# Constructive Arithmetics in Ore Localizations Enjoying Enough Commutativity 

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#### Abstract

This paper continues a research program on constructive investigations of non-commutative Ore localizations, initiated in our previous papers, and particularly touches the constructiveness of arithmetics within such localizations. Earlier we have introduced monoidal, geometric and rational types of localizations of domains as objects of our studies. Here we extend this classification to rings with zero divisors and consider Ore sets of the mentioned types which are commutative enough: such a set either belongs to a commutative algebra or it is central or its elements commute pairwise. By using the systematic approach we have developed before, we prove that arithmetic within the localization of a commutative polynomial algebra is constructive and give the necessary algorithms. We also address the important question of computing the local closure of ideals which is also known as the desingularization, and present an algorithm for the computation of the symbolic power of a given ideal in a commutative ring. We also provide algorithms to compute local closures for certain non-commutative rings with respect to Ore sets with enough commutativity.


Keywords: Ore localization; Noncommutative algebra; Algorithms

## Introduction

The algebraic technique of commutative localization has found applications across many areas of mathematics and beyond; it is instrumental everywhere from algebraic geometry to system and control theory. Among several possible generalizations to the non-commutative setting, Ore localization stands out as being approachable in a constructive manner by methods of modern computer algebra. This paper is a part of our broad program dedicated to realizing this approach. Starting point was the investigation of arithmetic operations with left and right fractions in Ore localizations of non-commutative domains in Hoffmann and Levandovskyy (2017a) and

[^0]its extended version Hoffmann and Levandovskyy (2017b). We have demonstrated that such arithmetic operations are based essentially on two algorithms, namely

- the computation of the kernel of a module homomorphism and
- the computation of the intersection of a left ideal with a monoid.

Especially the latter algorithm is hardly constructive in a broad generality, therefore we have introduced a partial classification of types of multiplicative monoids for which the intersection problem can be solved algorithmically. We recall an extended version of the classification in Definition 4.

In this paper we revisit the case of commutative polynomial algebras both on their own and as homomorphic images in a noncommutative ring as either central subalgebras or those which are generated by pairwise commuting elements. On the one hand we extend our framework to such algebras with zero divisors. On the other hand we also consider the important problem of the computation of the local closure of a submodule with respect to a given denominator set (also known as the desingularization), which is tightly connected with the generalized torsion submodule of a module.

Though some of the algorithms have been known in commutative algebra, they are scattered in the existing literature and are often deprived of proofs. We describe the problems in a systematic and self-contained way. In the collection of the algorithms we present, 4, 7, 8, 10, 11, and 12 are new. The following list summarizes the problems discussed in this paper with references to the corresponding algorithms:

Polynomial algebras: In a polynomial algebra $R=K[x] / J$, where $J$ is an ideal in the commutative polynomial ring $K[x]:=K\left[x_{1}, \ldots, x_{n}\right]$, we can compute the intersection of an ideal $I$ in $R$ with a multiplicative subset $S$ of $R$, if

- $S^{-1} R$ is monoidal and $S$ is finitely generated (Algorithm 4),
- $S^{-1} R$ is geometric (Algorithm 5), or
- $S^{-1} R$ is essential rational (Algorithm 6).

Furthermore, we can decide whether a multiplicative submonoid of $R$ contains 0 (Algorithm 2). It is important, since localizing $R$ at a submonoid $S$ containing 0 yields the trivial localization $S^{-1} R=\{0\}$.

Commutative rings: In an arbitrary commutative ring $R$ we can compute the closure of an ideal $I$ with respect to a multiplicative set $S$ via Algorithm 7 under the following conditions:
(1) The ideal $I$ is decomposable into primary ideals and such a decomposition is either known or computable.
(2) We can decide whether $Q \cap S=\emptyset$ for any primary ideal $Q$ in $R$.

In particular, we give Algorithm 8 for computing the symbolic power of a given ideal.
G-algebras: In a $G$-algebra we can compute the closure of an ideal $I$ with a left Ore set $S$, if

- $S^{-1} R$ is monoidal and $S$ is generated as a monoid by finitely many elements $f_{1}, \ldots, f_{k}$ that commute pairwise and $Z(A) \cap S$ contains a multiple of $f_{1} \cdot \ldots \cdot f_{k}$ (Remark 32), or
- $S^{-1} R$ is central essential rational (Algorithm 10).

In comparison to the ISSAC version of this paper (Hoffmann and Levandovskyy (2018)), the material has been expanded and slightly reworked (some proofs now contain more details). In particular, we have expanded Section 4 with criteria for emptiness of the intersection of primary ideals and multiplicative sets as well as with Algorithm 8 (computation of the symbolic power of an ideal), described Weyl closure algorithms in Remark 49, added Section 5.5 on the details of central Weyl closure including Algorithm 11, and finally added Section 5.6 with the new Algorithm 12 to compute the annihilator ideal of the important special function $f^{s}$.

## 1. The basics of (Ore) localization

All rings are assumed to be associative and unital, but not necessarily commutative.
Definition 1. A subset $S$ of a ring $R$ is called

- a multiplicative set if $1 \in S, 0 \notin S$ and for all $s, t \in S$ we have $s \cdot t \in S$.
- a left Ore set if it is a multiplicative set that satisfies the left Ore condition: for all $s \in S$ and $r \in R$ there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s} r=\tilde{r} s$.
- a left denominator set if it is a left Ore set that is additionally left reversible: for all $s \in S$ and $r \in R$ such that $r s=0$ there exists $\tilde{s} \in S$ satisfying $\tilde{s} r=0$.

For any subset $B$ of $R \backslash\{0\}$ we can consider the set $[B]$ consisting of all finite products of elements of $B$, where the empty product represents 1 . If $R$ is a domain then $[B]$ is always a multiplicative set called the multiplicative closure of $B$.

The main goal of localization can be seen from the following axiomatic definition:
Definition 2. Let $S$ be a multiplicative subset of a ring $R$. A ring $R_{S}$ together with a homomorphism $\varphi: R \rightarrow R_{S}$ is called a left Ore localization of $R$ at $S$ if:
(1) For all $s \in S$, the element $\varphi(s)$ is a unit in $R_{S}$.
(2) For all $x \in R_{S}$, there exist $s \in S$ and $r \in R$ such that $x=\varphi(s)^{-1} \varphi(r)$.
(3) We have $\operatorname{ker}(\varphi)=\{r \in R \mid \exists s \in S: s r=0\}$.

One can show that a left Ore localization of $R$ at $S$ exists if and only if $S$ is a left denominator set. In this case the localization is unique up to isomorphism. The classical construction is given by the following:

Theorem 3. Let $S$ be a left denominator set in a ring $R$. The relation $\sim$ on $S \times R$, given by

$$
\left(s_{1}, r_{1}\right) \sim\left(s_{2}, r_{2}\right) \Leftrightarrow \exists \tilde{s} \in S \exists \tilde{r} \in R: \tilde{s} s_{2}=\tilde{r} s_{1} \text { and } \tilde{s} r_{2}=\tilde{r} r_{1},
$$

is an equivalence relation. Now $S^{-1} R:=((S \times R) / \sim,+, \cdot)$ becomes a ring via

$$
\left(s_{1}, r_{1}\right)+\left(s_{2}, r_{2}\right):=\left(\tilde{s} s_{1}, \tilde{s} r_{1}+\tilde{r} r_{2}\right),
$$

where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s} s_{1}=\tilde{r} s_{2}$, and

$$
\begin{gathered}
\left(s_{1}, r_{1}\right) \cdot\left(s_{2}, r_{2}\right):=\left(\tilde{s} s_{1}, \tilde{r} r_{2}\right),
\end{gathered}
$$

where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s} r_{1}=\tilde{r} s_{2}$. Together with the localization map or structural homomorphism

$$
\rho_{S, R}: R \rightarrow S^{-1} R, \quad r \mapsto(1, r)
$$

the pair $\left(S^{-1} R, \rho_{S, R}\right)$ is the left Ore localization of $R$ at $S$.
The elements of $S^{-1} R$ are called left fractions and are denoted again as tuples $(s, r)$ which are identified with their equivalence class modulo $\sim$. The localizations that appear the most in applications are those with denominator sets of the following three types:

Definition 4. Let $K$ be a field, $R$ a $K$-algebra and $S$ a left denominator set in $R$. Then $S$ (and by extension, the localization $S^{-1} R$ ) might belong to one of the following non-exclusive types:

Monoidal $S$ is generated as a multiplicative monoid by at most countably many elements.
Geometric $S=(K[x] / J) \backslash \mathfrak{p}$ for some prime ideal $\mathfrak{p}$ in the polynomial algebra $K[x] / J \subseteq R$, where $J$ is an ideal in $K[x]:=K\left[x_{1}, \ldots, x_{n}\right]$.

Rational $S=T \backslash\{0\}$ for some $K$-subalgebra $T$ of $R$.
Special case: If $R$ is generated over $K$ by a set of variables $x=\left\{x_{1}, \ldots, x_{n}\right\}$ and $T$ is generated by a subset of $x$ we call $S$ an essential rational left denominator set.

Definition 5. Let $S$ be a left denominator set in a ring $R$ and $M$ a left $R$-module. Then the left Ore localization of $M$ at $S$ is defined as $S^{-1} M:=S^{-1} R \otimes_{R} M$.
Lemma 6 (e.g. Škoda (2006), 7.3). Let $S$ be a left denominator set in a ring $R$ and $M$ a left $R$-module. Any element of $S^{-1} M$ can be written in the form $(s, 1) \otimes m$ for some $s \in S$ and $m \in M$.

Lemma 6 allows us to write ( $s, m$ ) for an element in $S^{-1} M$ in analogy to the notation for elements of $S^{-1} R$.

Alternatively, one can define localization of modules similar to the axiomatic approach in Definition 2, prove its uniqueness and give an elementary construction like in Theorem 3.

Definition 7. Let $S$ be a left denominator set in a ring $R$ and $M$ a left $R$-module.

- The localization map of $M$ with respect to $S$ is the homomorphism of left $R$-modules

$$
\varepsilon:=\varepsilon_{S, R, M}: M \rightarrow S^{-1} M, \quad m \mapsto(1, m)
$$

with kernel $\{m \in M \mid \exists s \in S: s m=0\}$.

- Let $P$ be a left $R$-submodule of $M$. The $S$-closure or local closure of $P$ in $M$ with respect to $S$ is defined as $P^{S}:=\varepsilon_{S, R, M}^{-1}\left(S^{-1} P\right)$.

Let $S$ be a left Ore set in a domain $R$. In our paper Hoffmann and Levandovskyy (2017a) we introduced the notion of left saturation closure of $S$, given by

$$
\operatorname{LSat}(S):=\{r \in R \mid \exists w \in R: w r \in S\} .
$$

We proved that $\mathrm{LSat}(S)$ is a saturated left Ore set in $R$ (i.e. for all $s, t \in R$ such that $s \cdot t \in \operatorname{LSat}(S)$ we have $s, t \in \operatorname{LSat}(S))$ and that $S^{-1} R$ and $\operatorname{LSat}(S)^{-1} R$ are isomorphic rings via $(s, r) \mapsto(s, r)$, which shows that $\operatorname{LSat}(S)$ is a canonical form of $S$ with respect to the corresponding localization.

To describe the $S$-closure more directly we introduce a notion of left saturation closure similar to the one for left Ore sets:

Definition 8. Let $S$ be a left denominator set in a ring $R, M$ a left $R$-module and $P$ a left $R$ submodule of $M$. The left saturation closure of $P$ in $M$ with respect to $S$ is

$$
\operatorname{LSat}_{S}^{M}(P):=\{m \in M \mid \exists s \in S: s m \in P\}
$$

Note that both notions of left saturation closures are instances of a more general concept which will be explored in a future paper.

Lemma 9. Let $S$ be a left denominator set in a ring $R, M$ a left $R$-module and $P$ a left $R$ submodule of M. Then

$$
P^{S}=\operatorname{LSat}_{S}^{M}(P)
$$

Proof. Let $\varepsilon:=\varepsilon_{S, R, M}$. If $m \in P^{S}$, then $\varepsilon(m) \in S^{-1} P$, thus there exist $s \in S$ and $p \in P$ such that $(1, m)=\varepsilon(m)=(s, p)$. This implies the existence of $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s} \cdot 1=\tilde{r} s$ and $\tilde{s} m=\tilde{r} p \in P$, but the last equation implies $m \in \operatorname{LSat}_{S}^{M}(P)$. Now let $m \in \operatorname{LSat}_{S}^{M}(P)$, then there exists $s \in S$ such that $s m \in P$. But the $\varepsilon(m)=(1, m)=(s, s m) \in S^{-1} P$, thus $m \in \varepsilon^{-1}\left(S^{-1} P\right)=P^{S}$.

Lemma 10. Let $S$ be a left denominator set in a ring $R, M$ a left $R$-module and $\left\{P_{j}\right\}_{j \in J}$ a family of left $R$-submodules of $M$. Consider their intersection $P:=\bigcap_{j \in J} P_{j}$.
(a) We have $P^{S} \subseteq \bigcap_{j \in J} P_{j}^{S}$.
(b) If $J$ is finite, then $P^{S}=\bigcap_{j \in J} P_{j}^{S}$.

Proof. (a) Let $m \in P^{S}$, then there exists $s \in S$ such that $s m \in P=\bigcap_{j \in J} P_{j}$, thus $s m \in P_{j}$ and $m \in P_{j}^{S}$ for all $j \in J$, which implies $m \in \bigcap_{j \in J} P_{j}^{S}$.
(b) Let $m \in \bigcap_{j \in J} P_{j}^{s}$, then for all $j \in J$ there exists $s_{j} \in S$ such that $s_{j} m \in P_{j}$. Since $J$ is finite there exists a common left multiple $s \in S$ of the $s_{j}$ by the left Ore condition, which implies $s m \in P_{j}$ for all $j \in J$. Therefore, $s m \in \bigcap_{j \in J} P_{j}=P$ and $m \in P^{S}$.

## 2. Algorithmic toolbox

Let $K$ be a field and consider the two commutative polynomial rings $K[x]:=K\left[x_{1}, \ldots, x_{n}\right]$ and $K[y]:=K\left[y_{1}, \ldots, y_{m}\right]$ with the ideals $I={ }_{K[x]}\left\langle h_{1}, \ldots, h_{k}\right\rangle$ and $J={ }_{K[y]}\left\langle g_{1}, \ldots, g_{l}\right\rangle$. Let further $\varphi: K[x] / I \rightarrow K[y] / J, x_{i} \mapsto f_{i}$ be the ring map induced by elements $f_{1}, \ldots, f_{n} \in K[y]$. Algorithm 1 outlines a classical Gröbner-driven method for computing $\operatorname{ker}(\varphi)$ (for details see e.g. Greuel and Pfister (2008), Section 1.8.10).

Given a homomorphism of arbitrary rings $\psi: A \rightarrow B$ and a two-sided ideal $J$ in $B$, we have that $\psi^{-1}(J)=\operatorname{ker}(\varphi)$ for the induced homomorphism $\varphi: A \rightarrow B / J$. On the other hand the kernel of a homomorphism is the preimage of the zero ideal. Therefore computing kernels and preimages of two-sided ideals is equivalent. Note that this does not hold for preimages of left or right ideals, see Levandovskyy (2006).

```
Algorithm 1: KernelPolynomialAlgebra
    Input: \(K, I, J, \varphi\) as above.
    Output: \(\operatorname{ker}(\varphi)\).
    begin
        \(H:={ }_{K[x, y]}\left\langle h_{1}, \ldots, h_{k}, g_{1}, \ldots, g_{l}, x_{1}-f_{1}, \ldots, x_{n}-f_{n}\right\rangle ;\)
        compute \(H^{\prime}:=H \cap K[x]\) by eliminating \(y_{1}, \ldots, y_{m}\);
        return \(H^{\prime}\);
    end
```


## 3. Intersection of ideals with multiplicative sets in commutative polynomial algebras

The first problem we are interested in solving is the following:
Definition 11. Let $S$ be a left denominator set in a ring $R$ and $I$ a left ideal in $R$. The intersection problem is to decide whether $I \cap S=\emptyset$ and to compute an element contained in this intersection whenever the answer is negative.

In our paper Hoffmann and Levandovskyy (2017a) we have shown that this problem is integral to a constructive treatment of the Ore condition in $G$-algebras which in turn allows us to perform basic arithmetic operations in Ore localizations of $G$-algebras.

In the commutative setting it is an important ingredient for solving linear systems over commutative localizations (Posur (2018)).

Here we consider commutative polynomial algebras of the form $R:=K[x] / J$, where $J$ is an ideal in the commutative polynomial ring $K[x]:=K\left[x_{1}, \ldots, x_{n}\right]$. Furthermore, let $I$ be an ideal in $R$ and fix some suitable $g_{i}, h_{i} \in K[x]$ with $J={ }_{K[x]}\left\langle g_{1}, \ldots, g_{\ell}\right\rangle$ and $I={ }_{R}\left\langle h_{1}+J, \ldots, h_{k}+J\right\rangle$. In the following we give algorithms to solve the intersection problem for $I \cap S$, where $S$ is a multiplicative subset of $R$ belonging to one of the localization types described in Definition 4 with some computability restrictions.

### 3.1. Monoidal

In this subsection we start with algorithms in commutative rings and later proceed to noncommutative ones.

Suppose we are given a monoid $S \subseteq R$, finitely generated by a set $F=\left\{f_{1}+J, \ldots, f_{m}+J\right\}$. Then the monoid algebra $K[S]:=K[F] \subseteq R$ is a natural subalgebra of $R$. Moreover, consider $\psi: K\left[t_{1}, \ldots, t_{m}\right] \rightarrow K[x] / J, t_{i} \mapsto f_{i}+J$, then the monoid algebra $K[S]$ is a finitely presented $K$-algebra which is isomorphic to $K[t] / \operatorname{ker}(\psi)$. Since $R$ is commutative, but not necessarily a domain, we have to ensure that $S^{-1} R \neq\{0\}$, which is equivalent to $0 \notin S$. The latter property can be checked with Algorithm 2.

Proposition 12. Algorithm 2 terminates and is correct.
Proof. We have $0 \in S$ if and only if there exists $\alpha \in \mathbb{N}_{0}^{m}$ such that $f^{\alpha}=f_{1}^{\alpha_{1}} \cdot \ldots \cdot f_{m}^{\alpha_{m}} \in J$, which in turn is equivalent to the existence of $\alpha \in \mathbb{N}_{0}^{m}$ satisfying $t^{\alpha} \in \operatorname{ker}(\psi)=: H$. By e. g. Kreuzer and Robbiano (2005); Miller (2016) an ideal $H \subseteq K[t]$ contains a monomial if and only if the ideal $H:\left\langle t_{1} \cdot \ldots \cdot t_{m}\right\rangle^{\infty}$ contains 1 . Note that all operations involved are computable: the kernel $\operatorname{ker}(\psi)$ via Algorithm 1 and the saturation $H:\left\langle t_{1} \cdot \ldots \cdot t_{m}\right\rangle^{\infty}$ via Greuel and Pfister (2008), Section 1.8.9.

```
Algorithm 2: ZeroContainedinMonoid
    Input: A subset \(F=\left\{f_{1}+J, \ldots, f_{m}+J\right\} \subseteq R=K[x] / J\).
    Output: 1, if \(0 \in S=[F]\), and 0 otherwise.
    begin
        let \(\psi: K\left[t_{1}, \ldots, t_{m}\right] \rightarrow K\left[x_{1}, \ldots, x_{n}\right] / J, t_{i} \mapsto f_{i}+J ;\)
        \(H:=\operatorname{ker}(\psi) ; \quad / /\) preimage \(\psi^{-1}(0)\)
        \(M:=H:\left\langle t_{1} \cdot \ldots \cdot t_{m}\right\rangle^{\infty}\);
        if \(1 \in M\) then
            return 1;
        else
            return 0 ;
        end
    end
```

To solve the intersection problem in the monoidal case, we need to be able to determine the biggest monomial ideal contained in an ideal in a commutative polynomial algebra, which can be computed with Algorithm 3.

```
Algorithm 3: BiggestMonomialideal
    Input: An ideal \(L+J\) in \(R=K[x] / J\).
    Output: The biggest monomial ideal contained in \(L+J\).
    begin
        Let \(K\left[x, q^{ \pm 1}\right]:=K\left[x, q_{1}, q_{1}^{-1}, \ldots, q_{m}, q_{m}^{-1}\right] ;\)
        \(\varphi: K[x] \rightarrow K\left[x, q^{ \pm 1}\right], x_{i} \mapsto q_{i} x_{i} ; \quad\) // ring extension
        \(N:={ }_{K\left[x, q^{ \pm 1}\right]}\langle\varphi(L)\rangle \cap K[x]\); // contraction of an ideal
        return \((N+J) / J\);
    end
```

Proposition 13. Algorithm 3 terminates and is correct.
Proof. Termination is clear. Consider the Laurent polynomial ring

$$
K\left[q^{ \pm 1}\right]:=K\left[q_{1}, q_{1}^{-1}, \ldots, q_{m}, q_{m}^{-1}\right]
$$

and a homomorphism of $K$-algebras $\varphi: K[x] \rightarrow K\left[x, q^{ \pm 1}\right], x_{i} \mapsto q_{i} x_{i}$. By Kreuzer and Robbiano (2005); Miller (2016); Saito et al. (2000), the biggest monomial ideal contained in $L \subseteq K[x]$ is exactly $N$.

Since for all $m \in K[x]$ we have $m+J \in L+J$ if and only if $m \in L$, this is in particular true for monomials. Therefore the biggest monomial ideal of $L+J$ is the biggest monomial ideal of $L$ modulo $J$.

Now we have all the tools to consider the general situation.
Proposition 14. Let $A$ be an associative (but not necessarily commutative) unital $K$-algebra and $F=\left\{f_{1}, \ldots, f_{m}\right\} \subseteq A$ be a set of pairwise commuting elements in $A$. Moreover, let $S \subseteq A$ be

```
Algorithm 4: NCIdeal IntersectionWithMonoid
    Input: A left ideal \(I \subseteq A\), a generating set (of a monoid \(S\) ) \(F=\left\{f_{1}, \ldots, f_{m}\right\}\) in the
            \(K\)-algebra \(A\), such that \(f_{i} \in A\) commute pairwise.
    Output: \(I \cap S\) : either \(\emptyset\) or a finite set of monomial generators \(\left\{t^{\alpha}: \alpha \in \mathbb{N}_{0}^{n}\right\} \subseteq\left[t_{1}, \ldots, t_{m}\right]\).
    begin
        \(\psi: K\left[t_{1}, \ldots, t_{m}\right] \rightarrow A, t_{i} \mapsto f_{i} ;\)
        \(L:=\psi^{-1}(I) \subseteq K\left[t_{1}, \ldots, t_{m}\right] ; \quad / /\) preimage of \(I \subseteq A\)
        if \(\psi(L)=0\) then
            return \(\emptyset ; \quad / /\) since then \(\psi^{-1}(I)=\operatorname{ker}(\psi)\)
        end
        \(R:=K\left[t_{1}, \ldots, t_{m}\right] / \operatorname{ker}(\psi)\);
        \(M:=\operatorname{BigGestMonomiaLIdeal}(L, R)\);
        if \(M=\{0\}\) then
            return \(\emptyset\);
        end
        return \(M\);
    end
```

the monoid in A generated by $F$. Then Algorithm 4 correctly computes $I \cap S$. Furthermore, its termination depends solely on the termination of the computation of $\psi^{-1}(I)$, which in turn depends on $A, I$ and $F$.

Proof. The $K$-monoid algebra $K[S]=K\left[f_{1}, \ldots, f_{m}\right] \subseteq A$ is a $K$-subalgebra of $A$ and there is a natural homomorphism of $K$-algebras

$$
\psi: K\left[t_{1}, \ldots, t_{m}\right] \rightarrow A, t_{i} \mapsto f_{i}
$$

Then $K[S] \cong K\left[t_{1}, \ldots, t_{m}\right] / \operatorname{ker}(\psi)$, hence the monoid algebra $K[S]$ is a finitely presented commutative $K$-algebra. As soon as the preimage $\psi^{-1}(I)=I \cap K\left[t_{1}, \ldots, t_{m}\right]$ is computable we are left with the following problem: given an ideal $L \subseteq K\left[t_{1}, \ldots, t_{m}\right] / J$, compute an intersection of $L$ with the submonoid $\left[t_{1}, \ldots, t_{m}\right]$, which is solved by Algorithm 3 .

Corollary 15. Consider the situation of Proposition 14.

- If A is a commutative polynomial algebra, Algorithm 4 terminates for any I and $F$.
- If A is a GR-algebra, the ncPreimage algorithm from Levandovskyy (2006) either returns the preimage or reports that the computability condition is violated. Namely, ncPreimage assumes that a GR-algebra $A$ is equipped with an admissible elimination ordering. If a certain integer programming problem has a solution, such an ordering can be constructed from it, while infeasibility of the problem proves that no such ordering exists.


### 3.2. Geometric

Let $\mathfrak{p}={ }_{R}\left\langle p_{1}+J, \ldots, p_{m}+J\right\rangle$ be a prime ideal in $R$ with $p_{i} \in K[x]$ and consider the multiplicative set $S:=R \backslash \mathfrak{p}$. The preimage of $\mathfrak{p}$ under the canonical surjection $K[x] \rightarrow R$ is given by the ideal $\mathfrak{q}:={ }_{K[x]}\left\langle p_{1}, \ldots, p_{m}, g_{1}, \ldots, g_{\ell}\right\rangle$. Now there are two possible cases:

Case 1: $h_{i} \in \mathfrak{q}$ for all $i$. Then $h_{i}+J \in \mathfrak{p}$ for all $i$ and thus $I \subseteq \mathfrak{p}$, which implies $I \cap S=I \cap(R \backslash \mathfrak{p})=$ $I \backslash \mathfrak{p}=\emptyset$.

Case 2: $h_{i} \notin \mathfrak{q}$ for some $i$. Then $h_{i}+J \notin \mathfrak{p}$ and thus $I \cap S \neq \emptyset$.
Since ideal membership in polynomial rings can be decided with Gröbner basis tools, this observations lead to Algorithm 5, where $\operatorname{NF}\left(h_{i} \mid \mathfrak{q}\right)$ denotes the normal form of $h_{i}$ with respect to (a Gröbner basis of) the ideal $\mathfrak{q}$.

```
Algorithm 5: CommutativeGeometricIntersection
    Input: Ideals \(I, J, \mathfrak{p}\) and the multiplicative set \(S\) as above.
    Output: An element of \(I \cap S\) (if \(I \cap S \neq \emptyset\) ) or 0 (if \(I \cap S=\emptyset\) ).
    begin
        let \(\mathfrak{q}:={ }_{K[x]}\left\langle p_{1}, \ldots, p_{m}, g_{1}, \ldots, g_{\ell}\right\rangle \subseteq K[x] ;\)
        foreach \(i \in\{1, \ldots, k\}\) do
            if \(\operatorname{NF}\left(h_{i} \mid \mathfrak{q}\right) \neq 0\) then
                return \(h_{i}+J\);
            end
        end
        return 0 ;
    end
```


### 3.3. Rational

Let $r \in\{1, \ldots, n\}$ and consider $\hat{S}:=K\left[x_{1}+J, \ldots, x_{r}+J\right]$ as well as $S:=\hat{S} \backslash\{0\}$. Let $K[t]:=K\left[t_{1}, \ldots, t_{r}\right]$ and define the $\operatorname{map} \varphi: K[t] \rightarrow R$ by $t_{i} \mapsto x_{i}$ for $1 \leq i \leq r$.

Lemma 16. In the situation above we have $I \cap S=\emptyset$ if and only if $\varphi^{-1}(I) \subseteq \operatorname{ker}(\varphi)$.
Proof. Let $I \cap S=\emptyset$, then $I \cap \hat{S}=\{0\}$. Now $\varphi^{-1}(I) \subseteq \operatorname{ker}(\varphi)$ follows directly from $\varphi\left(\varphi^{-1}(I)\right) \subseteq$ $I \cap \operatorname{im}(\varphi)=I \cap \hat{S}=\{0\}$.

On the other hand, let $\varphi^{-1}(I) \subseteq \operatorname{ker}(\varphi)$ and choose an element $w \in I \cap \hat{S}=I \cap \operatorname{im}(\varphi)$. Then there exists $v \in K[t]$ such that $\varphi(v)=w \in I$ and thus $v \in \varphi^{-1}(I) \subseteq \operatorname{ker}(\varphi)$. This implies $w=\varphi(v)=0$ and therefore $I \cap \hat{S}=\{0\}$ or, equivalently, $I \cap S=\emptyset$.

Proposition 17. Algorithm 6 terminates and is correct.
Proof. Termination is obvious. The preimage $\varphi^{-1}(I)$ can be computed via Algorithm 1. Now we check whether $\varphi^{-1}(I)$ is contained in $\operatorname{ker}(\varphi)$ on the generators $w_{i}$. Correctness follows then from Lemma 16.

Another way to look at Algorithm 6 is the following: the preimage computation gives us ${ }_{K[x]}\left\langle g_{1}, \ldots, g_{\ell}, h_{1}, \ldots, h_{k}\right\rangle \cap K\left[x_{1}, \ldots, x_{r}\right]$; then we look for generators of this ideal that do not vanish modulo $J$.

```
Algorithm 6: CommutativeRational Intersection
    Input: \(I, J, r, S\) as above.
    Output: An element of \(I \cap S\) (if \(I \cap S \neq \emptyset\) ) or 0 (if \(I \cap S=\emptyset\) ).
    begin
        let \(\varphi: K[t] \rightarrow K[x] / J, t_{i} \mapsto x_{i}\);
        compute the preimage \(\varphi^{-1}(I)={ }_{K[t]}\left\langle w_{1}, \ldots, w_{m}\right\rangle\);
        foreach \(i \in\{1, \ldots, m\}\) do
            if \(\varphi\left(w_{i}\right) \neq 0\) then
                return \(\varphi\left(w_{i}\right)\);
            end
        end
        return 0 ;
    end
```


## 4. Applications to local closure in the commutative setting

Recall the following basic concepts of the theory of commutative rings: The radical of an ideal $I$ in a commutative ring $R$ is the ideal defined as $\sqrt{I}:=\left\{r \in R \mid r^{n} \in I\right.$ for some $\left.n \in \mathbb{N}\right\}$. A proper ideal $I$ of a commutative ring $R$ is called primary if for all $a, b \in R$ such that $a b \in I$ we have $a \in I$ or $b \in \sqrt{I}$. In a more symmetric view, $I$ is primary if and only if for all $a, b \in R$ with $a b \in I$ we have $a \in I$ or $b \in I$ or ( $a \in \sqrt{I}$ and $b \in \sqrt{I}$ ). The radical of a primary ideal is always a prime ideal. If $Q$ is primary with radical $P=\sqrt{Q}$, then $Q$ is also called $P$-primary. An ideal is called decomposable if it can be written as an intersection of finitely many primary ideals.

The goal of this section is to show how to compute the $S$-closure of an ideal $I$ in a commutative ring $R$ under two assumptions:
(1) We can decide whether $Q \cap S=\emptyset$ for any primary ideal $Q$ in $R$.
(2) The ideal $I$ is decomposable and we are either given a primary ideal decomposition of $I$ or are able to compute one. Note that a primary decomposition always exists in Noetherian rings. In polynomial algebras it can be algorithmically computed, see e.g. Greuel and Pfister (2008), though its embedded components are not unique.

The main ingredient is the following observation, which highlights the differences between primary ideals and arbitrary ideals:

Lemma 18. Let $S$ be a multiplicative set of a commutative ring $R$.
(a) If $I$ is an arbitrary ideal in $R$ such that $I \cap S \neq \emptyset$, then $I^{S}=R$.
(b) If $Q$ is a primary ideal in $R$ such that $Q \cap S=\emptyset$, then $Q^{S}=Q$.

Proof. Let $w \in I \cap S$, then $w \cdot 1=w \in I$, thus $1 \in I^{S}$ and therefore $I^{S}=R$. On the other hand, let $Q \cap S=\emptyset$ and $r \in Q^{S}$, then there exists $s \in S$ such that $s r \in Q$. Since $Q$ is primary we have $s \in Q$ or $r \in Q$ or $(s \in \sqrt{Q}$ and $r \in \sqrt{Q})$. But $s \notin \sqrt{Q}$ (and therefore $s \notin Q$ ), because otherwise $s^{n} \in Q \cap S=\emptyset$ for some $n \in \mathbb{N}$. Thus the only remaining option is $r \in Q$, which implies $Q^{S}=Q$.

Let $I$ be a decomposable ideal with a primary decomposition $I=\bigcap_{i=1}^{n} Q_{i}$. Then $I^{S}=\bigcap_{i=1}^{n} Q_{i}^{S}$ by Lemma 10. Combining this with Lemma 18 we can compute $I^{S}$ for any multiplicative set $S$ via Algorithm 7 if we can decide non-emptiness of the intersections $Q_{i} \cap S$.

```
Algorithm 7: CommutativeLocalClosureDecomp
    Input: A decomposable ideal \(I=\bigcap_{i=1}^{n} Q_{i}\) and a multiplicative set \(S\) in a commutative ring
            \(R\).
    Output: \(I^{S}\).
    begin
        foreach \(i \in\{1, \ldots, n\}\) do
            if \(Q_{i} \cap S=\emptyset\) then
                \(\tilde{Q}_{i}:=Q_{i} ;\)
            else
                \(\tilde{Q}_{i}:=R ;\)
            end
        end
        return \(\tilde{I}:=\bigcap_{i=1}^{n} \tilde{Q}_{i} ;\)
    end
```

In particular, we have the following:
Corollary 19. Let $S$ be a multiplicative set of a commutative ring $R$ and $I$ a decomposable ideal in $R$. Then there exists an ideal $J$ in $R$ satisfying $I=I^{S} \cap J$ and $S^{-1} J=R$.

Proof. Let $I=\bigcap_{i=1}^{n} Q_{i}$ be a decomposition of $I$ into primary ideals. Then $I^{S}$ is the intersection of all $Q_{i}$ such that $Q_{i} \cap S=\emptyset$. Define $J$ to be the intersection of all $Q_{i}$ such that $Q_{i} \cap S \neq \emptyset$, then the claim follows from the observations above.

The question remains how to decide whether $Q \cap S$ is empty or not. This can be reduced to the same question for prime ideals:

Lemma 20. Let $S$ be a multiplicative set of a commutative ring $R$ and $Q$ a $P$-primary ideal in $R$. Then $Q \cap S=\emptyset$ if and only if $P \cap S=\emptyset$.

Proof. If $P \cap S=\emptyset$, then $Q \cap S=\emptyset$ since $Q \subseteq P$. If $s \in P \cap S$, then $s^{n} \in Q \cap S$ for some $n \in \mathbb{N}$.

For certain localization types, the latter can be characterized in a way that allows for an algorithmic treatment:

Lemma 21. Let $S$ be a multiplicative set of a commutative ring $R$ and $P$ a prime ideal in $R$.
(a) Let $S=\left[s_{1}, \ldots, s_{n}\right]$ for some $s_{i} \in R$. Then $P \cap S=\emptyset$ if and only if $P \cap\left\{s_{1}, \ldots, s_{n}\right\}=\emptyset$.
(b) Let $S=R \backslash \mathfrak{p}$ for some prime ideal $\mathfrak{p} \subset R$. Then $P \cap S=\emptyset$ if and only if $P \subseteq \mathfrak{p}$.
(c) Let $S=T \backslash\{0\}$ for some subring $T$ of $R$. Then $P \cap S=\emptyset$ if and only if $P \cap T=\{0\}$.

Proof. The last two claims are obvious from the definitions, so assume $S=\left[s_{1}, \ldots, s_{n}\right]$. If $P \cap S=\emptyset$, then $P \cap\left\{s_{1}, \ldots, s_{n}\right\}=\emptyset$ since $\left\{s_{1}, \ldots, s_{n}\right\} \subseteq S$. If $s \in P \cap S$, then $s=\prod_{i=1}^{n} s_{i}^{v_{i}} \in P$ for some $v_{i} \in \mathbb{N}_{0}$, where for at least one $j \in\{1, \ldots, n\}$ we have $v_{j} \geq 1$ since $1 \notin P \cap S$. But then $s_{j} \in P$ since $P$ is prime.

In polynomial algebras, this enables computations, since ideal membership test and intersection with essential subalgebras are classical applications of Gröbner bases.

With the methods developed so far we can address the important notion of the symbolic power of an ideal (see Dao et al. (2017) for a modern overview of a vivid area of investigations). Let $I \subseteq R$ be an ideal in a Noetherian domain $R$. Suppose that $I=\bigcap_{i=1}^{r} Q_{i}$ is a primary decomposition with associated primes $\mathfrak{p}_{i}:=\sqrt{Q_{i}}$. Then the $n$-th symbolic power of $I$ is defined to be

$$
I^{(n)}=\bigcap_{i=1}^{r}\left(R_{\mathfrak{p}_{i}} I^{n} \cap R\right),
$$

where $R_{\mathfrak{p}_{i}}=\left(R \backslash \mathfrak{p}_{i}\right)^{-1} R$ is a localization of geometric type with respect to the Ore set $S_{i}:=R \backslash \mathfrak{p}_{i}$. Furthermore, $R_{p_{i}} I^{n} \cap R$ is exactly the local closure of $I^{n}$ with respect to $S_{i}$, which implies the inclusion $I^{n} \subseteq I^{(n)}$ for all $n$. In Example 23 we will see that equality does not hold in general. In the special case where $I=\mathfrak{p}$ is a prime ideal, the symbolic power $\mathfrak{p}^{(n)}$ is precisely the $\mathfrak{p}$-primary component of $\mathfrak{p}^{n}$.

These observations immediately lead to the following Algorithm 8 as an application of Algorithm 7 and utilizing Algorithm 5 to decide non-emptiness of the intersections.

```
Algorithm 8: SymbolicPower
    Input: A decomposable ideal \(I\) in a commutative domain \(R\) and \(n \in \mathbb{N}\).
    Output: \(I^{(n)}\).
    begin
        compute the associated primes \(\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{r}\) of \(I\);
        compute a primary decomposition \(I^{n}=\bigcap_{i=1}^{m} Q_{i}\) with associated primes \(\mathfrak{q}_{i}:=\sqrt{Q_{i}}\);
        foreach \(i \in\{1, \ldots, m\}\) do
            \(\tilde{Q}_{i}:=R\);
            foreach \(j \in\{1, \ldots, r\}\) do
                if \(\mathfrak{q}_{i} \subseteq \mathfrak{p}_{j}\) then
                \(\tilde{Q}_{i}:=\tilde{Q}_{i} \cap Q_{i} ;\)
            end
        end
        return \(\tilde{I}:=\bigcap_{i=1}^{m} \tilde{Q}_{i} ;\)
    end
```

Proposition 22. Algorithm 8 terminates and is correct.
Proof. Termination is obvious. The containment of ideals in Line 7 is equivalent to $Q_{i} \cap S_{j}=\emptyset$ via Lemma 20 and Lemma 21. If the latter condition fails, $Q_{i}$ can be ignored, since $Q_{i}^{S_{j}}=R$ by Lemma 18.

Example 23. Perhaps the most popular example for $\mathfrak{p}^{(2)} \neq \mathfrak{p}^{2}$ is given by

$$
\mathfrak{p}=\left\langle x^{4}-y z, y^{2}-x z, x^{3} y-z^{2}\right\rangle \subseteq \mathbb{Q}[x, y, z] .
$$

In this situation, $\mathfrak{p}^{2}$ has two associated primes: $\mathfrak{p}$ and $\langle x, y, z\rangle$. Only the $\mathfrak{p}$-primary component survives in the closure, giving $\mathfrak{p}^{2}=\mathfrak{p}^{(2)} \cap\left\langle z, y^{4}, x^{3} y^{3}, x^{4} y^{2}, x^{7} y, x^{8}\right\rangle$, where
$\mathfrak{p}^{(2)}=\left\langle y^{4}-2 x y^{2} z+x^{2} z^{2}, x^{3} y^{3}-x^{4} y z-y^{2} z^{2}+x z^{3}, x^{4} y^{2}-x^{5} z-y^{3} z+x y z^{2}, x^{7}+x^{2} y^{3}-3 x^{3} y z+z^{3}\right\rangle$.

## 5. Central closure of submodules

A central closure of a submodule is the local closure of a submodule with respect to a left Ore set which is contained in the center of the underlying ring. The goal of this section is to develop algorithms for certain central closures of submodules over $G$-algebras.

### 5.1. The class of G-algebras

Recall that a total ordering $\leq$ on $\mathbb{N}_{0}^{n}$ with least element 0 is called admissible if $\alpha \leq \beta$ implies $\alpha+\gamma \leq \beta+\gamma$ for all $\alpha, \beta, \gamma \in \mathbb{N}_{0}^{n}$.
Definition 24. Let $K$ be a field and $A$ a $K$-algebra generated by $x_{1}, \ldots, x_{n}$.

- The set of standard monomials of $A$ is

$$
\operatorname{Mon}(A):=\left\{x^{\alpha} \mid \alpha \in \mathbb{N}_{0}^{\eta}\right\}:=\left\{x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{n}^{\alpha_{n}} \mid \alpha_{i} \in \mathbb{N}_{0}\right\}
$$

- Let $\leq$ be an admissible ordering on $\mathbb{N}_{0}^{n}$. Any $f \in{ }_{K}\langle\operatorname{Mon}(A)\rangle \backslash\{0\}$ has a unique representation $f=\sum_{\alpha \in \mathbb{N}_{0}^{n}} c_{\alpha} x^{\alpha}$ for some $c_{\alpha} \in K$, where $c_{\alpha}=0$ for almost all $\alpha$, but $c_{\alpha} \neq 0$ for at least one $\alpha$. Now we define

$$
\begin{aligned}
& \text { - } \operatorname{le}_{\leq}(f):=\max _{\leq}\left\{\alpha \in \mathbb{N}_{0}^{n} \mid c_{\alpha} \neq 0\right\} \text {, the leading exponent of } f \text { with respect to } \leq, \\
& \text { - } \operatorname{lc}_{\leq}(f):=c_{\mathrm{le}_{\leq}(f)} \in K \backslash\{0\} \text {, the leading coefficient of } f \text { with respect to } \leq, \\
& \text { - } \operatorname{lm}_{\leq}(f):=x^{\operatorname{le}_{\leq}(f)} \in \operatorname{Mon}(A) \text {, the leading monomial of } f \text { with respect to } \leq .
\end{aligned}
$$

Definition 25. For $n \in \mathbb{N}$ and $1 \leq i<j \leq n$ consider the constants $c_{i j} \in K \backslash\{0\}$ and polynomials $d_{i j} \in K\left[x_{1}, \ldots, x_{n}\right]$. Suppose that there exists an admissible ordering $\leq$ on $\mathbb{N}_{0}^{n}$ such that for any $1 \leq i<j \leq n$ either $d_{i j}=0$ or $\mathrm{l}_{\leq}\left(d_{i j}\right)<\mathrm{l}_{\leq}\left(x_{i} x_{j}\right)$. The $K$-algebra

$$
A:=K\left\langle x_{1}, \ldots, x_{n} \mid\left\{x_{j} x_{i}=c_{i j} x_{i} x_{j}+d_{i j}: 1 \leq i<j \leq n\right\}\right\rangle
$$

is called a G-algebra if $\operatorname{Mon}(A)$ is a $K$-basis of $A$.
$G$-algebras (Levandovskyy and Schönemann (2003); Levandovskyy (2005)) are also known as algebras of solvable type (Kandri-Rody and Weispfenning (1990); $\operatorname{Kredel}(1993,2015)$ ) and as PBW algebras (Bueso et al. (2003)). $G$-algebras are left and right Noetherian domains that occur naturally in various situations and encompass algebras of linear functional operators modeling difference and differential equations.

Example 26. Let $K$ be a field, $q_{i} \in K \backslash\{0\}$ and $n \in \mathbb{N}$. Common $G$-algebras include the following examples, where only the relations between non-commutating variables are listed:

- The commutative polynomial ring $K\left[x_{1}, \ldots, x_{n}\right]$.
- The n-th Weyl algebra $A_{n}:=K\left\langle x_{1}, \ldots, x_{n}, \partial_{1}, \ldots, \partial_{n}\right\rangle$ with $\partial_{i} x_{i}=x_{i} \partial_{i}+1$ for all $1 \leq i \leq n$.
- The $n$-th shift algebra $S_{n}:=K\left\langle x_{1}, \ldots, x_{n}, s_{1}, \ldots, s_{n}\right\rangle$ with $s_{i} x_{i}=x_{i} s_{i}+s_{i}=\left(x_{i}+1\right) s_{i}$ for all $1 \leq i \leq n$.
- The $n$-th $q$-shift algebra $S_{n}^{(q)}:=K\left\langle x_{1}, \ldots, x_{n}, s_{1}, \ldots, s_{n}\right\rangle$ with $s_{i} x_{i}=q_{i} x_{i} s_{i}$ for all $1 \leq i \leq n$.
- The n-th $q$-Weyl algebra $A_{n}^{(q)}:=K\left\langle x_{1}, \ldots, x_{n}, \partial_{1}, \ldots, \partial_{n}\right\rangle$ with $\partial_{i} x_{i}=q_{i} x_{i} \partial_{i}+1$ for all $1 \leq i \leq n$.
- The $n$-th integration algebra $K\left\langle x_{1}, \ldots, x_{n}, I_{1}, \ldots, I_{n}\right\rangle$ with $I_{i} x_{i}=x_{i} I_{i}+I_{i}^{2}$ for all $1 \leq i \leq n$.

Furthermore, there exists a well-developed Gröbner basis theory for $G$-algebras which is close to the commutative case.

If considered over a field of prime characteristic, Weyl and shift algebras have very big centers. A similar situation happens if all quantum parameters $q_{i}$ are roots of unity for $q$-shift and $q$-Weyl algebras. Then the mentioned algebras are finitely generated modules over their centers and thus enjoy a lot of commutativity.

Note that any admissible ordering $\leq$ on $\mathbb{N}_{0}^{n}$ can be extended to an admissible ordering $\leq$ on $\{1, \ldots, r\} \times \mathbb{N}_{0}^{n}$, e.g. $\leq=\leq^{\text {POT }}$ :
Definition 27. Let $\leq$ be an admissible ordering on $\mathbb{N}_{0}^{n}$. The (ascending) position-over-term ordering extending $\leq$ is the ordering $\leq{ }^{\mathrm{POT}}$ on $\{1, \ldots, r\} \times \mathbb{N}_{0}^{n}$, defined via

$$
(i, \alpha) \leq{ }^{\mathrm{POT}}(j, \beta) \quad: \Leftrightarrow \quad i<j \text { or }(i=j \text { and } \alpha \leq \beta)
$$

for $\alpha, \beta \in \mathbb{N}_{0}^{n}$ and $i, j \in\{1, \ldots, r\}$.
We recall the definition of Gröbner bases and their characterization via left normal forms in preparation for the forthcoming Algorithm 10.

Definition 28. Let $A$ be a $G$-algebra, $r \in \mathbb{N}$, I a left $A$-submodule of $A^{r}, G$ a finite subset of $I$ and $\leq$ an admissible ordering on $\{1, \ldots, r\} \times \mathbb{N}_{0}^{n}$. Then $G$ is a left Gröbner basis of I with respect to $\leq$ if for all $f \in I \backslash\{0\}$ there exists $g \in G$ such that $\operatorname{lm}_{\preceq}(g) \mid \operatorname{lm}_{\preceq}(f)$.

Theorem 29. In the situation of Definition 28 the following are equivalent:
(1) G is a left Gröbner basis of I with respect to $\leq$.
(2) $\operatorname{LeftNF}_{\leq}(f \mid G)=0$ for all $f \in I$.

Further information can be found in e. g. Levandovskyy (2005) and Bueso et al. (2003).

### 5.2. Central saturation

Denote the center of a ring $R$ by $Z(R)$. Its elements are called central.
Definition 30. Let $R$ be a ring, $q \in Z(R), k \in \mathbb{N}$ and I a left $R$-submodule of $R^{k}$.

- The (central) quotient of I by $q$ is the left $R$-submodule

$$
I: q:=\left\{f \in R^{k} \mid q f \in I\right\}=\left\{f \in R^{k} \mid f q \in I\right\}
$$

```
Algorithm 9: LeftNF
    Input: A \(G\)-algebra \(A\) generated by variables \(x=\left\{x_{1}, \ldots, x_{n}\right\}, r \in \mathbb{N}, f \in A^{r}\), a finite
        subset \(G\) of \(A^{r}\), an admissible ordering \(\leq\) on \(\{1, \ldots, r\} \times \mathbb{N}_{0}^{n}\).
    Output: \(\operatorname{LeftNF}_{\preceq}(f \mid G)\).
    begin
        \(h:=f\);
        while \(h \neq 0\) and \(G_{h}:=\left\{g \in G: \operatorname{lm}_{\leq}(g) \mid \operatorname{lm}_{\leq}(h)\right\} \neq \emptyset\) do
            choose \(g \in G_{h}\);
            \((i, \alpha):=\mathrm{l}_{\leq}(h)\);
            \((i, \beta):=\mathrm{le}_{\leq}(g) ;\)
            \(h:=h-\frac{\operatorname{lc}_{c}(h)}{\operatorname{lc}_{\leq}\left(x^{\alpha-\beta} g\right)} x^{\alpha-\beta} g ;\)
        end
        return \(h\);
    end
```

- The central saturation of I by $q$ is the left $R$-submodule

$$
I: q^{\infty}:=\bigcup_{i \in \mathbb{N}_{0}}\left(I: q^{i}\right)=\left\{f \in R^{k} \mid \exists n \in \mathbb{N}_{0}: q^{n} f \in I\right\} .
$$

- The (central) saturation index of I by $q$ is

$$
\text { Satindex }(I, q):=\min \left(\left\{n \in \mathbb{N}_{0} \mid\left(I: q^{\infty}\right)=\left(I: q^{n}\right)\right\} \cup\{\infty\}\right)
$$

These saturations themselves are special cases of (generalized) left saturation closures, since $I: q=\operatorname{LSat}_{\{q\}}^{R^{k}}(I)$ and $I: q^{\infty}=\operatorname{LSat}_{[q]}^{R^{k}}(I)=I^{[q]}$ (by extending Definition 8 to arbitrary sets $S$ ).

Remark 31. In the situation of Definition 30, consider the left $R$-module homomorphism $\phi$ : $R^{k} \rightarrow R^{k} / I, f \mapsto f q+I$. We have

$$
\operatorname{ker}(\phi)=\left\{f \in R^{k} \mid q f+I=\phi(f)=0+I\right\}=I: q .
$$

Thus, if we can compute kernels of such left R-module homomorphisms, we can also compute central quotients. Furthermore, if we can decide equality of left $R$-modules, then we can also compute the central saturation iteratively, provided the saturation index is finite. The latter is always the case for Noetherian rings.

In particular, we have the following result for finitely generated monoidal central closures:
Remark 32. Let $S=\left[f_{1}, \ldots, f_{k}\right]$ be a left Ore set in a G-algebra A, I a left ideal in $A$ and $z \in Z(A) \cap S$. Then $I^{[z]}=\operatorname{LSat}_{[q]}(I)=I: q^{\infty}$ is computable. Since $[z] \subseteq S$ we have $I^{[z]} \subseteq I^{S}$. The other inclusion holds if $\operatorname{LSat}(S)=\operatorname{LSat}([z])$, which is equivalent to $f_{j} \in \operatorname{LSat}([z])$ for all $j$. A sufficient condition for this is that $f_{1}, \ldots, f_{k}$ commute pairwise and $z$ is a multiple of $f_{1} \cdot \ldots \cdot f_{k}$. This also includes the special case where $f_{1}, \ldots, f_{k} \in Z(A)$.

The restriction to central $q$ allows straightforward generalizations of classical commutative results, e.g. regarding the decomposition of ideals:

Lemma 33. Let $I$ be a left ideal in a ring $R$ and $q \in Z(R)$. If $n:=\operatorname{Satindex}(I, q)<\infty$, then $I={ }_{R}\left\langle I, q^{n}\right\rangle \cap\left(I: q^{n}\right)$.

Proof. Let $J:={ }_{R}\left\langle I, q^{n}\right\rangle \cap\left(I: q^{n}\right)$. Since $I \subseteq{ }_{R}\left\langle I, q^{n}\right\rangle$ and $I \subseteq\left(I: q^{n}\right)$ we clearly have $I \subseteq J$. On the other hand, let $a \in J$, then $q^{n} a \in I$ (since $a \in\left(I: q^{n}\right)$ ) and $a=b+r q^{n}$ for some $b \in I$ and $r \in R$ (since $a \in_{R}\left\langle I, q^{n}\right\rangle$ ). Now

$$
q^{2 n} r=q^{n} r q^{n}=q^{n}(a-b)=q^{n} a-q^{n} b \in I
$$

shows that $r \in\left(I: q^{2 n}\right)=\left(I: q^{n}\right)$, which implies $r q^{n}=q^{n} r \in I$, thus $a=b+r q^{n} \in I$.

### 5.3. Antiblock orderings

In preparation for the upcoming Algorithm 10 we need the notion of antiblock orderings as well as some basic results which are included here for the sake of completeness.

Let $n, m, r \in \mathbb{N}$.
Definition 34. Let $\leq$ be an admissible ordering on $\mathbb{N}_{0}^{n}$. The (ascending) position-over-term ordering extending $\leq$ is the ordering $\leq{ }^{\mathrm{POT}}$ on $\{1, \ldots, r\} \times \mathbb{N}_{0}^{n}$, defined via

$$
(i, \alpha) \leq^{\mathrm{POT}}(j, \beta) \quad: \Leftrightarrow \quad i<j \text { or }(i=j \text { and } \alpha \leq \beta)
$$

for $\alpha, \beta \in \mathbb{N}_{0}^{n}$ and $i, j \in\{1, \ldots, r\}$.
Lemma 35. Let $\leq=\left(\leq_{1}, \leq_{2}\right)$ be an ( $n, m$ )-antiblock ordering on $\mathbb{N}_{0}^{n+m} \cong \mathbb{N}_{0}^{n} \times \mathbb{N}_{0}^{m}$ and $\leq:=\leq^{\text {POT }}$. Let $\left(\alpha_{1}, \alpha_{2}\right),\left(\beta_{1}, \beta_{2}\right) \in \mathbb{N}_{0}^{n+m}$ and $i, j \in\{1, \ldots, r\}$ such that we have $\left(i,\left(\alpha_{1}, \alpha_{2}\right)\right) \leq\left(j,\left(\beta_{1}, \beta_{2}\right)\right)$, then $\left(i, \alpha_{2}\right) \leq_{2}{ }^{\mathrm{POT}}\left(j, \beta_{2}\right)$.

Proof. We have

$$
\begin{aligned}
\left(i,\left(\alpha_{1}, \alpha_{2}\right)\right) \leq\left(j,\left(\beta_{1}, \beta_{2}\right)\right) & \Leftrightarrow\left(i,\left(\alpha_{1}, \alpha_{2}\right)\right) \leq^{\mathrm{POT}}\left(j,\left(\beta_{1}, \beta_{2}\right)\right) \\
& \Leftrightarrow i<j \text { or }\left(i=j \text { and }\left(\alpha_{1}, \alpha_{2}\right) \leq\left(\beta_{1}, \beta_{2}\right)\right) \\
& \Rightarrow i<j \text { or }\left(i=j \text { and } \alpha_{2} \leq_{2} \beta_{2}\right) \\
& \Leftrightarrow\left(i, \alpha_{2}\right) \leq_{2}{ }^{\mathrm{POT}}\left(j, \beta_{2}\right) .
\end{aligned}
$$

### 5.4. Central essential rational closure

In this section, let $K$ be a field, $n, m, r \in \mathbb{N}$ and $A$ a $G$-algebra over $K$ generated by two blocks of variables $x=\left\{x_{1}, \ldots, x_{n}\right\}$ and $y=\left\{y_{1}, \ldots, y_{m}\right\}$ such that $x$ generates a sub- $G$-algebra $B$ of $A$ with $B \subseteq Z(A)$. Then $S:=B \backslash\{0\}$ is a left Ore set in $B$ as well as in $A$ since it is a multiplicative set consisting of central elements. Furthermore, let $\leq=\left(\leq_{1}, \leq_{2}\right)$ be an ( $n, m$ )-antiblock ordering satisfying the ordering condition for $G$-algebras on $A$ and $\leq:=\leq^{\text {POT }}$. Finally, let $\varepsilon:=\varepsilon_{S, A, A^{r}}$, $\rho:=\rho_{S, A}$ and $\leq_{2}=\leq_{2}{ }^{\mathrm{POT}}$. Observe the following:

- We have $\operatorname{ker}(\varepsilon)=\left\{m \in A^{r} \mid \exists s \in S: s m=0\right\}=\{0\}$ since $A$ is a domain, thus $\varepsilon$ is injective.
- Since $B \subseteq Z(A)$ we can identify the subring $B$ with the commutative polynomial ring $K[x]=K\left[x_{1}, \ldots, x_{n}\right]$. Then we have $S^{-1} B \cong K(x)$.
- We can view $S^{-1} A$ as a $G$-algebra over the field $K(x)$ in the variables $y_{1}, \ldots, y_{m}$ with the relations inherited from $A$, thus the Gröbner basis theory of $G$-algebras applies.
- The monomials in the module $S^{-1}\left(A^{r}\right) \cong\left(S^{-1} A\right)^{r}$ are of the form $\varepsilon\left(y^{\alpha} e_{i}\right)=\rho\left(y^{\alpha}\right) \varepsilon\left(e_{i}\right)$. Let $(s, f) \in S^{-1} A^{r}$, then $(s, f)$ and $(1, f)=\varepsilon(f)$ have the same leading exponent and the same leading monomial with respect to $\leq_{2}$.

Lemma 36. Let $f \in A^{r} \backslash\{0\}$ and $\mathrm{l}_{\leq}(f)=\left(i,\left(\alpha_{1}, \alpha_{2}\right)\right)$, then $\mathrm{le}_{\leq_{2}}(\varepsilon(f))=\left(i, \alpha_{2}\right)$.
Proof. Define $\underline{r}:=\{1, \ldots, r\}$ and let

$$
f=\sum_{(j,(\beta, \gamma)) \in \underline{x} \times \mathbb{N}_{0}^{n+m}} c_{(j,(\beta, \gamma))} x^{\beta} y^{\gamma} e_{j}=\sum_{(j, \gamma) \in \underline{r} \times \mathbb{N}_{0}^{m}} \underbrace{\left.\sum_{\beta \in \mathbb{N}_{0}^{n}} c_{(j,(\beta, \gamma))} x^{\beta}\right)}_{=: \tilde{c}_{(j, \gamma)} \in K[x]} y^{\gamma} e_{j}
$$

with $c_{(j,(\beta, \gamma))} \in K$, then $(j,(\beta, \gamma)) \leq\left(i,\left(\alpha_{1}, \alpha_{2}\right)\right)=\operatorname{le}_{\leq}(f)$ whenever $c_{(j,(\beta, \gamma))} \neq 0$. Furthermore,

$$
\varepsilon(f)=\varepsilon\left(\sum_{(j, \gamma) \in \underline{-} \times \mathbb{N}_{0}^{m}} \tilde{c}_{(j, \gamma)} y^{\gamma} e_{j}\right)=\sum_{(j, \gamma) \in \underline{\underline{x}} \times \mathbb{N}_{0}^{m}} \varepsilon\left(\tilde{c}_{(j, \gamma)} y^{\gamma} e_{j}\right)=\sum_{(j, \gamma) \in \underline{-} \times \mathbb{N}_{0}^{m}} \rho\left(\tilde{c}_{j, \gamma}\right) \cdot \varepsilon\left(y^{\gamma} e_{j}\right)
$$

implies that it suffices to show that $(j, \gamma) \leq_{2}\left(i, \alpha_{2}\right)$ whenever $\tilde{c}_{(j, \gamma)} \neq 0$. The last condition implies that there is some $\beta \in \mathbb{N}_{0}^{n}$ such that $c_{(j,(\beta, \gamma))} \neq 0$. Now $(j,(\beta, \gamma)) \leq\left(i,\left(\alpha_{1}, \alpha_{2}\right)\right)=\operatorname{le}_{\leq}(f)$ implies $(j, \gamma) \leq_{2}\left(i, \alpha_{2}\right)$ by Lemma 35, thus le $\leq_{2}(\varepsilon(f))=\left(i, \alpha_{2}\right)$.

Proposition 37. Let I be a left $A$-submodule of $A^{r}$ and $G$ a left Gröbner basis of I with respect to $\leq$. Then $\varepsilon(G)$ is a left Gröbner basis of $J:=S^{-1} I$ with respect to $\leq_{2}=\leq_{2}{ }^{\mathrm{POT}}$.
Proof. Let $z \in J \backslash\{0\}$, then $z=(s, f)$ for some $s \in S$ and $f \in I$. Since $G$ is a left Gröbner basis of $I$ there exists $g \in G$ such that $\operatorname{lm}_{\leq}(g) \mid \operatorname{lm}_{\leq}(f)$. In terms of leading exponents, where $\left(i,\left(\alpha_{1}, \alpha_{2}\right)\right)=\mathrm{le}_{\leq}(g)$ and $\left(j,\left(\beta_{1}, \beta_{2}\right)\right)=\mathrm{le}_{\leq}(f)$, this means $i=j$ and $\left(\alpha_{1}, \alpha_{2}\right) \leq\left(\beta_{1}, \beta_{2}\right)$, in particular, we have $\alpha_{2} \leq_{2} \beta_{2}$. Since $\mathrm{le}_{\leq_{2}}(\varepsilon(g))=\left(i, \alpha_{2}\right)$ and $\mathrm{le}_{\leq}((s, f))=\mathrm{le}_{\leq_{2}}(\varepsilon(f))=\left(j, \beta_{2}\right)$ by the previous Lemma 36, we have $\operatorname{lm}_{\leq_{2}}(\varepsilon(g)) \mid \operatorname{lm}_{\leq_{2}}(z)$.
Definition 38. Consider a polynomial $f \in K[x] \backslash K$. Since $K[x]$ is a unique factorization domain, $f$ has a representation as a product of finitely many irreducible elements. The square-free part of $f$, denoted $\sqrt{f}$, is the product of all unique irreducible elements up to associativity that occur in this factorization.
Remark 39. Algorithm 10 is based on its commutative special case which can be found in Becker and Weispfenning (1993), Table 8.8, as algorithm ЕхтCont.

In the situation of Algorithm 10, the candidate $h$ is constructed such that for any $g \in H$ there exists $\ell \in \mathbb{N}$ satisfying $\mathrm{lc}_{\leq_{2}}(\varepsilon(g)) \mid h^{\ell}$.
Proposition 40. Algorithm 10 terminates and is correct.
Proof. The saturation index computation is finite since all $G$-algebras are Noetherian, thus termination of the whole algorithm is ensured. To prove correctness we have to show that $I^{S}=I: h^{k}$.

First, let $f \in I: h^{k}$, then $h^{k} f \in I$ and $\varepsilon(f)=(1, f)=\left(h^{k}, h^{k} f\right) \in S^{-1} I$. Thus we have $f \in \varepsilon^{-1}\left(S^{-1} I\right)=I^{S}$, which implies $I: h^{k} \subseteq I^{S}$.

For the other inclusion, let $f \in I^{S}=\varepsilon^{-1}\left(S^{-1} I\right)$, then $\varepsilon(f) \in S^{-1} I$. Now $\varepsilon(H)$ is a left Gröbner basis of $S^{-1} I$ with respect to $\leq_{2}$ by Proposition 37. Furthermore, Theorem 29 implies that $\operatorname{LeftNF}(\varepsilon(f) \mid \varepsilon(H))=0$. We now prove $f \in I: h^{k}$ by an induction on the minimal number $N \in \mathbb{N}$ of steps necessary in the left normal form algorithm given in Algorithm 9 to reduce $\varepsilon(f)$ to zero:

```
Algorithm 10: CentralEssential RationalClosure
    Input: A left \(A\)-submodule \(I\) of \(A^{r}\).
    Output: A left Gröbner basis \(G \subseteq A^{r}\) of \(I^{S}\) with respect to \(\leq\).
    begin
        \(H:=\operatorname{LeftGröbnerBasis}(I, \leq) ;\)
        \(h:=\sqrt{\prod_{g \in H} \operatorname{lc}_{ธ_{2}}(\varepsilon(g))} \in K[x] \backslash\{0\} ;\)
        \(k:=\operatorname{Satindex}(I, h)\);
        \(G:=\operatorname{LeftGröbNERBASIS}\left(I: h^{k}, \leq\right)\);
        return \(G\);
    end
```

Induction base: If $N=0$, then $\varepsilon(f)=0$. Since $\varepsilon$ is injective we have $f=0$, which trivially implies $f \in I: h^{\infty}$.

Induction hypothesis: Assume that for any $\tilde{f} \in I^{S}$, such that $\varepsilon(\tilde{f})$ can be reduced to zero in $N-1$ steps by the left normal form algorithm with respect to $\varepsilon(H)$, we have $\tilde{f} \in I: h^{k}$.

Induction step: Let $f \in I^{S}$ such that the left normal form algorithm needs at least $N$ steps to reduce $\varepsilon(f)$ to zero with respect to $\varepsilon(H)$. Then there exists $g \in H$ such that $\operatorname{lm}_{\leq_{2}}(\varepsilon(g)) \mid$ $\operatorname{lm}_{\leq_{2}}(\varepsilon(f))$. Let $\left(i_{f}, \alpha\right)=\mathrm{l}_{\leq_{2}}(\varepsilon(f))$ and $\left(i_{g}, \beta\right)=\mathrm{l}_{\leq_{2}}(\varepsilon(g))$, then $i_{g}=i_{f}$ and

$$
t:=\varepsilon(f)-\frac{\mathrm{lc}_{\leq_{2}}(\varepsilon(f))}{\operatorname{lc}_{\leq_{2}}\left(\varepsilon\left(y^{\alpha-\beta} g\right)\right)} \rho\left(y^{\alpha-\beta}\right) \varepsilon(g) \in S^{-1} A^{r}
$$

can be reduced to zero in $N-1$ steps with respect to $\varepsilon(H)$. Since the relations between the variables in $A$ have the form $y_{j} y_{i}=c_{i j} y_{i} y_{j}+d_{i j}$ for some $c_{i j} \in K \backslash\{0\}$ and $d_{i j} \in A$ such that $\mathrm{le}_{\leq}\left(d_{i j}\right)<\mathrm{le}_{\leq}\left(y_{i} y_{j}\right)$, we have

$$
\mathrm{lc}_{\leq_{2}}\left(\varepsilon\left(y^{\alpha-\beta} g\right)\right)=u \cdot \mathrm{lc}_{\leq_{2}}(\varepsilon(g))
$$

for some $u \in K \backslash\{0\}$, which is just the product of all $c_{i j}$ that occur while bringing $\varepsilon\left(y^{\alpha-\beta} g\right)$ in standard monomial form, and thus

$$
t=\varepsilon(f)-\frac{\mathrm{lc}_{\leq_{2}}(\varepsilon(f))}{u \mathrm{lc}_{\leq_{2}}(\varepsilon(g))} \rho\left(y^{\alpha-\beta}\right) \varepsilon(g) .
$$

Since $\operatorname{lc}_{\leq_{2}}(\varepsilon(g))$ divides a power of $h$, there exists $\ell \in \mathbb{N}$ such that

$$
c:=\frac{h^{\ell}}{u \operatorname{lc}_{\unlhd_{2}}(\varepsilon(g))} \in K[x] \backslash\{0\}
$$

and therefore

$$
\tilde{f}:=h^{\ell} f-c \operatorname{lc}_{\leq_{2}}(\varepsilon(f)) y^{\alpha-\beta} g \in I^{S},
$$

since $f \in I^{S}$ by assumption and $g \in H \subseteq I \subseteq I^{S}$. Now

$$
\begin{aligned}
h^{\ell} t & =h^{\ell} \varepsilon(f)-\frac{h^{\ell}}{u \operatorname{lc}_{\leq_{2}}(\varepsilon(g))} \operatorname{lc}_{\leq_{2}}(\varepsilon(f)) \rho\left(y^{\alpha-\beta}\right) \varepsilon(g) \\
& =h^{\ell} \varepsilon(f)-c \operatorname{lc}_{\leq_{2}}(\varepsilon(f)) \rho\left(y^{\alpha-\beta}\right) \varepsilon(g) \\
& =\varepsilon\left(h^{\ell} f-c \operatorname{c}_{\leq_{2}}(\varepsilon(f)) y^{\alpha-\beta} g\right) \\
& =\varepsilon(\tilde{f}),
\end{aligned}
$$

thus we can apply the induction hypothesis: we have $\tilde{f} \in I^{S}$ such that $\varepsilon(\tilde{f})=h^{\ell} t$ can be reduced to zero in $N-1$ steps with respect to $\varepsilon(H)$, since $h^{\ell} \in S$ is invertible in $S^{-1} A$ and thus does not change the reducibility of $t$. This gives us $\tilde{f} \in I: h^{k}$ or $h^{k} \tilde{f} \in I$. Now

$$
h^{\ell+k} f=h^{k} \tilde{f}+h^{k} c \mathrm{c}_{\leq_{2}}(\varepsilon(f)) y^{\alpha-\beta} g \in I
$$

implies $f \in I: h^{\ell+k}=I: h^{k}$, which shows $I^{S} \subseteq I: h^{k}$.
Remark 41. Let $R$ be a commutative principal ideal domain, which is a computable ring with the field of fractions $Q$. By replacing $K[x]$ with $R$ in Algorithm 10, we observe that the same proof can be applied to the following more general situation: suppose that A is a G-algebra over $Q$ which contains $Q[x]$. Assume further that $c_{i j}$ and all the coefficients of $d_{i j}$ are in $R$, then we define $A_{R}$ to be an $R$-algebra subject to the same relations as $A$.

Consider the algorithm applied for a G-algebra $A$ over $Q$, a left submodule $I \subseteq A^{r}$, an algebra $A_{R}$ over $R$, and $S=R \backslash\{0\}$. We replace $K[x]$ with $R$ and do not need to employ an antiblock ordering. After computing a left Gröbner basis $H$ of I over the field $Q$ we can assume that no denominators are present in $H$. Now the candidate $h \in R \backslash\{0\}$ and the rest of the algorithm is the same. Also the proof carries almost verbatim with only one modification: since $c:=\frac{h^{l}}{u \mathrm{c}_{\leq_{2}}(\varepsilon(g))} \in Q \backslash\{0\}$ is a fraction, while $h, u, \mathrm{lc}_{\leq_{2}}(\varepsilon(g)) \in R$, we just have to replace $\tilde{f}$ with $\hat{f}:=u \cdot \tilde{f} \in I^{S}$. Of course, the computations of the saturation index and the final left Gröbner basis happen over $R$ (which would require a special implementation in comparison to the case of ground fields). A very natural application of the described algorithm is for $R=\mathbb{Z}$.

We implemented Algorithm 10 using the computer algebra system Singular:Plural (Greuel et al. (2016)) and used it on problems coming from e.g. $D$-module theory:

Example 42. In $\mathcal{D}_{3}[s]$, the third Weyl algebra over the field $K=\mathbb{Q}$ with an additional commutative variable $s$, we compute the $K[s] \backslash\{0\}$-closure of the left ideal $L_{1}$ which is generated by the elements of order 1 in the derivatives

$$
x \partial_{x}+y \partial_{y}-5 s, x z \partial_{z}+y \partial_{z}-x s, y^{2} z^{2} \partial_{z}+y^{3} \partial_{x}+x^{3} \partial_{y}-y^{2} z s-x^{2} \partial_{z}
$$

The candidate used for saturation is $25 s^{2}+25 s+6=(5 s+2)(5 s+3)$ and the saturation is reached after one step taking barely any time. The resulting ideal $L$ is a part of the annihilating ideal I of the special function $\left((x z+y)\left(x^{4}-y^{4}\right)\right)^{s}$. Notably, the factor $5 s+2$ is still present among the leading coefficients of generators of $L$. Moreover, $L_{1} \subsetneq L$ shows that I cannot be generated by the elements of order 1 only.

The following establishes a sufficient condition for computing local closures iteratively:
Lemma 43. Let $S_{1}$ and $S_{2}$ be left denominator sets in a ring $R, M$ a left $R$-module and $P$ a left $R$-submodule of M. If $S_{1} S_{2}=S_{2} S_{1}$, then $P^{S_{1} S_{2}}=\left(P^{S_{1}}\right)^{S_{2}}$.
Proof. Since $S_{1}$ and $S_{2}$ are subsets of $S_{1} S_{2},\left(P^{S_{1}}\right)^{S_{2}} \subseteq P^{S_{1} S_{2}}$ immediately follows. For the other inclusion, if $m \in P^{S_{1} S_{2}}$, then there exist $s_{1} \in S_{1}$ and $s_{2} \in S_{2}$ such that $s_{1} s_{2} m \in P$, thus $m \in\left(P^{S_{1}}\right)^{S_{2}}$.

Remark 44. Let $R$ be a commutative principal ideal domain and $R[x]$ a polynomial ring with field of fractions $Q(x)$. Consider a left ideal L in the single Ore extension $Q(x)[\partial ; \sigma, \delta]$, then in Zhang (2016) one finds an algorithm for computing the contraction $Q(x)[\partial ; \sigma, \delta] L \cap R[x][\partial ; \sigma, \delta]$.

We recognize the latter as the $R[x] \backslash\{0\}$-closure of $L$. In the general setting, addressed in Remark 41, we can compute the $R[x] \backslash\{0\}$-closure of a submodule of $A^{r}$ in two steps: let $S_{1}=Q[x] \backslash\{0\}$ and $S_{2}=R \backslash\{0\}$. Then $I^{R[x] \backslash 0\}}=\left(I^{S_{1}}\right)^{S_{2}}$ holds by the following result. The left submodule $I^{S_{1}}$ can be computed with the modified Algorithm 10 as explained in Remark 41.

### 5.5. Central Weyl closure

Let $K$ be a field and $A_{n}^{K}$ the $n$-th Weyl algebra over $K$ in the variables $x=\left\{x_{1}, \ldots, x_{n}\right\}$ and $\partial=\left\{\partial_{1}, \ldots, \partial_{n}\right\}$. In this section, we utilize the central closure algorithm presented above in combination with the famous Weyl closure algorithm from Tsai (2000) to give an algorithm to compute the $K[x, s] \backslash\{0\}$-closure of a left ideal in the algebra $A_{n}^{K}[s]:=A_{n}^{K} \otimes_{K} K[s]$, where $s=\left\{s_{1}, \ldots, s_{m}\right\}$ is a set of additional commutative indeterminates and $S:=K[s] \backslash\{0\}$. Lastly, let $\rho:=\rho_{S, A_{n}^{K}[s]}$.

Remark 45. Note that the extended ring $A_{n}^{K}[s]:=A_{n}^{K} \otimes_{K} K[s]$ is no longer a Weyl algebra, but still a G-algebra. Localizing $A_{n}^{K}[s]$ at $S$ yields $A_{n}^{K}(s)$, which is isomorphic to $A_{n}^{K(s)}$, the n-th Weyl algebra over the field $K(s)$. The localization map $\rho$ is injective, since $A_{n}^{K}[s]$ is a domain.

Moreover, most of computations are done over $K[s]$, though mathematically we work over $K(s)$; retaining more generators with coefficients in $K[s]$ is a classical strategy, while working with localizations.

Definition 46. Let I be a left ideal in $A_{n}^{K}[s]$. Define

$$
G:=I^{K[x, s] \backslash\{0\}}=\operatorname{LSat}_{K[x, s] \backslash\{0\}}^{A_{n}^{K}[s]}(I)=\left\{r \in A_{n}^{K}[s] \mid \exists w \in K[x, s] \backslash\{0\}: w r \in I\right\}
$$

and

$$
H:=\operatorname{LSat}_{K(s)[x] \backslash\{0\}}^{A_{R}^{K(s)}}\left(\left(I^{S}\right)^{e}\right)=\left\{r \in A_{n}^{K(s)} \mid \exists w \in K(s)[x] \backslash\{0\}: w r \in\left(I^{S}\right)^{e}\right\} .
$$

Lemma 47. We have $H=G^{e}$ with respect to $\rho$.
Proof. Let $r \in G$, then there exists $w \in K[x, s] \backslash\{0\}$ such that $w r \in I$. Now

$$
\rho(w) \rho(r)=\rho(w r) \in \rho(I) \subseteq \rho\left(I^{S}\right) \subseteq\left(I^{S}\right)^{e}
$$

and $\rho(w) \in \rho(K[x, s] \backslash\{0\}) \subseteq K(s)[x] \backslash\{0\}$, thus $\rho(r) \in H$ and therefore $\rho(G) \subseteq H$. Since $H$ is a left ideal in $A_{n}^{K(s)}$ it contains $G^{e}$, the left ideal generated by $\rho(G)$, thus $G^{e} \subseteq H$.
Now let $(t, r) \in H$, where $t \in S$ and $r \in A_{n}^{K}[s]$, then there exists $(q, w) \in K(s)[x] \backslash\{0\}$, where $q \in S$ and $w \in K[x, s] \backslash\{0\}$, such that $(q, w) \cdot(t, r) \in\left(I^{S}\right)^{e}$. Since $w$ and $t$ commute we have $(q, w) \cdot(t, r)=(t q, w r)$. Now $(t q, w r) \in\left(I^{S}\right)^{e}$, therefore there exist $\tilde{t} \in S$ and $p \in I^{S}$ such that $(t q, w r)=(\tilde{t}, p)$. This implies the existence of $\bar{t} \in S$ and $\bar{r} \in A_{n}^{K}[s]$ such that $\bar{t} t q=\bar{r} \tilde{t}$ and $\bar{t} w r=\bar{r} p \in I^{S}$, thus there exist $\hat{t} \in S$ satisfying $\hat{t} \bar{t} w r \in I$. Since $\hat{t} \bar{t} w \in K[x, s] \backslash\{0\}$ we have $r \in G$, thus $(t, r)=(t, 1) \cdot(1, r)=(t, 1) \cdot \rho(r) \in G^{e}$ and therefore $H \subseteq G^{e}$.

```
Algorithm 11: CentralWeylClosure
    Input: A left ideal \(I\) of \(A_{n}^{K}[s]\).
    Output: The \(K[x, s] \backslash\{0\}\)-closure of \(I\).
    begin
        compute \(I^{S}\) via central closure in \(A_{n}^{K}[s]\); // central closure
        extend \(I^{S}\) from \(A_{n}^{K}[s]\) to \(A_{n}^{K(s)}\); // extension
        compute the \(K(s)[x] \backslash\{0\}\)-closure \(H=\left\langle\rho\left(h_{1}\right), \ldots, \rho\left(h_{k}\right)\right\rangle\) of \(\left(I^{S}\right)^{e}\) in \(A_{n}^{K(s)}\) via Weyl
            closure; // Weyl closure
        let \(F:=\left\langle h_{1}, \ldots, h_{k}\right\rangle \subseteq A_{n}^{K}[s] ; \quad / /\) primitive contraction
        compute \(F^{S}\) via central closure in \(A_{n}^{K}[s] ; \quad / /\) central closure
        return \(F^{S}\);
    end
```



Figure 1: Idea of Algorithm 11.

Proposition 48. Algorithm 11 terminates and is correct.
Proof. Termination is obvious. Since $A_{n}^{K(s)}$ is Noetherian, the ideal $H$ is finitely generated: let $H=\left\langle\left(a_{1}, h_{1}\right), \ldots,\left(a_{k}, h_{k}\right)\right\rangle$ for some $a_{i} \in S$ and $h_{i} \in A_{n}^{K}[s]$. Since $\left(a_{i}, 1\right)$ is a unit in $A_{n}^{K(s)}$ for all $i$, we have $H=\left\langle\left(1, h_{1}\right), \ldots,\left(1, h_{k}\right)\right\rangle=\left\langle\rho\left(h_{1}\right), \ldots, \rho\left(h_{k}\right)\right\rangle$. It remains to show that $G=F^{S}$.
First, let $g \in G$, then $\rho(g) \in G^{e}=H$ by Lemma 47, thus there exist $t_{i} \in S$ and $r_{i} \in A_{n}^{K}[s]$ such that

$$
\rho(g)=\sum_{i=1}^{k}\left(t_{i}, r_{i}\right) \rho\left(h_{i}\right)=\sum_{i=1}^{k}\left(t_{i}, r_{i} h_{i}\right)=\left(t, \sum_{i=1}^{k} \tilde{r}_{i} r_{i} h_{i}\right)
$$

for some common left denominator $t \in S$ and $\tilde{r}_{i} \in A_{n}^{K}[s]$. Now

$$
\rho(t g)=\rho(t) \rho(g)=\rho(t)\left(t, \sum_{i=1}^{k} \tilde{r}_{i} r_{i} h_{i}\right)=\rho\left(\sum_{i=1}^{k} \tilde{r}_{i} r_{i} h_{i}\right) .
$$

Since $\rho$ is injective we have $t g=\sum_{i=1}^{k} \tilde{r}_{i} r_{i} h_{i} \in F$, therefore $g \in F^{S}$ and thus $G \subseteq F^{S}$.
For the second inclusion let $y \in F^{S}$, then there exists $t \in S$ such that $t y \in F$. Now $\rho(t y) \in F^{e} \subseteq H$, since $F \subseteq H^{c}$ implies $F^{e} \subseteq\left(H^{c}\right)^{e}=H$. The ideal $H$ is left $K(s)[x] \backslash\{0\}$-saturated by construction, thus $\rho(t) \in K(s)[x] \backslash\{0\}$ implies $\rho(y) \in G^{e}$. Then $y \in\left(G^{e}\right)^{c}=G^{S}=G$ since $G$ is left $S$-saturated, therefore $F^{S} \subseteq G$.

Remark 49. Let $I \subseteq A_{n}^{K}=$ : D be a left ideal. In Tsai (2000), two Weyl closure algorithms have been presented. The optimized one only works for ideals of finite holonomic rank (i.e. those ideals $J$, such that $\operatorname{dim}_{K(x, s)} S^{-1} D / S^{-1} D J<\infty$ for $\left.S=K[x, s] \backslash\{0\}\right)$ and is based on a D-module-theoretic (monoidal) localization algorithm. It has been implemented in Macaulay2 and in Singular: Plural.

The general one, which works for any ideal I, is much harder, since it relies on a complicated algorithm to determine monoidal $[f]$-torsion of certain finitely generated $D$-modules associated with I. Notably, the general algorithm has not been implemented in any computer algebra system.

### 5.6. Application: computing $\operatorname{Ann}_{D[s]} f^{s}$ via CentralWeylClosure

The algorithm above has a nice application: namely, the computation of the annihilator of the special function $f^{s}:=f_{1}^{s_{1}} \cdot \ldots \cdot f_{r}^{s_{r}}$ in the algebra $A_{n}^{K}[s]=D[s]$, where $f_{i} \in K[x]$. It is a left ideal, denoted by $\operatorname{Ann}_{D[s]} f^{s}$. For simplicity of the presentation, let $r=1$, i. e. $f=f_{1} \in K[x]$ and $s_{1}=s$. We refer the reader to Andres et al. (2010) for details. Analytic considerations deliver the following additional information:

Proposition 50. With notations as above, let $S=K[x, s] \backslash\{0\} \subseteq D[s]$ and $S^{\prime}=K[x] \backslash\{0\}$.
(a) The left ideal $\mathrm{Ann}_{D[s]} f^{s}$ is $S$-closed.
(b) Both $G_{1}:=\left\{\left.f \partial_{i}-\frac{\partial f}{\partial x_{i}} s \right\rvert\, 1 \leq i \leq n\right\}$ and $G_{2}$ (see below) generate localized ideals $S^{-1} \mathrm{Ann}_{D[s]} f^{s}$ and $S^{\prime-1} \mathrm{Ann}_{D[s]} f^{s}$, respectively.

Consider the $K[x, s]$-module of syzygies of the tuple $\left(f, s \frac{\partial f}{\partial x_{1}}, \ldots, s \frac{\partial f}{\partial x_{n}}\right)$. Since it is finitely generated, let $\left\{g_{1}, \ldots, g_{t}\right\} \subseteq K[x, s]^{n+1}$ be its generating set. Then

$$
G_{2}=\left\{a_{0}+a_{1} \partial_{1}+\cdots+a_{n} \partial_{n} \mid\left(a_{0}, a_{1}, \ldots, a_{n}\right)=g_{i}, 1 \leq i \leq n\right\} .
$$

Note that $G_{2}$ is generated by vector fields of order one in $D[s]$. Also, $\operatorname{Ann}_{D[s]} f^{s}$ is of holonomic rank 1. Therefore we can at long last provide the following alternative to the algorithm of Briançon-Maisonobe (Briançon and Maisonobe (2002)):

```
Algorithm 12: AnNFsViaWeylClosure
    Input: A set \(G \subseteq D[s]\) and \(f^{s}\) as above.
    Output: A generating set of \(\mathrm{Ann}_{D[s]} f^{s} \subseteq D[s]\).
    begin
        ensure that \(G\) generates \((K[x, s] \backslash\{0\})^{-1} \mathrm{Ann}_{D[s]} f^{s}\);
        compute \(F:=\operatorname{CentralWeylClosure}(G)\);
        return \(F\);
    end
```


### 5.7. Central geometric closure

Consider the setting from Section 5.4, but we are now interested in computing the closure $I^{T}$, where $T:=K[x] \backslash \mathfrak{p}$ for some prime ideal $\mathfrak{p}$ in $K[x]$. By construction we have $I \subseteq I^{T} \subseteq I^{S}$ and we can characterize when the second inclusion is in fact an equality:

Lemma 51. We have $I^{T}=I^{S}$ if and only if $\operatorname{Ann}_{T}\left(I^{S} / I\right) \neq \emptyset$.

Proof. Recall that for an $A$-module $I$ and a subset $P$ of $A, \operatorname{Ann}_{P}(I):=\{p \in P \mid p I=0\}$. Let $W \in\{S, T\}$, then $I^{W}$ is finitely generated by some elements $f_{1}, \ldots, f_{k} \in A$ and we have that $\operatorname{Ann}_{W}\left(I^{W} / I\right)$ is non-trivial: since $f_{i} \in I^{W}$ there exist $w_{i} \in W$ such that $w_{i} f_{i} \in I$, so $w_{1} \cdot \ldots \cdot w_{k} \in$ $\operatorname{Ann}_{W}\left(I^{w} / I\right)$ due to $W$ being central in $A$. If $I^{S}=I^{T}$ then $\operatorname{Ann}_{T}\left(I^{S} / I\right)=\operatorname{Ann}_{T}\left(I^{T} / I\right) \neq \emptyset$. On the other hand, let $t \in \operatorname{Ann}_{T}\left(I^{S} / I\right)$ and $r \in I^{S}$, then $t r \in I$ and thus $r \in I^{T}$, which shows $I^{S}=I^{T}$.

Note that $\operatorname{Ann}_{T}\left(I^{S} / I\right) \neq \emptyset$ is equivalent to $\operatorname{Ann}_{B}\left(I^{S} / I\right) \nsubseteq \mathfrak{p}$ (recall, that $B$ is identified with $K[x]$ in our setup from Section 5.4) and the latter can be checked algorithmically, since $I^{S}$ is computable via Algorithm 10.

Nevertheless there are situations where neither inclusion is strict:
Example 52. Consider $I={ }_{A}\langle x(x-1) \partial\rangle$, where $A$ is the first Weyl algebra in $x$ and $\partial$. Then $I^{S}={ }_{A}\langle\partial\rangle$ and $I^{T}={ }_{A}\langle x \partial\rangle$, if we choose $\mathfrak{p}={ }_{K[x]}\langle x\rangle$, which leads to $I \subsetneq I^{T} \subsetneq I^{S}$.

Further advances towards an algorithm for computing $I^{T}$ are the subject of ongoing research.

## 6. Conclusion

We have provided several algorithms for solving the intersection problem and for computing local closure in various settings with respect to Ore sets with enough commutativity. In particular, it follows that arithmetic within the localization of a commutative polynomial algebra is constructive and can be used also in homomorphic images of such algebras inside non-commutative algebras.

At the end of our paper Hoffmann and Levandovskyy (2018), we posed the following questions: does there exist an algorithm to compute...

- the closure in the case of geometric localization without invoking primary decomposition?
- the central geometric closure?
- the geometric closure in the Weyl algebra tensored with a commutative polynomial ring?

As it turns out, the first question has been answered since then in Ishihara and Yokoyama (2018), which was published just a few months later than Hoffmann and Levandovskyy (2018). The main tool used is the double ideal quotient $I:(I: P)$, where $P$ is a prime ideal, and its variants. This opens a perspective towards better versions of algorithms, which rely on primary decomposition; among other, for the computation of the symbolic power of an ideal.

However, the other questions we posed are still open.

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