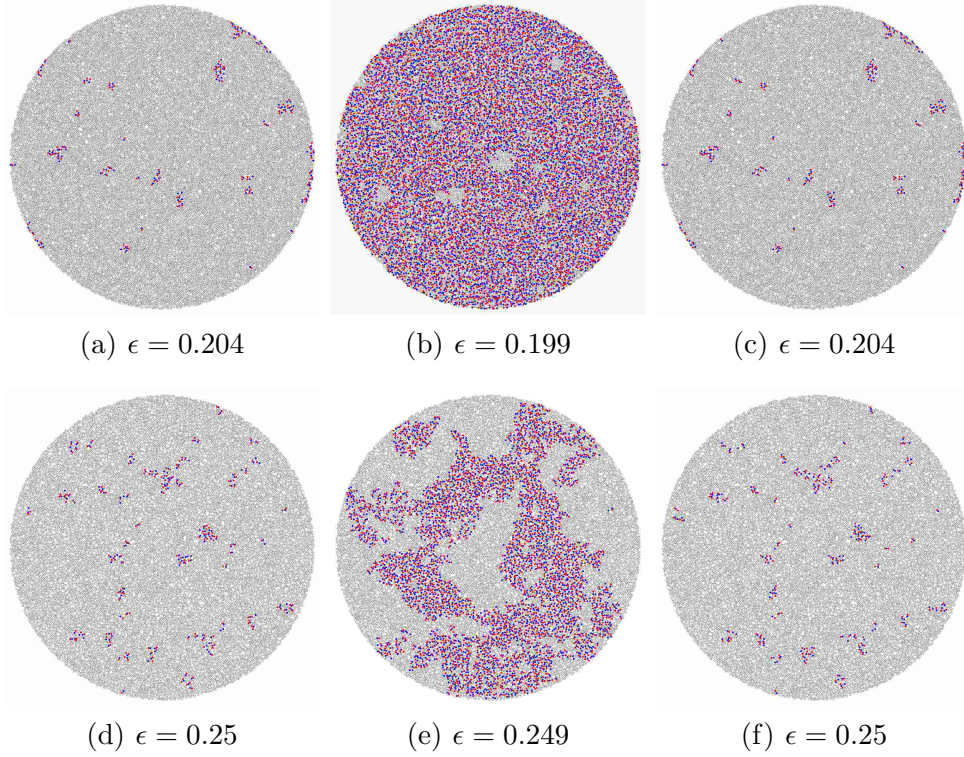


Fig. 16. Stability of oscillators under changes of ϵ : (a)–(c) $\beta = 1$, (d)–(f) $\beta = 2$.

nodes with two or four neighbors are second dominants in these skeletons.

4.4 Stability of localized oscillators

Finding 6 *Localized excitations, or oscillators, are stable under changes of relative excitability threshold.*

The finding is illustrated for $\beta = 1$ and $\beta = 2$ in Fig. 16. We choose values of threshold ϵ at the edge of excitability — $\epsilon = 0.204$ for $\beta = 1$ and $\epsilon = 0.25$ for $\beta = 2$ — and perturb β -skeleton with a random configuration of excitations. Few localized oscillators emerge (Fig. 16ad). Then we decrease excitability thresholds — dynamically, without resetting the skeleton to resting state — to $\epsilon = 0.199$ ($\beta = 1$) and $\epsilon = 0.249$ ($\beta = 2$). The oscillators start to generate excitation wave-fragments. The wave-fragments emitted by the oscillators fill the skeletons (Fig. 16be). When we reduce excitability thresholds back to their original levels 0.204 and 0.25 traveling wave-fragments disappear but oscillators remain untouched, they stay where they were before relative excitability threshold changed (Fig. 16cf).

5 Discussion

We studied discrete excitation dynamics on a smoothly parameterised family of planar proximity graphs — β -skeletons. The skeletons were constructed on a set of uniform discs packed into a larger disc. Thus each skeleton is characterised by β and density of node packing. We considered rules of absolute excitability (a resting node excites if number of its neighbors exceeds certain threshold) and relative excitability (a resting node excites if a ratio of its excited neighbors to a total number of neighbors exceeds certain threshold).

In computational experiments we found that overall level of activity in an absolutely excitable β -skeleton is proportional to node packing density and inversely proportional to β . We demonstrated that space-time dynamics of absolutely excitable β -skeletons can be classified as follows: spiral and target waves, disordered dynamics with irregularly traveling wave-fragments, slowly spreading domains of excitation, and small localized stationary domains of oscillatory activity. Transition of a skeleton between these classes is controlled by changing β and density of node packing. Both β and density affect distribution of node degrees. We provided a parameterization of the classes using just one parameter — average node degree.

There are five classes of space-time activity of the relatively excitable β -skeletons with tightly packed nodes (density 0.407, 15000 nodes): spiral and target waves, omnidirectional propagating patterns of disordered excitation, localized domains of excitations, branching domains of activity and tiny oscillators. The classes are controlled by β and the relative excitability threshold ϵ . Tiny oscillators usually emerge just below edge of excitability. The tiny oscillators are stable under relative excitability threshold. Decreasing threshold we can increase number of excited loci around the oscillators but we can not modify oscillators themselves. As soon as we increase the threshold all excitation activity but original oscillators disappear.

The class of branching domains is a transitional class occupying a part of $\beta - \epsilon$ -space between full excitability and non-excitability. With regards to propagating patterns, waves are pronounced for $\beta = 1$, and are alike classical excitation waves in discrete media. Increase of β to 1.5 causes wave-fronts to break up into separate wave-fragments.

We believe our results will find their applications in analysis of spreading crowd activities, e.g. riots, on a city's streets; distributed containment of traffic jams on motorways networks; study of foraging patterns of social insects and myxomycetes; control of excitation dynamics and propagation of defects in soft matter; design and simulation of conglomerates of simple excitable elements.

6 Acknowledgment

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References

- [1] Adamatzky A. and Holland O. Reaction-diffusion and ant-based load balancing of communication networks. *Kybernetes* 31 (2002) 667–681.
- [2] Adamatzky A. Developing proximity graphs by *Physarum Polycephalum*: Does the plasmodium follow Toussaint hierarchy? *Parallel Processing Letters* 19 (2008) 105–127.
- [3] Adamatzky A. and Jones J. Road planning with slime mould: If *Physarum* built motorways it would route M6/M74 through Newcastle *Int J Bifurcation and Chaos* (2009), in press. See also [arXiv:0912.3967v1](#).
- [4] Beavon D. J. K., Brantingham P. L. and Brantingham P. J. The influence of street networks on the patterning of property offenses. www.popcenter.org/library/CrimePrevention/Volume_02/06beavon.pdf
- [5] Billiot J. M., Corset F., Fontenas E. Continuum percolation in the relative neighborhood graph. [arXiv:1004.5292](#)
- [6] Dale M. R. T. *Spatial Analysis in Plant Ecology* (Cambridge University Press, 2000).
- [7] Dale M. R. T., Dixon P., Fortin M.-J., Legendre P., Myers D. E. and Rosenberg M. S. Conceptual and mathematical relationships among methods for spatial analysis. *Ecography* 25 (2002) 558–577.
- [8] Delaunay B. Sur la sphère vide, *Izvestia Akademii Nauk SSSR, Otdelenie Matematicheskikh i Estestvennykh Nauk*, 7 (1934) 793–800.
- [9] Gabriel K. R. and Sokal R. R. A new statistical approach to geographic variation analysis. *Systematic Zoology* 18 (1969) 259–270.
- [10] Jaromczyk J. W. and G. T. Toussaint, Relative neighborhood graphs and their relatives. *Proc. IEEE* 80 (1992) 1502–1517.
- [11] Jombart T., Devillard S., Dufour A.-B., Pontier D. Revealing cryptic spatial patterns in genetic variability by a new multivariate method. *Heredity* 101 (2008) 92–103.
- [12] Kirkpatrick D.G. and Radke J.D. A framework for computational morphology. In: Toussaint G. T., Ed., *Computational Geometry* (North-Holland, 1985) 217–248.

- [13] Legendre P. and Fortin M.-J. Spatial pattern and ecological analysis. *Vegetatio* 80 (1989) 107–138.
- [14] Li X.-Y. Application of computation geometry in wireless networks. In: Cheng X., Huang X., Du D.-Z. (Eds.) *Ad Hoc Wireless Networking* (Kluwer Academic Publishers, 2004) 197–264.
- [15] Magwene P. W. Using correlation proximity graphs to study phenotypic integration. *Evolutionary Biology*. 35 (2008) 191–198.
- [16] Matula D. W. and Sokal R. R. Properties of Gabriel graphs relevant to geographic variation research and clustering of points in the plane. *Geogr. Anal.* 12 (1980) 205–222.
- [17] Muhammad R. B. A distributed graph algorithm for geometric routing in ad hoc wireless networks. *J Networks* 2 (2007) 49–57.
- [18] Runions A., Fuhrer M., Lane B., Federl P., Rolland-Lagan A.-G., and Prusinkiewicz P. Modeling and visualization of leaf venation patterns. *ACM Transactions on Graphics* 24 (2005) 702–711.
- [19] Santi P. *Topology Control in Wireless Ad Hoc and Sensor Networks* (Wiley, 2005).
- [20] Sokal R. R. and Oden N. L. Spatial autocorrelation in biology 1. Methodology. *Biological Journal of the Linnean Society* 10 (2008) 199–228.
- [21] Song W.-Z., Wang Y., Li X.-Y. Localized algorithms for energy efficient topology in wireless ad hoc networks. In: *Proc. MobiHoc 2004* (May 24–26, 2004, Roppongi, Japan).
- [22] Sridharan M. and Ramasamy A. M. S. Gabriel graph of geomagnetic Sq variations. *Acta Geophysica* (2010) 10.2478/s11600-010-0004-y
- [23] Toroczkai Z. and Guclu H. Proximity networks and epidemics. *Physica A* 378 (2007) 68. [arXiv:physics/0701255v1](https://arxiv.org/abs/physics/0701255v1)
- [24] Wan P.-J., Yi C.-W. On the longest edge of Gabriel Graphs in wireless ad hoc networks. *IEEE Trans. on Parallel and Distributed Systems* 18 (2007) 111–125.
- [25] Watanabe D. A study on analyzing the road network pattern using proximity graphs. *J of the City Planning Institute of Japan* 40 (2005) 133–138.
- [26] Watanabe D. Evaluating the configuration and the travel efficiency on proximity graphs as transportation networks. *Forma* 23 (2008) 81–87.