ON SOME CLASSES OF IRREDUCIBLE POLYNOMIALS

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Abstract. One of the fundamental tasks of Symbolic Computation is the factorization of polynomials into irreducible factors.

The aim of the paper is to produce new families of irreducible polynomials, generalizing previous results in the area. One example of our general result is that for a near-separated polynomial, i.e., polynomials of the form $F(x, y) = f_1(x)f_2(y) - f_2(x)f_1(y)$, then $F(x, y) + r$ is always irreducible for any constant r different from zero.

We also provide the biggest known family of HIP polynomials in several variables. These are polynomials $p(x_1, \ldots, x_n) \in \mathbb{K}[x_1, \ldots, x_n]$ over a zero characteristic field $\mathbb{K}$ such that $p(h_1(x_1), \ldots, h_n(x_n))$ is irreducible over $\mathbb{K}$ for every n-tuple $h_1(x_1), \ldots, h_n(x_n)$ of non constant one variable polynomials over $\mathbb{K}$. The results can also be applied to fields of positive characteristic, with some modifications.

1. Introduction

Let $\mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables $(x_1, \ldots, x_n) = \underline{x}$ over a field $\mathbb{K}$. A polynomial $p(x_1, \ldots, x_n) = p(\underline{x}) \in \mathbb{K}[x_1, \ldots, x_n] = \mathbb{K}[\underline{x}]$ is called a Hereditarily Irreducible Polynomial (HIP) if $p(h_1(x_1), \ldots, h_n(x_n))$ is irreducible in $\mathbb{K}[\underline{x}]$ for every n-tuple $h_1(x_1), \ldots, h_n(x_n)$ of non constant one variable polynomials over $\mathbb{K}$, see [1, 9]. In [9] the authors present a class of HIP polynomials only in two variables x and y over the complex number field $\mathbb{C}$, namely the polynomials $p(x, y) = a(x)y + 1$ such that $a(x)$ is a square free polynomial of degree at least two. Later, [3] provided an extension of this class, namely polynomials over a zero characteristic field $\mathbb{K}$ of the form: $p(x, y) = a(x)y + b(x)$ such that $a(x)$ has at least two simple roots and $(a(x), b(x)) = 1$.

One of the main task of this paper is to produce new families of HIP. In particular, we extend in Section 2 the class of the polynomials in [3] to primitive polynomials of the form $p(x, y) = a(x)c(x, y) + b(x) \in \mathbb{K}[x, y]$ such that $c(x, y) = \sum_{i=0}^{d} c_i(x)y^i \in \mathbb{K}[x][y]$, $\gcd(a(x), c_d(x)b(x)) = 1$ and $a(x)$ has at least two simple roots.

There are several results on the so called difference or separated polynomials, which are polynomials in two variables x, y of the form $P(x, y) = p(x) - p(y)$, (see the excellent book [8]). The authors of [4] showed that $P(x, y) + r$ is an irreducible polynomial, where r is a nonzero element of a zero characteristic field $\mathbb{K}$. In a similar way it is defined a near-separated polynomial as one of the form $F(x, y) = f_1(x)f_2(y) - f_2(x)f_1(y)$, see for instance [2, 10]. In the last section of this paper we provide a slight generalization of the celebrated Eisenstein criteria and, with it, we can show that $F(x, y) + r$ is absolutely irreducible for any constant

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$r \neq 0$. Our criterion also allows us to address other problems on irreducibility. In particular, we are able to simplify and generalize some results of [5].

Through this paper, we denote by $\mathbb{K}$ an arbitrary field and by $\overline{\mathbb{K}}$ its algebraic closure. A polynomial $p(\underline{x}) \in \mathbb{K}[\underline{x}]$ is absolutely irreducible if it is irreducible over $\overline{\mathbb{K}}[\underline{x}]$.

2. Hereditarily Irreducible Polynomial (HIP)

We start this section with a simple application of the well known Eisenstein's criterion in [6].

Proposition 1. *Let $a(\underline{x}) \in \mathbb{K}[\underline{x}]$ be with an irreducible factor of multiplicity 1. Then any primitive polynomial $p(\underline{x}, y) = a(\underline{x})c(\underline{x}, y) + b(\underline{x}) \in \mathbb{K}[\underline{x}][y]$ is absolutely irreducible, for any pair of polynomials $c(\underline{x}, y) = \sum_{i=0}^{d} c_i(\underline{x})y^i \in \mathbb{K}[\underline{x}, y]$ with $b(\underline{x}) \in \mathbb{K}[\underline{x}]$ and $\gcd(a(\underline{x}), c_d(\underline{x})b(\underline{x})) = 1$.*

Proof. We take $q(\underline{x})$ in the algebraic closure of $\mathbb{K}$ to be an irreducible factor of multiplicity 1 of the polynomial $a(\underline{x})$, which exists by hypothesis. Consider $p(\underline{x}, y) = \sum_{i=0}^{d} f_i(\underline{x})y^i$ where $f_i \in \mathbb{K}[\underline{x}][y]$. To prove the proposition we will apply Eisenstein criterion for $q(\underline{x})$ in $\bar{\mathbb{K}}[\underline{x}]$. Indeed, we can apply the criterion since $p(\underline{x}, y)$ is primitive as polynomial in $\mathbb{K}[\underline{x}][y]$, $q(\underline{x})|f_i(\underline{x})$ for $i = 1, \dots, d$, $q(\underline{x}) \nmid f_0(\underline{x}) = c_0(\underline{x})a(\underline{x}) + b(\underline{x})$, since $\gcd(a(\underline{x}), b(\underline{x})) = 1$ and $q(\underline{x})^2 \nmid f_d(\underline{x}) = a(\underline{x})c_d(\underline{x})$ since $\gcd(a(\underline{x}), c_d(\underline{x})) = 1$ and $q(\underline{x})$ is an irreducible factor of multiplicity 1. □

In order to describe classes of HIP polynomials and after the above result it is quite natural to introduce the following concept:

Definition 2. *A polynomial $a(\underline{x}) \in \mathbb{K}[x]$ is called a Near Hereditarily Irreducible Polynomial (NHIP) if $a(h_1(x_1), \dots, h_n(x_n))$ has an irreducible factor of multiplicity 1, for every n-tuple $h_1(x_1), \dots, h_n(x_n)$ of non constant one variable polynomials.*

We need the following result that appears in [7]:

Lemma 3. *Let $a(\underline{x}), b(\underline{x})$ be polynomials in $\mathbb{K}[\underline{x}]$ with $\gcd(a(\underline{x}), b(\underline{x})) = 1$. Then for every n-tuple $h_1(x_1), \dots, h_n(x_n)$ of non constant one variable polynomials over $\mathbb{K}$ we have $\gcd(a(h_1(x_1), \dots, h_n(x_n), b(h_1(x_1), \dots, h_n(x_n)) = 1$.*

And consequence we have the following:

Lemma 4. *Let $a(\underline{x}, y) \in \mathbb{K}[\underline{x}][y]$ be a primitive polynomial, then for every $n+1$-tuple $h_1(x_1), \dots, h_n(x_n), g(y)$ of non constant one variable polynomials over $\mathbb{K}$ $p(h_1, \dots, h_n, g(y)) = p(\underline{h}, g(y))$ is also primitive in $\mathbb{K}[\underline{x}][y]$.*

Proof. Let $p(\underline{x}, y) = \sum_{i=0}^{n} p_i(\underline{x})y^i$ with $\gcd(p_0(\underline{x}), \dots, p_n(\underline{x})) = 1$. Then, by previous Lemma 3 $\gcd(p_0(\underline{h}), \dots, p_n(\underline{h})) = 1$, and hence $p(\underline{h}, y) = \sum_{i=0}^{n} p_i(\underline{h})y^i$ is primitive. Now suppose that $p(\underline{h}, g(y))$ is not primitive in $\mathbb{K}[\underline{x}][y]$. Then there exist a non constant polynomial $q(\underline{x}) \in \mathbb{K}[\underline{x}]$ such that $p(\underline{h}, g(y)) = q(\underline{x})R(\underline{x}, y)$, for certain polynomial $R(\underline{x}, y) \in \mathbb{K}[\underline{x}, y]$. But then, taking $n+1$ different values of $g(y_l) = a_l \in \bar{K}$, we get $p(\underline{h}, a_l) = \sum_{i=0}^{n} p_i(\underline{h})a_l^i = q(\underline{x})R(\underline{x}, y_l)$ for $l = 0, \dots, n$ and multiplying by the inverse of the Vandermonde matrix $A = (a_l^k)$ we get

$$p_i(\underline{h}) = q(\underline{x})A^{-1}R(\underline{x}, y_l)$$

hence $q(\underline{x})$ divides $p_i(\underline{h})$, for $i = 0, \dots, n$ which is a contradiction. □

Now, we are in the conditions to show this elementary result.

Corollary 5. *Let $a(\underline{x}) \in \mathbb{K}[\underline{x}]$ be a NHIP polynomial. Then any primitive polynomial $p(\underline{x}, y) = a(\underline{x})c(\underline{x}, y)+b(\underline{x}) \in \mathbb{K}[\underline{x}][y]$ is HIP for any pair $c(\underline{x}, y) = \sum_{i=0}^{d} c_i(\underline{x})y^i \in \mathbb{K}[\underline{x}, y]$ with $b(\underline{x}) \in \mathbb{K}[\underline{x}]$ and $\gcd(a(\underline{x}), c_d(\underline{x})b(\underline{x})) = 1$.*

Proof. The proof is an immediate consequence of Proposition 1 and the above result. Observe that primitiveness is maintained thanks the above Lemma 4. □

Obviously, any HIP polynomial is a NHIP one. The following interesting result will help us to provide NHIP univariate polynomials and HIP polynomials.

Lemma 6. *Let $f(x) \in \mathbb{K}[x]$ be a polynomial with formal derivative $f'(x) \neq 0$, and $a, b \in \mathbb{K}$ with $a \neq b$. Then the polynomial $(f(x) - a)(f(x) - b)$ has at least two simple roots in $\overline{\mathbb{K}}$.*

Proof. We write $q(x) = (f(x) - a)(f(x) - b) = \prod_{i=1}^{d}(x - a_i)^{e_i}$ where $a_i \in \overline{\mathbb{K}}$, for $i = 1, \ldots, d$. And derivating, we have:

$$q'(x) = f'(x)(2f(x) - a - b) = h(x)g(x),$$

where:

$$h(x) = \prod_{i=1}^{d}(x - a_i)^{e_i-1}, \quad g(x) = \sum_{i=1}^{d} e_i \prod_{i\neq j}(x - a_j)$$

We state that $h(x)$ divides $f'(x)$. In order to prove this, we distinguish two cases, depending on the characteristic of the field $\mathbb{K}$.

Now, if the characteristic of $\mathbb{K}$ is 2, then $0 \neq (a+b)f'(x) = g(x)h(x)$. Otherwise, $\gcd(h(x), 2f(x) - a - b) = 1$ and we get $h(x)$ divides $f'(x)$. Indeed, note that the roots of $h(x)$ are a_i, the roots of $q(x)$, and since $f(a_i) = a$ or $f(a_i) = b$. We can not have $2f(a_i) - a - b = 0$ since $a \neq b$ by hypothesis. Then,

$$\frac{\deg q(x)}{2} - 1 \geq \deg f'(x) \geq \deg h(x) = \deg q(x) - d$$

It implies that $\deg q(x) \leq 2d - 2$ and finishes the proof. □

Remark. The previous result is an improvement of Lemma 3 of [3].

The following example illustrates that the hypothesis $f'(x) \neq 0$ can not be omitted.

Example 7. *Let $f(x) = x^p \in \mathbb{F}_p[x]$, then $(f(x) - 1)(f(x) + 1) = (x^2 - 1)^p$*

An immediate consequence of Lemma 6 is:

Corollary 8. *If $h(x) \in \mathbb{K}[x]$ has $n > 1$ simple roots and $f(x) \in \mathbb{K}[x]$ is so that $f'(x) \neq 0$, then $h(f(x))$ has at least n simple roots in $\overline{\mathbb{K}}$.*

Proof. Suppose h has $a_i, i = 1, \ldots, n$ as simple roots and $h(x) = p(x) \prod_{i=1}^{n}(x - a_i)$ for $p(x) \in \bar{\mathbb{K}}[x]$. The result is trivial if each of the factors $f(x) - a_i$ has at least one simple root. Otherwise, suppose $f(x) - a_1$ has no simple roots. Then, applying Lemma 6 to $(f(x) - a_1)(f(x) - a_i)$ for $i = 2, \ldots, n$, we conclude that $f(x) - a_i$ must have two simple roots and these simple roots are necessarily all distinct, so in total $h(f(x))$ would have $2n-2 \geq n$ simple roots. □

Remark 9. *Note that in more than one variable, a polynomial can have factors of multiplicity one, while the composition could have all the factors with multiplicity bigger than one, as one can see for example taking the polynomial* $p(x_1,\ldots,x_n) = x_1\cdots x_n$ *and considering the* n*-tuple* $\bar{h}(\underline{x}) = (x_1^2,\ldots,x_n^2)$ *since then* $p(\bar{h}(\underline{x})) = x_1^2\ldots x_n^2$.

We now show the main result of this section as consequence of Corollary 5 and Lemma 6.

Theorem 10. *Let* $\mathbb{K}$ *be a field of characteristic zero and* $p(x,y)$ *be a primitive bivariate polynomial* $\mathbb{K}[x][y]$ *given by* $p(x,y) = a(x)c(x,y) + b(x)$ *with* $c(x,y) = \sum_{i=0}^{d} c_i(x)y^i \in \mathbb{K}[x][y]$ *such that* $\gcd(a(x), c_d(x)b(x)) = 1$*, and* $a(x)$ *has at least two simple roots. Then* $p(x,y)$ *is a HIP polynomial.*

Proof. First note that since we are in a field of characteristic zero, any non constant polynomial $f(x)$ has non zero derivative. Hence, if $a(x) = (x-a)(x-b)m(x)$ where a, b are two different simple roots, then $a(f(x)) = (f(x)-a)(f(x)-b)m(f(x))$ will also have at least two simple roots by Lemma 6. Moreover the polynomial $p(f(x), g(y)) \in \mathbb{K}[x][y]$ is primitive by Lemma 4. On the other hand,

$$p(f(x), g(y)) = A(x)C(x,y) + B(x)$$

for certain polynomials $C(x,y) = \sum_{i=0}^{D} C_i(x)y^i \in \mathbb{K}[x][y]$, $A(x) = a(f(x))$, $B(x) = b(f(x))$ and $C_D(x) = \alpha c_d(f(x))$ with $0 \neq \alpha \in \mathbb{K}$. Since $\gcd(a(x), c_d(x)b(x)) = 1$ then by Lemma 3 we have $\gcd(A(x), C_D(x)B(x)) = 1$. So to get the result, we just need to apply Proposition 1 to the polynomial $p(f(x), g(y))$.

□

Remark. The result can not be generalized to fields of positive characteristic in full generality. For example $(x^2-1)y - 1$ is not HIP in $\mathbb{F}_p$, since $(x^{2p}-1)y^p - 1 = ((x^2-1)y)^p - 1$ is reducible in $\mathbb{F}_p$.

As consequence of Theorem 10 and Corollary 5 we are able to construct HIP polynomials in any arbitrary number of variables. For example, by Theorem 10 we take a HIP bivariate polynomial $a(x_1, x_2) \in \mathbb{K}[x_1, x_2]$ then it is NHIP polynomial, now choosing polynomials $c(x_1, x_2, y), b(x_1, x_2)$ as in Corollary 5 we get that $a(x_1, x_2)c(x_1, x_2, y) + b(x_1, x_2) \in \mathbb{K}[x_1, x_2, y]$ is HIP.

3. Eisenstein criterion and some classes of irreducible polynomials in two variables

In this section we deal with the irreducibility of certain classes of polynomials in two variables.

Proposition 11. *Let* $\phi(x,y) = (u(x,y), v(x,y))$ *be an automorphism of* $\mathbb{K}[x,y]$ *and* $F(x,y) = uQ(u,v) + r(u)$ *be a primitive polynomial such that* $r(0) \neq 0$ *and* $Q(u,v) = \sum_{j=0}^{d} p_j(u)v^i$ *verifies* $u \nmid p_d(u)$*. Then* $F(x,y)$ *is absolutely irreducible.*

Proof. Just apply Eisenstein criterion with the prime $u \in \overline{\mathbb{K}}[u]$ to obtain the irreducibility of $F(x,y) = \tilde{F}(u,v)$ in the ring $\overline{\mathbb{K}}[u][v] = \overline{\mathbb{K}}[u,v] = \overline{\mathbb{K}}[x,y]$. Note that $\tilde{F}(u,v) \in \mathbb{K}[u][v]$ is primitive, and $r(0) \neq 0$ implies $u \nmid r(u)$ by Lemma 3.

□

Remark 12. *The previous result somehow gives us a generalization of Eisenstein criterion since it allows us to consider primes in the full ring $\mathbb{K}[x,y]$ in which we want to know irreducibility of certain polynomial. Somehow is like proving that the polynomial $a+bx$ is irreducible over $\mathbb{K}[x]$ by noticing that x, prime in $\mathbb{K}[x]$, divides all the terms except the first, and x^2 does not divide the leading term.*

We include now other less obvious applications of the result. The first is a generalization of Proposition 1.5 in [4].

Theorem 13. *Let $f_1(t), f_2(t) \in \mathbb{K}[t]$ two distinct polynomials of respective degree $d_1 > 0$ and $d_2 > 0$, $0 \neq r \in \mathbb{K}$ and $F(x,y) = f_1(x)f_2(y) - f_2(x)f_1(y) + r$. We have,*

- *If char $\mathbb{K} = 0$ then $F(x,y)$ is absolutely irreducible.*
- *If char $\mathbb{K} = p$ and $p \nmid d_1 - d_2$, then $F(x,y)$ is absolutely irreducible.*

Proof. Observe that we can suppose $d_1 \neq d_2$ since otherwise, if $d_1 = d_2$, then $f_1(t) = af_2(t) + h(t)$ with $\deg(h(t)) < \deg(f_2(t))$ and $a \in \mathbb{K}$, and

$$F(x,y) = (af_2(x)+h(x))f_2(y) - (af_2(y)+h(y))f_2(x) + r = h(x)f_2(y) - h(y)f_2(x) + r.$$

We write $f_l(t) = \sum_{i=0}^{d_l} a_{j,l} t^j$ $(j = 1, 2)$, and taking $u = x - y$, $v = y$. Then, by Taylor expansion, (by abuse of notation we will use $\frac{f_l^{(j)}(y)}{j!}$ for the j-th hasse derivative in positive characteristic), we have

$$f_l(u+v) = \sum_{j=0}^{d_l} a_{j,l}(u+v)^j = \sum_{j=0}^{d_l} a_{j,l} \sum_{i=0}^{j} \binom{j}{i} u^i v^{j-i} = \sum_{i=0}^{d_l} \left(\sum_{j=i}^{d_l} a_{j,l} \binom{j}{i} v^{j-i} \right) u^i$$

and so $F(x,y)$ can be written as a polynomial in the variables u, v as

$$F(x,y) = \sum_{i=0}^{d_1} \left(\sum_{j=i}^{d_1} a_{j,1} \binom{j}{i} v^{j-i} \right) u^i f_2(v) - \sum_{i=0}^{d_2} \left(\sum_{j=i}^{d_2} a_{j,2} \binom{j}{i} v^{j-i} \right) u^i f_1(v) + r.$$

Now observe that $\deg_v \left(\sum_{j=i}^{d_l} a_{j,l} \binom{j}{i} v^{j-i} \right) u^i f_m(v) \leq d_1 + d_2 - i$, and so we can write

$$F(x,y) = u(f_1'(v)f_2(v) - f_1(v)f_2'(v)) + uH(v,u) + r$$

with $\deg_v H(u,v) < d_1 + d_2 - 1$, and

$$f_1'(v)f_2(v) - f_1(v)f_2'(v) = (d_1 - d_2)cy^{d_1+d_2-1} + R(y)$$

for some $c \neq 0$ and $\deg(R(y)) < d_1 + d_2 - 1$. The result follows from Proposition 11 by choosing the automorphism $\phi = (u,v)$.

□

Another trivial application of Proposition 11 gives the following result. We will use the following notation: we let $H(x,y) = \sum_{i=0}^{d} a_i(x) y^i$ be a polynomial in $\mathbb{K}[x,y]$ with coefficients $a_i(x) = \sum_{j=0}^{d_i} a_{ij} x^j$ and consider $f(x) = \sum_{j=0}^{d_f} f_j x^j$. We will let $M = \{0 \leq i \leq d \,:\, d_i + id_f = m\}$, where $m = \max\{d_j + jd_f\}$.

Corollary 14. *Let $H(x,y)$ and $f(x)$ as above and consider $F(x,y) = (y-f(x))H(x,y)+r$ with $0 \neq r \in \mathbb{K}$. Then if $\sum_{i\in M} a_{id_i}(f_{d_f})^i \neq 0$, then $F(x,y)$ is absolutely irreducible.*

Proof. We observe that $H(x, f(x)+u) = \sum_{j=0}^{d} \frac{\partial^j H(x,f(x))}{\partial y^j} \frac{u^j}{j!}$ and

$$\deg \frac{\partial^j H(x,f(x))}{\partial y^j} \leq \max_{i\leq j}\{d_i + (i-j)d_f\}.$$

Hence, $\deg_x H(x, f(x)+u) \leq \deg H(x, f(x)) = \max_{i\geq j}\{d_i + id_f\}$ and we have $H(x, f(x)+h) = \left(\sum_{i\in M} a_{id_i}(f_{d_f})^i\right) x^m + R(x)$ with $\deg R(x) < m$. The result follows by applying Proposition 11 with $u = y - f(x)$ and $v = x$. □

It would be interesting, from the computational point of view, to recognize whether a given polynomial $F(x,y)$ is of the form considered in Corollary 14. It comes the following question:

Open problem 15. *Given $F(x,y) \in \mathbb{K}[x,y]$, determine wether it exists a polynomial $f(x) \in \mathbb{K}[x]$ and a constant $r \in \mathbb{K}$ such that $F(x,y) = (y-f(x))H(x,y)+r$ for some polynomial $H(x,y)$.*

Observe that this is equivalent to find $f(x) \in \mathbb{K}[x]$ such that $F(x, f(x))$ is constant.

We intend to address this in a future work, but here we would like to include a simple example. Suppose $F(x,y) = \sum_{i=0}^{d} b_i(x)y^i$ for some polynomials $b_i(x) \in K[x]$. Note that, in case that there are $f(x)$ and r in the above conditions, then by substituting $f(x) = \sum_{i=0}^{d} f_i x^i$ into $F(x,y)$ we get

$$r = F(x, f(x)) = \sum_{i=0}^{d} b_i(x) f(x)^i = b_0(x) + f(x)G(x),$$

for some $G(x) \in \mathbb{K}[x]$. In particular, $f(x)$ divides $b_0(x) - r$, and $\deg f(x) \leq \deg b_0(x)$. Consider $F(x,y) = -x^5 - x^3y^2 + x^2y + y^3 + 1$. First we write it as a polynomial in $\mathbb{K}[x][y]$ as $F(x,y) = y^3 - x^3y^2 + x^2y - x^5 + 1$. So $\deg f(x) \leq 5$. Now, substituting $f(x) = \sum_{i=0}^{d} f_i x^i$ into $F(x,y)$ we see, using SAGE, that the only possible solutions are $f(x) = x^3$ or $f(x) = \pm ix$, and $r = 1$, and we get the expression $F(x,y) = (y - x^3)(y^2 + x^2) + 1$.

It is interesting to observe that there could be more that one expression for $F(x,y) = (y - f(x))H(x,y) + r$ with the same constant r as it is the case for example for the polynomial $F(x,y) = \prod_{i=1}^{n}(y - f_i(x))H(x,y) + r$. However, if $(y - f_1(x))H_1(x,y) + r_1 = (y - f_2(x))H_2(x,y) + r_2$ with $r_1 \neq r_2$, then $f_1(x) = f_2(x) + r$ for some constant r. Indeed, substituting y by $f_1(x)$ inrthe previous identity gives the desired result.

The polynomials in Corollary 14 have already been considered in the literature. In [5] the authors consider the polynomial $F(x,y) = h(x)\prod_{i=1}^{n}(y - f_i(x)) + g(x)$ and they are interested on the irreducibility over the rational function field $\mathbb{K}(x)$ with certain conditions on the degree of $f_i(x)$. We can drop all those conditions in two different ways:

Corollary 16. *Let $F(x,y)$ be the polynomial defined as above, with distinct n polynomials $f_i(x)$ and such that with $\sum_{i=1}^{n} deg(f_i(x)) > 0$.*

a) *If $h(x)$ has a simple root $\alpha \in \overline{\mathbb{K}}$ and $\gcd(h(x), g(x)) = 1$ then $F(x,y)$ is absolutely irreducible.*
b) *If $0 \neq g(x) = r \in \mathbb{K}$ and $f_n(x) - f_i(x)) \notin \mathbb{K}$, for $i = 1, \ldots, n-1$, then $F(x,y)$ is absolutely irreducible.*

Proof. a) is an immediate consequence of Proposition 1.

To prove b) we will apply Proposition 11 with $u = y - f_n(x), v = x$. Indeed,

$$F(x,y) = uh(x) \prod_{i=1}^{n-1} (u - g_i(x)) + r,$$

where $g_i(x) = f_i(x) - f_n(x)$. Then

$$\prod_{i=1}^{n-1} (u - g_i(x)) = \sum_{j=0}^{n-1} p_j(x) u^j,$$

where

$$p_j(x) = (-1)^{n-1-j} \sum_{1 \leq i_1 < \cdots < i_{n-1-j} \leq n-1} \prod_{l=1}^{n-1-j} g_{i_l},$$

in particular $\deg(p_j(x)) < \deg(p_0(x))$ since

$$\left(\prod_{l=1}^{n-1-j} g_{i_l} \right) \Bigg| \left(\prod_{l=1}^{n-1} g_l \right)$$

for any set of sudindexes, and $f_n(x) - f_i(x) \notin \mathbb{K}$ (for $i = 1, \ldots, n-1$), and hence $F(x,y)$ is primitive in $\mathbb{K}[u][x]$ and $u \nmid p_d(u)$. □

Remark. Observe that b) is also considered in [5].

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