

Unexpected hypersurfaces with multiple fat points

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October 1, 2019

Abstract

Starting with the ground-breaking work of Cook II, Harbourne, Migliore and Nagel, there has been a lot of interest in unexpected hypersurfaces. In the last couple of months a considerable number of new examples and new phenomena has been observed and reported on. All examples studied so far had just one fat point. In this note we introduce a new series of examples, which establishes for the first time the existence of unexpected hypersurfaces with multiple fat points. The key underlying idea is to study Fermat-type configurations of points in projective spaces.

1 Introduction

In the ground-breaking paper [4] the authors introduced the notion of unexpected curves.

Definition 1. We say that a reduced finite set of points $Z \subset \mathbb{P}^2$ *admits an unexpected curve* of degree $j + 1$ if there is an integer $j > 0$ such that, for a general point P , jP fails to impose the expected number of conditions on the linear system of curves of degree $j + 1$ containing Z . That is, Z admits an unexpected curve of degree $j + 1$ if

$$h^0(\mathcal{I}_{Z+jP}(j+1)) > \max \left\{ h^0(\mathcal{I}_Z(j+1)) - \binom{j+1}{2}, 0 \right\}.$$

This notion has been generalized to hypersurfaces in projective spaces of arbitrary dimension in the subsequent paper [7].

Definition 2. We say that a reduced set of points $Z \subset \mathbb{P}^N$ *admits an unexpected hypersurface* of degree d if there exists a sequence of integers m_1, \dots, m_s such that for general points P_1, \dots, P_s the zero-dimensional subscheme $m_1P_1 + \dots + m_sP_s$ fails to impose independent conditions on forms of degree d vanishing along Z .

A well known example of this kind is provided by 7 general double points in \mathbb{P}^4 . There is a twisted quartic curve passing through these 7 points and its secant variety is a threefold of degree 3 vanishing to order 2 along the quartic curve, in particular in the 7 general points. This threefold is unexpected, see [1].

This example has two deficiencies. First, the singular points are not isolated. Second, the set Z is empty.

The first example of an unexpected surface Q_R in \mathbb{P}^3 admitted by a non-empty set Z has been discovered and described in [2, Theorem 1]. In that example there is an isolated point R of multiplicity 3 which is a general point. The surface Q_R has 4 additional double points, whose coordinates depend on R , and no other singularities. In the present note we show the following result.

Keywords: fat points, linear systems, postulation problem, unexpected hypersurfaces, Fermat-type arrangements.

Mathematics Subject Classification (2010): MSC 14C20 MSC 14J26 MSC 14N20 MSC 13A15 MSC 13F20.

Main Theorem. *There exists a non-empty set Z in P^N with $N = 2k+1$ for $k \geq 1$ which admits an unexpected hypersurface for multiplicities $3, \underbrace{2, \dots, 2}_{k-1}$. This hypersurface has only isolated singularities.*

For $N = 3$ we have a computer-free proof. Our construction in higher dimensional projective spaces builds upon the existence of an unexpected surface of degree 4 in \mathbb{P}^3 with a general point of multiplicity 3. Given a general point R in \mathbb{P}^5 we project it to a coordinate \mathbb{P}^3 (i.e. codimension 2 flat Π defined by the vanishing of two coordinates) obtaining the point R_Π . Then the $N = 3$ case gives an unexpected surface in Π which vanishes to order 3 at R_Π . Taking a cone in \mathbb{P}^5 with vertex in the coordinate line spanned by the coordinate points corresponding to the two coordinates which defined Π we get a 4-fold of degree 4 in \mathbb{P}^5 which in particular vanishes to order 3 at R . Of course this 4-fold has non-isolated singularities. However, taking the linear system of quartics generated by all such cones in \mathbb{P}^5 we are able to identify its 6 generators. Given another general point P in \mathbb{P}^5 we find an explicit formula for coefficients in front of the 6 generators (depending on P) so that the resulting 4-fold has only isolated singularities and in particular has multiplicity 3 at R and multiplicity 2 at P . Once the equation is in place, all our claims can be (in principle) checked by direct computation. We provide a Singular script [13] to ease their verification. The construction in higher dimensions is performed along the same lines. So far it is justified only by computer computations with random points rather than arbitrary as in the case of Theorem 10.

The main idea here is to explore Fermat-type configurations of points in projective spaces, see [12] for a recent survey on Fermat-type hyperplane arrangements and their applications in algebra and geometry.

2 Fermat-type point configurations

By the way of warm-up we begin with points in \mathbb{P}^2 . In Theorem 5 we show a peculiar family of unexpected curves, which degree grows while the multiplicity in the general point remains fixed, equal 4. Passing then to the general case of \mathbb{P}^N we show identify generators of the ideal of Fermat-type configuration of points. This is crucial for the subsequent part.

2.1 Fermat point configurations in \mathbb{P}^2

Let $n \geq 1$ be a positive integer and let $I_{2,n}$ be the complete intersection ideal in $\mathbb{C}[x, y, z]$ viewed as the graded ring of \mathbb{P}^2 generated by

$$x^n - y^n, \quad y^n - z^n.$$

Let $F_{2,n}$ be the ideal

$$F_{2,n} = I_{2,n} \cap (x, y) \cap (x, z) \cap (y, z).$$

The support of $I_{2,n}$ is the set of n^2 points

$$(1 : \varepsilon^\alpha : \varepsilon^\beta)$$

where ε is a primitive root of unity of order n and $1 \leq \alpha, \beta \leq n$. The support of $F_{2,n}$ is the union of the support of $I_{2,n}$ and the 3 coordinate points $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$.

The ideals $F_{2,n}$ have appeared recently in various guises, most notably in the context of the containment problem, see [11] for an introduction to this circle of ideas and [9], [8] for specific applications. Their algebraic properties have been studied in depth by Nagel and Secreanu [10].

For $n = 3$ the set of points defined by $F_{2,3}$ is the set of singular points (i.e. points where two or more arrangement lines meet) of the dual Hesse arrangement of 9 lines with 12 triple points at

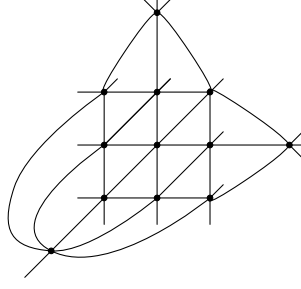


Figure 1: Dual Hesse arrangement

the zeroes of $F_{2,3}$. This arrangement is depicted in Figure 1. The lines are indicated by curved segments as it is not possible to embed this configuration in the real projective plane due to the celebrated Sylvester-Gallai Theorem, see [3]. It has been proved already in [6, Lemma 2.1] that the ideal $F_{2,3}$ is generated in degree 4. More precisely we have

Lemma 3. *The ideal $F_{2,3}$ is generated by the binomials*

$$x(y^3 - z^3), y(z^3 - x^3), z(x^3 - y^3).$$

In fact exactly the same proof works for any $n \geq 3$, so that we have

Lemma 4. *The ideal $F_{2,n}$ is generated by the binomials*

$$x(y^n - z^n), y(z^n - x^n), z(x^n - y^n).$$

This ideal is supported on n^2 points which form a complete intersection and the 3 coordinate points. It turns out that this set of points admits an unexpected curve (Definition 1) for any $n \geq 3$.

Theorem 5 (Unexpected curves with a point of multiplicity 4). *Let Z be the configuration of points in \mathbb{P}^2 defined by $F_{2,n}$ for $n \geq 3$. Let $R = (a : b : c)$ be a general point in \mathbb{P}^2 . With $u = \binom{n}{2} - 1$, $v = \binom{n-1}{2}$, $w = \binom{n+1}{2}$, the polynomial*

$$\begin{aligned} Q_P(x : y : z) = & -cxy((ub^n + vc^n)(z^n - x^n) + (ua^n + vc^n)(y^n - z^n)) \\ & -bxz((ua^n + vb^n)(y^n - z^n) + (uc^n + vb^n)(x^n - y^n)) \\ & -ayz((ub^n + va^n)(z^n - x^n) + (uc^n + va^n)(x^n - y^n)) \\ & + wa^{n-1}bcx^2(y^n - z^n) + wab^{n-1}cy^2(z^n - x^n) \\ & + wabc^{n-1}z^2(x^n - y^n) \end{aligned} \tag{1}$$

- *vanishes at all points of Z ,*
- *vanishes to order 4 at R ,*
- *defines an unexpected curve of degree $n + 2$ for Z with respect to R .*

Proof. This claim is easy verify by direct calculations, which we omit. □

Theorem 5 is of interest as it exhibits a first family of unexpected curves, where the degree of curves grows but the multiplicity at the general point remains constant.

2.2 Fermat-type configurations of points in \mathbb{P}^N

For a positive integer n let $I_{N,n}$ be the complete intersection ideal defined by the binomials

$$x_0^n - x_1^n, x_1^n - x_2^n, \dots, x_{N-1}^n - x_N^n. \quad (2)$$

The set $Z_{N,n}$ of zeroes of $I_{N,n}$ is the set of N^n points of the form

$$(1 : \varepsilon^{\alpha_1} : \varepsilon^{\alpha_2} : \dots : \varepsilon^{\alpha_N}),$$

where ε is a primitive root of unity of order n and

$$1 \leq \alpha_1, \alpha_2, \dots, \alpha_N \leq n.$$

Adding to this set of points the coordinate points of \mathbb{P}^N , we obtain the set $W_{N,n}$ defined by the ideal $F_{N,n}$.

In the sequel we will deal with the index $n = 3$, so we drop the degree from the notation. We also use the convention that the indices are understood modulo $N + 1$. For example $x_{N+2} = x_1$. Moreover it is convenient to introduce the following notation

$$[a, b] = a^3 - b^3,$$

which we use for numbers and for monomials. Note that the symbol $[a, b]$ is anti-symmetric and satisfies a Jacobi-type identity

$$a^3[b, c] + b^3[c, a] + c^3[a, b] = 0 \quad \text{for all } a, b, c. \quad (3)$$

Lemma 3 generalizes in the following way.

Lemma 6. *The ideal F_N is generated by the binomials of the form*

$$x_i[x_{i+1}, x_j], \quad (4)$$

where $i \in \{0, 1, \dots, N\}$ and $j \in \{0, 1, \dots, N\} \setminus \{i, i+1\}$.

Proof. Let J be the ideal generated by binomials in (4). It is clear that elements in J vanish in all points of W_N , so there is $J \subset F_N$.

It is also easy to see that J contains all binomials of the form $x_i[x_j, x_k]$, where the indices are mutually distinct. Indeed, we have

$$x_i[x_j, x_k] = x_i[x_{i+1}, x_k] - x_i[x_{i+1}, x_j].$$

It is also clear that the ideal I_N defined in (2) is generated also by binomials

$$[x_0, x_1], [x_0, x_2], \dots, [x_0, x_N],$$

which are slightly more convenient to work with in this proof.

Let $f \in F_N$ be an arbitrary polynomial. We want to show the containment $f \in J$ or equivalently $f = 0 \pmod{J}$. Since f vanishes in particular in all points of Z_N , there are homogeneous polynomials g_1, \dots, g_N such that

$$f = \sum_{i=1}^N g_i[x_0, x_i]. \quad (5)$$

from now on, we work \pmod{J} . Since for $j \neq 0, i$ the binomials $x_j[x_0, x_i]$ are in J , we may assume that each polynomial g_i depends only on x_0 and x_i . Moreover, for a fixed $j \neq 0, i$ we have

$$x_0 x_i[x_0, x_i] = x_i \cdot x_0[x_j, x_i] - x_0 \cdot x_i[x_j, x_0] \in J.$$

Thus it must be

$$g_i = a_i x_0^d + b_i x_i^d \pmod{J}$$

for some $d \geq 0$ and some scalars $a_1, \dots, a_N, b_1, \dots, b_N \in \mathbb{C}$. Evaluating (5) in the coordinate points we obtain

$$a_1 + a_2 + \dots + a_N = 0 \text{ and } b_1 = b_2 = \dots = b_N = 0. \quad (6)$$

Hence (5) reduces to

$$f = a_1 x_0^d [x_0, x_1] + a_2 x_0^d [x_0, x_2] + \dots + a_N x_0^d [x_0, x_N] \pmod{J}. \quad (7)$$

If $d = 0$, then evaluating (7) in the coordinate points, we obtain

$$a_1 = a_2 = \dots = a_N = 0$$

and we are done.

If $d > 0$, then, using first part of (6), we can rewrite (7) as

$$f = a_1 x_0^d [x_N, x_1] + a_2 x_0^d [x_N, x_2] + \dots + a_{N-1} x_0^d [x_N, x_{N-1}],$$

which is clearly an element of J and we are done again. \square

3 Unexpected quartic surfaces in \mathbb{P}^3 with a triple point

The story of unexpected hypersurfaces begins with unexpected curves discovered by Di Gennaro, Ilardi and Vallès in [5] and studied systematically by Cook II, Harbourne, Migliore and Nagel in [4]. In [2] the first example of a higher dimensional unexpected hypersurface (a surface in \mathbb{P}^3) has been described by Bauer, Malara, Szemberg and the present author. Shortly after that Harbourne, Migliore, Nagel and Teitler in [7] constructed examples of unexpected hypersurfaces in projective spaces of arbitrary dimension. All these examples have just one singular general point.

Here we recall the construction of the unexpected quartic in \mathbb{P}^3 . This example has not only motivated our construction of unexpected hypersurfaces with multiple fat points but it serves as a building fundament for this construction.

According to Lemma 6, in \mathbb{P}^3 the ideal $F_{3,3}$ has 8 generators:

$$g_{0,2} = x_0[x_1, x_2], \quad g_{0,3} = x_0[x_1, x_3], \quad g_{1,3} = x_1[x_2, x_3], \quad g_{1,0} = x_1[x_2, x_0],$$

$$g_{2,0} = x_2[x_3, x_0], \quad g_{2,1} = x_2[x_3, x_1], \quad g_{3,1} = x_3[x_0, x_1], \quad g_{3,2} = x_3[x_0, x_2].$$

It is convenient to introduce the following notation. Given mutually distinct numbers $i, j \in \{0, 1, 2, 3\}$, we denote by k the index $\{0, 1, 2, 3\} \setminus \{i, i+1, j\}$. As usually, the indices are considered modulo 4. With this notation we have (compare [2, Theorem 6]).

Theorem 7. *For a general point $R = (a_0 : a_1 : a_2 : a_3)$, the quartic*

$$Q_R(x_0 : x_1 : x_2 : x_3) = \sum_{i=0}^3 \sum_{j=i+2}^{i+3} (-1)^k a_i^2 [a_{i+1}, a_k] \cdot g_{i,j}$$

has a triple point at R and satisfies $Q_R \in F_{3,3}$.

Proof. The second property is obvious. Checking that Q_R vanishes to order 3 at R is a straightforward computation. However for further reference we will present some computations which make heavy use of available symmetries.

$$\begin{pmatrix} 0 & 0 & -a_1 & 0 & -a_2 & 0 & a_3 & a_3 \\ a_1^2 & a_1^2 & -a_0^2 & 0 & 0 & 0 & 0 & 0 \\ a_2^2 & 0 & 0 & 0 & a_0^2 & 0 & 0 & 0 \\ 0 & -a_3^2 & 0 & 0 & 0 & 0 & a_0^2 & a_0^2 \\ a_0 & a_0 & 0 & 0 & 0 & -a_2 & -a_3 & 0 \\ 0 & 0 & a_2^2 & a_2^2 & 0 & -a_1^2 & 0 & 0 \\ 0 & 0 & 0 & a_3^2 & 0 & 0 & a_1^2 & 0 \\ -a_0 & 0 & a_1 & a_1 & 0 & 0 & 0 & -a_3 \\ 0 & 0 & 0 & 0 & a_3^2 & a_3^2 & 0 & -a_2^2 \\ 0 & -a_0 & 0 & -a_1 & a_2 & a_2 & 0 & 0 \end{pmatrix}$$

Table 1: Interpolation matrix for a point of multiplicity 3 in \mathbb{P}^3 .

To begin with, we check that the first order derivatives of Q_R vanish at R . By symmetry it is enough to check just one derivative. We compute the one with respect to x_0 and obtain

$$\frac{\partial}{\partial x_0} Q_R = ([a_1 a_2][x_1 x_3] - [a_1 a_3][x_1 x_2]) a_0^2 + 3x_0^2 (a_1^2 x_1 [a_2 a_3] + a_2^2 x_2 [a_3 a_1] + a_3^2 x_3 [a_1 a_2]).$$

Vanishing of the first bracket when evaluating at R is straightforward. The vanishing of the second bracket follows from the Jacobi-type identity (3).

For the second order derivative with respect to x_0 we have

$$\frac{\partial^2}{\partial x_0^2} Q_R = -6x_0 \cdot \sum_{k=1}^3 (-1)^k a_j^2 a_l^2 (a_l x_j - a_j x_l),$$

where the indices j, l are chosen in dependence on k so that $j < l$ and $\{j, k, l\} = \{1, 2, 3\}$.

For the mixed derivative of order 2 we may again use the symmetry and look at just one other variable, here x_1 . We have

$$\frac{\partial^2}{\partial x_1 \partial x_0} Q_R = 3[a_2 a_3](a_1^2 x_0^2 - a_0^2 x_1^2).$$

Clearly, in both cases we obtain zero, when evaluating at R .

Instead of computing derivatives one by one, we can also use the whole interpolation matrix at once. We present this matrix in Table 1, since this is a baby case for our argument in the proof of Theorem 10, see Table 2. The columns of the table correspond to the generators of $F_{3,3}$, whereas the rows stand for order 2 partial derivatives. Taking into account that the point R is general, we have divided out common factors appearing in some rows. This is the reason why not all rows have entries of the same degree. Nevertheless, the matrix as entire, remains of course homogeneous. \square

4 Unexpected quartic 4-folds in \mathbb{P}^5 with a triple and double point

In this section we present in detail unexpected hypersurfaces in \mathbb{P}^5 , which have 2 general fat points, one of multiplicity 3 and the other one of multiplicity 2.

According to Lemma 6 the ideal $F_{5,3}$ has 24 generators and its zero locus consists of 3^5 Fermat points and 6 coordinate points, so that there are altogether 249 points in $W_{5,3}$. The linear system of quartics in \mathbb{P}^5 has (affine) dimension 126, so the points in $W_{5,3}$ do not impose independent conditions on quartics. Of course there exists a subset of 102 points in $W_{5,3}$ which impose independent conditions on quartics in \mathbb{P}^5 but to determine such a set is not relevant for our considerations here.

Since vanishing to order 3 imposes 21 conditions, it is expected that the system of quartics in \mathbb{P}^5 vanishing along $W_{5,3}$ and having a triple point in a general point $R = (a_0 : a_1 : \dots : a_5)$ has dimension 3. It turns out however that this system has in fact higher dimension.

Proposition 8. *Let $I(R)$ denote the ideal of the point R . For*

$$V_{5,3} = H^0(\mathbb{P}^5; \mathcal{O}_{\mathbb{P}^5}(4) \otimes F_{5,3} \otimes I(R)^3)$$

we have $\dim V_{5,3} = 6$.

Proof. We verify our claim computing the interpolation matrix at R . More precisely, we compute all partial derivatives of order 2 (there are 21 of them) of all 24 generators of $F_{5,3}$. The matrix we get has a lot of zeros and thus its rank is easy to determine. The rank is the number of *independent* conditions imposed on quartics in $F_{5,3}$ by vanishing at R to order 3.

0	0	0	0	$-a_1$	0	0	0	$-a_2$	0	0	0	$-a_3$	0	0	0	$-a_4$	0	0	0	a_5	a_5	a_5	a_5
a_1^2	a_1^2	a_1^2	a_1^2	$-a_0^2$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
a_2^2	0	0	0	0	0	0	0	a_0^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	a_3^2	0	0	0	0	0	0	0	0	0	0	a_0^2	0	0	0	0	0	0	0	0	0	0	
0	0	a_4^2	0	0	0	0	0	0	0	0	0	0	0	0	0	a_0^2	0	0	0	0	0	0	
0	0	0	$-a_5^2$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	a_0^2	a_0^2	a_0^2	
a_0	a_0	a_0	a_0	0	0	0	0	0	$-a_2$	0	0	0	$-a_3$	0	0	0	$-a_4$	0	0	$-a_5$	0	0	0
0	0	0	0	a_2^2	a_2^2	a_2^2	a_2^2	0	$-a_1^2$	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	a_3^2	0	0	0	0	0	0	0	a_1^2	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	a_4^2	0	0	0	0	0	0	0	0	0	0	a_1^2	0	0	0	0	0	
0	0	0	0	0	0	0	a_5^2	0	0	0	0	0	0	0	0	0	0	0	0	a_1^2	0	0	
$-a_0$	0	0	0	a_1	a_1	a_1	a_1	0	0	0	0	0	0	$-a_3$	0	0	0	$-a_4$	0	0	$-a_5$	0	0
0	0	0	0	0	0	0	0	a_3^2	a_3^2	a_3^2	a_3^2	0	0	$-a_2^2$	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	a_4^2	0	0	0	0	0	0	0	a_2^2	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	a_5^2	0	0	0	0	0	0	0	0	0	a_2^2	0	
0	$-a_0$	0	0	0	$-a_1$	0	0	a_2	a_2	a_2	a_2	0	0	0	0	0	0	0	$-a_4$	0	0	$-a_5$	
0	0	0	0	0	0	0	0	0	0	0	0	a_4^2	a_4^2	a_4^2	a_4^2	0	0	0	$-a_3^2$	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	a_5^2	0	0	0	0	0	a_3^2	0	
0	0	$-a_0$	0	0	0	$-a_1$	0	0	0	$-a_2$	0	a_3	a_3	a_3	a_3	0	0	0	0	0	0	$-a_5$	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	a_5^2	a_5^2	a_5^2	a_5^2	0	0	$-a_4^2$	
0	0	0	$-a_0$	0	0	0	$-a_1$	0	0	0	$-a_2$	0	0	0	$-a_3$	a_4	a_4	a_4	a_4	0	0	0	

Table 2: Interpolation matrix for a point of multiplicity 3 in \mathbb{P}^5 .

□

Since a double point in \mathbb{P}^5 imposes 6 conditions on forms of arbitrary degree, we do not expect that there is a non-zero section in $V_{5,3}$ vanishing to order 2 at an additional general point. However such unexpected hypersurface does exist!

We define first the following 6 cones in \mathbb{P}^5 . For indices $i, j \in \{0, \dots, 5\}$, with $i < j$, we denote by s, t, u, v the remaining 4 indices in the growing order. Then, for $i, j \in \{0, 1, 2, 3\}$ with $i < j$ we set

$$R_{i,j} = (a_s : a_t : a_u : a_v)$$

and

$$J_{i,j} = Q_{R_{i,j}}(x_s : x_t : x_u : x_v),$$

where Q_R is taken from Theorem 7.

Remark 9. It can be shown that $J_{0,1}, \dots, J_{2,3}$ generate $V_{5,3}$ but this is irrelevant for the further assertions and we omit the verification of this claim.

Theorem 10. *Let $P = (b_0 : b_1 : \dots : b_5)$ be a general point in \mathbb{P}^5 . Then there exists a unique quartic $Q_{R,P} \in V_{5,3}$ vanishing at*

- all points of the Fermat-type configuration W_5 ,
- the point $R = (a_0 : a_1 : \dots : a_5)$ to order 3,
- the point P to order 2.

Proof. We are able to write down explicit equation of the quartic

$$\begin{aligned} Q_{R,P}(x_0 : \dots : x_5) = & J_{2,3}(P) \cdot J_{0,1}(x_0 : \dots : x_5) - J_{1,3}(P) \cdot J_{0,2}(x_0 : \dots : x_5) \\ & + J_{0,3}(P) \cdot J_{1,2}(x_0 : \dots : x_5) + J_{1,2}(P) \cdot J_{0,3}(x_0 : \dots : x_5) \\ & - J_{0,2}(P) \cdot J_{1,3}(x_0 : \dots : x_5) + J_{0,1}(P) \cdot J_{2,3}(x_0 : \dots : x_5). \end{aligned}$$

The sign in the front of a summand of $Q_{R,P}$ depends on whether there is a pair of consecutive numbers in the indices of involved J 's or not. If there isn't, i.e., for pairs (0, 2) and (1, 3) we get a minus.

Note that the first two assertions of the Theorem follow straightforward from the way we defined the cones $J_{0,1}, \dots, J_{2,3}$. The last assertion can be checked by direct computations as in the proof of Theorem 7.

The most tricky part of the proof was to find the coefficients in front of the cones. \square

5 Unexpected quartic $2k$ -folds in \mathbb{P}^{2k+1} with a triple point and $k - 1$ double points

Before we conclude we collect a couple of facts towards the proof of our Main Theorem. We begin with an observation which parallels Proposition 8

Proposition 11. *Let $N = 2k + 1$ for some $k \geq 1$. Let $R = (a_0 : \dots : a_N)$ be a general point in \mathbb{P}^N and let $I(R)$ be its saturated ideal. Let*

$$V_{N,3} = H^0(\mathbb{P}^N; \mathcal{O}_{\mathbb{P}^N}(4) \otimes F_{N,3} \otimes I(R)^3).$$

Then $\dim V_{N,3} = \binom{N-1}{2}$.

Remark 12. The statement of Proposition 11 means that a general triple point always imposes exactly 3 conditions less on generators of $F_{N,3}$ than expected. We don't know how to verify this theoretically.

The final step in proving Main Theorem is the next statement.

Proposition 13. *Let P_1, \dots, P_{k-1} be general points in \mathbb{P}^N with $N = 2k + 1$. Vanishing to order 2 in these points imposes $2k^2 - k - 1$ conditions on elements in $V_{N,3}$.*

Remark 14. Note that vanishing to order 2 at a general point is expected to impose $N + 1 = 2k + 2$ conditions. Thus $(k - 1)$ points are expected to impose $2k^2 - 2$ conditions. Interestingly, computer experiments show that the first $(k - 2)$ points do impose on $V_{N,3}$ the expected number of conditions, namely $2k^2 - 2k - 4$. It is only the last point which imposes $k + 3$ instead of $2k + 2$ conditions on the system

$$H^0(\mathbb{P}^N; \mathcal{O}_{\mathbb{P}^N}(4) \otimes F_{N,3} \otimes I(R)^3 \otimes I(P_1)^2 \otimes \dots \otimes I(P_{k-2})^2).$$

We do not have a theoretical explanation of this phenomena.

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