

Symmetric Rendezvous With Advice: How to Rendezvous in a Disk ^{*}

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Abstract. In the classic Symmetric Rendezvous problem on a Line (SRL), two robots at known distance 2 but unknown direction execute the same randomized algorithm trying to minimize the expected rendezvous time. A long standing conjecture is that the best possible rendezvous time is 4.25 with known upper and lower bounds being very close to that value. We introduce and study a geometric variation of SRL that we call Symmetric Rendezvous in a Disk (SRD) where two robots at distance 2 have a common reference point at distance ρ . We show that even when ρ is not too small, the two robots can meet in expected time that is less than 4.25. Part of our contribution is that we demonstrate how to adjust known, even simple and provably non-optimal, algorithms for SRL, effectively improving their performance in the presence of a reference point. Special to our algorithms for SRD is that, unlike in SRL, for every fixed ρ the worst case distance traveled, i.e. energy that is used, in our algorithms is finite. In particular, we show that the energy of our algorithms is $O(\rho^2)$, while we also explore time-energy tradeoffs, concluding that one may be efficient both with respect to time and energy, with only a minor compromise on the optimal termination time.

1 Introduction

In a rendezvous game two players reside at unknown locations in a given domain and they wish to minimize the (expected) meeting (rendezvous) time. Various rendezvous problems have been studied intensively, with applications in computer science and real-world modeling, such as the search for a mate problem in which species with a low spatial density try to find suitable partners [9]. Rendezvous problems can be classified as

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asymmetric, in which each agent may use a different strategy, or *symmetric*, in which each agent follows the same algorithm; moreover, strategies can be classified as *mixed*, incorporating randomness, or *pure* which are deterministic.

In this paper, we discuss *symmetric rendezvous with advice*. Two speed-1 robots (mobile agents) start at known distance but at unknown locations and they are trying to meet (rendezvous). At any time, robots have the option to meet at a known immobile reference point that is initially placed ρ away from both agents. The goal is to design mixed strategies so as to minimize the expected rendezvous time, i.e. the expected value of the first time that robots meet. After scaling, our problem can be equivalently described as a Symmetric Rendezvous problem in a unit Disk (SRD), where mobile agents lie at the perimeter of disk at known arc distance 2α , having the option to always meet at the origin.

SRD is a geometric variation of the well-studied Symmetric Rendezvous problem on a Line (SRL) where no reference point is available, and for which a long-standing conjecture stipulates that it can be solved in expected time 4.25. Critical differences between the two problems is that in SRD (a) the rendezvous can always be realized deterministically, (b) the performance can be much better than the distance from the reference point ρ and better than the conjectured 4.25 even for not too small values of ρ and (c) the worst case rendezvous time can be bounded in ρ even when one tries to minimize the expected rendezvous time. The latter is an important property, since if the two agents are vehicles with limited fuel, our strategies can be used to guarantee rendezvous before the fuel runs out.

1.1 Related Work

The rendezvous problem is a special type of a search game where two or more agents (robots) attempt to occupy the same location at the same time in a domain. Search games and rendezvous have a long history; see [9] and [4] for a thorough introduction to the area, and [3] for a not so recent survey. The challenge of the task (search or rendezvous) is induced by limitations related to communication, coordination, synchronization, mobility, visibility, or other types of resources, whereas examples of rendezvous domains include networks, discrete nodes and geometric environments. Notably, each of the aforementioned specifications, along with combinations of them, have given rise to a long list of publications, a short representative list of which we discuss below.

The rendezvous problem was first proposed informally by Alpern [1] in 1976, and received attention due to the seminal works of Anderson and Weber [12] for discrete domains and of Alpern [2] for continuous domains. Our work is a direct generalization of the special and so-called Symmetric Rendezvous Search Problem on a Line (SRL) proposed by Alpern [2] in 1995. In that problem, two blind agents are at known distance 2 on a line, and they can perform the same synchronized randomized algorithm (with no shared randomness). The original algorithm of Alpern [2] had performance (expected rendezvous time) 5, which was later improved to 4.5678 [13], then to 4.4182 [15], then to 4.3931 [35], and finally to the best performance known of 4.2574 [28] by Han et al. Similarly, a series of proven lower bounds [8], [35] have lead to the currently best value known of 4.1520 [28].

A number of variations of SRL have been exhaustively studied, and below we mention just a few. The symmetric rendezvous problem with unknown initial distance or with partial information about it has been considered in [17] and [16]. A number of different topologies have been considered including labeled network [10], labeled line [18], ring [31], [27] (see survey monograph [30]), torus [29], planar lattice [5], and high dimensional host spaces [7]. We note here that the topology we consider in this work follows a long list studies of relevant search/rendezvous-type problems in the disk. The rendezvous problem with faulty components has been studied in [24] and [25]. Asynchronous strategies have been explored in [34] and [33]. Studied variations of robots capabilities include sense of direction [6], [14], memory [20], visibility [22], speed [26], power consumption [11] and location awareness [19]. Interesting variations of communication models between agents have been studied in [23] (whiteboards), [21] (tokens), [30] (mobile tokens), and [33] (look-compute-move model). Finally, [32] is a comprehensive survey in deterministic rendezvous in networks.

1.2 Formal Definitions, Notation & Terminology

Problem Definition In the Symmetric Rendezvous problem in a Disk (SRD) two agents (robots) are initially placed on the plane at known distance from each other but at unknown location. A common reference point O is at known distance and known location to both robots. The robots can move at speed 1 anywhere on the plane, and they detect each other only if they are at the same location, i.e. when they meet. Given that robots run the same (randomized) and synchronized algorithm, the

goal is to design trajectory movements so as to minimize the (expected) meeting, also known rendezvous, time.

The natural way to model SRD is to have robots start on the perimeter of disk, where its center serves as the common reference point. We adopt two equivalent parameterizations of the problem that arise by either normalizing robots' initial distance or the radius of the disk. In SRD^ρ the disk has radius ρ , and the robots have Euclidean distance 2, while in SRD_α robots start on the perimeter of a unit disk and their arc distance is 2α .

As we explain below, SRD^ρ is the natural extension of the well-studied rendezvous on a line problem, while SRD_α is convenient for analyzing the performance of trajectory movements. We will use both perspectives of the problem interchangeably. Clearly, the initial Euclidean distance of the two robots in SRD_α is $2 \sin(\alpha)$. Hence, after scaling the instance by $1/\sin(\alpha)$, the initial distance of the robots becomes 2, and the reference point (the origin) is at distance $\rho = 1/\sin(\alpha)$. Therefore, SRD^ρ and SRD_α are equivalent under transformation $\alpha = \arcsin(1/\rho)$. Moreover, we will silently assume that $0 < \alpha < \pi/4$ as otherwise SRD_α is degenerate, or that $\rho > \sqrt{2}$ for SRD^ρ .

The Related Rendezvous on a Line Problem In the well-studied Rendezvous problem on a Line (SRL), two robots, with the same specifications as in SRD are placed at known distance 2, but at unknown locations on the line. The objective is again to minimize the (expected) rendezvous time. Note that SRL is exactly the same as SRD^∞ .

Natural randomized algorithms for solving SRL are so-called *k-Markovian Strategies*, i.e. random processes that iterate indefinitely, so that in every iteration each robot follows a partial trajectory of total length k (or k times more than the original distance of the agents). The simplest *2-Markovian Strategy* achieves expected rendezvous time 7: each robot with probability $1/2$ moves distance 1 to the left and then to the right, back to its original position (and robot follows the symmetric trajectory to the right with the complementary probability). Note that robots meet with probability $1/4$ after time 1, and otherwise they repeat the experiment after moving distance 2. If f denotes the expected meeting time, then clearly $f = \frac{1}{4} + \frac{3}{4}(2 + f)$ from which we obtain $f = 7$.

An elegant refinement was proposed by Alpern [2] and achieves expected rendezvous time 5. In this *3-Markovian Strategy* each robot with probability $1/2$ moves distance 1 to the left, then to the right back to its original position and then further right at distance (and robot follows the

symmetric trajectory to the right with the complementary probability). This time, robots meet with probability $1/4$ after time 1, and with probability $1/4$ after time 3, otherwise they repeat the same process. If f denotes the expected meeting time, then $f = \frac{1}{4} + \frac{1}{4}3 + \frac{1}{2}(3 + f)$ from which we obtain $f = 5$. Interestingly, this is also the best possible 3-Markovian strategy.

Alpern's algorithm above is a distance-preserving algorithm, that is, after each iteration robots either meet or they preserve their original distance (but not their original locations). After a series of improvements, this idea was fruitfully generalized to *k-Markovian Strategies* by Han et al. [28] giving the best known rendezvous time 4.2574 (for $k = 15$). Notably, the best lower bound known is 4.1520 [28], which has resulted into the believable conjecture that 4.25 is the best rendezvous time possible.

Measures of Efficiency SRD and SRL can be viewed as online problems, where robots attempt to solve the problem only with partial input information. The natural measure of efficiency of any proposed *online algorithm* is the so-called competitive ratio, defined as the ratio between the (expected) online algorithm performance over the best possible performance achievable by an *offline* algorithm that knows the input. With this terminology in mind, it is immediate that Alpern's Algorithm [2] for SRL is 5-competitive, while the conjecture above stipulates that 4.25 is the best possible competitive ratio for the problem.

Using the terminology above, the best offline algorithm can solve SRD^ρ in time 1, and SRD_α in time $\sin(\alpha)$, hence for our competitive analysis we will always scale the expected performance of our randomized algorithms accordingly. As a result, the competitive ratio of our algorithms will be described by functions of ρ and α for SRD^ρ and SRD_α , respectively, that are at least 1 for all values of the parameters.

Our main goal will be to beat the psychological threshold of 4.25 for SRD^ρ , even for not too small values of ρ , demonstrating this way both the usefulness of a reference point and the effectiveness of our algorithms. In order to quantify this more explicitly, we introduce one more alternative measure of efficiency: an algorithm for SRD^ρ will be called *δ -effective*, if δ is the largest value of ρ for which the expected rendezvous time is no more than 4.25. If such ρ does not exist, i.e. if the algorithm has expected rendezvous time at least 4.25 for all $\rho > \sqrt{2}$, then we call the algorithm 0-effective. To conclude, apart from calculating the competitive ratio of our algorithms for SRD^ρ , we will complementarily comment also on the effectiveness, with the understanding that the higher their value is,

the better the algorithm is. Note for example that the naive algorithm that simply has robots go to the reference point is ρ -competitive and 4.25-effective.

Finally, we also consider the worst case performance of our algorithms that we call *energy*. Formally, the energy of a rendezvous algorithm is defined as the supremum of the time by when the rendezvous is realized with probability 1. Note that any algorithm for SRL is bound to have infinite energy, whereas we show in this paper a family of algorithms for SRD that have bounded energy.

1.3 Our Results

Techniques Outline Our main contribution is the exploration of 3-Markovian strategies for SRD. In particular, we adjust Alpern’s optimal 3-Markovian algorithm [2] so as to take advantage of the reference point. Similar to the algorithm for SRL, our algorithm uses infinitely many random bits. In each random step, robots attempt to meet twice. If the rendezvous is not realized, then the projection of their trajectory to the perimeter of the original disk has length 3, however agents reside in a smaller disk but still at the same arc-distance. Then, robots repeat the process, so that, overall, the distances of the possible meeting points to the origin are strictly decreasing, i.e. the disk is sequentially shrinking. The trajectories of the robots are determined by *two critical angles*, that determine the distance of the possible meeting points to the origin, i.e. how much the disk are shrunk.

If in each iteration, the disk is shrunk “a lot”, then robots move much more than half their Euclidean distance in order to meet, however when they repeat the experiment, they are solving a simpler problem since they are at the same arc-distance but the reference point is closer. If, on the other hand, the new disk is comparable to the original one, then robots attempt to greedily rendezvous as fast as possible, however if the meeting is not realized, robots have to solve an identical rendezvous problem (and such a strategy is bound to have a competitive ratio no better than 5, i.e. the ratio of the original SRL). Hence, the heart of the difficulty is to determine the two critical angles so that the instance that robots have to solve in each step shrinks by the right amount. Part of our contribution is that we demonstrate how to model the latter problem as a non-trivial non-linear optimization problem, which we also solve.

High Level Contributions As it is typical in online algorithmic problems, the impossibility of achieving optimal solutions is due to the un-

known input (in our case the exact location of the robots). Our work contributes toward the fundamental algorithmic question as to whether additional resources (partial information about the unknown input - in our case a reference point) could yield improved upper bounds. Not only we answer this question in the positive, and we quantify properly our findings, but our trajectories also demonstrate how a rendezvous can be realized in 2 dimensions, even though the detection visibility of the robots is one dimensional. Part of our contribution is to also demonstrate how to adjust known algorithms for SRL so as to solve SRD. In particular, our methods can be generalized and induce improved competitive ratio upper bounds when the starting rendezvous algorithm is some other k -Markovian trajectory, $k > 3$ (see [28]). However, each such adaptation requires the determination of more than two critical angles, and the induced non-linear optimization problems would be possible to solve only numerically, rather than analytically as we do in this work. At the end, our algorithms are simple, yet powerful enough to induce good performance for a wide range of SRD instances.

Discussion on Energy We also consider the *worst case* rendezvous time for our algorithms that we deliberately call *energy*. In real-life applications, robots are bound to run only for limited time due to restricted resources (e.g. fuel). Assuming that the actual energy spent (fuel burnt) by a robot is proportional to its operation time, we view the worst-case running time of our algorithms as the minimum *energy* required by the robots that ensures that the execution of the algorithm terminates successfully with probability 1. Note that in the original SRL problem, and for any feasible rendezvous strategy, there is a positive probability (though exponentially small) that the rendezvous is arbitrarily large. Given that mobile robots should have access to bounded *energy* (fuel), the probability that the rendezvous is never realized is positive. In contrast, we show that our algorithms for SRD require bounded energy, that there is a finite time by when the rendezvous is realized with probability 1. We show that this property holds true under mild conditions for our algorithms, and in particular it holds true for our algorithm that minimizes the expected rendezvous time. For the latter algorithm we show that the energy required is $\Theta(\rho^2)$. Finally, and somehow surprising, we also show that by compromising slightly on the expected termination time, the required energy becomes $\Theta(\rho)$.

Paper Organization Section 2 is devoted to the optimization problem of minimizing the expected rendezvous time. First, in Section 2.1 we introduce some simple rendezvous algorithms that are mostly used as benchmark results for what will follow. Section 2.2 introduces the first non-trivial refinement, by providing a single random bit 1-Markovian algorithm. Our observations and results of that section are later used in Section 2.3, where we discuss general 3-Markovian strategies. Our main contribution is the determination of optimal critical angles, as well as of the induced competitive ratio, and induced effectiveness. We also provide the asymptotic behavior of the critical angles, as well as the convergence to competitive ratio 5, as the distance ρ of the reference point goes to infinity. Then, in Section 3 we study the worst case rendezvous time induced by our most efficient algorithm for SRD. In particular, the main contribution of Section 3.1 is the asymptotic analysis of the worst case rendezvous time for our algorithm that is meant to minimize the expected rendezvous time, and is shown to be $\Theta(\rho^2)$. Motivated by this, we study in Section 3.2 time-energy tradeoffs. More specifically, we show that asymptotically in ρ , the expected termination time can stay optimal achieving improved but still $\Theta(\rho^2)$ energy, while only slightly suboptimal termination time allows for $\Theta(\rho)$ energy. Our expected rendezvous time positive results for SRD are summarized in Figure 1. Many of our calculations throughout the paper are assisted by computer symbolic software (MATHEMATICA), but all our results are rigorous. Appendix A contains additional technical lemmata (and their proofs) omitted altogether from the main body, and which are invoked throughout this paper.

2 Rendezvous Algorithms in a Disk

2.1 Some Immediate Benchmark Upper Bounds

First we establish some immediate positive results that can be used as benchmarks for rendezvous trajectories that we will present in subsequent sections. Recall that the naive “go-to-origin” algorithm is 4.25-effective.

The first attempt is to blindly implement the 4.2574-competitive algorithm of [28] for SRL. Indeed, given instance SRD_α , robots can be restricted to move on the perimeter of the disk. It is clear that the resulting algorithm has expected rendezvous time α , and hence competitive ratio $4.2574 \frac{\alpha}{\sin(\alpha)}$ for SRD_α (note that $\frac{\alpha}{\sin(\alpha)} \geq 1$). However, one can slightly improve upon this by making robots move along chords instead. Indeed, the algorithm of [28] for SRL has the property that robots always move and attempt to meet at integral points, assuming that one of the robots

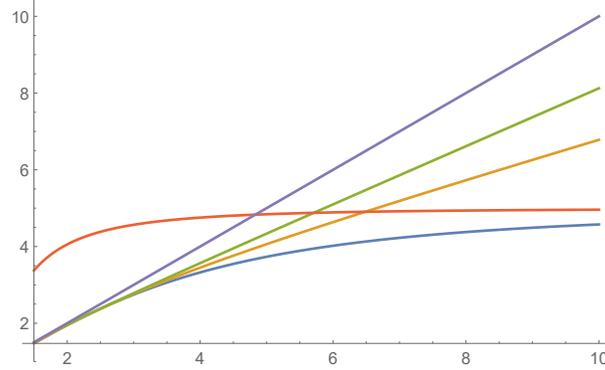


Fig. 1. A comparison between the competitive ratio of the discussed algorithms for SRD^ρ . The horizontal axis corresponds to ρ , and the vertical to the competitive ratio. The curves, along with the corresponding theorems that establish each result are as follows: purple curve is the naive “go-to-origin” ρ -competitive 4.25-effective algorithm, green curve is the 4.888-effective 1-random bit Algorithm due to Theorem 3, yellow curve is the 5.3236-effective Algorithm due to Theorem 4, red curve is the 2.57-effective Algorithm due to Theorem 2, and blue curve is the 7.1367-effective Algorithm due to Theorem 5.

starts from the origin of the real line. Now for problem SRD^ρ in the disk, and given any initial location of the robots, consider an infinite sequence of clockwise and of counterclockwise arcs of length 2, along with their corresponding chords of length $2 \sin(1)$. Any integral movement of robots in the line can be simulated by movements on the chords by multiples of $\sin(1)$, while $\sin(1)$ is also the optimal offline solution. Therefore, we immediately obtain the following.

Theorem 1. SRD^ρ admits an online algorithm which is 4.2574-competitive and 0-effective.

Next we show that Theorem 1 admits an easy refinement using a simple 3-Markovian process, which is a direct application of [2].

Theorem 2. SRD^ρ admits an online algorithm which is $\left(\frac{7\rho^2+8\sqrt{\rho^2-1}\rho-3}{3\rho^2+1}\right)$ -competitive and 2.57-effective.

Proof. We introduce the language of SRD_α . The main idea of the algorithm is that each robot iteratively tries to greedily meet her peer in the “middle point of their locations”. More specifically, in each iteration, each robot tosses a coin, which advises the robot whether her peer is at arc

distance α cw or ccw. Call the current robot's location A , say on a unit disk, and let B be a point at cw distance α . Then the robot attempts to meet her peer in the middle point M of A, B , and this succeeds with probability $1/4$. If this fails, it might be due to that the other robot was actually in the opposite direction and her random ccw move brought her at the corresponding point M' . Then the two robots attempt to meet in the middle point of M, M' , and again this meeting is realized with probability $1/4$. In the complementary event, with probability $1/2$ both robots choose to move in the same direction in their first move. Still after their second move, and given they have not met, they are still at arc-distance α , but now they reside on a smaller disk, and they repeat the process.

Denote by \mathcal{R} the expected rendezvous time of the algorithm, given that agents start on the perimeter of a radius-1 disk. If robots do not meet, they are still at arc distance α in a disk that is scaled by $\cos(\alpha)$. Therefore, their Euclidean distance in the resulting disk is $2 \cos(\alpha) \sin(\alpha) = \sin(2\alpha)$.

Notice that with probability $1/4$ robots meet at time $\sin(\alpha)$. With probability $1/4$, they meet at time $\sin(\alpha) + \frac{1}{2} \sin(2\alpha)$. Otherwise, they have already walked distance $\sin(\alpha) + \frac{1}{2} \sin(2\alpha)$, and they are at arc distance α of a radius- $\cos(\alpha)$ disk, when they repeat the process. Therefore,

$$\mathcal{R} = \sin(\alpha) + \frac{3}{4} \sin(2\alpha) + \frac{1}{2} \cos(\alpha) \mathcal{R}.$$

Solving for \mathcal{R} gives expected rendezvous time $\mathcal{R} = \frac{\sin(a)(2+3\cos(a))}{2-\cos(a)}$. Hence,

the competitive ratio for SRD_α is $\frac{2+3\cos(a)}{2-\cos(a)}$ and for SRD^ρ it is $\frac{7\rho^2+8\sqrt{\rho^2-1}\rho-3}{3\rho^2+1}$. Note that the competitive ratio becomes 4.25 exactly for $\rho = \frac{29}{\sqrt{165}} \approx 2.25765$. \square

2.2 Rendezvous with Minimal Randomness

Theorems 1 and 2 were obtained by algorithms that use infinitely many random bits. This section is devoted into showing that even with 1 random bit, we can perform better than the naive “go-to-origin” algorithm, as well as of the algorithms of Theorems 1 and 2, at least for certain values of α, ρ . This will also help as a warm-up for our later results.

Consider instance SRD_α and mobile agents at arc distance α as in Figure 2. Each of them knows that their peer is α away either clockwise or counterclockwise, and consider the corresponding arcs. Notice that in both algorithms of Theorems 1 and 2 robots attempt to meet at the bisectors of the two arcs. Given a fixed angle β , each robot, and at each iteration

chooses uniformly at random either the cw or the ccw direction, and moves in that direction with respect to the origin till the bisector is hit. We call this move a *random β -darting*. Notice that 0-darting corresponds to going to the origin, while the algorithm of Theorem 2 we have $\beta = \pi/2 - \alpha$. The main idea behind our 1-random bit algorithm 1RB with parameter β is to choose the optimal $\beta \in [0, \pi/2 - \alpha)$ that minimizes the expected termination time.

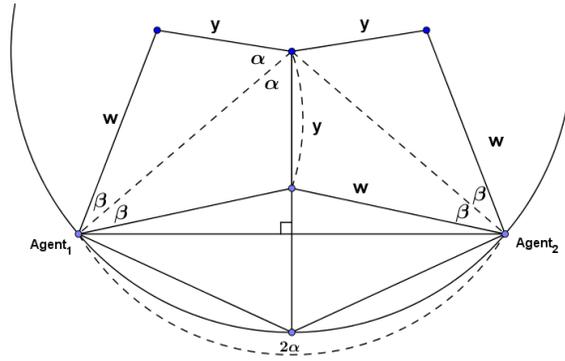


Fig. 2. Geometry of Algorithm 1RB $_{\beta}$.

Algorithm 1 1RB $_{\beta}$

- 1: Do a random β -darting.
 - 2: Go to origin (if peer is not already met).
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Lemma 1. *The expected rendezvous time $R(\beta)$ of 1RB $_{\beta}$ is*

$$\mathcal{R}(\beta) = \sin(\alpha) \csc(\alpha + \beta) + \frac{3}{4} \sin(\beta) \csc(\alpha + \beta).$$

Proof. For fixed α , let $w = w(\beta)$ be the length of the line segment between the position of a robot and the possible meeting point at the bisector of the critical arcs. Let also $y = y(\beta)$ denote the distance of the possible meeting point from the origin (see also Figure 2).

Clearly, with probability 1/4 robots move after time w , and otherwise they meet at time $y + w$. Hence

$$\mathcal{R}(\beta) = \frac{1}{4}y + \frac{3}{4}(y + w) = y + \frac{3}{4}w.$$

The proof follows by noticing that $w = \sin(\alpha) \csc(\alpha + \beta)$ and that $y = \sin(\beta) \csc(\alpha + \beta)$, which is obtained by a simple geometric argument based on the Law of sines. \square

Theorem 3. *The optimal $1RB_\beta$ algorithm uses*

$$\bar{\beta} = \max\left(0, -\sin(1/\rho) + \arccos\left(\frac{3}{4}\right)\right)$$

in which case the algorithm is $\frac{3\sqrt{\rho^2-1}+\sqrt{7}}{4}$ -competitive and 4.88813-effective.

Proof. For convenience, we analyze the performance on SRD_α instead. Using Lemma 1, we find the critical values of the expected rendezvous time $\mathcal{R}(\beta)$ by calculating

$$\frac{d}{d\beta} \mathcal{R}(\beta) = \left(\frac{3}{4} - \cos(\alpha + \beta)\right) \csc^2(\alpha + \beta) \sin(\alpha).$$

Observe that as $0 < \alpha \leq \alpha + \beta \leq \frac{\pi}{2}$ we have $\cos(\alpha + \beta) \leq \cos \alpha$ and thus, for $\alpha > \arccos \frac{3}{4}$ we see that $\mathcal{R}(\beta)$ is increasing. Hence, $\mathcal{R}(\beta)$ is minimized at $\mathcal{R}(0) = \sin \alpha \csc \alpha = 1$.

For $\alpha \leq \arccos \frac{3}{4}$, $\mathcal{R}(\beta)$ is decreasing when $\beta < \bar{\beta}$ and increasing when $\beta > \bar{\beta}$, where $\bar{\beta} = -\alpha + \arccos\left(\frac{3}{4}\right)$. Thus, $\mathcal{R}(\beta)$ is minimized at $\bar{\beta}$ and by some straightforward trigonometric calculations we see that $\mathcal{R}(\bar{\beta}) = \cos\left(\arccos\left(\frac{3}{4}\right) - \alpha\right)$. Since $\arccos \frac{3}{4} - \alpha \in [0, \frac{\pi}{2}]$, then this cost is no more than 1.

When the problem is not degenerate, we conclude that $1RB_{\bar{\beta}}$ is $\frac{\cos(\arccos(\frac{3}{4})-\alpha)}{\sin(\alpha)}$ -competitive. Our claim now follows for SRD^ρ using transformation $\alpha = \sin(1/\rho)$. Finally note that $\frac{3\sqrt{\rho^2-1}+\sqrt{7}}{4}$ is increasing, and it is equal to 4.25 when $\rho = \frac{1}{3}\sqrt{305 - 34\sqrt{7}} \approx 4.88813$. \square

2.3 Improved Rendezvous with 3-Markovian Trajectories

In this section we generalize the algorithm of Section 2 in two ways; first we allow more random bits, and second, in every random trial, we allow robots trajectories two darting attempts (recall that Algorithm $1RB_\beta$ allows for only one darting attempts. In the language of the established results for SRL we will adopt Alpern's 3-Markovian trajectory [2].

The main idea behind our new algorithms is as follows

At every random step, robots will reside at the perimeter of a disk, and they will be at constant arc distance α . As in $1RB_\beta$, each robot is associated with two bisectors in which robot will make an attempt to

meet her peer. A fixed angle β along with a random bit will determine the direction (cw or ccw) of the random β -darting that will bring the robot in one of the bisectors. Note that due to the symmetry imposed by the trajectory, a meeting is realized in this step with probability $1/4$. If the rendezvous is not realized, the robot will attempt a deterministic γ -darting to the other bisector, and the meeting is realized in this step with probability $1/4$ as well. If the rendezvous fails again, then the process repeats or robots go to the origin to meet. A process that involves k random bits (and hence $2k$ possible meeting points) will be referred to as k -step 3-Markovian. Note that we allow $k = \infty$. The formal description of the algorithm is as follows.

Algorithm 2 k -RB $_{\beta,\gamma}$

- 1: Repeat k times
 - 2: Do a random β -darting.
 - 3: Do a γ -darting in the opposite direction
 - 4: Go to origin (if peer is not already met).
-

Observe that the algorithm of Theorem 2 can be alternatively described as ∞ -RB $_{\pi/2-\alpha/2,\pi/2-\alpha/2}$, while 1RB $_{\beta}$ is equivalent to 1-RB $_{\beta,0}$. Next we analyze k -RB $_{\beta,\gamma}$ for all values of k, β, γ . Our goal is to analyze the expected rendezvous time, denoted by $\mathcal{R}_k(\beta, \gamma)$. We adopt the language either of SRD $^{\rho}$ or of SRD $_{\alpha}$ depending on what is more convenient, in which case $\mathcal{R}_k(\beta, \gamma)$ will be either a function of ρ or of α . To make this more explicit in our notation, and in order to remove any ambiguity, we will be writing $\mathcal{R}_k^{\rho}(\beta, \gamma)$ and $\mathcal{R}_k^{\alpha}(\beta, \gamma)$ for the expected running time in SRD $^{\rho}$ and SRD $_{\alpha}$, respectively. Note that $\mathcal{R}_k^{\rho}(\beta, \gamma) = \frac{\mathcal{R}_k^{\alpha}(\beta, \gamma)}{\sin(\alpha)}$.

Lemma 2. *For every fixed α , the performance of k -RB $_{\beta,\gamma}$ for SRD $_{\alpha}$, when $k = 1, \infty$, is*

$$\mathcal{R}_1^{\alpha}(\beta, \gamma) = \frac{1}{2} \csc(\alpha + \beta) (\sin(\beta) \csc(2\alpha + \gamma) (3 \sin(\alpha) \cos(\alpha) + \sin(\gamma)) + 2 \sin(\alpha)) \quad (1)$$

$$\mathcal{R}_{\infty}^{\alpha}(\beta, \gamma) = \frac{\sin(\alpha) (3 \sin(\alpha - \beta) - 3 \sin(\alpha + \beta) - 4 \sin(2\alpha + \gamma))}{-2 \cos(\alpha - \beta + \gamma) + 2 \cos(3\alpha + \beta + \gamma) + \cos(\beta - \gamma) - \cos(\beta + \gamma)} \quad (2)$$

Proof. Note that each random step of k -RB $_{\beta,\gamma}$ involves two darting moves. For fixed α , and a disk of radius 1, let $w = w(\beta)$ be the length of the line

segment between the position of a robot and the possible meeting point at the bisector of the critical arcs in the first darting move. Let also $y = y(\beta)$ denote the distance of the possible meeting point from the origin (see also Figure 3). The values for w, y are obtained as in the proof of Lemma 1 and are summarized below. Notice that after the first darting move, robots are

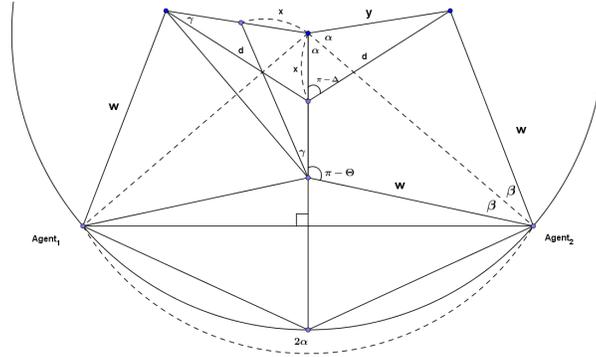


Fig. 3. Geometry of Algorithm $k\text{-RB}_{\beta, \gamma}$, where $\Theta = \alpha + \beta$ and $\Delta = 2\alpha + \gamma$.

in a disk of radius y . Similarly, let $d = d(\beta, \gamma)$ be the length of the line segment between the position of a robot after the first darting attempt and the possible meeting point at the bisector of the critical arcs in the second darting move. Let also $x = x(\beta, \gamma)$ denote the distance of the second possible meeting point from the origin. Overall, we have

$$w = \sin(\alpha) \csc(\alpha + \beta) \quad (3)$$

$$y = \sin(\beta) \csc(\alpha + \beta) \quad (4)$$

$$x = y \sin(\gamma) \csc(2\alpha + \gamma) \quad (5)$$

$$d = y \sin(2\alpha) \csc(2\alpha + \gamma). \quad (6)$$

Now consider an iteration of the algorithm, where the radius of the disk is 1. The probability of the agents meeting at this iteration is $\frac{1}{2}$, and given that they meet the distance travelled is equally likely to be w or $w + d$, giving a contribution to the mean cost equal to $w + \frac{1}{2}d$. If robots do not meet at this step, then they are at distance x from the origin. So they either go to the origin, if number of iterations has exceeded k , or they

repeat. Hence,

$$\begin{aligned}
\mathcal{R}_k^\alpha(\beta, \gamma) &= \sum_{i=0}^{k-1} \left(\frac{1}{2}\right)^{i+1} \left[x^i(w + \frac{1}{2}d) + \sum_{j=0}^{i-1} (w+d)x^j \right] + \left(\frac{1}{2}\right)^k \left(x^k + \sum_{j=0}^{k-1} (w+d)x^j \right) \\
&= \sum_{i=0}^{k-1} \left(\frac{1}{2}\right)^{i+1} \left[x^i(w + \frac{1}{2}d) + (w+d)\frac{x^{i+1}-x}{x^2-x} \right] + \left(\frac{1}{2}\right)^k \left(x^k + (w+d)\frac{x^k-1}{(x-1)} \right) \\
&= \left(\frac{1}{2}\right)^{k+1} \frac{x^k(3d+4w+2x-4) - 2^k(3d+4w)}{x-2},
\end{aligned}$$

Setting $k = 1$ and taking the limit $k \rightarrow \infty$ (note that $0 < x < 1$) gives

$$\mathcal{R}_1^\alpha(\beta, \gamma) = w + \frac{3}{4}d + \frac{1}{2}x,$$

and

$$\mathcal{R}_\infty^\alpha(\beta, \gamma) = \frac{3d+4w}{2(2-x)}.$$

Then, the statement of the Lemma follows after elementary trigonometric manipulations. \square

Theorem 4. Consider problem SRD $^\rho$. If $\rho < \csc\left(\frac{1}{2}\cos^{-1}\left(\frac{2}{3}\right)\right) \approx 2.44949$, then the optimal 1-RB $_{\beta,\gamma}$ algorithm is obtained for $\bar{\gamma} = 0$, and the algorithm is identical to the optimal 1RB $_\beta$ algorithm (see Theorem 3).

If $\rho \geq \csc\left(\frac{1}{2}\cos^{-1}\left(\frac{2}{3}\right)\right)$, then the optimal 1-RB $_{\beta,\gamma}$ is obtained for the following parameters

$$\begin{aligned}
\bar{\gamma} &= \cos^{-1}\left(\frac{2}{3}\right) - 2\sin^{-1}\left(\frac{1}{\rho}\right) \\
\bar{\beta} &= \cos^{-1}\left(\frac{3}{4}\cos\left(\cos^{-1}\left(\frac{2}{3}\right) - 2\sin^{-1}\left(\frac{1}{\rho}\right)\right)\right) - \sin^{-1}\left(\frac{1}{\rho}\right)
\end{aligned}$$

For the optimal parameters, the algorithm has competitive ratio $\cos(\bar{\beta})$ which equals

$$\frac{1}{2} \left(-\frac{\sqrt{5}}{\rho^2} + \sqrt{\rho^2-1} - \frac{2\sqrt{\rho^2-1}}{\rho^2} + 2\sqrt{1 - \frac{\left(\rho\left(\sqrt{5-\frac{5}{\rho^2}} + \rho\right) - 2\right)^2}{4\rho^4}} + \sqrt{5} \right)$$

and it is 5.32366-effective.

Proof. Lemma 2 gives us performance $\mathcal{R}_1^\alpha(\beta, \gamma)$ of 1-RB $_{\beta, \gamma}$ for problem SRD $_\alpha$, and the competitive ratio is obtained by scaling by $\sin(\alpha)$. The critical points of $\mathcal{R}_1^\alpha(\beta, \gamma)$ are $\bar{\beta}, \bar{\gamma}$ satisfying equations

$$\bar{\gamma} = \arccos \frac{2}{3} - 2\alpha \quad (7)$$

$$\bar{\beta} = \arccos\left(\frac{3}{4} \cos(\bar{\gamma})\right) - \alpha, \quad (8)$$

as shown in Lemmata 7, 8. These equations have unique solutions in $[0, \frac{\pi}{2} - \alpha]$ if and only if $\alpha \leq \frac{1}{2} \arccos \frac{2}{3}$; otherwise, they have no solution. As $\mathcal{R}_1^\alpha(\beta, \gamma)$ has at most one critical point and is locally convex at that point (see Lemma 9), these equations minimize $\mathcal{R}_1^\alpha(\beta, \gamma)$ for $\alpha < \frac{1}{2} \arccos \frac{2}{3}$. Now we substitute (7) and (8) in (1) to obtain, after straightforward manipulations, that at its minimum $\mathcal{R}_1^\alpha(\bar{\beta}, \bar{\gamma}) = \cos \bar{\beta}$.

For $\alpha \geq \frac{1}{2} \arccos \frac{2}{3}$, $\mathcal{R}_1^\alpha(\beta, \gamma)$ is monotone increasing with respect to γ , and thus is optimized at $\gamma = 0$, at which the strategy becomes identical to the one-step algorithm described before. Optimal parameters and run times can be calculated accordingly for SRD $^\rho$ using transformation $\alpha = \arcsin(1/\rho)$ and simplifying trigonometric expressions. \square

We can now compute also the optimal parameters for ∞ -RB $_{\beta, \gamma}$. Since the competitive ratio becomes a lengthy expression in ρ for SRD $^\rho$, we choose to only comment on the effectiveness of the resulting algorithm. The competitive ratio will be explicit from our calculations.

Theorem 5. *For all $\rho \geq 1/\sin(1/2) \approx 2.08583$, the optimal ∞ -RB $_{\beta, \gamma}$ algorithm for SRD $^\rho$ uses parameters $\bar{\beta}, \bar{\gamma}$ satisfying equations*

$$\frac{3}{4} \cos(\bar{\gamma}) = \cos(\arcsin(1/\rho) + \bar{\beta}) \quad (9)$$

$$\frac{2}{3} \cos(\bar{\beta}) = \cos(2 \arcsin(1/\rho) + \bar{\gamma}). \quad (10)$$

In particular, we have

$$\bar{\beta} := \arctan \left(\frac{-v + \sqrt{v^2 - \left(\frac{9}{4} \cos^2 \alpha - 1\right) \left(\frac{5}{4} - v^2\right)}}{\frac{9}{4} \cos^2 \alpha - 1} \right) \quad (11)$$

$$\bar{\gamma} := \arccos \left(\frac{4}{3} \cos(\alpha + \bar{\beta}) \right). \quad (12)$$

where $v := (2 \cos \alpha - \cos 2\alpha) \csc 2\alpha$ and $\alpha = \arcsin(1/\rho)$. The competitive ratio of the algorithm can be computed by substituting $\bar{\beta}, \bar{\gamma}$ in (2). Also for these values of $\bar{\beta}, \bar{\gamma}$, the algorithm is 7.13678-effective.

Proof. For convenience we adopt the language of SRD_α . The nonlinear system (9), (10) characterizes the critical points of function $\mathcal{R}_\infty^\alpha(\beta, \gamma) : \mathbb{R}^2 \mapsto \mathbb{R}$, i.e. it is obtained by requiring that

$$\frac{\partial}{\partial \beta} \mathcal{R}_\infty^\alpha(\beta, \gamma) = \frac{\partial}{\partial \gamma} \mathcal{R}_\infty^\alpha(\beta, \gamma) = 0.$$

We prove this in Lemma 10.

Now observe that equations (9), (10) is just a system of polynomial equations in $\cos(\beta)$, $\cos(\gamma)$. In fact, substituting one for the other results in a degree 4 polynomial equation that can be solved analytically. Only one of the solutions satisfies conditions $0 \leq \beta \leq \pi/2 - \alpha$, which is the $\bar{\beta} = \bar{\beta}(\alpha)$ described in (11). The value of $\bar{\gamma}$ is calculated using (9) as $\bar{\gamma} := \arccos\left(\frac{4}{3} \cos(\alpha + \bar{\beta})\right)$.

For all $\alpha < 3/4$, we show in Lemma 11 that $0 \leq \bar{\beta}, \bar{\gamma} \leq \pi/2 - \alpha$. Finally, in Lemma 12 we show that, for all $\alpha < 1/2$, the aforementioned values of $\bar{\beta}, \bar{\gamma}$ do indeed correspond to a minimizer for $\mathcal{R}_\infty^\alpha(\beta, \gamma)$ by showing that $\nabla^2 \mathcal{R}_\infty^\alpha(\bar{\beta}, \bar{\gamma})$ is positive definite.

Overall, we conclude that $\bar{\beta}, \bar{\gamma}$ do minimize $\mathcal{R}_\infty^\alpha(\beta, \gamma)$, in which case the competitive ratio becomes $\mathcal{R}_\infty^\alpha(\bar{\beta}, \bar{\gamma}) / \sin(\alpha)$. Equating the last expression with 4.25, and solving for $\rho = 1/\sin(\alpha)$ gives numerical value $\rho = 7.13678$. \square

We conclude this section by providing some asymptotic analysis for the optimal parameters $\bar{\beta}, \bar{\gamma}$ of Algorithm $\infty\text{-RB}_{\beta, \gamma}$ as $\rho \rightarrow \infty$. As expected, both $\bar{\beta}, \bar{\gamma}$ tend to $\pi/2$, as well as $\mathcal{R}_\infty^\rho(\bar{\beta}, \bar{\gamma})$ tends to 5 (the competitive ratio of the SRL algorithm we are extending). This is what we make explicit with the next theorem, by also providing the rate of convergence.

Theorem 6. *For the optimal parameters $\bar{\beta} = \bar{\beta}(\rho), \bar{\gamma} = \bar{\gamma}(\rho)$ of Algorithm $\infty\text{-RB}_{\beta, \gamma}$, we have*

$$\lim_{\rho \rightarrow \infty} \frac{\pi/2 - \bar{\beta}}{\arcsin(1/\rho)} = 5 \tag{13}$$

$$\lim_{\rho \rightarrow \infty} \frac{\pi/2 - \bar{\gamma}}{\arcsin(1/\rho)} = \frac{16}{3} \tag{14}$$

Moreover,

$$\lim_{\rho \rightarrow \infty} \rho^2(5 - \mathcal{R}_\infty^\rho(\bar{\beta}, \bar{\gamma})) = 289/6.$$

Proof. We use the language of SRD_α , and in particular we consider $\bar{\beta} = \bar{\beta}(\alpha), \bar{\gamma} = \bar{\gamma}(\alpha)$, and $\alpha \rightarrow 0$. By Theorem 5, and using (11), it is easy to see

that $\lim_{\alpha \rightarrow 0} \bar{\beta}(\alpha) = \pi/2$. Some straightforward but tedious calculations also show that $\lim_{\alpha \rightarrow 0} \frac{\pi/2 - \bar{\beta}(\alpha)}{\alpha} = 5$. The statement for $\bar{\gamma}$ follows similarly using again Theorem 5 and in particular that $\bar{\gamma} = \arccos\left(\frac{4}{3} \cos(\alpha + \bar{\beta})\right)$.

Note that the expected rendezvous time $\mathcal{R}_\infty^\rho(\bar{\beta}, \bar{\gamma})$ in SRD^ρ is also the competitive ratio of the problem. In the language of SRD_α the competitive ratio is $\mathcal{R}_\infty^\alpha(\bar{\beta}, \bar{\gamma}) / \sin(\alpha)$.

By (13) we know that as α tends to 0, $\bar{\beta}$ behaves similar to $\pi/2 - 5\alpha$, and by (14) that $\bar{\gamma}$ behaves similar to $\pi/2 - \frac{16}{3}\alpha$. Using now (2) of Lemma 2, we have that

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \frac{\mathcal{R}_\infty^\alpha(\bar{\beta}, \bar{\gamma})}{\sin(\alpha)} &= \lim_{\alpha \rightarrow 0} \frac{\mathcal{R}_\infty^\alpha(\pi/2 - 5\alpha, \pi/2 - \frac{16}{3}\alpha)}{\sin(\alpha)} \\ &= \lim_{\alpha \rightarrow 0} \frac{-4 \cos\left(\frac{10\alpha}{3}\right) - 3(\cos(4\alpha) + \cos(6\alpha))}{\cos\left(\frac{\alpha}{3}\right) - 2\left(\cos\left(\frac{2\alpha}{3}\right) + \cos\left(\frac{22\alpha}{3}\right)\right) + \cos\left(\frac{31\alpha}{3}\right)} = 5. \end{aligned}$$

Similarly, we can show that $\lim_{\alpha \rightarrow 0} \frac{5 - \frac{\mathcal{R}_\infty^\alpha(\bar{\beta}, \bar{\gamma})}{\sin(\alpha)}}{\sin^2(\alpha)} = 289/6$. \square

3 Energy-Efficient Rendezvous

3.1 Energy Analysis of our Infinite-Step Rendezvous Algorithm

A unique feature of the SRD problem is that, unlike in SRL, the worst case rendezvous time can be finite. As before we distinguish whether we calculate the energy of $\infty\text{-RB}_{\beta, \gamma}$ in SRD^ρ or in SRD_α by writing $\mathcal{E}_\infty^\rho(\beta, \gamma)$ and $\mathcal{E}_\infty^\alpha(\beta, \gamma)$, respectively.

Lemma 3. *The energy $\mathcal{E}_\infty^\alpha(\beta, \gamma)$ of $\infty\text{-RB}_{\beta, \gamma}$ for SRD_α is finite if and only if $\sin(\beta) \sin(\gamma) < \sin(\alpha + \beta) \sin(2\alpha + \gamma)$. Moreover*

$$\mathcal{E}_\infty^\alpha(\beta, \gamma) := \frac{\sin(\alpha) \csc(\alpha + \beta) + \sin(\beta) \csc(\alpha + \beta) \sin(2\alpha) \csc(2\alpha + \gamma)}{1 - \sin(\beta) \csc(\alpha + \beta) \sin(\gamma) \csc(2\alpha + \gamma)}. \quad (15)$$

Proof (of Lemma 3). For convenience, we analyze the performance for SRD_α . As in the proof of Lemma 2 (see also Figure 3), in every iteration of $\infty\text{-RB}_{\beta, \gamma}$ agents walk a distance equal to $w + d$ and the radius of their disk is shrunk by x , so that the energy of $\infty\text{-RB}_{\beta, \gamma}$ is calculated as

$$(w + d) \sum_{j=0}^{\infty} x^j,$$

where w, x, d are as in (3), (5) and (6), respectively. Clearly, the sum of the energy converges if and only if $x < 1$, or equivalently

$$\frac{\sin(\beta) \sin(\gamma)}{\sin(\alpha + \beta) \sin(2\alpha + \gamma)} < 1.$$

When the energy sum converges, it equals

$$\frac{w + d}{1 - x}.$$

Now we use (3), (5), (6), and the claim follows. \square

Lemma 4. *For any fixed ρ , the energy $\mathcal{E}_\infty^\rho(\bar{\beta}, \bar{\gamma})$ of the optimal ∞ -RB $_{\bar{\beta}, \bar{\gamma}}$ is finite.*

Proof. Translating Theorem 5 to the language of SRD $_\alpha$ we know that parameters $\bar{\beta}, \bar{\gamma}$ satisfy

$$\frac{3}{4} \cos(\bar{\gamma}) = \cos(\alpha + \bar{\beta})$$

and

$$\frac{2}{3} \cos(\bar{\beta}) = \cos(2\alpha + \bar{\gamma})$$

or equivalently that

$$\sin(\alpha + \bar{\beta}) = \sqrt{1 - \frac{9}{16} \cos^2(\bar{\gamma})},$$

and

$$\sin(2\alpha + \bar{\gamma}) = \sqrt{1 - \frac{4}{9} \cos^2(\bar{\beta})}.$$

But then, it is immediate that $\sin(\alpha + \bar{\beta}) > \sin(\bar{\gamma})$ and that $\sin(2\alpha + \bar{\gamma}) > \sin(\bar{\beta})$. Multiplying side-wise the latter two inequalities shows that the condition of Lemma 3 is satisfied. Hence, the energy of ∞ -RB $_{\bar{\beta}, \bar{\gamma}}$ is finite for every ρ . \square

Using values $\bar{\beta}, \bar{\gamma}$ (see (11) and (12) of Theorem 5), and substituting in (15) of Lemma 3 we obtain an explicit, yet complicated, function of α (or equivalently of $\rho = 1/\sin(\alpha)$) for $\mathcal{E}_\infty^\rho(\bar{\beta}, \bar{\gamma})$. Using MATHEMATICA we can observe graphically that $\mathcal{E}_\infty^\rho(\bar{\beta}, \bar{\gamma})$ is strictly increasing (which is also expected), and that $\mathcal{E}_\infty^\rho(\bar{\beta}, \bar{\gamma})/\rho^2$ is strictly decreasing in $\rho > 2$. However a formal proof is eluding us due to the complication of the formulas. Nevertheless, we can find the asymptotic behaviour of the energy as ρ tends to infinity.

Theorem 7. For the optimal parameters $\bar{\beta} = \bar{\beta}(\rho), \bar{\gamma} = \bar{\gamma}(\rho)$ of Algorithm ∞ -RB $_{\beta,\gamma}$, we have

$$\lim_{\rho \rightarrow \infty} \frac{\mathcal{E}_{\infty}^{\rho}(\bar{\beta}, \bar{\gamma})}{\rho^2} = \frac{18}{79}.$$

Proof. We adopt the language of SRD $_{\alpha}$, and we invoke Theorem 6. By (13) we know that as α tends to 0, $\bar{\beta}$ behaves similar to $\pi/2 - 5\alpha$, and by (14) that $\bar{\gamma}$ behaves similar to $\pi/2 - \frac{16}{3}\alpha$.

Using now (2) of Lemma 2, we have that

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \mathcal{E}_{\infty}^{\alpha}(\bar{\beta}, \bar{\gamma}) \sin(\alpha) &= \lim_{\alpha \rightarrow 0} \mathcal{E}_{\infty}^{\alpha}(\pi/2 - 5\alpha, \pi/2 - \frac{16}{3}\alpha) \sin(\alpha) \\ &= \lim_{\alpha \rightarrow 0} \frac{\sin(\alpha) (\sin(\alpha) + \sin(2\alpha) \cos(5\alpha) \sec(\frac{10\alpha}{3}))}{\cos(4\alpha) - \cos(5\alpha) \cos(\frac{16\alpha}{3}) \sec(\frac{10\alpha}{3})}. \end{aligned}$$

The latter limit can be computed in MATHEMATICA and it equals 18/79. \square

An immediate corollary of Theorem 7 is that $\mathcal{E}_{\infty}^{\rho}(\bar{\beta}, \bar{\gamma}) = \Theta(\rho^2)$. As long as the rendezvous between the two agents is not realized, both follow random-walk-like trajectories (see Figure 4).

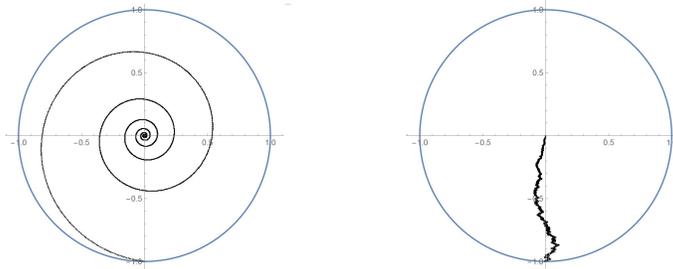


Fig. 4. Possible trajectories of one agent in Algorithm ∞ -RB $_{\beta,\gamma}$ in SRD $_{\alpha}$ when $\alpha = 0.01$. The figure on the left depicts the trajectory in which the agent always attempts to meet her peer first by moving ccw and then cw, resulting in a spiral. The figure on the right depicts a random trajectory. Both trajectories have the same length which is approximately 22.7911.

3.2 Expected Rendezvous Time - Energy Tradeoffs

In this section we attempt to understand how energy constraints can impact the performance of ∞ -RB $_{\beta,\gamma}$. By Theorem 6 we know that the op-

timal ∞ -RB $_{\bar{\beta}, \bar{\gamma}}$ Algorithm induces competitive ratio 5, asymptotically in $\rho \rightarrow \infty$. By Theorem 7 we know that the same algorithm (with the same parameters) requires $\Theta(\rho^2)$ energy. In the other extreme, if the energy is less than ρ , then the problem admits no solution (and if the energy equals ρ , then the best rendezvous is attained when robots go directly to the reference point). Hence, we are motivated to study the problem of minimizing the expected rendezvous time in SRD $^\rho$ given that agents' energy is between ρ and $\frac{18}{79}\rho^2$. Somehow surprisingly, we show below that for every $\epsilon > 0$ we can preserve a competitive ratio of 5 and energy no more than $\epsilon\rho^2 + o(\rho^2)$ or competitive ratio $5 + \epsilon$ and energy no more than $\frac{2}{\sqrt{\epsilon}}\rho + o(\rho)$, both asymptotically in ρ .

Theorem 8. *The following claims are true asymptotically for SRD $^\rho$ as $\rho \rightarrow \infty$. For every $\epsilon > 0$, there exist β_1, γ_1 so that the competitive ratio of ∞ -RB $_{\beta_1, \gamma_1}$ is 5, as well as $\mathcal{E}_\infty^\rho(\beta_1, \gamma_1) / \rho^2 \leq \epsilon$. Moreover, for every $\delta > 0$, there exist β_2, γ_2 so that the competitive ratio of ∞ -RB $_{\beta_2, \gamma_2}$ is $5 + \delta$, as well as $\mathcal{E}_\infty^\rho(\beta_1, \gamma_1) / \rho \leq 2/\sqrt{\delta}$.*

The two claims of Theorem 8 follow directly from the two lemmata below. In particular, Lemma 5 shows that the competitive ratio of ∞ -RB $_{\beta, \gamma}$ can stay 5, even if the energy needed to solve SRD $^\rho$ is $\epsilon\rho^2$, for each $\epsilon > 0$. Lemma 6 shows that if one is willing to have competitive ratio $5 + \epsilon$, then that would be possible with linear energy in ρ , and in particular no more than $2\rho/\sqrt{\epsilon}$, again for every $\epsilon > 0$.

Lemma 5. *For every positive $\epsilon > 0$, there exist β, γ , such that for the performance of ∞ -RB $_{\beta, \gamma}$ for SRD $^\rho$, we have that*

$$\lim_{\rho \rightarrow \infty} \rho (\mathcal{R}_\infty^\rho(\beta, \gamma) - 5) = \frac{27}{11\epsilon^2} - \Theta(1/\epsilon),$$

and

$$\lim_{\rho \rightarrow \infty} \frac{\mathcal{E}_\infty^\rho(\beta, \gamma)}{\rho^2} = \epsilon.$$

Proof. We use the language of SRD $_\alpha$. For some positive constants k, m , we use $\beta = \pi/2 - k\alpha$ and $\gamma = \pi/2 - m\alpha$. First, using Lemma 2, it is easy to see that

$$\begin{aligned} & \lim_{\alpha \rightarrow 0} \frac{\mathcal{R}_\infty^\alpha(\beta, \gamma)}{\sin(\alpha)} \\ &= \lim_{\alpha \rightarrow 0} \frac{-6 \cos(\alpha) \cos(\alpha k) - 4 \cos(\alpha(m-2))}{\cos(\alpha(k-m)) - 2(\cos(\alpha(k-m+1)) + \cos(\alpha(k+m-3))) + \cos(\alpha(k+m))} \\ &= 5. \end{aligned}$$

Some more elaborate calculations can show in fact that

$$\lim_{\alpha \rightarrow 0} \frac{\frac{\mathcal{R}_\infty^\alpha(\beta, \gamma)}{\sin(\alpha)} - 5}{\sin^2(\alpha)} = k^2 - 10k + \frac{3m^2}{2} - 16m + \frac{39}{2} \quad (16)$$

By the proof of Lemma 3

$$\begin{aligned} & \lim_{\alpha \rightarrow 0} \sin(\alpha) \mathcal{E}_\infty^\alpha(\beta, \gamma) \\ &= \lim_{\alpha \rightarrow 0} \frac{\sin^2(\alpha) \csc\left(\frac{\alpha}{2}\right) (\cos(\alpha(k+1)) + \cos(\alpha - \alpha k) + \cos(\alpha(m-2)))}{2 \cos\left(\frac{\alpha}{2}\right) \sin(\alpha(k+m-1)) - 2 \cos(\alpha - \alpha k) \sin\left(\alpha\left(\frac{3}{2} - m\right)\right)} \\ &= \frac{6}{2k + 4m - 5}. \end{aligned} \quad (17)$$

The claim follows by choosing $k = \frac{31\epsilon + 18}{22\epsilon}$ and $m = \frac{6\epsilon + 12}{11\epsilon}$. These values are obtained by requiring that (17) equals ϵ and minimizing (16). In particular, substituting m, k in (16) we obtain

$$\lim_{\alpha \rightarrow 0} \frac{\frac{\mathcal{R}_\infty^\alpha(\beta, \gamma)}{\sin(\alpha)} - 5}{\sin^2(\alpha)} = \frac{27}{11\epsilon^2} - \frac{237}{11\epsilon} - \frac{39}{44}.$$

Finally, substituting m, k in (17) we obtain

$$\lim_{\alpha \rightarrow 0} \sin(\alpha) \mathcal{E}_\infty^\alpha(\beta, \gamma) = \epsilon.$$

□

Lemma 6. *For every positive $\epsilon > 0$, there exist β, γ , such that for the performance of ∞ -RB $_{\beta, \gamma}$ for SRD $^\rho$, we have that*

$$\lim_{\rho \rightarrow \infty} \mathcal{R}_\infty^\rho(\beta, \gamma) = 5 + \epsilon,$$

and

$$\lim_{\rho \rightarrow \infty} \frac{\mathcal{E}_\infty^\rho(\beta, \gamma)}{\rho} \leq \frac{2}{\sqrt{\epsilon}}.$$

Proof. We start with some simple observations. Using Lemma 2 we have

$$\lim_{\alpha \rightarrow 0} \frac{\mathcal{R}_\infty^\alpha(\beta, \gamma)}{\sin(\alpha)} = \lim_{\alpha \rightarrow 0} \frac{\csc(\alpha + \beta)(3 \cos(\alpha) \sin(\beta) \csc(2\alpha + \gamma) + 2)}{2 - \sin(\beta) \sin(\gamma) \csc(\alpha + \beta) \csc(2\alpha + \gamma)} = \frac{2}{\sin(\beta)} + \frac{3}{\sin(\gamma)}$$

By the proof of Lemma 3 and after simple manipulations, we have

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \mathcal{E}_\infty^\alpha(\beta, \gamma) &= \lim_{\alpha \rightarrow 0} \frac{2 \cos\left(\frac{\alpha}{2}\right) (-\sin(\alpha - \beta) + \sin(\alpha + \beta) + \sin(2\alpha + \gamma))}{(2 \cos(\alpha) + 1) \sin\left(\frac{3\alpha}{2} + \beta + \gamma\right) - \sin\left(\frac{\alpha}{2} - \beta + \gamma\right)} \\ &= \frac{\sin(\gamma) + 2 \sin(\beta)}{\cos(\beta) \sin(\gamma) + 2 \cos(\gamma) \sin(\beta)} \end{aligned}$$

Now use abbreviation $b = \sin(\beta)$ and $c = \sin(\gamma)$. Set $b = c = \frac{5}{5+\epsilon}$, and observe that

$$\lim_{\alpha \rightarrow 0} \frac{\mathcal{R}_{\infty}^{\alpha}(\beta, \gamma)}{\sin(\alpha)} = 5 + \epsilon$$

while

$$\lim_{\alpha \rightarrow 0} \mathcal{E}_{\infty}^{\alpha}(\beta, \gamma) = \frac{\epsilon + 5}{\sqrt{\epsilon(\epsilon + 10)}} \leq \frac{2}{\sqrt{\epsilon}}.$$

Finally, we note that we can achieve the same competitive ratio, and slightly improve the required energy. For this, we need to set alternatively $b = \frac{2}{\lambda\epsilon+2}$ and $c = \frac{3}{-\lambda\epsilon+\epsilon+3}$. For each $\lambda \in [0, 1]$ it is easy to see that $2/b + 3/c = 5 + \epsilon$. Choosing also $\lambda = 3/11$ minimizes the energy, which becomes

$$\lim_{\alpha \rightarrow 0} \mathcal{E}_{\infty}^{\alpha}(\beta, \gamma) = \frac{41\epsilon + 198}{3\sqrt{3}\sqrt{\epsilon(3\epsilon + 44)} + 16\sqrt{\epsilon(4\epsilon + 33)}}.$$

□

4 Conclusion

We introduced and studied a new geometric variant of symmetric rendezvous that we call Symmetric Rendezvous in a Disk (SRD). Our main contribution pertains to the algorithmic reduction of known suboptimal algorithms for the classic Symmetric Rendezvous problem on a Line (SRL) to SRD. Since SRD can also be interpreted as a variant of SRL in which agents are equipped with additional advice, our results demonstrate how this advice can be beneficial to the expected rendezvous time, beating in some cases the conjectured best possible time for SRL. Special to SRD is also that, unlike in SRL, our algorithms induce bounded worst case (energy) performance. Motivated by this, we also studied energy-efficiency tradeoffs, and we showed that, somehow surprisingly, one can achieve rendezvous with limited energy (and with probability 1) by compromising only slightly on the expected rendezvous time.

Our techniques can be generalized for all known improved rendezvous protocols for SRL, however optimal reductions will be challenging to obtain. Nevertheless, it is interesting to investigate heuristic reductions, which we leave as an open research direction. Other interesting variants of our problem include the introduction of more agents, or relaxations of the notion of advice that we are using.

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A Omitted Lemmata (and Their Proofs)

Lemma 7. *Let $(\bar{\beta}, \bar{\gamma})$ be a critical point of \mathcal{R}_1^α , and set $\Delta = 2\alpha + \bar{\gamma}$. Then $\cos \Delta = \frac{2}{3}$.*

Proof. Let $\bar{\beta}, \bar{\gamma}$ be a critical point of \mathcal{R}_1^α . Then

$$\frac{\partial \mathcal{R}_1^\alpha(\bar{\beta}, \bar{\gamma})}{\partial \gamma} = \frac{d \csc \Delta}{4} (-3 \cos \Delta + 2) = 0,$$

and thus as d is nonzero, it must be true that $\cos \Delta = \frac{2}{3}$. □

Lemma 8. Any critical point $\bar{\beta}, \bar{\gamma}$ of \mathcal{R}_1^α satisfies

$$\cos(\Theta) = \frac{3}{4} \cos(\bar{\gamma}), \quad (18)$$

where $\Theta = \alpha + \bar{\beta}$.

Proof. Let $\bar{\beta}, \bar{\gamma}$ be a critical point of \mathcal{R}_1^α . Then

$$\frac{\partial \mathcal{R}_1^\alpha(\bar{\beta}, \bar{\gamma})}{\partial \beta} = w \csc \beta \left(-y \cos \Theta + \frac{3}{4}d + \frac{1}{2}x \right) = 0,$$

and thus $\cos \Theta = \frac{3}{4} \left(\frac{d}{y} + \frac{2x}{3y} \right) = \frac{3}{4} \left(\sin 2\alpha \csc \Delta + \frac{2}{3} \sin \gamma \csc \Delta \right)$. Substituting in $\frac{2}{3} = \cos \Delta$ (by Lemma 7) gives

$$\cos \Theta = \frac{3}{4} (\sin 2\alpha + \cos \Delta \sin \gamma) \csc \Delta,$$

which simplifies to $\cos \Theta = \frac{3}{4} \cos \gamma$. \square

Lemma 9. The critical points $\bar{\beta}, \bar{\gamma}$ of Lemma 8 are local minima of \mathcal{R}_1^α .

Proof. Let $\bar{\beta}, \bar{\gamma}$ be a critical point of \mathcal{R}_1^α . Now, taking the second derivative of $\mathcal{R}_1^\alpha(\bar{\beta}, \bar{\gamma})$ with respect to γ gives

$$\frac{\partial^2 \mathcal{R}_1^\alpha(\bar{\beta}, \bar{\gamma})}{\partial \gamma^2} = d \left(\frac{3}{4} \cot^2 \Delta - \csc \Delta \cot \Delta + \frac{3}{4} \csc^2 \Delta \right),$$

the right hand side of which simplifies to $2d \cot \Delta \left(-\frac{1}{2} \csc \Delta + \frac{3}{4} \cot \Delta \right) + \frac{3}{4}d$. Observe that $-\frac{1}{2} \csc \Delta + \frac{3}{4} \cot \Delta = \frac{\partial \mathcal{R}_1^\alpha(\bar{\beta}, \bar{\gamma})}{\partial \gamma} = 0$ at a critical point, and thus

$$\frac{\partial^2 \mathcal{R}_1^\alpha(\bar{\beta}, \bar{\gamma})}{\partial \gamma^2} = \frac{3}{4}d.$$

Now,

$$\frac{\partial \mathcal{R}_1^\alpha(\bar{\beta}, \bar{\gamma})}{\partial \beta} = w \left(-\cot \Theta + \left(\frac{3}{4}d + \frac{1}{2}x \right) \csc \beta \right),$$

which simplifies to

$$\frac{\partial \mathcal{R}_1^\alpha(\bar{\beta}, \bar{\gamma})}{\partial \beta} = w \csc \beta \left(\mathcal{R}_1^\alpha(\bar{\beta}, \bar{\gamma}) - \cos \beta \right).$$

Differentiating and rearranging once more obtains

$$\frac{\partial^2 \mathcal{R}_1^\alpha(\bar{\beta}, \bar{\gamma})}{\partial \beta^2} = -2 \cot \Theta \frac{\partial \mathcal{R}_1^\alpha(\bar{\beta}, \bar{\gamma})}{\partial \beta} + w = w$$

Similarly,

$$\frac{\partial^2 \mathcal{R}_1^\alpha(\bar{\beta}, \bar{\gamma})}{\partial \beta \partial \gamma} = w \csc \beta \frac{\partial \mathcal{R}_1^\alpha(\bar{\beta}, \bar{\gamma})}{\partial \gamma} = 0.$$

Then the determinant of the Hessian of $\mathcal{R}_1^\alpha(\bar{\beta}, \bar{\gamma})$ is equal to $\frac{3}{4}wd$, which is positive; thus, $\mathcal{R}_1^\alpha(\bar{\beta}, \bar{\gamma})$ is locally convex and attains a minimum at its unique critical point. \square

Lemma 10. *The critical points of $\mathcal{R}_\infty^\alpha(\beta, \gamma) : \mathbb{R}^2 \mapsto \mathbb{R}$ are the solutions to the system*

$$\begin{aligned} \cos \Theta &= \frac{3}{4} \cos \gamma \\ \cos \Delta &= \frac{2}{3} \cos \beta, \end{aligned}$$

where $\Delta = \beta + \gamma$ and $\Theta = 2\alpha + \gamma$.

Proof. Taking the first derivative of $\mathcal{R}_1^\alpha(\beta, \gamma)$ with respect to β gives

$$\frac{\partial \mathcal{R}_1^\alpha(\beta, \gamma)}{\partial \beta} = \frac{1}{2(2-x)} \left(4 \frac{\partial w}{\partial \beta} + 3 \frac{\partial d}{\partial \beta} + 2\mathcal{R}_1^\alpha(\bar{\beta}, \bar{\gamma}) \frac{\partial x}{\partial \beta} \right).$$

By substituting in the appropriate derivatives $\frac{\partial w}{\partial \beta} = -w \cot(\alpha + \beta)$, $\frac{\partial d}{\partial \beta} = wd \csc \beta$, and $\frac{\partial x}{\partial \beta} = wx \csc \beta$, we attain

$$\frac{\partial \mathcal{R}_1^\alpha(\beta, \gamma)}{\partial \beta} = \frac{w \csc(\beta)}{2(2-x)} (4y \cos(\Theta) + 3d + 2x\mathcal{R}_1^\alpha(\beta, \gamma)).$$

This derivative is zero at β when and only when

$$\mathcal{R}_1^\alpha(\beta, \gamma) = \frac{4y \cos(\Theta) + 3d}{2x} \quad (19)$$

Similarly, as $\frac{\partial w}{\partial \gamma} = 0$, $\frac{\partial d}{\partial \gamma} = -d \cot(\Delta)$, and $\frac{\partial x}{\partial \gamma} = d \csc(\Delta)$, then

$$\frac{\partial \mathcal{R}_1^\alpha(\beta, \gamma)}{\partial \gamma} = \frac{d \csc(\Delta)}{2(2-x)} (-3 \cos(\Delta) + 2\mathcal{R}_1^\alpha(\beta, \gamma)).$$

At a critical point, this derivative is zero and thus

$$\mathcal{R}_1^\alpha(\beta, \gamma) = \frac{3}{2} \cos(\Delta). \quad (20)$$

Assume that both equations (19) and (20) hold; then, equating both formulas for $\mathcal{R}_1^\alpha(\beta, \gamma)$ gives

$$\frac{4y \cos(\Theta) + 3d}{2x} = \frac{3}{2} \cos(\Delta).$$

As x is nonzero, y must also be nonzero and thus solving for $\cos(\Theta)$ gives

$$\cos(\Theta) = \frac{3}{4} \left(\frac{x \cos \Delta + d}{y} \right).$$

Substituting in formulas for d, x , and y gives

$$\cos(\Theta) = \frac{3}{4} \cos \gamma. \quad (21)$$

Now, returning to equation (20), substituting in a formula for $\mathcal{R}_1^\alpha(\beta, \gamma)$ and solving for w gives

$$w = \frac{3}{4}((2-x) \cos \Delta - d).$$

Dividing both sides by y , substituting in explicit formulas, and simplifying gives

$$\frac{\sin \alpha}{\sin \beta} = \frac{3}{4} \left(\frac{2 \sin \Theta \cos \Delta}{\sin \beta} - \cos \gamma \right),$$

which solves for

$$\sin \alpha = \frac{3}{2} \sin \Theta \cos \Delta - \left(\frac{3}{4} \cos(\gamma) \right) \sin \beta.$$

Using (21), we obtain

$$\sin \alpha = \frac{3}{2} \sin \Theta \cos \Delta - \cos \Theta \sin \beta,$$

which with some minor trigonometric manipulation gives

$$\cos \Delta = \frac{2}{3} \cos \beta.$$

□

Lemma 11. *Let $\bar{\beta}, \bar{\gamma}$ be as described in the statement of Theorem 5. Then, for all $\alpha \in (0, 3/4)$, we have that*

$$0 \leq \bar{\beta}, \bar{\gamma} \leq \pi/2 - \alpha.$$

Proof. The lemma is established numerically by plotting $\bar{\beta}, \bar{\gamma}$ against $\pi/2 - \alpha$, see Figure 5.

□

Lemma 12. *Let $\bar{\beta}, \bar{\gamma}$ be as described in the statement of Theorem 5. Then both eigenvalues of $\nabla^2 \mathcal{R}_\infty(\bar{\beta}, \bar{\gamma})$ are strictly positive for all $0 < \alpha < 1/2$, and hence critical values $\bar{\beta}, \bar{\gamma}$ minimize $\mathcal{R}_\infty(\bar{\beta}, \bar{\gamma})$.*

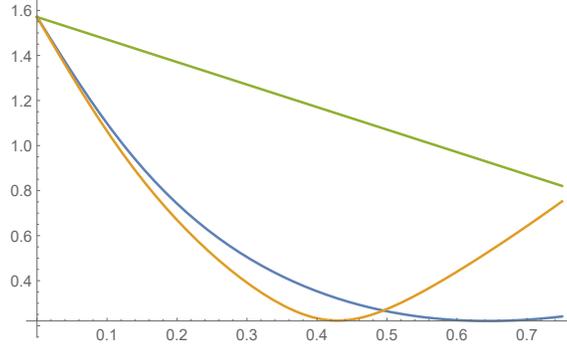


Fig. 5. Horizontal axis corresponds to α . Graph depicts the behaviour of $\bar{\beta} = \bar{\beta}(\alpha)$ (blue curve) and of $\bar{\gamma} = \bar{\gamma}(\alpha)$ (yellow curve) as described in (11), (12), respectively, of Theorem 5, against $\pi/2 - \alpha$ (green line).

Proof. $\mathcal{R}_\infty^\alpha(\beta, \gamma)$ is given by (2) of Lemma 2, so for all β, γ , we can compute $\nabla^2 \mathcal{R}_\infty^\alpha(\beta, \gamma)$. In the resulting 2×2 matrix, we substitute the values $\bar{\beta}, \bar{\gamma}$, as in (11), (11) of Theorem 5 to obtain $\nabla^2 \mathcal{R}_\infty^\alpha(\bar{\beta}, \bar{\gamma})$ whose entries depends exclusively on α . Using symbolic software, we calculate both eigenvalues of $\nabla^2 \mathcal{R}_\infty^\alpha(\bar{\beta}, \bar{\gamma})$, and we verify that they are both strictly positive, for all $0 < \alpha < 1/2$, see Figure 6.

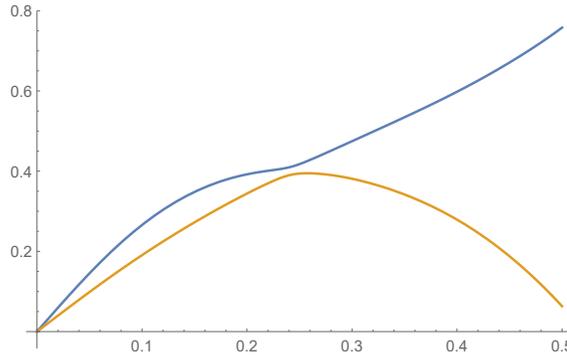


Fig. 6. The two eigenvalues of $\nabla^2 \mathcal{R}_\infty^\alpha(\bar{\beta}, \bar{\gamma})$ as a function of α .

□