CATALAN-MANY TROPICAL MORPHISMS TO TREES; PART I: CONSTRUCTIONS

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ABSTRACT. We investigate the *tree gonality* of a genus-g metric graph, defined as the minimum degree of a tropical morphism from any tropical modification of the metric graph to a metric tree. We give a combinatorial constructive proof that this number is at most $\lceil g/2 \rceil + 1$, a fact whose proofs so far required an algebro-geometric detour via special divisors on curves. For even genus, the tropical morphism which realizes the bound belongs to a family of tropical morphisms that is pure of dimension $3g - 3$ and that has a generically finite-to-one map onto the moduli space of genus-g metric graphs. Our methods focus on the study of such families. This is part I in a series of two papers: in part I we fix the combinatorial type of the metric graph to show a bound on tree-gonality, while in part II we vary the combinatorial type and show that the number of tropical morphisms, counted with suitable multiplicities, is the same Catalan number that counts morphisms from a general genus-g curve to the projective line.

Introduction. We investigate the *tree gonality* of metric graphs. This concept has its roots in algebraic geometry. The *gonality* of a genus-g curve X is the minimum degree of a non-constant morphism from X to $\mathbb{P}^1$. The non-constant morphisms $\phi : X \to \mathbb{P}^1$ are in one-to-one correspondence with linear series of degree d and dimension 1 on X, so equivalently the gonality of X is the minimum degree of such linear series. Brill-Noether theory for algebraic curves establishes that the gonality of X is at most $\lceil g/2 \rceil + 1$. The set of curves for which this bound is sharp is dense in the moduli space of genus-g curves $\mathcal{M}_g$.

A striking interplay between graphs and algebraic curves has been discovered over the last two decades. In tropical geometry the objects that play the role of algebraic curves and of the projective line are metric graphs and metric trees, respectively. This goes beyond a formal analogy of concepts: specialization lemmas and correspondence theorems between both settings are an active field of research at the interface between tropical and non-archimedean geometry (see, e.g., [Bak08; ABBR15a; ABBR15b]).

There are two definitions of gonality for (metric) graphs which are relevant to our story. A chip-firing game on a (metric) graph yields a divisor theory (see [BN07; HKN13]). The *divisorial gonality* of a (metric) graph is the minimum degree of a rank-1 divisor. On the other hand, certain maps between metric graphs with suitable balancing conditions bahave similarly to morphisms between curves (see [BN09; BBM11; Cha13; Cap14; Mik07]). We call these maps *tropical morphisms*. The *tree gonality* of a metric graph is the minimum degree of a tropical morphism from any *tropical modification* of the metric graph to a metric tree. Tropical modification is an operation which retracts or attaches edges ending in a valency-1 vertex, and is useful to satisfy balancing conditions.

Divisorial gonality mimics the definition of gonality of X via linear series, while tree gonality mimics the one via non-constant morphisms. Unlike the algebro-geometric setting,

in the tropical world divisorial gonality and tree-gonality do not coincide. If $\Phi : \Gamma \to \Delta$ is a tropical morphism to a tree, then the divisor $\sum_{x' \in \Phi^{-1}(x)} m_\varphi(x') x'$ has rank at least 1 for any choice of x in Δ (see Subsection 2.2 for notation). So tree gonality is an upper bound of divisorial gonality. On the other hand, there are families of graphs with divisorial gonality equal to 2, and tree gonality equal to $\lceil g/2 \rceil + 1$, where g is the first Betti number of the graph. At the time of writing, perhaps the most famous open question on divisorial gonality is the following:

Conjecture 1 (Baker; Conjecture 3.10 in [Bak08])**.** *The divisorial gonality of a finite connected graph G is at most $\lceil g/2 \rceil + 1$, where g is the first Betti number of G.*

This conjecture has attracted a sizeable amount of attention from the community. If one replaces G with a metric graph Γ, then the result is proven in [Bak08, Theorem 3.12]. The proof requires an algebro-geometric detour via special divisors on curves. There is wide interest for a purely combinatorial proof, as it is hoped it may yield methods better suited to tackle Conjecture 1. Our main contribution is such a purely combinatorial proof:

Theorem 2. *The tree gonality of a genus-g metric graph Γ is at most $\lceil g/2 \rceil + 1$.*

In fact, we manage to construct tropical morphisms that witness the bound $\lceil g/2 \rceil + 1$ on tree-gonality, so beyond being combinatorial, the proof is also *effective*. There is some computer code and visualizations available at [Dra].

Proof idea. The main idea behind our proof is short, elegant, and ubiquitous: we first analyse a favourable case and then continuously deform the result. We need an initial family of graphs Γ for which we have a construction of a tropical morphism $\Phi : \Gamma \to \Delta$ attaining the gonality bound; and a procedure to deform Φ into $\bar{\Phi} : \bar{\Gamma} \to \bar{\Delta}$ for any given $\bar{\Gamma}$ via a deformation path in $\mathcal{M}_g^{\mathrm{trop}}$, the moduli space of metric graphs. We argue that for this strategy to work we need that Φ, and all but finitely many of the points in the deformation path to $\bar{\Phi}$, belong to cones of tropical morphisms that we call *full-dimensional*. Intuitively, these morphisms *depend* on the right number of parameters to cover $\mathcal{M}_g^{\mathrm{trop}}$.

We use this idea for both parts of this series of articles, only the ingredients change. In Part I we use constructions from [CD18] as initial families, which gives us a tropical morphism for each possible trivalent graph H (this H is the *combinatorial type*, defined in Subsection 1.3). Thus, the deformation procedure only has to handle paths where the combinatorial type is the same throughout the deformation path in $\mathcal{M}_g^{\mathrm{trop}}$. Setting up the deformation requires analysing 16 cases and a *balancing relation* between certain determinants associated to the tropical morphisms. This balancing relation, closely related to the theory of morphisms between tropical fans [GKM09], ensures that we can always proceed further in our desired path from Γ to $\bar{\Gamma}$.

In Part II the initial point of the deformation is our own family of graphs, which we call *caterpillars of loops*. We understand them better than the constructions of [CD18] in that we are able to prove that if Γ is a caterpillar of loops, then there are exactly Catalan-many tropical morphisms realizing the gonality bound. We also strengthen the deformation procedure. By analyzing 8 further cases we can deform past changes of combinatorial type. We show that with suitable multiplicity the count of tropical morphisms realizing the gonality bound remains constant thorough deformation.

To carry out this simple idea we introduce a technical framework to describe what is a full-dimensional cone and its properties, set up the deformation procedure, and support

the case work required to prove the correctness of deformation. The road through Parts I and II is long, but it comes with two rewards: we achieve a combinatorial proof of the gonality bound; at the finish line we arrive at a tropical space that parametrizes tropical morphisms attaining the gonality bound, which surjects onto $\mathcal{M}_g^{\mathrm{trop}}$, is connected through codimension 1, and has fibres with Catalan-many points. This mirrors the beautiful classical story due Eisenbud and Harris [EH87]. It would be interesting to understand how our work relates to the tropicalization of the space of admissible covers from [CMR16], but this is beyond the scope of these combinatorial papers.

Organization of article. Section 1 recalls several concepts on graphs, metric graphs, and $\mathcal{M}_g^{\mathrm{trop}}$. Section 2 develops a formalism for tropical morphisms that highlights the combinatorics. Here we introduce full-dimensional cones of tropical morphisms. Section 3 introduces the main combinatorial gadget of this article, the gluing datum. We use it to study the properties of full-dimensional cones. Section 4 is a brief pause from developing the theory, with a fully worked out example suggesting the deformation procedure—in fact, we encourage the reader to go through this example before going through the more technical parts of the article! Section 5 resumes the theory with a study of the deformation operation, and a proof of the main result. This proof depends on many constructions that we carry out in Section 6.

1. Background

1.1. **Graphs.** A graph G is a pair $(V(G), E(G))$ of disjoint sets, the vertex and edge sets respectively, and a map ι_G defined on $E(G)$ encoding incidences, whose values are multisets of two elements of $V(G)$. Given an edge e with $\iota_G(e) = \{A, B\}$ we call A, B the *ends* of e, and say that A and e are *incident*. A *loop* is an edge e such that its two ends are equal. A *subgraph* of G is a graph G' such that $V(G') \subset V(G)$, $E(G') \subset E(G)$, and $\iota_{G'}$ equals the restriction $\iota_G|_{E(G')}$. A subset S of $V(G)$ induces a subgraph by taking for edge set all the edges of G with both ends in S. Likewise, a subset S of $E(G)$ induces a subgraph by taking for vertex set the ends of all the edges in S. A graph G is *finite* if both $V(G)$ and $E(G)$ are finite sets; and *loopless* if it has no loops.

A *path* of G is a sequence $\langle A_0, e_1, A_1, \ldots, e_\mu, A_\mu \rangle$ of alternately vertices and edges, where no element is repeated, such that consecutive elements are incident and $\mu \geq 1$. The *ends* of a path are the vertices A_0, A_μ. The remaining vertices $A_1, \ldots, A_{\mu-1}$ are *interior*. We call μ the *length* of P. We write $x \in P$ to mean that x is an element in the sequence P. If P is a path with ends A, B, and e in $E(G) \setminus P$ is an edge that also has ends A, B, then we call the subgraph consisting of e together with P a *cycle*. Given a vertex A, its *connected component* is the subgraph induced by the set of ends of all paths with $A \in P$. The vertex set $V(G)$ is partitioned into connected components. A graph is *connected* if it has a single connected component.

Let G be a finite, connected graph. Set $g(G) = |E(G)| - |V(G)| + 1$. Following a convention due to [BN07], we call this number the *genus* of G. If G has genus 0, then we call G a *tree*. Write $x \in G$ for $x \in V(G) \cup E(G)$. Let A be a vertex. Write $E_G(A)$ for the subset of $E(G)$ incident to A. The *valency* $\mathrm{val}_G A$ of A is the number of edges in $E_G(A)$, with loops counting twice. The *minimum valency* of G is the number $\min_{A \in V(G)} \mathrm{val}_G A$. We write $E(A)$ and $\mathrm{val}\, A$ when G is clear from the context. A vertex is *monovalent*, *divalent*, *trivalent*, or *n-valent* if its valency equals 1, 2, 3, or n, respectively. We call a monovalent vertex of a tree a *leaf*.

Let $S \subset E(G)$. The *deletion of S*, denoted by $G \setminus S$, is the graph with edge set $E(G) \setminus S$ and incidence map $\iota_G|_{E(G) \setminus S}$. The *contraction of S*, denoted by G/S, is the graph $(V(G)/\sim, E(G) \setminus S)$, where $\sim$ identifies the vertices A and B if they are the ends of some e in S, with incidence map $\pi \circ \iota_G|_{E(G) \setminus S}$ with $\pi : V(G) \to V(G)/\sim$ the canonical projection. The *subdivision* of an edge e yields the graph $(V(G) \cup \{C\}, (E(G) \setminus \{e\}) \cup \{e_1, e_2\})$, where the ends of e_1 are A, C, and of e_2 are C, B.

Let G, G' be two graphs. A *graph morphism* is a map $\gamma : V(G) \cup E(G) \to V(G') \cup E(G')$ such that: $\gamma(V(G)) \subseteq V(G')$; if $\gamma(e) \in V(G')$ for an edge e in $E(G)$ with ends A and B, then $\gamma(e) = \gamma(A) = \gamma(B)$; if $\gamma(e) \in E(G')$ for an edge e in $E(G)$ with ends A and B, then the ends of $\gamma(e)$ are $\gamma(A)$ and $\gamma(B)$. In essence, a morphism is an incidence preserving map that can contract edges. If $\gamma(E(G)) \subseteq E(G')$, then we call γ a *homomorphism*. An *isomorphism* is a bijective homomorphism; its inverse is then a homomorphism as well.

An edge (resp. vertex) labelling is an injective map λ from $E(G)$ (resp. $V(G)$) to a set S. A total order $\leq_S$ on S induces a total order on $E(G)$ by letting $e \leq_{E(G)} e'$ when $\lambda(e) \leq_S \lambda(e')$. We use $\mathbb{N}$ as labelling set, with its natural order. In the next section we consider elements from the vector space $\mathbb{R}^{E(G)}$ (real valued functions on $E(G)$). By choosing an edge labelling λ this space is identified with $\mathbb{R}^{|E(G)|}$: we write $e_j \in E(G)$ for $\lambda^{-1}(j)$ (we use this notation whenever a λ has been chosen); we take as ordered basis of $\mathbb{R}^{E(G)}$ the functions ℓ_i that map e_j to 1 if $i = j$, and zero otherwise.

1.2. **Metric graphs.** A *length function* for G is a map $\ell : E(G) \to \mathbb{R}_{>0}$. The pair (G, ℓ) gives rise to a one-dimensional CW-complex Γ by gluing the disjoint union of closed real intervals $\Gamma_e = [0, \ell(e)]$, one for each e in $E(G)$, along their endpoints in the manner prescribed by G. We call the first Betti number of the CW-complex Γ the *genus* of Γ, denoted by $g(\Gamma)$. It coincides with $g(G)$. This CW-complex is equipped with the shortest-path metric in which the intervals Γ_e have lengths $\ell(e)$. This metric space is called a *metric graph*. We call G a *model*, and the pair (G, ℓ) a *realization*, of Γ.

Let $\Gamma = (G, \ell_G)$. There are infinitely many realizations of Γ. For example, we often consider the model G' obtained by subdividing e in $E(G)$ into e_1, e_2, and taking as length function some ℓ' with $\ell'(e_1) + \ell'(e_2) = \ell(e)$, so (G', ℓ') also realizes Γ. A *vertex set* of Γ is a finite subset $S \subset \Gamma$ of points such that $\Gamma \setminus S$ is homeomorphic to a disjoint union of open real intervals. Every realization (G, ℓ_G) of Γ induces a vertex set: the set of endpoints of the intervals Γ_e. We refer to the intervals Γ_e as the *edges* of Γ. Every vertex set S induces a realization (G_S, ℓ_S) of Γ: the vertices of G_S are the points in S; the edges are the intervals $\Gamma \setminus S$; the length function ℓ_S is given by the lengths of $\Gamma \setminus S$. Thus, a choice of model G for Γ amounts to a choice of vertex set, and therefore edge set as well.

We single out a particular realization for Γ. Consider points x in Γ such that for all $\varepsilon > 0$ the open ball $B(x, \varepsilon)$ with centre x and radius ε is not isometric to the interval $(-\varepsilon, \varepsilon)$. We call these *essential vertices*. The naming is due to the fact that if x is an essential

vertex, then x cannot be in the topological interior of an edge Γ_e. Thus, the set of essential vertices is contained in the set of endpoints of edges Γ_e, and so there are finitely many. On the other hand, we call the metric graphs with an empty set of essential vertices *metric loops*. They have genus 1 and admit the graph with one vertex and one loop as a model.

Lemma 3. *Let Γ be a metric graph distinct from a metric loop, $\mathcal{E}$ the essential vertices of Γ, and $S \subset \Gamma$ a finite set of points. Then S is a vertex set if and only if $\mathcal{E} \subset S$.*

The valency of a point x in Γ is the valency of x in G_S for S a vertex set with x in S. If Γ is not a metric loop, then we call the realization $(G_{\mathcal{E}}, \ell_{\mathcal{E}})$ the *essential realization*. Thus, by Lemma 3 the *essential model* $G_{\mathcal{E}}$ is minimal in the sense that every other model arises from a sequence of edge subdivisions of $G_{\mathcal{E}}$. Hence, all the realizations are refinements of the essential realisation. This shows $\mathrm{val}\, x$ to be well defined.

1.3. **Tropical modification.** In the literature, *tropical modification* of a metric graph $\Gamma = (G, \ell)$ is an operation that deletes or attaches a segment ending in a monovalent point. For investigating tropical morphisms we consider metric graphs to be equivalent under a sequence of these operations. This allows to meet certain *balancing conditions*, and does not alter Γ in a significant manner. Our approach to tropical modification is based on the notion of dangling elements in the model G:

Definition 4. An edge e of G is *dangling* if deleting e produces two connected components and one of those is a tree. A vertex A of G is dangling if all e in $E(A)$ are dangling.

This definition detects points contained in a segment that has been attached to Γ. Fix a model G for Γ. Let x be in Γ. If x is in the vertex set induced by G, then we say that x is dangling when its corresponding vertex in G is dangling. Otherwise x is in the interior of an edge Γ_e and we say that x is dangling if e is dangling. It is straighforward to see that this definition is independent of the chosen realization (G, ℓ).

Let $\widetilde{G}$ be the graph obtained by deleting all the dangling elements of G. By Definition 4 we have that $g(G) = g(\widetilde{G})$. Let $\widetilde{\Gamma} = (\widetilde{G}, \ell|_{\widetilde{G}})$, where $\ell|_{\widetilde{G}}$ is the restriction of ℓ to $E(\widetilde{G})$. We identify G with $\widetilde{G}$ and Γ with $\widetilde{\Gamma}$ under tropical modification. Let H be the essential model of $\widetilde{\Gamma}$. The construction of $\widetilde{\Gamma}$ and Lemma 3 together imply that:

Lemma 5. *Let Γ be a metric graph of genus at least 2, and Γ' a tropical modification of Γ, such that Γ' has a model G' with minimum valency at least three. Then Γ' is isometric to $\widetilde{\Gamma}$ and G' is isomorphic to H.*

Thus, $\widetilde{\Gamma}$ is a canonical representative of the equivalence class under isomorphism and tropical modification of Γ. We call a graph with minimum valency at least three a *combinatorial type*. This is an important invariant under tropical modification, so we say that H is the combinatorial type of Γ.

1.4. **Moduli space of metric graphs.** We give a barebones account of the moduli space of genus-g metric graphs $\mathcal{M}_g^{\mathrm{trop}}$. If all the vertices of a graph are trivalent, we call it a *trivalent combinatorial type*, or simply *trivalent*. We denote by $\mathbb{G}_g$ the family of genus-g trivalent combinatorial types. For the construction of $\mathcal{M}_g^{\mathrm{trop}}$ we use the following straightforward lemma:

Lemma 6. *Let $g \geq 2$ and H_0 be a genus-g combinatorial type. There exists H in $\mathbb{G}_g$ such that there is a sequence of edge contractions of H that yield a graph isomorphic to H_0.*

Given a graph G, we denote by C_G the set of length functions defined on G. This set has a cone structure, as it equals the positive orthant $\mathbb{R}_{>0}^{E(G)}$. We identify each point ℓ of C_G with the metric graph (G, ℓ) to obtain the (rational polyhedral) cone C_G of metric graphs with model G.

Motivated by Lemma 6 we add some of the boundary points of C_G with the following convention: given a map $\ell : E(G) \to \mathbb{R}_{\geq 0}$, contract all the edges of G for which $\ell(e) = 0$, to get G'; the pair (G, ℓ) refers to the metric graph $(G', \ell|_{G'})$. Let $\overline{C}_G \subset \mathbb{R}_{\geq 0}^{E(G)}$ be the set of maps ℓ for which the metric graph (G, ℓ) has genus g, that is, all cycles of G have positive length. gluing together these completed cones gives rise to a space where points bijectively correspond to classes under tropical modification of genus-g metric graphs:

Definition 7. The moduli space of genus-g metric graphs is the space

$$\mathcal{M}_g^{\mathrm{trop}} = \left(\bigsqcup_{H \in \mathbb{G}_g} \overline{C}_H \right) / \cong,$$

where $\cong$ identifies points ℓ_H and $\ell_{H'}$ for which (H, ℓ_H) and $(H', \ell_{H'})$ are isometric. It is given the quotient topology.

Given a metric graph Γ, by Lemma 5 and 6 there is a point in $\mathcal{M}_g^{\mathrm{trop}}$ corresponding to a graph $\widetilde{\Gamma}$ that is equivalent to Γ under tropical modification. By construction this point is unique.

Remark 8. The reader might find in the literature a different use for the symbol $\mathcal{M}_g^{\mathrm{trop}}$, albeit for constructions related to ours. They differ by imposing additional structure on the metric graph: marked points, legs of infinite length, weights on the vertices, and more. These are not required for our purposes, as they typically arise to encode more algebro-geometric information in a graph. See [Koz09] and [ACP15] for in-depth studies. $\triangle$

Remark 9. Identifying isometric points complicates the geometry of $\mathcal{M}_g^{\mathrm{trop}}$. We do not obtain a polyhedral cone complex, as one might expect. This issue is addressed in [ACP15] with the introduction of *abstract polyhedral cone complexes*. See their Section 4.3 for a proof that $\mathcal{M}_g^{\mathrm{trop}}$ is an abstract polyhedral cone complex. $\triangle$

Let H be a combinatorial type. From $2|E(H)| = \sum_{A \in V(H)} \mathrm{val}\, A$ one can conclude that H is trivalent if and only if $|E(H)| = 3g - 3$. Hence, $\mathcal{M}_g^{\mathrm{trop}}$ is pure of dimension $3g - 3$.

1.5. **Notation.** In the next four sections the following letters, and small variations thereof, are used to denote specific objects. Metric graphs: Γ for metric graphs with $g(\Gamma) \geq 2$; Δ for metric trees (due to tree being δέντρο in Greek), that is genus 0 metric graphs; and $\widetilde{\Gamma}$ for the deletion of the dangling elements of Γ. Finite graphs: G for a finite connected graph with $g(G) \geq 2$, A for its vertices and e for its edges; T for a finite tree, v for its vertices and t for its edges; and H for the combinatorial type of G. Morphisms: φ for maps between finite graphs; and Φ for maps between metric graphs. We write $[d]$ for the set $\{1, 2, \ldots, d\}$, and $x \in G$ for $x \in V(G) \cup E(G)$.

2. Tropical morphisms

2.1. **DT-morphisms.** We begin with a map encapsulating the combinatorial information of tropical morphisms. These are a special class of graph morphisms, augmented with an index map.

Definition 10 (discrete tropical morphism)**.** Let G, G' be loopless graphs, $\varphi : G \to G'$ a morphism, and $m_\varphi : G \to \mathbb{Z}_{\geq 0}$ a map.

(a) m_φ is an *index map* for φ if for every e in $E(G)$ we have that $\varphi(e) \in V(G')$ if and only if $m_\varphi(e) = 0$. That is, only index zero edges are contracted by φ.

(b) φ is *harmonic* with index map m_φ if it satisfies the *balancing condition*: for every A in $V(G)$ and e' incident to $\varphi(A)$ we have that

$$m_\varphi(A) = \sum_{e \in E(A) \cap \varphi^{-1}(e')} m_\varphi(e).$$

Note that this makes the sum independent of the choice of e'.

(c) φ is *non-degenerate* if $m_\varphi(x) \geq 1$ for all x in G.

(d) φ satisfies the *Riemann-Hurwitz condition* (RH-condition) if for every A in $V(G)$:

$$r_\varphi(A) = (\operatorname{val} A - 2) - (\operatorname{val} \varphi(A) - 2) \cdot m_\varphi(A) \geq 0.$$

(e) A *discrete tropical morphism* is a pair (φ, m_φ) consisting of a non-degenerate harmonic morphism φ with index map m_φ that satisfies the RH-condition. We write DT-morphism to shorten.

Remark 11. The non-degeneracy condition ensures that no edge is collapsed. $\triangle$

Remark 12. Definition 10 is similar to Definition 2.1 of [Cap14], but there are differences: condition (c) of Definition 2.1 in the cited work is required only for vertices of G (see discussion in Section 1.3 of [CD18]); an alternative formula for r_φ is given on Equation (10) of the cited work (see Lemma 13); and we do not have a vertex labelling representing extra genus. $\triangle$

Lemma 13 (formula for r_φ)**.** *Let φ be a DT-morphism with index map m_φ. Then*

$$r_\varphi(A) = 2(m_\varphi(A) - 1) - \sum_{e \in E(A)} (m_\varphi(e) - 1).$$

Proof. By the balancing condition $m_\varphi(A) \cdot \operatorname{val} \varphi(A) = \sum_{e \in E(A)} m_\varphi(e)$. Thus,

$$r_\varphi(A) = \operatorname{val} A - 2 - m_\varphi(A) \cdot (\operatorname{val} \varphi(A) - 2)$$
$$= 2(m_\varphi(A) - 1) + \operatorname{val} A - \sum_{e \in E(A)} m_\varphi(e)$$
$$= 2(m_\varphi(A) - 1) - \sum_{e \in E(A)} (m_\varphi(e) - 1). \qquad \square$$

We always denote the index map of φ by m_φ. So we write "*φ is a DT-morphism*" instead of "*φ is a DT-morphism with index map m_φ*". We call G the *source*, and G' the *target*, of φ. The index m_φ can be understood as a local degree: for A in $V(G)$ restricting φ to the subgraph G_A induced by $E(A)$ gives a DT-morphism onto the subgraph induced by $\varphi(E(A))$; any fibre of $\varphi|_{G_A}$ has $m_\varphi(A)$ elements, each element counted with multiplicity m_φ. Since G is connected this local degree gives rise to a global degree, where the constant obtained in the count of preimages with multiplicities is called the *degree* of φ:

$$\deg \varphi = \sum_{\varphi(A) = A'} m_\varphi(A);$$

this is independent of A' by the connectedness of G (see proof of Lemma 2.4 of [BN09]).

Example 14. Figure 1 illustrates two DT-morphisms φ_A, φ_B of degrees 3 and 4, respectively. The targets are drawn below, the sources above, and the fibres with dotted lines. The edges have index 1, unless noted otherwise with a number above the edge.

φ_A φ_B

FIGURE 1

This setup is parallel to the classical theory of maps between Riemann surfaces. There is a Riemann-Hurwitz formula relating the genera of the source and the target.

Lemma 15 (Riemann-Hurwitz formula). *Let $\varphi\colon G \to G'$ be a DT-morphism. Then*

$$2g(G) - 2 = \deg\varphi \cdot (2g(G') - 2) + \sum_{A \in V(G)} r_\varphi(A).$$

Proof. Recall that in a graph the sum of all the valencies is twice the number of edges:

$$\sum_{A \in V(G)} r_\varphi(A) = \sum_{A \in V(G)} \left(\operatorname{val} A - 2 - m_\varphi(A) \cdot (\operatorname{val}\varphi(A) - 2) \right)$$
$$= 2|E(G)| - 2|V(G)| - \deg\varphi \cdot (2|E(G')| - 2|V(G')|)$$
$$= 2g(G) - 2 - \deg\varphi \cdot (2 \cdot g(G') - 2).$$

Rearrange terms to obtain the result. $\qquad\square$

The RH-condition implies that the sum $\sum_{A \in V(G)} r_\varphi(A)$ is positive, thus $g(G) \geq g(G')$ by Lemma 15. As a side remark, consider Equation (15) of [Cap14]. By taking degrees of the divisors we recover Lemma 15. Thus, one of the motivations for the RH-condition is a necessary and sufficient condition to have the ramification divisor be effective, as in the classical setting.

2.2. **Tropical morphisms.** By choosing a length function for the target G' we obtain a *tropical morphism* of metric graphs, in the sense of [CD18].

Definition 16 (tropical morphism). Let $\varphi\colon G \to G'$ be a DT-morphism and $\ell_{G'}$ in $C_{G'}$. The corresponding *tropical morphism* is the unique map Φ from the source metric graph $\Gamma = (G,\ e \mapsto \ell_{G'}(\varphi(e))\,/\,m_\varphi(e)\,)$ to the target $\Gamma' = (G', \ell_{G'})$, whose restriction to Γ_e is a bijective linear function to $\Gamma'_{\varphi(e)}$, and that agrees with φ on vertex points.

Call φ and $(\varphi, \ell_{G'})$ a *model* and a *realization* of Φ, respectively. From the connectedness of G and G' we get that Φ is continuous. The length function for the source is constructed so the linear map $\Phi|_{\Gamma_e}$ has integral slope $m_\varphi(e)$. For a point x in the interior of an edge the slope is the multiplicity, and if x is a vertex then the multiplicity is computed with the balancing condition. With this, the count with multiplicities of points in the fibre $\Phi^{-1}(x')$ is independent of x' in Γ', and moreover agrees with the degree of any model of Φ.

Definition 16 differs from [CD18] because here our focus is on combinatorial construc-
tions. In the previous paragraph we have argued that the map $\Phi = (\varphi, \ell_{G'})$ is a tropical
morphism in the sense of [CD18]. Conversely, given a tropical morphism $\Phi : \Gamma \to \Gamma'$, we
choose vertex sets $\mathcal{G}$, $\mathcal{T}$ on Γ, Γ' such that Φ induces a DT-morphism φ_{ess}.

Construction 17 (essential model). Let $\Phi : \Gamma \to \Gamma'$ be a tropical morphism, and $\mathcal{E}$, $\mathcal{E}'$ the
essential vertices of Γ, Γ' respectively. Set $\mathcal{T} = \mathcal{E}' \cup \Phi(\mathcal{E})$ and $\mathcal{G} = \Phi^{-1}(\mathcal{T})$. By Lemma 3
these are vertex sets, so let $(G, \ell), (G', \ell')$ be the induced realizations of Γ, Γ'. Define a
map $\varphi_{\mathrm{ess}} : G \to G'$ by sending a vertex of Γ to the corresponding vertex in Γ', and for
each edge e in $E(G)$ choose a point x in Γ_e and map $e \mapsto e'$, where e' is such that $\Phi(x)$ is
in $\Gamma'_{e'}$. As index map take $m_{\varphi_{\mathrm{ess}}} = \ell' \circ \varphi(e)/\ell(e)$.

Lemma 18. *The pair (φ_{ess}, ℓ') realizes Φ, and is minimal in the sense that for any other
model of Φ, the induced vertex sets on Γ, Γ' arise as edge subdivisions of $\mathcal{G}, \mathcal{T}$.*

Proof. We claim that φ_{ess} is well defined and is a DT-morphism. Let x in Γ_e and e' in
$E(G')$ such that $\Phi(x)$ is in $\Gamma'_{e'}$. We show that e' only depends on e. To see this, take x_1,
x_2 in Γ_e. There is a topological path P (the image of a continuous map $\omega : [0, 1] \to \Gamma$)
from x_1 to x_2 contained in the interior of Γ_e, so in particular it does not pass through any
vertex of $\mathcal{G}$. Thus, $\Phi(P)$ does not pass through any vertex x' of $\mathcal{T}$ (otherwise P would pass
through some vertex in $\Phi^{-1}(x') \subset \mathcal{G}$). Hence, $\Phi(x_1)$ and $\Phi(x_2)$ are in the same $\Gamma'_{e'}$. The
index map $m_{\varphi_{\mathrm{ess}}}$ records the slope of $\Phi|_{\Gamma_e}$, thus the balancing condition, non-degeneracy,
and the RH-condition are satisfied. It is then clear that $(\varphi_{\mathrm{ess}}, \ell')$ realizes Φ.

Regarding minimality, as DT-morphisms are graph morphisms we get that $\varphi(V(G)) \subset
V(G')$. Thus, in every model φ of Φ the vertex set of Γ' has to contain $\Phi(\mathcal{E})$ (and $\mathcal{E}'$).
Furthermore, the non-degeneracy condition of φ implies that the fibre of a vertex of G' is
a subset of $V(G)$. So a vertex set of Γ has to contain $\Phi^{-1}(\mathcal{T})$. $\square$

2.3. **Isomorphisms, tropical modification, combinatorial type.** Recall that in the
space $\mathcal{M}_g^{\mathrm{trop}}$ we consider metric graphs in an equivalence class under isometry and tropical
modification. This setup extends naturally to tropical morphisms.

Definition 19 (isomorphism of graph morphisms). Let $\varphi_1 : G_1 \to G_1'$ and $\varphi_2 : G_2 \to G_2'$
be graph morphisms. An isomorphism from φ_1 to φ_2 is given by graph isomorphisms
(γ, γ') for which Diagram 20.a commutes. If additionally φ_1 and φ_2 are DT-morphisms,
then (γ, γ') is an isomorphism of DT-morphisms if the index map m_{φ_2} pulls back to m_{φ_1},
namely Diagram 20.b commutes. Let Φ_1 and Φ_2 be tropical morphisms. An isomorphism
from Φ_1 to Φ_2 is given by isometries Ψ and Ψ' for which Diagram 20.c commutes.

$$
\begin{array}{ccc}
G_1 \xrightarrow{\varphi_1} G_1' & \qquad & \Gamma_1 \xrightarrow{\Phi_1} \Gamma_1' \\
\gamma \downarrow \quad \downarrow \gamma' & \gamma \downarrow \searrow^{m_{\varphi_1}} & \Psi \downarrow \quad \downarrow \Psi' \\
G_2 \xrightarrow{\varphi_2} G_2' & G_2 \xrightarrow{m_{\varphi_2}} \mathbb{Z}_{\geq 0} & \Gamma_2 \xrightarrow{\Phi_2} \Gamma_2'
\end{array}
$$

$$
\text{(a)} \qquad\qquad \text{(b)} \qquad\qquad \text{(c)}
$$

Diagram 20

We now define equivalence under tropical modification for a DT-morphism $\varphi : G \to G'$.
We say that the fibre $\varphi^{-1}(x')$, for x' in G', is *dangling* if all its elements are dangling
in G. Let $D \subset E(G')$ be the set of edges with dangling fibres. Consider the subgraph $\hat{G}'$
of G' induced by $E(G') \setminus D$. The preimage $\varphi^{-1}(\hat{G}')$ is not necessarily a connected graph

(see Example 23), but the definition of dangling implies that there is only one connected component with non-zero genus. Call it $\hat{G}$. Restricting φ to $\hat{G}$ gives a DT-morphism $\hat{\varphi} : \hat{G} \to \hat{G}'$ that we call the *deletion of dangling fibres* of φ. See the Section *Modifications* in [CD18] for some examples. Tropical modification identifies φ with $\hat{\varphi}$.

The same definitions and constructions can be carried out on the metric side for a tropical morphism $\Phi = (\varphi, \ell)$. A fibre $\Phi^{-1}(x)$ is dangling if all its points are dangling, and the deletion of dangling fibres $\hat{\Phi}$ is the restriction of Φ to the connected component with non-zero genus which remains after deleting all points belonging to dangling fibres. One can check that $(\hat{\varphi}, \ell|_{\hat{G}'})$ is a realization of $\hat{\Phi}$. Tropical modification identifies Φ with $\hat{\Phi}$. It is straightforward to see that if Φ_1 and Φ_2 are equivalent under tropical modification, then $\hat{\Phi}_1$ and $\hat{\Phi}_2$ are isomorphic, thus $\hat{\Phi}$ is a canonical representative for the equivalence class of Φ under isomorphism and tropical modification. Thus, it is natural to use a model of $\hat{\Phi}$ as the *combinatorial type* of the class of Φ:

Definition 21. Let Φ be a tropical morphism and $\hat{\Phi}$ its deletion of dangling fibres. The *combinatorial type* of Φ is the model $\hat{\varphi}_{\text{ess}}$ of $\hat{\Phi}$ constructed using Construction 17.

Alternatively, one can use Construction 17 on Φ to get $(\varphi_{\text{ess}}, \ell_{\text{ess}})$ and delete all dangling fibres to get $\widehat{\varphi_{\text{ess}}}$. It is straightforward to see that $\widehat{\varphi_{\text{ess}}}$ is isomorphic to $\hat{\varphi}_{\text{ess}}$. So $(\hat{\varphi}_{\text{ess}}, \ell_{\text{ess}}|_{\hat{G}'_{ess}})$ realizes $\hat{\Phi}$. We call the values of $\ell_{\text{ess}}|_{\hat{G}'_{ess}}$ the *parameters* of Φ. Thus, we say that Φ moves in the cone C_φ with dimension $\dim C_\varphi$ and depends on $|E(\hat{G}'_{ess})|$ many parameters.

2.4. **Cone of sources.** We associate to a DT-morphism φ the set C_φ of sources (modulo tropical modification) obtained by varying the parameters. By Section 2.3 this is an invariant under isomorphism and tropical modification. We say that φ *realizes* Γ if $\widetilde{\Gamma} \in C_\varphi$ (recall that $\widetilde{\Gamma}$ is obtained by deletion of dangling points). We introduce an important linear map to see that C_φ has the structure of a polyhedral cone. Another important invariant under tropical modification of φ is the combinatorial type of its source, which we denote by $H(\varphi)$.

Let $\varphi : G \to G'$ be a DT-morphism, $H = H(\varphi)$, and $g = g(H)$. Consider a length function $\ell_{G'}$ in $C_{G'}$; the tropical morphism $\Phi = (\varphi, \ell_{G'})$; and $\widetilde{\Gamma} = (H, \ell_H)$ where (H, ℓ_H) is the deletion of dangling elements of the source of Φ. Extend the map sending $\ell_{G'}$ to ℓ_H (defined from $\mathbb{R}_{>0}^{E(G')}$ to $\mathbb{R}_{>0}^{E(H)}$) to a linear map A_φ from $\mathbb{R}^{E(G')}$ to $\mathbb{R}^{E(H)}$. We call A_φ the *edge-length map*. Recall from Section 1.1 that by choosing labellings λ_H and $\lambda_{G'}$ we identify $\mathbb{R}^{E(H)}$ and $\mathbb{R}^{E(G')}$ with $\mathbb{R}^{|E(H)|}$ and $\mathbb{R}^{|E(G')|}$. This gives an $|E(H)| \times |E(G')|$-matrix for A_φ.

We define the *cone of sources* C_φ to be $A_\varphi(\mathbb{R}_{>0}^{E(G')})$. It parametrizes, up to tropical modification, the sources of tropical morphisms with model φ. Let Φ be a tropical morphism with model φ. We say that Φ *moves* in the cone C_φ.

Note that $\dim C_\varphi = \operatorname{rk} A_\varphi \leq 3g - 3$. We are interested in tropical morphisms Φ that move in a cone whose dimension equals the number of parameters defining Φ. Namely, those for which $\operatorname{rk} A_\varphi = |E(G')|$. We say that these cones, and DT-morphisms, have *full-rank*. If φ has full-rank then there is a unique preimage $A_\varphi^{-1}(\ell_H)$ for ℓ_H. In other words, a unique tropical morphism $(\varphi, A_\varphi^{-1}(\ell_H))$ whose source retracts to $\widetilde{\Gamma} = (H, \ell_H)$.

2.5. **Combinatorial types.** Let $\varphi : G \to G'$ be a DT-morphism, and Φ a tropical morphism with model φ. Recall that the combinatorial type of Φ is the essential model $\hat{\varphi}_{\mathrm{ess}}$ of the deletion of dangling points $\hat{\Phi}$. Now we give an analogue of Lemma 5 for tropical morphisms; namely we give combinatorial conditions that characterize the DT-morphisms that show up as combinatorial types. We also show that DT-morphisms with full-rank satisfy these conditions.

Definition 22. A *combinatorial type of DT-morphisms* is a DT-morphism $\varphi : G \to G'$ without dangling fibres and such that $\sum_{A \in \varphi^{-1}(v)} r_\varphi(A) \geq 1$ for every divalent v in $V(G')$.

Example 23. Figure 2 illustrates two DT-morphisms φ_A, φ_B. They are equivalent under tropical modification, and φ_B is a combinatorial type of DT-morphisms. Note that the degree is not invariant.

φ_A $\qquad\qquad$ φ_B

FIGURE 2

Lemma 24. *Let Φ be a tropical morphism, and $\bar{\Phi} : \bar{\Gamma} \to \bar{\Gamma}'$ be a tropical modification of Φ, such that $\bar{\Phi}$ has a model $\bar{\varphi} : \bar{G} \to \bar{G}'$ that is a combinatorial type of DT-morphisms. Then $\bar{\Phi}$ is isomorphic to $\hat{\Phi}$ and $\bar{\varphi}$ is isomorphic to $\hat{\varphi}_{ess}$.*

Proof. By definition $\bar{\varphi}$ has no dangling fibres. So by Lemma 18 we only need to show that the points of $\bar{\Gamma}'$ corresponding to vertices of $\bar{G}'$ are in the set $\mathcal{T}$, with $\mathcal{T}$ as in Construction 17 (the union of the set of essential vertices of $\bar{\Gamma}'$ with the image of the set of essential vertices of $\bar{\Gamma}$). This means proving that for every divalent v in $V(\bar{G}')$ there is a vertex A in $\bar{\varphi}^{-1}(v)$ such that $\mathrm{val}\, A \geq 3$. Note that for A in $\bar{\varphi}^{-1}(v)$ we have $r_{\bar{\varphi}}(A) = \mathrm{val}\, A - 2 - m_{\bar{\varphi}}(A)(\mathrm{val}\, v - 2) = \mathrm{val}\, A - 2$. By the balancing condition there are no monovalent vertices in $\bar{\varphi}^{-1}(v)$. Thus, $\sum_{A \in \bar{\varphi}^{-1}(v)} r_{\bar{\varphi}}(A) = \sum_{A \in \bar{\varphi}^{-1}(v)} \mathrm{val}\, A - 2 \geq 1$ if and only if there is at least one vertex A in $\bar{\varphi}^{-1}(v)$ with $\mathrm{val}\, A \geq 3$. $\qquad\square$

Combinatorial types of DT-morphisms are canonical representatives in the equivalence class under tropical modification: the discussion after Definition 21 tells us how to construct them, and Lemma 24 ensures their uniqueness.

We call the sum $\sum_{A \in \varphi^{-1}(v)} r_\varphi(A)$ the *change* at v, and denote it by $\mathrm{ch}\, v$. We have seen in the proof of Lemma 24 that divalent v with $\mathrm{ch}\, v = 0$ arise as edge-subdivisions of the essential model. Thus, for divalent v if t, t' are the edges incident to v, then morally the fibres of t, t' are equal (each A in $\varphi^{-1}(v)$ has valency 2, and the two edges have the same index m_φ). This implies the following result, which is our motivation for calling this quantity "change".

Lemma 25. *Let v in $V(G')$ be divalent, and $E(A) = \{t, t'\}$. If $\mathrm{ch}\, v = 0$, then the columns of A_φ corresponding to t, t' are equal.*

Lemma 26. *If $\varphi : G \to G'$ has full-rank, then φ is a combinatorial type of DT-morphisms.*

Proof. Let H be the combinatorial type of G, and $A_\varphi : C_{G'} \to C_H$ be the edge-length map. Suppose t in $E(G')$ has a dangling fibre. Then varying $\ell_{G'}(t)$ does not affect $A_\varphi(\ell_{G'})$, contradicting that A_φ has full-rank. So φ has no dangling fibres. Moreover, by Lemma 25 we have that φ has no divalent vertices with $\operatorname{ch} v = 0$. $\square$

Finally, we prove a combinatorial property that puts the condition $\operatorname{ch} v \geq 1$, for divalent vertices, in a wider context.

Lemma 27. *If φ is a combinatorial type, then $\operatorname{ch} v + \operatorname{val} v \geq 3$ for all v in $V(G')$.*

Proof. We only need to show that $\operatorname{ch} v \geq 2$ for monovalent vertices v of G'. Let v be monovalent. Note that a monovalent vertex of G is dangling. Since G does not have dangling fibres, pick A in $\varphi^{-1}(v)$ with $\operatorname{val} A \geq 2$. For such A we have that $r_\varphi(A) = \operatorname{val} A - 2 - m_\varphi(A)(\operatorname{val} v - 2) = \operatorname{val} A - 2 + m_\varphi(A)$. Since v is monovalent we have that $m_\varphi(A) \geq \operatorname{val} A$, thus $r_\varphi(A) \geq 2$ and by the RH-condition this implies that $\operatorname{ch} v \geq 2$. $\square$

2.6. **Dimension Formula.** Let $\varphi : G \to G'$ be a DT-morphism. We prove a formula relating the number $|E(G')|$, the degree $d = \deg \varphi$, and the genera of the source and the target of φ. Note that $|E(G')|$ is an upper bound for $\dim C_\varphi$. Hence, we call this result *the dimension formula.*

Lemma 28 (total change). *Let $\varphi : G \to G'$ be a DT-morphism. Then*

$$\sum_{v \in V(G')} \operatorname{ch} v = 2g(G) - 2g(G') \cdot d + 2d - 2.$$

Proof. This is a consequence of Lemma 15 (Riemann-Hurwitz formula). $\square$

Proposition 29 (dimension formula). *Let $\varphi : G \to G'$ be a DT-morphism. Then*

$$|E(G')| + \sum_{v \in V(G')} (\operatorname{ch} v + \operatorname{val} v - 3) = 2g(G) - g(G') \cdot (2d - 3) + 2d - 5.$$

Proof. The ingredients are the fact that the sum of valencies gives twice the number of edges, and Lemma 28 (total change). We compute:

$$|E(G')| + \sum_{v \in V(G')} (\operatorname{ch} v + \operatorname{val} v - 3) = 3|E(G')| - 3|V(G')| + \sum_{v \in V(G')} \operatorname{ch} v$$

$$= 2g(G) - g(G') \cdot (2d - 3) + 2d - 5. \qquad \square$$

We regard the second term on the left hand side in Proposition 29 as a *correction term.* Lemma 27 gives us a choice in the equivalence class under tropical modification of φ for which the correction term is non-negative. With this we get an upper bound:

Corollary 30. *Let $\varphi : G \to G'$ be a DT-morphism. Then*

$$\dim C_\varphi \leq 2g(G) - g(G') \cdot (2d - 3) + 2d - 5.$$

Proof. Let $\hat{\varphi} : \hat{G} \to \hat{G}'$ be a combinatorial type of DT-morphisms that is equivalent under tropical modification to φ. By Lemma 27 and the dimension formula we get

$$\dim C_{\hat{\varphi}} \leq |E(\hat{G}')| \leq 2g(\hat{G}) - g(\hat{G}') \cdot (2d - 3) + 2d - 5.$$

By the properties of tropical modification we have that $\dim C_{\hat{\varphi}} = \dim C_\varphi$, $g(\hat{G}) = g(G)$, and $g(\hat{G}') = g(G')$, proving the desired bound. $\square$

2.7. **Tree gonality.** To close this section we use the bound given by the dimension formula to reprove a result on tree gonality that appeared in [CD18]. We go on to outline what is the central problem for the rest of this article, and our strategy to tackle it.

Definition 31. The *tree gonality* of a metric graph Γ is the minimum degree of a tropical morphism from some tropical modification of Γ to some metric tree.

Let $\varphi : G \to T$ be a DT-morphism to a tree. Every metric graph Γ in C_φ has tree gonality at most $\deg \varphi$. From now on we only work with DT-morphisms $\varphi : G \to T$ to trees. For such maps Corollary 30 gives that $\dim C_\varphi \leq 2g + 2d - 5$. Thus:

Corollary 32. *Let g, d be positive integers. The locus of metric graphs in $\mathcal{M}_g^{trop}$ with tree gonality at most d has dimension at most $\min(2g + 2d - 5, 3g - 3)$.*

Proof. Let $\widetilde{\Gamma}$ be in $\mathcal{M}_g^{trop}$, Γ a tropical modification of $\widetilde{\Gamma}$, and $\Phi : \Gamma \to \Delta$ a tropical morphism to a tree with $\deg \Phi \leq d$. Then $\widetilde{\Gamma} \in C_\varphi$, where $\varphi : G \to T$ is the combinatorial type of Φ. By the preceding discussion and since $\dim \mathcal{M}_g^{trop} = 3g - 3$ we get $\dim C_\varphi \leq \min(2g + 2d - 5, 3g - 3)$. Note also that $|E(T)| \leq \min(2g + 2d - 5, 3g - 3)$, and that $|E(G)| \leq \deg \varphi \cdot |E(T)| \leq d \cdot |E(T)|$, so there are finitely many possibilities for G, T, and consequently also for φ. This gives the result. $\square$

Since $3g - 3 \leq 2g + 2d - 5$ implies that $\lceil g/2 + 1 \rceil \leq d$, we get that most metric graphs in $\mathcal{M}_g^{trop}$ have tree gonality at least $\lceil g/2 + 1 \rceil$. This is one of the two inequalities that comprises Theorem 1 in [CD18]. The other inequality, namely that metric graphs have gonality at most $\lceil g/2 + 1 \rceil$, was shown constructively for certain families of graphs, one family for each combinatorial type:

Theorem 33. *Let H be a trivalent combinatorial type and $g = g(H)$. Then there exists a DT-morphism φ such that $\deg \varphi = \lceil g/2 + 1 \rceil$, $H(\varphi) = H$ and $\dim C_\varphi = 3g - 3$.*

Theorem 33 is proven constructively in Section 4 of [CD18]. When g is even, these families have the following property:

Definition 34. A DT-morphism φ, with $g = g(H(\varphi))$, is *full dimensional* if
$$\dim C_\varphi = 3g - 3 = 2g + 2d - 5.$$

The pairwise equalities in the definition of full-dimensional have the following consequences:
- $\dim C_\varphi = 3g - 3$ implies that $H(\varphi)$ is trivalent and the tropical morphism moves in a space of the right dimension, a necessary condition to hope to realize all sources in $C_{H(\varphi)}$.
- $\dim C_\varphi = 2g + 2d - 5$, $\dim C_\varphi = 3g - 3$, and the dimension formula together imply that $|E(T)| = 3g - 3$, that φ has full-rank, and $\text{val}\, v + \text{ch}\, v = 3$ for all v in $V(T)$.
- $3g - 3 = 2g + 2d - 5$ implies that g is even and $d = g/2 + 1$.

These observations give a practical criteria to check whether φ is full-dimensional:

Lemma 35. *A DT-morphism φ is full-dimensional if and only if A_φ is a nonsingular $(3g - 3) \times (3g - 3)$ matrix.*

Example 36. Lemma 35 and the calculation done in the fifth case of Example 62 shows that the map φ_A in Example 14 is full-dimensional.

Let H be a trivalent combinatorial type with even genus, and φ the full-dimensional DT-morphism given by Theorem 33. Our aim here, in Part I of this series of two articles is to introduce a deformation procedure that, when applied to φ, yields full-dimensional DT-morphisms $\varphi^{(1)}$, $\varphi^{(2)}$, and so on, with cones $C^{(1)}$, $C^{(2)}$, and so on, which together cover C_H, showing that all metric graphs with combinatorial type H have gonality at most $g/2 + 1$. For the remaining we direct our efforts to the construction of this deformation procedure. We are not limited by the condition that g is even, since the odd genus case follows from the even genus case, see Subsection 5.4.

As a sneak peek, we mention that in Part II of this series of two articles the main problem is the study of the set $\mathbb{FD}_g$ of full-dimensional DT-morphisms whose source has genus g. A full-dimensional DT-morphism has full-rank, so it is a combinatorial type. By the argument given at the end of the proof of Corollary 32 we have that $\mathbb{FD}_g$ is finite.

We glue together the cones corresponding to elements in $\mathbb{FD}_g$ in a construction akin to that of $\mathcal{M}_g^{\mathrm{trop}}$ and obtain a space $\Phi^d \mathcal{M}_g^{\mathrm{trop}}$. There is a projection Π from $\Phi^d \mathcal{M}_g^{\mathrm{trop}}$ to $\mathcal{M}_g^{\mathrm{trop}}$ where a tropical morphism Φ is mapped to its source. We show that Π is a surjection, that $\Phi^d \mathcal{M}_g^{\mathrm{trop}}$ is connected in codimension-1, and that the fibres of Π have Catalan-many points when counted with certain multiplicity. This opens the possibility of proving the upper bound on gonality without using Theorem 33: instead, we only need to study the fibre $\Pi^{-1}(\widetilde{\Gamma})$ for one particular kind of metric graph $\widetilde{\Gamma}$, and the rest is deformation.

3. GLUING DATUMS

Let $\varphi \colon G \to T$ be a DT-morphism to a tree. That the fibre $\varphi^{-1}(x)$ has $\deg \varphi$ elements counted with multiplicity, and that φ is a graph homomorphism, leads us to regard G as the graph resulting from taking $\deg \varphi$ copies of T, and identifying together certain vertices and edges between copies. In this view the index map m_φ records how many copies of T were glued together in a particular place. We make this notion precise with the combinatorial gadget of a *gluing datum*.

3.1. **The gluing datum.** gluing datums are a tool to ease the visualization of DT-morphisms; to do book-keeping on edges of G in the process of deformation of DT-morphisms (see Sections 5 and 6); and to write computer programs (see [Dra] for code).

Definition 37 (gluing datum). Let T be a finite tree, d be a positive integer, and $\sim$ be an equivalence relation on $W = T \times [d]$, where $[d] = \{1, \ldots, d\}$. Write $\overline{(x, k)}$ for the classes of $W/\sim$, where $x \in V(T) \cup E(T)$. The triple $(T, d, \sim)$ is a *gluing datum* when $\sim$ satisfies these properties:

(1) Verticality: If $(x, i) \sim (x', i')$, then $x = x'$. Each x in T defines a relation $\sim_x$ on $[d]$ with $i \sim_x j$ if $(x, i) \sim (x, j)$. We say that $\sim_x$ is the *gluing relation above x*. By verticality these relations determine $\sim$.

(2) Refinement: If v is in $V(T)$, and t is incident to v, then $\sim_t$ is a refinement of $\sim_v$.

(3) Connectedness: For any two classes $\overline{(x, k)}$ and $\overline{(x', k')}$ there is a sequence $\overline{(x_0, i_0)} = \overline{(x, k)}, \overline{(x_1, i_1)}, \ldots, \overline{(x_r, i_r)} = \overline{(x', k')}$ such that for each q the elements x_q, x_{q+1} are incident (so one of them is a vertex and the other an edge), and there is at least one j such that $(x_q, j) \in \overline{(x_q, i_q)}$ and $(x_{q+1}, j) \in \overline{(x_{q+1}, i_{q+1})}$.

FIGURE 3. On the left, a gluing datum giving rise to the theta graph. In the centre, an allowed set of gluing relations. On the right, a set of gluing relations not allowed because of the RH-condition.

(4) Riemann-Hurwitz condition: For v in $V(T)$ and i in $[d]$ let $A \subset [d]$ be the class of i under the relation $\sim_v$. Let $l = \operatorname{val} v$ and $E(v) = \{t_1, \ldots, t_l\}$. By the refinement property the relation $\sim_{t_q}$ above t_q partitions A into k_q sets. Then

$$(k_1 + k_2 + \cdots + k_l) - 2 \geq |A| \cdot (l - 2).$$

Let $M = (T, d, \sim)$ be a gluing datum. Write $|\overline{(x,k)}|$ for the cardinality of the equivalence class $\overline{(x,k)}$. We abuse notation and say i in $\overline{(x,k)}$ to mean (x,i) in $\overline{(x,k)}$. By verticality this causes no confusion. A consequence is to regard $\overline{(x,k)}$ as a subset of $[d]$, for the purpose of comparing classes. We do, but to avoid confusion it is pointed out in every such instance of this use that $\overline{(x,k)}$ is momentarily being regarded as a subset of $[d]$, instead of a class of $W/\sim$.

We call T the *base tree* of M. We visualize gluing datums by drawing d copies of T on top of each other, and imagining all of them above T. Curved lines and shaded regions indicate where vertices or edges are identified. A class $\overline{(x,k)}$ is said to be *above* x. By the verticality condition, one can only glue elements of W that are above the same element of the base tree, see Figure 3.

The quotient $G = W/\sim$ has the following natural structure of a loopless graph: the classes $\overline{(x,k)}$ in $W/\sim$ with x in $V(T)$ are the vertices of G; those with x in $E(T)$ are the edges of G; an edge $\overline{(t,i)}$ in $E(G)$ is incident to a vertex $\overline{(v,k)}$ in $V(G)$ if and only if t and v are incident in T and $i \in \overline{(v,k)}$. This is a well defined graph, by the verticality and the refinement properties. It is connected, by the connectedness property. It has no loops by the verticality property and the fact that T is a tree. Consider the natural map $\varphi_M : G \to T$ given by $\varphi_M \overline{(x,k)} = x$, and the index map $m_{\varphi_M} \overline{(x,k)} = |\overline{(x,k)}|$.

Lemma 38 (Prop. 5, [CD18]). *For a gluing datum $M = (T, d, \sim)$, the map $\varphi_M \overline{(x,k)} = x$ with index map $m_{\varphi_M} \overline{(x,k)} = |\overline{(x,k)}|$ is a DT-morphism of degree d.*

Proof. We prove that φ is a DT-morphism by showing the balancing condition; the other requirements are immediate. Let A be a vertex of G. Set $v = \varphi(A)$. Fix t in $E(v)$. Recall that by the refinement property the relation $\sim_t$ above t partitions A. Thus, if $e_1, \ldots, e_r$ are the classes in $E(A)$ above t, then $|A| = |e_1| + \cdots + |e_r|$. Since $m_\varphi(e) = |e|$ we get that

$$m_\varphi(A) = |A| = \sum_{\substack{e \in E(A) \\ \varphi(e) = t}} m_\varphi(e). \qquad \square$$

We illustrate the previous concepts in the following example.

Example 39. Let $M = (T, 3, \sim)$ be a gluing datum with T and $\sim$ as in Figure 4.

$$\sim_{v_1} = \sim_{v_2} = \{\{1,2\},\{3\}\}$$
$$\sim_{v_4} = \sim_e = \{\{1\},\{2,3\}\},$$
$$\sim_{v_1} = \sim_{v_2} = \{\{1,3\},\{2\}\},$$
$$\text{else } \sim_x = \{\{1\},\{2\},\{3\}\}.$$

T $\sim$ A visualization of M

FIGURE 4

In the visualization we have rendered dangling edges with dotted lines and lighter colour. The copies of T are numbered from top to bottom. We also have labelled with $A, B, \ldots, F$ the vertices of G that have non-dangling valency equal to 3. The other non-dangling vertices have non-dangling valency equal to 2. Note that the map φ_M is isomorphic as a DT-morphism to the map φ_A of Example 14 (see also Example 45).

We denote by nd-$E(A)$ the set of non-dangling edges of G incident to a vertex A and define the *non-dangling valency* nd-val A as $|\text{nd-}E(A)|$. Observe that nd-val $A = 0$ if and only if A is dangling, otherwise nd-val $A \geq 2$. In the visualization of M in Example 39 the darker coloured edges make up the following graph:

Construction 40. Let $H(M)$ be the following graph: $V(H(M))$ is the set of vertices A of G such that nd-val $A \geq 3$; and $E(H(M))$ is the set of paths of G whose ends are in $V(H(M))$ and interior vertices have non-dangling valency equal to 2.

Recall that $H(\varphi_M)$ is the combinatorial type of the source of the DT-morphism φ_M associated to M. It is straightforward to see that $H(M)$ and $H(\varphi_M)$ are isomorphic. Observe also that the edge e of G is either dangling, or there is a unique $h(e)$ in $E(H(M))$ such that $e \in h(e)$. We say that h in $E(H(M))$ *passes through* a class $\overline{(x,k)}$ of G if $\overline{(x,k)} \in h$ and $\overline{(x,k)}$ is not and end of h. If h passes through $\overline{(x,k)}$, then we say that h *passes above* x. This gives a natural way to also visualize $H(M)$ above T, as in Example 39.

Example 41. Figure 5 illustrates G and $H(M)$ for the gluing datum M of Example 39. We kept the labelling for vertices with non-dangling valency greater than 2 from the visualization of M in Example 39, for ease of comparison.

G $H(M)$

FIGURE 5

Remark 42. Unlike our approach here where a gluing datum gives rise to a DT-morphism φ, the definition of gluing datum in [CD18] gives rise to a tropical morphism $\Phi : \Gamma \to \Delta$. To go from a gluing datum as in [CD18] to one as in our setting we need to remove the metric information by taking a model T of Δ where for every edge t of T the relation above each interior point x in Δ_t equals some fixed relation $\sim_t$. In the terminology of the Definition 4 of [CD18] a possibility for such a model is the set of points from the essential model of Δ together with the image under Φ of the set of monovalent points in the metric forest

$$\{u \in \Gamma \mid \exists x \in \Delta \text{ with } \psi_i(x) \sim \psi_j(x) \text{ for distinct } i, j; \text{ and } u = \psi_i(x)/\sim\},$$

where the map ψ_i sents Δ to its i-th copy. $\triangle$

Construction 43. Given a degree-d DT-morphism $\varphi : G \to T$ we construct a gluing datum M with base tree T such that φ is isomorphic to φ_M. We use four steps:

(1) Choose a leaf v of T and a map $\chi_v : \varphi^{-1}(v) \to \mathcal{P}([d])$, where $\mathcal{P}([d])$ is the power set of $\{1, \ldots, d\}$, such that $|\chi_v(A)| = m_\varphi(A)$ for $A \in \varphi^{-1}(v)$ and $\text{im}(\chi_v)$ is a partition of $[d]$.

(2) Choose incident x, x' in T such that a map χ_x has been chosen for x and no map has been chosen for x' yet.

(3) Choose a map $\chi_{x'} : \varphi^{-1}(x') \to \mathcal{P}([d])$ such that $|\chi_{x'}(X')| = m_\varphi(X')$ for $X' \in \varphi^{-1}(x')$, $\text{im}(\chi_{x'})$ is a partition of $[d]$; and for every pair of incident elements (X', X) in $\varphi^{-1}(x') \times \varphi^{-1}(x)$ we have that $\chi_{x'}(X') \subset \chi_x(X)$ if x' is an edge, or $\chi_{x'}(X') \supset \chi_x(X)$ otherwise.

(4) Repeat step 2 and 3 until every x in T has a χ_x.

Take $M = (T, d, \sim)$, where $\sim_x$ is defined by the partition $\text{im}(\chi_x)$.

In Step (3) of Construction 43 the choice is possible since φ satisfies the balancing condition. It is straightforward to check that the gluing datum $(T, d, \sim)$ has the properties we desire. This construction is essentially the one given in Proposition 6 of [CD18].

3.2. **Isomorphism classes of DT-morphisms and gluing datums.** The choices that are crucial in Construction 43 happen at Steps (1) and (3). We identify all the possibilities using a notion of isomorphism of gluing datums. It turns out that with the right notion of isomorphism there is a bijection between isomorphism classes of DT-morphisms and isomorphism classes of gluing datums. This completes the setup to study DT-morphisms via gluing datums.

Observe that applying a fixed permutation π of $[d]$ on all relations $\sim_x$ gives a different gluing datum, but does not change the isomorphism type of G, nor the index map for the edges of G, nor which edges of G lie above a particular edge of T. We call this operation a *tree-swap*. It enables us to relate two distinct choices done at Step (1) of Construction 43.

Choose a vertex v of T, and let S be the vertex set of one of the connected components of $T \setminus \{v\}$. Let T_{br} be the graph induced by the vertex set $S \cup \{v\}$. We call it a *branch* of T. Choose i and j such that $i \sim_v j$ and swap i with j in all relations above T_{br}. We call this a *branch-swap*. A series of branch-swaps enables us to relate two distinct choices done at Step (3) of Construction 43. This motivates the notion of isomorphism of gluing datums. In the following definition we write $\overline{(x, i)}_M$ for a class of M, to avoid confusions, and in general we regard $\overline{(x, i)}_M$ as a subset of $[d]$.

Definition 44 (gluing datum isomorphism)**.** An isomorphism from $M = (T, d, \sim)$ to $M' = (T', d, \sim')$ consists of a graph isomorphism $\tau : T \to T'$, and for each x in T a permutation π_x of $[d]$ satisfying two properties:

(1) class-preserving: $\pi_x\left(\overline{(x,i)}_M\right)$ and $\overline{(\tau(x), \pi_x(i))}_{M'}$ are equal for all x in T, i in $[d]$.

(2) incidence-preserving: for all v in $V(T)$, t in $E(v)$, and i in $[d]$ we have that $\overline{(t,j)}_M \subset \overline{(v,i)}_M$ if and only if $\overline{(\tau(t), \pi_t(j))}_{M'} \subset \overline{(\tau(v), \pi_v(i))}_{M'}$.

Example 45. Figure 6 presents three isomorphic gluing datums. On the left, M from Example 39. Next, in M' we swapped the middle and the bottom tree. Finally, in M'' we also swapped the middle and the bottom tree, but only above the branch that is right to the vertex C. They all give rise to isomorphic DT-morphisms.

$$M \qquad\qquad\qquad M' \qquad\qquad\qquad M''$$

Figure 6

Lemma 46. *The isomorphism classes of gluing datums are in bijection with the isomorphism classes of DT-morphisms from a graph to a tree.*

Proof. Let M, M' be gluing datums giving rise to graphs G, G' respectively, and $\varphi = \varphi_M$, $\varphi' = \varphi_{M'}$. First suppose that τ, $\{\pi_x\}_{x \in T}$ is an isomorphism from M to M'. Consider the natural map $\gamma : G \to G'$ given by $\gamma(\overline{(x,k)}_M) = \overline{(\tau(x), \pi_x(k))}_{M'}$. By the class-preserving property we have that γ is bijective. By the incidence-preserving property we have that both γ, γ^{-1} are graph homomorphisms. We claim that (γ, τ) is an isomorphism from φ to φ'. Diagram 20.a commutes because

$$\varphi' \circ \gamma\left(\overline{(x,k)}_M\right) = \varphi'\left(\overline{(\tau(x), \pi_x(k))}_{M'}\right) = \tau(x),$$
$$\tau \circ \varphi\overline{(x,k)} = \tau(x).$$

Diagram 20.b commutes because by class-preserving we get that

$$m_\varphi\left(\overline{(x,k)}_M\right) = |\overline{(x,k)}_M| = |\overline{(\tau(x), \pi_x(k))}_{M'}|,$$
$$m_{\varphi'} \circ \gamma\left(\overline{(x,k)}_M\right) = m_{\varphi'}\left(\overline{(\tau(x), \pi_x(k))}_{M'}\right) = |\overline{(\tau(x), \pi_x(k))}_{M'}|.$$

Now suppose that γ, τ is an isomorphism from φ to φ'. Since $m_\varphi = m_{\varphi'} \circ \gamma$, we get that $|\overline{(x,k)}_M| = |\gamma(\overline{(x,k)}_M)|$. Thus, for each x in T we can choose a permutation π_x of $[d]$ such that $\pi_x(\overline{(x,k)}_M) = \gamma(\overline{(x,k)}_M)$. We claim that τ, $\{\pi_x\}_{x \in T}$ is an isomorphism from M to M'. We have the class-preserving property by construction. Since γ, γ^{-1} are graph homomorphisms, we have that $\overline{(t,j)}_M$, $\overline{(v,i)}_M$ are incident if and only if $\gamma(\overline{(t,j)}_M)$, $\gamma(\overline{(v,i)}_M)$ are incident. Since $\varphi' \circ \gamma = \tau \circ \varphi$, we get that $\gamma(\overline{(x,i)}) = \overline{(\tau(x), \pi_x(i))}$. The previous two facts and the construction of G and G' together imply that $\overline{(t,j)}_M \subset \overline{(v,i)}_M$ if and only if $\overline{(\tau(t), \pi_t(j))}_{M'} \subset \overline{(\tau(v), \pi_v(i))}_{M'}$.

Thus, we have shown that the map that sends the isomorphism class of M to the isomorphism class of φ_M is well defined and injective. Construction 43 shows that the map is also surjective. $\square$

Now that the correspondence between DT-morphisms and gluing datums is set up, we use the following notation for convenience:

- $C_M = C_{\varphi_M}$, the cone of sources.
- $A_M = A_{\varphi_M}$, the edge-length map.
- $(M, \ell_T) = (\varphi_M, \ell_T)$, a tropical morphism.
- We say that M has full-rank, or is full-dimensional, or is a combinatorial type if φ_M has full-rank, or is full-dimensional, or is a combinatorial type, respectively.

We also define the *genus of a gluing datum* to be $g(H(M))$. Likewise, the genus of a DT-morphism is the genus of its source.

3.3. **Local properties.** Let $M = (T, d, \sim)$ be a genus-g gluing datum and φ be $\varphi_M : G \to T$. Ideally, a deformation procedure would modify a local part of M, namely the gluing relations above a particular edge of T and its endpoints. The resulting M' should be full-dimensional if we began with a full-dimensional M. The obstruction to this idea is that a local change in M leads to a change in A_M that may make the rank drop. Controlling the rank of A_M is a global condition, which is harder to check than local conditions.

We defer the hard problem of constructing full-dimensional gluing datums to Section 6, and explore a different question here: what conditions are necessary for a gluing datum to be full-dimensional. It turns out that full-dimensional gluing datums have some remarkable combinatorial properties, which are easier to check than the rank. This allows us to easily discard many potential constructions as not full-dimensional. We present these combinatorial properties in Definition 48, after giving some auxiliary definitions.

Definition 47. Let $M = (T, d, \sim)$ be a gluing datum.
- A vertex v of T is *change-minimal* if $\operatorname{ch} v + \operatorname{val} v = 3$.
- A vertex A of G satisfies the *no-return* condition if there are at least two non-dangling edges in $E(A)$ above different edges of T; in other words $|\varphi(\operatorname{nd-}E(A))|$ is at least 2.
- Let $h = \langle A_0, e_1, \ldots, e_\mu, A_\mu \rangle$ be an edge of $H(M)$ (recall that this means that h is a path in G, both ends of h have nd-val value at least 3, and the inner vertices of h have nd-val value equal to 2). We say that h satisfies the *pass-once condition* if φ restricted to the set $\{e_i \in h \mid \varphi(e_i)$ is not incident to a leaf$\}$ is injective.

Definition 48. Let $M = (T, d, \sim)$ be a gluing datum.
- M is *change-minimal* if all vertices of T are change-minimal.
- M satisfies the *dangling-no-glue* condition if $|\overline{(x, k)}| = 1$ for all dangling $|\overline{(x, k)}|$.
- M satisfies the *no-return* condition if all non-dangling vertices A of G such that $\varphi(A)$ is not monovalent satisfy the no-return condition.
- M satisfies the *pass-once* condition if all edges of $H(M)$ satisfy the pass-once condition.

We proceed to describe informally the intuition behind each property and give an example before beginning with proofs. Being change-minimal is equivalent to saying that the correction term in the dimension formula equals zero, namely that $|E(T)| = 2g + 2d - 5$. By

the discussion after Definition 34 we have that full-dimensional implies change-minimal. Dangling-no-glue says that no identification is used in parts of G that do not affect $\ell_{H(M)}$. This is a sort of "efficiency" condition. No-return forbids edges of $H(M)$ to change copies of T in any place except above a leaf of T. Pass-once eases the calculation of A_φ (see Proposition 61).

Example 49. The gluing datum of Example 39 is full-dimensional (see the calculation at 5.a of Example 62). It can be checked that this gluing datum is change-minimal, satisfies dangling-no-glue, no-return, and pass-once; this agrees with our claim. Now we point out instances in the examples that do not fullfill some of the items in Definition 48.

- Consider the map φ_B from Example 14. The second vertex from left to right in the target is the only change-minimal vertex, and above it the no-return condition is violated. The pass-once condition is violated by the edge passing above the middle edge of the target. The dangling-no-glue condition is violated by all of the dangling elements.
- Consider the map φ_A from Example 23. It does not satisfy dangling-no-glue because of the leftmost vertex in the target, and is not change-minimal. It satisfies no-return and pass-once.

Instead of showing that full-dimensional implies dangling-no-glue, no-return and pass-once, we derive these conditions from being change-minimal and having full-rank. This opens the possibility of using these results in future work for studying those special graphs that have gonality less than $\lceil g/2 \rceil + 1$. We begin with a generalization of Lemma 25.

Lemma 50 (zero change). *Let v be in $V(T)$, h an edge of $H(M)$ such that $\varphi(h) \cap E(v)$ contains two edges t_1 and t_2, and $A_1, \ldots, A_r$ the vertices in $h \cap \varphi^{-1}(v)$. If the following conditions are true:*

(a) $r_\varphi(A_q) = 0$ for all $1 \le q \le r$,
(b) the ends of h are not above v,
(c) $|e| = 1$ for dangling e in $E(A_q)$ for all $1 \le q \le r$,

then in the edge-length matrix we have $a_{ht_1} = a_{ht_2}$.

Before proceeding to the proof, note that the coefficient a_{ht}, for h in $E(H(M))$ and t in $E(T)$, can be calculated as follows:

$$\ell_{H(M)}(h) = \sum_{e \in E(G):\, e \in h} \ell_G(e) = \sum_{t \in E(T)} \left(\sum_{e \in h:\, \varphi(e) = t} \frac{1}{|e|} \right) \ell_T(t) = \sum_{t \in E(T)} a_{ht} \ell_T(t),$$

where the sum in the parenthesis equals zero if the index set is empty.

Proof. Since A_q is not an end of h we have that nd-val $A_q = 2$. Let $A_q = \overline{(v, k_q)}$, and e, e' be the two edges in nd-$E(A_q)$. Lemma 13 and conditions (a) and (c) imply that $2|A_q| = |e| + |e'|$, and since $|A_q| \ge |e|, |e|$ we get that $|A_q| = |e| = |e'|$, which makes the classes $\overline{(t_1, k_q)}$, A_q, $\overline{(t_2, k_q)}$ equal as subsets of $[d]$. Thus, $a_{ht_1} = 1/\overline{(t_1, k_1)} + \cdots + 1/\overline{(t_1, k_r)} = 1/\overline{(t_2, k_1)} + \cdots + 1/\overline{(t_2, k_r)} = a_{ht_2}$. $\square$

Note that Condition (c) of Lemma 51 is implied by dangling-no-glue. Recall that if val $v = 2$, then $r_\varphi(A_q) = 0$ gives that val $A_q = 2$. Hence, A_q is not an end of h, and there are no dangling edges incident to A_q, fulfilling Conditions (b) and (c). Now we prove dangling-no-glue.

Remark 51 (change-minimal leaves)**.** Let v in $V(T)$ be a leaf that is change-minimal, A in $\varphi^{-1}(v)$ non-dangling, and t the edge incident to v. Since A is non-dangling val $A \geq 2$, and as val $v = 1$ we get that $|A| \geq$ val A. Thus, $r_\varphi(A) = $ val $A - 2 - |A|($val $v - 2) \geq 2$. Since ch $v = 2$ we get $r_\varphi(A) = 2$, val $A = 2$, and $|A| = 2$. Thus, there is a unique non-dangling vertex above v, $\sim_t$ is trivial, and exactly one edge of $H(M)$ passes above v and t.

This implies that in the edge-length matrix A_M the column corresponding to t has only one non-zero entry; it is a 2 in the row corresponding to the edge of $H(M)$ passing above v. Thus, if M is change-minimal and has full-rank, then each edge of $H(M)$ passes above at most one leaf; for otherwise A_M would have two equal columns, a contradiction. $\triangle$

Lemma 52. *Let M be change-minimal and full-rank, and A in $V(G)$ dangling. Then $r_\varphi(A) = 0$.*

Proof. Let $v = \varphi(A)$. By change-minimal $r_\varphi(A)$ equals 0, 1, or 2. If $r_\varphi(A) = 1$, then Remark 51 (change-minimal leaves) implies that val $v = 2$, and so ch $v = 1$. Thus, if h in $E(H(M))$ passes through A' above w_0, then $r_\varphi(A') = 0$, because A' is non-dangling, so $A' \neq A$ and $r_\varphi(A') + r_\varphi(A) \leq$ ch $v = 1$. Hence, Lemma 50 (zero change) implies that the columns of A_M corresponding to t_1, t_2 incident to v are equal, contradicting that M has full-rank. If $r_\varphi(A) = 2$, Remark 51 implies that A is non-dangling, a contradiction. $\square$

Lemma 53. *A change-minimal and full-rank M satisfies the dangling-no-glue condition.*

Proof. Observation I: Let A be a vertex such that $r_\varphi(A) = 0$ and all elements of $E(A)$ have cardinality 1, except possibly one element e. Then by Lemma 13 (formula for r_φ) we get that $r_\varphi(A) = 2(|A| - 1) - (|e| - 1)$. As $r_\varphi(A) = 0$ we get $|e| + 1 = 2|A|$. Hence, $|A| = |e| = 1$ since $|A| \geq |e|$.

Now suppose that e_0 is dangling and $|e_0| > 1$. Choose an end A_0 of e_0. If A_0 is dangling then Lemma 52 allows us to apply Observation I to choose e_1 in $E(A_0)$, distinct from e_0, such that $|e_1| > 1$. Repeat the previous step with e_1, and continue to construct a sequence of vertices A_0, A_1, and so on; since G is finite, either we arrive to a non-dangling A_r, or a vertex repeats in the sequence. Either way this means that after deleting e_0 the connected component of A_0 has a cycle. This argument can be applied to the other end of e_0 as well, giving a contradiction that e_0 is dangling. Thus, if e is dangling, then $|e| = 1$. If A is dangling, all edges in $E(A)$ are dangling, so they have cardinality 1, and the result follows from another application of Lemma 52 and Observation I. $\square$

Now we prove the no-return condition.

Lemma 54 (nd. r_φ formula)**.** *Let M satisfy dangling-no-glue, and A be in $V(G)$. If $e_1, e_2, \ldots, e_r$ in $E(G)$ are the non-dangling edges incident to A, then*

$$r_\varphi(A) = \text{nd-val } A - 2 + 2|A| - (|e_1| + \ldots + |e_r|).$$

Proof. This is a consequence of Lemma 13 and the dangling-no-glue condition. $\square$

Lemma 55 (r1 implies no-return)**.** *Let M satisfy dangling-no-glue, and A in $V(G)$ be non-dangling with val $\varphi(A) \geq 2$. If $r_\varphi(A) \leq 1$, then A satisfies no-return.*

Proof. If $|A| = 1$ the result is clear. Assume that $|A| \geq 2$. Let $e_1, \ldots, e_r$ be the edges in nd-$E(A)$. Note that $r \geq 2$, since A is non-dangling. Assume that all e_q are above the same edge t, so $|e_1| + \ldots + |e_r| \leq |A|$. Lemma 54 (nd. r_φ formula) implies that $r_\varphi(A) = r - 2 + 2|A| - (|e_1| + \ldots + |e_r|) \geq r - 2 + |A| \geq 2$, a contradiction. $\square$

Corollary 56. *A change-minimal M that satisfies the dangling-no-glue condition also satisfies the no-return condition.*

Proof. Since M is change-minimal by Remark 51 if $r_\varphi(A) \geq 2$, then $\varphi(A)$ is a leaf. The corollary follows then from Lemma 55. $\square$

By dangling-no-glue, G looks like T locally around dangling vertices. We now give a local description for non-dangling vertices, assuming a change-minimal M. To make coming references to this result easier, we state all the possible cases, though several are straightforward corollaries of previous results.

Proposition 57 (local properties). *Let M be change-minimal and satisfying dangling-no-glue, A in $V(G)$ with nd-val A either 2 or 3, $v = \varphi(A)$, $r =$ nd-val A, and $\{e_1, \ldots, e_r\} =$ nd-$E(A)$. Then exactly one of the following cases happens:*

- *(r0-nd3): $r_\varphi(A) = 0$ and nd-val $A = 3$. Then φ restricted to nd-$E(A)$ is injective; val v is 3; and $|e_1| + |e_2| + |e_3| = 2|A| + 1$.*
- *(r0-nd2): $r_\varphi(A) = 0$ and nd-val $A = 2$. Then φ restricted to nd-$E(A)$ is injective; val v is 2 or 3; and $|e_1| = |A| = |e_2|$.*
- *(r1-nd3): $r_\varphi(A) = 1$ and nd-val $A = 3$. Then there is a labelling of the e_i where $\varphi(e_1) = \varphi(e_2) \neq \varphi(e_3)$; val v is 2; and $|e_1| + |e_2| = |A| = |e_3|$.*
- *(r1-nd2): $r_\varphi(A) = 1$ and nd-val $A = 2$. Then φ restricted to nd-$E(A)$ is injective; val v is 2; and there is a labelling of the e_i where $|e_1| = |A|$ and $|e_2| = |A| - 1$.*
- *(r2-nd2): $r_\varphi(A) = 2$. Then $\varphi(e_1) = \varphi(e_2)$; val v is 1; $|e_1| = |e_2| = 1$; and $|A| = 2$.*

Proof. Change-minimal implies that $r_\varphi(A)$ equals 0, 1, or 2, and that val v equals 1, 2, 3 for v in $V(T)$. By Remark 51 (change-minimal leaves) $r_\varphi(A) = 2$ and nd-val $A = 3$ cannot be. So the described cases are all the possibilities. The same remark proves (r2-nd2). Note that M satisfies the no-return condition by Corollary 56.

For the case (r0-nd3), no-return implies that either the three non-dangling edges are above different edges of T, or (up to labelling) e_2 and e_3 are above the same edge and e_1 is above another. In the latter case Lemma 54 and $|e_1| \leq |A|$ imply $0 = 1 + 2|A| - |e_1| - |e_2| - |e_3| \geq 1 + |A| - |e_2| - |e_3|$, but $|e_2| + |e_3| \leq |A|$, a contradiction. Hence e_1, e_2, e_3 are above distinct edges of T, so val $v = 3$. Lemma 54 implies the last formula.

The remaining follows from change-minimal, dangling-no-glue, Lemma 54, and the gluing datum axioms. $\square$

Corollary 58 (image of a path). *Let M be change-minimal and satisfy dangling-no-glue. Let $P = \langle A_0, e_1, \ldots, e_\mu, A_\mu \rangle$ be a path of G such that A_i is non-dangling and val $\varphi(A_i) \geq 2$ for each i. Then $\langle \varphi(A_0), \varphi(e_1), \ldots, \varphi(e_\mu), \varphi(A_\mu) \rangle$ is a path of T.*

Proof. Since φ is a graph morphism, and by the local properties of Proposition 57 , we get that consecutive elements of $\varphi(P) := \langle \varphi(A_0), \varphi(e_1), \ldots, \varphi(e_\mu), \varphi(A_\mu) \rangle$ are incident, and distinct. Since consecutive elements of $\varphi(P)$ are distinct, if $\varphi(P)$ repeats an element, then we get a cycle. As T is a tree we conclude that $\varphi(P)$ is a path. $\square$

Now we move on to the pass-once condition.

Lemma 59. *Let M be change-minimal and have full-rank, u and v be divalent and monovalent vertices of T, respectively, with $E(u) = \{t', t\}$, $E(v) = \{t\}$. Then there is exactly one edge of H above t and one above t'; they are a loop and a bridge, respectively.*

Proof. Swap trees so that $1 \sim_v 2$. Remark 51 (change-minimal leaves) yields that $e_1 = \overline{(t,1)}$ and $e_2 = \overline{(t,2)}$ are two distinct edges of G, with $|e_1| = |e_2| = 1$, and are the only non-dangling edges above t. This implies, by the no-return condition, that $\overline{(u,1)}$ and $\overline{(u,2)}$ are the only non-dangling classes above u (it is possible that $\overline{(u,1)} = \overline{(u,2)}$). Let h be the edge of H containing e_1, e_2. If both $\overline{(u,1)}$ and $\overline{(u,2)}$ have non-dangling valency equal to 2, then the only edge of H passing above t, and t' is h. Thus, a_{ht} and $a_{ht'}$ are the only non-zero entries in the columns of A_M corresponding to t and t', respectively, a contradiction. Therefore, assume without loss of generality that nd-val $\overline{(u,1)} = 3$. As val $u = 2$ and $|e_1| = 1$, by Case (r1-nd3) of the local properties we must have that nd-$E(\overline{(u,1)})$ has two edges above t; namely e_1, e_2, that make the loop; and one edge above t', the bridge. $\square$

Lemma 60. *A change-minimal and full-rank M satisfies the pass-once condition*

Proof. Let $h = \langle A_0, e_1, \ldots, e_\mu, A_\mu \rangle$ be an edge of $H(M)$. If h does not pass above a leaf, then we are done by Corollary 58. Assume h passes above a leaf. By Remark 51 (change-minimal leaves) there is a single vertex $A_i \in h$ such that $v := \varphi(A_i)$ is a leaf. Thus, h passes at most twice above any edge of T by Corollary 58. We also get that $\varphi(e_i) = \varphi(e_{i+1})$ from Remark 51. Let $t_1 = \varphi(e_i)$. Suppose that h violates the pass-once condition. That is, for some $j, k \notin \{i, i+1\}$ we have $\varphi(e_j) = \varphi(e_k)$. We get that $\varphi(e_{i-1}) = \varphi(e_{i+2})$, because otherwise $\varphi(\langle A_0, \ldots A_i \rangle)$ and $\varphi(\langle A_i, \ldots A_\mu \rangle)$ would intersect only at t_1 by Corollary 58 and the fact that T is a tree, which contradicts that $\varphi(e_j) = \varphi(e_k)$. Let $t_2 = \varphi(e_{i-1})$.

Label the copies of T such that $1 \sim_v 2$. Label h with 1. Lemma 59 implies that val $w = 3$. Let w be the other end of t_1, and t_3 be the other edge incident to w. We investigate the classes above t_1, t_2 and t_3. Observe that $\overline{(t_3, 1)}$ and $\overline{(t_3, 2)}$ are dangling. Change-minimal and Remark 51 imply that $|\overline{(t_1, 1)}| = |\overline{(t_1, 2)}| = 1$. All the vertices of G above w have r_φ-value zero. Thus, case (r0-nd2) of the local properties on $\overline{(w,1)}$ and $\overline{(w,2)}$ implies that $|\overline{(t_2, 1)}| = |\overline{(t_2, 2)}| = 1$. Hence, $a_{1,1} = 2, a_{1,2} = 2, a_{1,3} = 0$. As $\overline{(t_1, k)}$ is dangling for $k \geq 3$, the other non-dangling vertices above w have nd-val equal to 2. Lemma 50 (zero change) implies that $a_{i,2} = a_{i,3}$ for $i \geq 2$. See Figure 7, dangling edges shown dotted.

$$a_{1,1} = 2 \qquad a_{1,2} = 2 \qquad a_{1,3} = 0$$
$$a_{i,1} = 0, \qquad\qquad a_{i,2} = a_{i,3},$$
$$\text{for } i \geq 2.$$

FIGURE 7

Let $\mathbf{a}_1, \mathbf{a}_2,$ and $\mathbf{a}_3$ be the first three columns of A_M. Note that $\mathbf{a}_2 = \mathbf{a}_1 + \mathbf{a}_3$, contradicting that M has full-rank. $\square$

Proposition 61 (edge-length map is local)**.** *Let $M = (T, d, \sim)$ satisfy that the leaves of T are change-minimal. Then M satisfies the pass-once condition if and only if for all h in $E(H)$, and t in $E(T)$ the entry a_{ht} is:*

(a) 2 if h passes above a leaf v in $V(T)$, and t is incident to v.
(b) $1/|\overline{(t,k)}|$ if h passes through some edge $\overline{(t,k)}$ of G, t not incident to a leaf.
(c) 0 otherwise.

Proof. Part (a) follows from Remark 51. Part (b) is true if and only if pass-once is. $\square$

4. Examples

Example 62 (3 loops on a loop). Let $\widetilde{\Gamma} = (H, \ell_H)$ be a loop of 3 loops (as in [LPP12]), with H, vertex and edge labellings shown in Figure 8. Set $y_i = \ell_H(h_i)$, $z_i = \ell_T(t_i)$.

H

Figure 8

Assume without loss of generality that $y_3 = \max(y_1, y_2, y_3)$, $y_4 \leq y_5$, $y_6 \leq y_7$, and $y_8 \leq y_9$. We show 9 cases of glueing datums which realize $\widetilde{\Gamma}$. These depend on how large the length y_3 is, compared to several other linear combinations of lengths. Each case depicts: the base tree T with an edge labelling; the glueing datum which realizes $\widetilde{\Gamma}$; and its edge-length matrix using as labelling for H the one shown in Figure 8. The odd cases are full-dimensional, the even cases are not. In two consecutive odd cases the edge-length matrices differ only by one column. We compute $\det A_M$ to show these are indeed full-dimensional.

• First case: if $y_3 < y_1 + y_2$, then M_1 in Figure 9 realizes $\widetilde{\Gamma}$.

T_1 M_1 $\det A_1 = 128$

Figure 9

• Second case: if $y_3 = y_1 + y_2$, then M_2 in Figure 10 realizes $\widetilde{\Gamma}$. Note that edge t_1 was contracted.

FIGURE 10

- Third case: if $y_1 + y_2 < y_3 < y_1 + y_2 + y_4$, Figures 11 and 12 show realizations of $\widetilde{\Gamma}$.
  - Case 3.a

FIGURE 11

  - Case 3.b

FIGURE 12

Here $\widetilde{\Gamma}$ is realized by two tropical morphisms (distinct ones in general, as we argue next), yet at the combinatorial level the gluing datum is the same. We drew M_3 twice to illustrate which of y_4, y_5 is longer; and to show the labelling of the base tree. The two length functions $z^{(a)}$, $z^{(b)}$ are related by swapping z_4 and z_5.

We claim that $\Phi^{(a)} = (\varphi_3, z^{(a)})$ and $\Phi^{(b)} = (\varphi_3, z^{(b)})$ are distinct tropical morphisms if and only if z_4, z_5 are distinct lengths. Set $v = \varphi(B)$. Since z_4, z_5 are distinct, one of the two edges between B, C is longer than the other, and there is a preimage of v under $\Phi^{(a)}$ in this longer edge. This is not true for $\Phi^{(b)}$. Conversely, it is clear that if z_4, z_5 are equal then swapping the lengths has no effect.

This situation persists for the remaining gluing datums in this example.

- Fourth case: if $y_3 = y_1 + y_2 + y_4$, then Figures 13 and 14 show realizations of $\widetilde{\Gamma}$. The edge t_4 was contracted. The drawing shows two different gluing datums (which reflects a bifurcation in the shrinking and regrowing sequence so far), but by swapping some branches one can see them to be isomorphic, so we label both by M_4. This observation applies as well to the remaining cases.
  - Case 4.a

$$T_4^{(a)} \qquad\qquad M_4 \qquad\qquad A_4$$

FIGURE 13

– Case 4.b

$$T_4^{(b)} \qquad\qquad M_4 \qquad\qquad A_4$$

FIGURE 14

- Fifth case:
  – Case 5.a: if $y_1 + y_2 + y_4 < y_3 < y_1 + 2y_2 + y_4$, then M_5 in Figure 15 realizes $\widetilde{\Gamma}$.

$$T_5^{(a)} \qquad\qquad M_5 \qquad\qquad \det A_5 = 16$$

FIGURE 15

  – Case 5.b: if $y_1 + y_2 + y_4 < y_3 < 2y_1 + y_2 + y_4$, then M_5 in Figure 16 realizes $\widetilde{\Gamma}$.

$$T_5^{(b)} \qquad\qquad M_5 \qquad\qquad A_5$$

FIGURE 16

- Sixth case:
  – Case 6.a: if $y_3 = y_1 + 2y_2 + y_4$, then M_6 in Figure 17 realizes $\widetilde{\Gamma}$. The edge t_2 was contracted.

$$T_6^{(a)} \qquad M_6 \qquad A_6$$

$$A_6 = \begin{pmatrix} 0 & 1 & 0 & & & & & \\ 0 & 0 & 1/2 & & & & & \\ 1 & 1 & 1 & & & & & \\ 1 & & & & & & & \\ 1 & & & 2 & & & & \\ & & & & 2 & & & \\ & & & & & 2 & & \\ & & & & & & 2 & \end{pmatrix}$$

FIGURE 17

– Case 6.b: if $y_3 = 2y_1 + y_2 + y_4$, then M_6 in Figure 18 realizes $\widetilde{\Gamma}$. The edge t_3 was contracted.

$$T_6^{(b)} \qquad M_6 \qquad A_6$$

$$A_6 = \begin{pmatrix} 0 & 0 & 1/2 & & & & & \\ 0 & 1 & 0 & & & & & \\ 1 & 1 & 1 & & & & & \\ 1 & & & & & & & \\ 1 & & & 2 & & & & \\ & & & & 2 & & & \\ & & & & & 2 & & \\ & & & & & & 2 & \end{pmatrix}$$

FIGURE 18

- Seventh case:
  – Case 7.a: if $y_1 + 2y_2 + y_4 < y_3 < y_1 + 2y_2 + y_4 + y_6$, then M_7 in Figure 19 realizes $\widetilde{\Gamma}$. To obtain this gluing datum from 6.a we have swapped branch copies 2 and 3 in the branch above t_6, and then regrown t_2.

$$T_7^{(a)} \qquad M_7 \qquad \det A_7 = -16$$

FIGURE 19

  – Case 7.b: if $2y_1 + y_2 + y_4 < y_3 < 2y_1 + y_2 + y_4 + y_8$, then M_7 in Figure 20 realizes $\widetilde{\Gamma}$. This was obtained from 6.b by swapping branch copies 2 and 3 in the branch above t_9, and then regrowing t_2.

$$T_7^{(b)} \qquad\qquad M_7 \qquad\qquad A_7$$

FIGURE 20

- Eighth case:
  - Case 8.a: if $y_3 = y_1 + 2y_2 + y_4 + y_6$, then M_8 in Figure 21 realizes $\widetilde{\Gamma}$. The edge t_6 was contracted.

$$T_8^{(a)} \qquad\qquad M_8 \qquad\qquad A_8$$

FIGURE 21

  - Case 8.b: if $y_3 = 2y_1 + y_2 + y_4 + y_8$, then M_8 in Figure 22 realizes $\widetilde{\Gamma}$. The edge t_8 was contracted.

$$T_8^{(b)} \qquad\qquad M_8 \qquad\qquad A_8$$

FIGURE 22

- Ninth case:
  - Case 9.a: if $y_1 + 2y_2 + y_4 + y_6 < y_3$, then M_9 in Figure 23 realizes $\widetilde{\Gamma}$.

$$T_9^{(a)} \qquad\qquad M_9 \qquad\qquad \det A_9 = 16$$

FIGURE 23

– Case 9.b: if $2y_1 + y_2 + y_4 + y_8 < y_3$, then M_9 in Figure 24 realizes $\widetilde{\Gamma}$.

$$T_9^{(b)} \qquad\qquad M_9 \qquad\qquad A_9$$

FIGURE 24

5. DEFORMATION

We visualize Example 62 as a *movie* featuring a continuous deformation of the length function ℓ_H (for an actual movie go here[1]). The deformation path grows the length y_3 while leaving fixed the remaining lengths. The path moves in and through nine different cones of sources C_M. We use the full-rank property of A_M to calculate $\ell_T = A_M^{-1}\ell_H$, which defines the tropical morphism (M, ℓ_T). As we grow the length y_3, some lengths of ℓ_T shrink down to zero. We call the cases where a length of ℓ_T is zero a *limit* in the deformation. Since we do not allow zero lengths in an edge-length function, we contract an edge of T instead.

If we were to walk further past a limit while using the same gluing datum we would get negative entries in $A_M^{-1}\ell_H$, which is not allowed in a length function. In order to go beyond we pass to a different gluing datum. In Example 62 the even cases are *limit gluing datums*, as an edge is contracted. Around these limits there is a change in the gluing relations. In this section we show that understanding this change of gluing relations sets up a deformation procedure and gives a proof of Theorem 2.

5.1. **Limits.** We set up a framework to study limits. Let $M = (T, d, \sim)$ be a gluing datum, and t an edge of T.

Definition 63 (Limit gluing datum). The *limit* of M by contracting t, in short *the limit at t*, is the gluing datum $M_0 = (T_0, d, \sim_0)$ given by the following data:

- T_0 is obtained by contracting t in T. The ends u, v of t get identified with a vertex w_0 of T_0. Edges and vertices of T different from u, v, and t correspond in a natural manner with edges and vertices of T_0.
- $\sim_{0,w_0}$ equals the finest common coarsening of $\sim_u$ and $\sim_v$; and $\sim_{0,x}$ equals $\sim_x$ for $x \neq w_0$.

For the remainder of this section, let $M_0 = (T_0, d, \sim_0)$ be the limit at t of M, giving rise to G_0, and u, v be the ends of t that contract to w_0 in $V(T_0)$. Fix an edge t of T. Let $\varphi_0 : G_0 \to T_0$ be the DT-morphism that arises from $\varphi_M : G \to T$ by contracting t in T to obtain T_0, and contracting all the edges in $\varphi^{-1}(t)$ to obtain G_0. It is straightforward to see that φ_0 is canonically isomorphic to φ_{M_0}, where M_0 is the limit of M at t. Namely, the classes representing vertices and edges of G *contract* to the classes representing vertices

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[1]https://mathsites.unibe.ch/jdraisma/MovieGenus4.mp4

and edges of G_0. Note that a class $\overline{(x,i)}$ of G contracts to $\overline{(x_0,j)}$ of G_0 if and only if $\overline{(x,i)} \subseteq \overline{(x_0,j)}$ and x contracts to x_0.

To prove that M_0 is indeed a gluing datum, we observe that verticality, refinement and connectedness are inherited. We write r_0 for r_{φ_0}. The RH-condition amounts to proving that $r_0(A_0) \geq 0$ for all A_0 in $V(G_0)$. This is true because r_0 is additive under contraction:

Proposition 64 (r_φ under contraction)**.** *Let A_0 be in $\varphi_0^{-1}(w_0)$, and $A_1, \ldots, A_r$ be the vertices of G that contract to A_0. If $g(H(M)) = g(H(M_0))$, then:*

$$r_0(A_0) = r_\varphi(A_1) + \cdots + r_\varphi(A_r),$$

$$\mathrm{ch}_{M_0}\, w_0 = \mathrm{ch}_M\, u + \mathrm{ch}_M\, v.$$

Proof. Let G_A be the subgraph defined by the vertices and edges of G that contract to A_0. Namely, $V(G_A) = \{A_1, \ldots, A_r\}$. It is a forest; otherwise we would have $g(H(M_0)) < g(H(M))$; and also connected, hence a tree. Restrict φ to G_A to get a map φ_A onto the graph on $\{u, v\}$, joined by the single edge t. This map is a DT-morphism of degree $|A_0|$. By Lemma 28 the total change of φ_A is $\sum_{B \in V(G_A)} r_{\varphi_A}(B) = 2|A_0| - 2$. Using Lemma 13 we calculate:

$$\sum_{i=1}^{r} r_\varphi(A_i) = \sum_{i=1}^{r}(2|A_i| - 2) - \sum_{i=1}^{r} \sum_{\substack{e \in E(A_i) \\ \varphi(e) \neq t}} (|e| - 1) - 2 \cdot \sum_{e \in E(G_A)} (|e| - 1)$$

$$= \sum_{B \in V(G_A)} r_{\varphi_A}(B) - \sum_{i=1}^{r} \sum_{\substack{e \in E(A_i) \\ \varphi(e) \neq t}} (|e| - 1)$$

$$= 2|A_0| - 2 - \sum_{e \in E(A_0)} (|e| - 1) = r_0(A).$$

Apply this formula over all vertices above w_0 to get the second result. $\qquad\square$

5.2. **Inherited properties.** Now we study which properties of Definition 48 are inherited by limits. Let M_0, G_0, u, v, w_0, φ_0 and r_0 be as in Subsection 5.1.

Lemma 65 (dangling in the limit)**.** *If $g(H(M)) = g(H(M_0))$, then a vertex A_0 in $\varphi_0^{-1}(w_0)$ is dangling if and only if all the $A_1, \ldots, A_r$ in $V(G)$ that contract to A_0 are dangling.*

Proof. Since $g(H(M)) = g(H(M_0))$, no cycle of G is contracted in G_0. This implies that A_0 is not contained in a cycle of G_0 if and only if none of $A_1, \ldots, A_r$ are contained in a cycle of G. $\qquad\square$

Lemma 66. *If $g(H(M)) = g(H(M_0))$ and M satisfies the dangling-no-glue condition, then M_0 satisfies the dangling-no-glue condition.*

Proof. We only need to check that a dangling vertex A_0 above w_0 satisfies $|A_0| = 1$. Since $\sim_{w_0}$ is the finest common coarsening of $\sim_u$ and $\sim_v$, we have $|A_0| > 1$ only if for some A in $V(G)$ contracting to A_0 we have $|A| > 1$. By Lemma 65 there is no such A. $\qquad\square$

Lemma 67 (non-dangling union)**.** *Assume that $g(H(M)) = g(H(M_0))$. Let M satisfy the dangling-no-glue condition, let A_0 in $\varphi_0^{-1}(w_0)$ be non-dangling, and let $A_1, \ldots, A_r$ in $V(G)$ be the non-dangling vertices that contract to A. Then $A_0 = A_1 \cup \cdots \cup A_r$, as subsets of $[d]$.*

Proof. This is equivalent to proving that for every k in A_0 at least one of $B_u = \overline{(u,k)}$, $B_v = \overline{(v,k)}$ in $V(G)$ is non-dangling. Assume not, so by dangling-no-glue $|B_u| = |B_v| = 1$, thus B_u, B_v are not incident to other vertices above u or v. Hence these two dangling classes are the only ones that contract to A_0. By Lemma 65 A_0 is dangling, a contradiction. $\square$

Note that in the previous lemmas we have used the fact that the gluing relations of M_0 agree with those of M outside of w_0. From this fact we also get that if M has an edge labelling $\lambda = (\lambda_T, \lambda_H)$, with $\lambda_T : E(T) \to [|E(T)|]$ and $\lambda_H : E(H(M)) \to [|E(H(M))|]$ (recall that $[m]$ means $\{1, \ldots, m\}$), then they induce edge labellings $\lambda_{T,0}$ and $\lambda_{H,0}$ on M_0. It is clear how to do this for T_0. For calculating $\lambda_{H,0}(h_0)$, with h_0 in $E(H(M_0))$, choose any $e_0 \in h_0$. As edges of G not above u, v, or t correspond to the edges of G_0, there is an edge e in G that corresponds to e_0. Recall that $h(e)$ is the unique edge of $H(M)$ containing e. Set $\lambda_{H,0}(h_0) = \lambda_H(h(e))$. It is straightforward to see that this is well defined.

Going one step further, if two gluing datums share a limit, then loosely speaking their combinatorial information agrees outside the contracted edge and its ends. We define a condition on the edge orders that allows us to relate the edge-length matrices. Recall from the proof of Lemma 46 that an isomorphism (τ, π_x) from M to some gluing datum M' gives rise to a graph isomorphism $\gamma : G \to G'$. This γ descends to a graph isomorphism $\widetilde{\gamma} : H(M) \to H(M')$.

Definition 68. Let $M^{(q)} = (T^{(q)}, d, \sim^{(q)})$ be gluing datums with edge labellings $\lambda_T^{(q)} : E(T^{(q)}) \to [|E(T^{(q)})|]$ and $\lambda_H^{(q)} : E(H(M^{(q)})) \to [|E(H(M^{(q)}))|]$, for $q = 1, 2$. Let $M_0^{(q)}$ be the limit at $t_k = (\lambda_T^{(q)})^{-1}(k)$ of $M^{(q)}$, with $\lambda_{T,0}^{(q)}$ and $\lambda_{H,0}^{(q)}$ the induced labellings. If $|E(H(M_0))| \geq |E(H(M^{(q)}))| - 1$, then we say that the labellings of $M^{(1)}$ and $M^{(2)}$ are *compatible at* t_k if there is an isomorphism (τ, π_x) from $M_0^{(1)}$ to $M_0^{(2)}$ such that $\lambda_{T,0}^{(2)} = \lambda_{T,0}^{(1)} \circ \tau^{-1}$ and $\lambda_{H,0}^{(2)} = \lambda_{H,0}^{(1)} \circ \widetilde{\gamma}^{-1}$.

The condition $|E(H(M_0))| \geq |E(H(M^{(q)}))| - 1$ ensures that in M_0 at most one edge of $H(M^{(q)})$ is contracted. This is needed to get unicity at the end of Remark 72.

Lemma 69 (limit matrix). *Let $M^{(1)}$, $M^{(2)}$ be gluing datums with labellings compatible at t_k, and edge-length matrices (a_{ij}), (b_{ij}) respectively. Then $a_{ij} = b_{ij}$ for $j \neq k$.*

Proof. Restrict the edge-length map of $M^{(1)}$ to the subspace $\{x_k = 0\}$ to get the edge-length map of $M_0^{(1)}$. This amounts to deleting the k-th column of (a_{ij}). The matrix of the edge-length map of $M_0^{(1)}$ coincides with that of the edge length-map of $M_0^{(2)}$ because of the isomorphism and compatibility of labellings. Since the matrix of the edge length map of $M_0^{(2)}$ is obtained by deleting the k-th column of (b_{ij}), we get the result. $\square$

Lemma 70. *If M has full-rank, then every limit of M has full-rank.*

Lemma 71. *If M satisfies pass-once, then every limit of M satisfies pass-once.*

Proof. This is a consequence of Lemma 69 and Proposition 61. $\square$

Remark 72. Note that if the limit of M at t, and the limit of M' at t' are isomorphic, then a labelling λ on M induces a unique labelling λ' on M' such that the two labellings are compatible at t_k with $k = \lambda_T(t)$. Indeed, induce λ_0 on M_0 from λ; the isomorphism from M_0 to M_0' gives a labelling λ_0' from λ_0; and there is a unique λ' that induces λ_0'

on M_0' such that $\lambda_{T'}'(t') = \lambda_T(t)$; and if edges h', h of $H(M')$, $H(M)$, respectively, were contracted, then $\lambda_{H'}'(h') = \lambda_H(h)$ $\triangle$

Lemma 73. *If M satisfies no-return and pass-once, then a limit M_0 satisfies no-return if either w_0 is a leaf or $\operatorname{ch} w_0 \leq 2$.*

Proof. Suppose that A_0 in $V(G_0)$, with $r = \operatorname{nd-val} A \geq 2$, violates no-return. Since M satisfies no-return, then A_0 is above w_0. If $\operatorname{val} w_0 = 1$ there is nothing to check, so assume w_0 is not a leaf. If $r_0(A_0) \leq 1$, we get a contradiction with Lemma 55 (r1 implies no-return). Since $\operatorname{ch} w_0 \leq 2$, we have that $r_0(A_0) = 2$. As A_0 violates no-return, all the non-dangling edges $e_1, \ldots, e_r$ incident to A_0 are above a single edge of T, thus $|e_1| + \cdots + |e_r| \leq |A|$, so $|A| \geq 2$. Hence, $r_0(A) = 2 = r - 2 + 2|A| - (|e_1| + \cdots + |e_r|) \geq r - 2 + |A|$ gives $|A| = 2$ and $r = 2$. So there is an edge of H_0 passing above w_0 by going through A_0, and twice above t. Since w_0 is not a leaf, M_0 does not satisfy pass-once. On the other hand, Lemma 71 implies that M_0 satisfies pass-once, a contradiction. $\square$

The condition of having $\operatorname{ch} w_0 \leq 2$ when w_0 is not a leaf is satisfied when M is change-minimal and has full-rank; in particular it is satisfied by full-dimensional M. Indeed, Proposition 64 and the equality $\operatorname{val} w_0 = \operatorname{val} u + \operatorname{val} v - 2$ together imply that

(C) $\operatorname{ch} w_0 + \operatorname{val} w_0 - 3 = (\operatorname{ch} u + \operatorname{val} u - 3) + (\operatorname{ch} v + \operatorname{val} v - 3) + 1 = 1.$

This means that $\operatorname{ch} w_0$ is 0,1,2 or 3. The last value implies that w_0 is a leaf. Thus:

Lemma 74. *Let M be change-minimal with full-rank, and M_0 be a limit of M with $g(H(M)) = g(H(M_0))$. Then M_0 has full-rank and satisfies dangling-no-glue, no-return, and pass-once.*

Proof. This is a consequence of Lemmas 66, 70, 71, 73 and Equation (C). $\square$

5.3. **Deformation.** We introduce the deformation procedure. Let ω be a deformation path in the interior of some cone C_H. The deformation procedure proves that upon encountering a limit while deforming a tropical morphism Φ along ω, as in Example 62, it is possible to continue further by changing the combinatorial structure of the tropical morphism. This means passing to another gluing datum. The majority of the proof of this fact is deferred to Section 6. Here we use Lemma 69 to reduce the proof to the problem of constructing full-dimensional gluing datums with a given limit.

Recall that $\mathbb{FD}_g$ is the set of full-dimensional DT-morphisms whose sources have genus g. We abuse notation and write M in $\mathbb{FD}_g$ for φ_M in $\mathbb{FD}_g$. Let M_0 be the limit at t_1 of some M in $\mathbb{FD}_g$. Denote by $\operatorname{Star}(M_0)$ the set of isomorphism classes of gluing datums in $\mathbb{FD}_g$ that contract to a limit in the isomorphism class of M_0. By Remark 72 a choice of an edge labelling on M induces an edge labelling compatible at t_1 on each element of $\operatorname{Star}(M_0)$.

Since ω is contained in a cone C_H, namely our walk does not change the combinatorial type of the source, assume that $H(M_0)$ and $H(M)$ are isomorphic. Thus, C_{M_0} is a subset of $C_{H(M)}$. Denote by $|\ell_1 - \ell_2|$ the euclidean distance in $\mathbb{R}^{E(H)}$. We wish to prove:

Proposition 75 (deformation I). *Let M_0 be the limit at t_1 of some full-dimensional gluing datum M, with $H(M_0)$ isomorphic to $H = H(M)$. Let $y_0 \in C_{M_0} \subset C_H$. There exists a constant $\varepsilon(M_0, y_0) > 0$ such that if $\hat{y}$ is an element of $C_H \setminus C_{M_0}$ with $|y_0 - \hat{y}| < \varepsilon(M_0, y_0)$, then there is $\hat{M}$ in $\operatorname{Star}(M_0)$ with $H(\hat{M})$ isomorphic to H and $\hat{y} \in C_{\hat{M}}$.*

Proposition 75 is called *deformation I* because in the part II of this series of articles we enhance it to show that the number of distinct ways one can walk into a limit coincides with the number of ways one can walk out of it, under a suitable notion of multiplicity. This notion of multiplicity is suggested by the *balancing condition* in Section 6 of this article.

We compute $\varepsilon(M_0, y_0)$ to have the property that if $y \in \mathbb{R}^{E(H)}$ is in a ball B centered at y_0 with radius $\varepsilon(M_0, y_0)$ and the labellings of the elements of $\mathrm{Star}(M_0)$ are compatible at t_1, then given $\hat{M} = (\hat{T}, d, \sim)$ in $\mathrm{Star}(M_0)$ the only obstruction for $\hat{y} \in B$ to also be in $C_{\hat{M}}$ is that $\hat{y}(t_1)$ is positive. For $\hat{M}$ in $\mathrm{Star}(M_0)$ write $\hat{A}$ for $A_{\hat{M}}$, and let $\hat{z}_0 = \hat{A}^{-1}(y_0) \in \mathbb{R}^{E(\hat{T})}$. Denote by $\min^+ y$ the smallest positive entry of y. Every element of the open ball $B(\hat{z}_0, \min^+ \hat{z}_0) \subset \mathbb{R}^{E(\hat{T})}$ has positive entries, except possibly the one corresponding to t_1. By continuity of $\hat{A}^{-1}$ we can choose a number $\varepsilon(\hat{M}, M_0, y_0) > 0$ such that the image under $\hat{A}^{-1}$ of the open ball with centre y_0 and radius $\varepsilon(\hat{M}, M_0, y_0) > 0$ lies in $B(\hat{z}_0, \min^+ \hat{z}_0)$. We define
$$\varepsilon(M_0, y_0) = \min_{\hat{M} \in \mathrm{Star}(M_0)} \varepsilon(\hat{M}, M_0, y_0).$$
This is well defined and positive since $\mathbb{FD}_g$ is finite. By construction:

Lemma 76. *Let $\hat{M}$ be in $\mathrm{Star}(M_0)$ and $\hat{y}$ in C_H. Set $\hat{z} = \hat{A}^{-1}(\hat{y})$. If $|y_0 - \hat{y}| < \varepsilon(M_0, \ell_0)$, then $\hat{y} \in C_{\hat{M}}$ if and only if $\hat{z}(t_1) > 0$.*

Now we compute a formula for $\hat{z}(t_1)$, showing that its sign depends only on M_0, $\hat{y}$, and crucially only on $\det \hat{A}$. Let $\hat{A} = (\hat{a}_{ij})$. Append a column of zeroes at the beginning of the edge-length matrix of M_0 to obtain $A_0 = (a_{ij})$. From Lemma 69 (limit matrix) it follows that $\hat{a}_{ij} = a_{ij}$ for all i and $j \geq 2$. Hence, the first row of the adjugate matrix $\mathrm{adj}(\hat{A})$ depends only on M_0. Let $(c_1, c_2, \ldots, c_{3g-3})$ be that first row. Recall that the adjugate matrix satisfies:
$$\mathrm{adj}(\hat{A}) \cdot \hat{A} = \det(\hat{A}) \cdot I = \hat{c} \cdot I,$$
where $\hat{c} = \det(\hat{A})$. This formula holds in general for gluing datums with square edge-length matrices. This gives that:

$$(*) \qquad \hat{c} = \sum_{i=1}^{3g-3} c_i \hat{a}_{i1},$$

$$(**) \qquad 0 = \sum_{i=1}^{3g-3} c_i \hat{a}_{ij} = \sum_{i=1}^{3g-3} c_i a_{ij}, \ \text{for } j \geq 2.$$

Moreover, since $\hat{M}$ is full-dimensional we have $\hat{c} \neq 0$, so $\hat{A}^{-1} = \frac{1}{\hat{c}} \mathrm{adj}(\hat{A})$. Thus,

$$(\mathrm{L1}) \qquad \hat{z}(t_1) = \frac{1}{\hat{c}} \sum_{i=1}^{3g-3} c_i \, \hat{y}(h_i).$$

Hence, the sum depends only on $\hat{c}$, $\hat{y}$ and on the c_i (which depend on M_0). Write $c = \det A_M$ as well. Applying Equation (L1) to $z(t_1) = (A_M^{-1}\hat{y})(t_1)$ we get that

$$z(t_1) = \frac{1}{c} \sum_{i=1}^{3g-3} c_i \, \hat{y}(h_i).$$

Using Lemma 76 and comparing signs we obtain:

Lemma 77. *Assume that* $|y_0 - \hat{y}| < \varepsilon(M_0, \ell_0)$ *and* $\hat{y} \notin C_{M_0}$. *If* $\operatorname{sgn}\hat{c} = \operatorname{sgn}c$, *then* $\hat{y} \in C_{\hat{M}}$ *if and only if* $\hat{y} \in C_M$. *If* $\operatorname{sgn}\hat{c} \neq \operatorname{sgn}c$, *then* $\hat{y} \in C_{\hat{M}}$ *if and only if* $\hat{y} \notin C_M$.

So we would like to construct full-dimensional gluing datums with prescribed limit at t_1 and a prescribed sign for $\det A_M$.

Lemma 78. *If* M *is in* $\operatorname{Star}(M_0)$, *the edge labelling on* $\operatorname{Star}(M_0)$ *is compatible at* t_1, *and* $H(M_0)$ *is isomorphic to* $H(M)$, *then there is* $\hat{M}$ *in* $\operatorname{Star}(M_0)$ *with* $\operatorname{sgn} \det A_M \neq \operatorname{sgn} \det A_{\hat{M}}$.

Our constructive proof of Lemma 78 is the main achievement of this article. This is carried out in Section 6. Proposition 75 (deformation I) follows easily from Lemma 78:

Proof of Proposition 75 (deformation I). If $\hat{y} \in C_M$, then we are done, just take $\hat{M} = M$. Otherwise $\hat{y} \notin C_M$. By Lemma 78 there is a gluing datum $\hat{M}$ in $\operatorname{Star}(M_0)$ with $\operatorname{sgn}\hat{c} \neq \operatorname{sgn}c$. By Lemma 77 we have that $\hat{y} \in C_{\hat{M}}$. □

5.4. **Proof of main result.** To close this section, we put together the deformation procedure of Proposition 75 and the initial families of Theorem 33 to derive a proof of Theorem 2. Our deformation machinery is designed to work with full-dimensional gluing datums, which implies even genus. The odd genus case follows from the usual trick of attaching a lollipop. For executing the trick we need a converse to Lemma 59.

Lemma 79. *(loop and bridge) Let* M *be change-minimal and full-rank. Let* A *in* $V(H(M))$ *be trivalent, incident to a bridge* h_b *and a loop* h_l. *Then* h_b *and* h_l *are above a path of* T *of length 2 that leads to a leaf. Moreover, they are the only edges of* H *above this path.*

Proof. Regard A as a vertex of G. Let $v = \varphi(A)$. Suppose that $r_\varphi(A) = 0$. By Lemma 53 we can apply Proposition 57 (local properties). By the case (r0-nd3) the three non-dangling edges incident to A are above three distinct edges of T. Since two of the edges of G incident to A are in h_l, we have that h_l passes above at least two leaves of T, contradicting Remark 51 (change-minimal leaves). The possibility $r_\varphi(A) = 2$ is ruled out since nd-val $A = 3$. Thus, $r_\varphi(A) = 1$. The case (r1-nd3) implies that v is divalent and there is an edge t in $E(T)$ incident to v such that there are non-dangling edges e_1 and e_2 in $E(A)$ above t. Both e_1, e_2 are in h_l, for otherwise there is a contradiction again. Hence, h_l passes twice above t, so by pass-once t leads to a leaf. The path is made by t and the other edge incident to v, and now we can apply Lemma 59 to conclude uniqueness. □

Let $M = (T, d, \sim)$ be a genus-g gluing datum. We add some boundary points to $C_M \subset C_{H(M)}$, in a process akin to the one done in Section 1.4 that added some boundary points to C_H, giving rise to $\overline{C}_H$. For a given ℓ_T in $\mathbb{R}_{\geq 0}^{E(T)}$ contract all the edges of T for which $\ell_T(e) = 0$ to get M_0 (this is independent of contraction order); the pair (M, ℓ_T) refers to the tropical morphism $(M_0, \ell|_{T_0})$. Denote by $\overline{C}_M$ the image under A_M of the points ℓ_T in $\mathbb{R}_{\geq 0}^{E(T)}$ such that the source of (M, ℓ_T) has genus g; namely points where the genus does not drop.

Remark 80. The cone $\overline{C}_M \subset \overline{C}_{H(M)}$ is closed in the topology that $\mathbb{R}^{E(H(M))}$ induces on $\overline{C}_{H(M)}$. △

Let H be a genus-g combinatorial type. We denote by $\mathbb{FD}_H$ the subset of gluing datums M in $\mathbb{FD}_g$ such that $H(M)$ is isomorphic to H. To prove Theorem 2 we show that it is

possible to cover $\overline{C}_H$ with the cones $\overline{C}_M$ of the M in $\mathbb{FD}_H$, by identifying $\overline{C}_H$ with points in $\mathbb{R}_{\geq 0}^{3g-3}$. Recall that an edge labelling (λ_T, λ_H) on M identifies $\overline{C}_M$ with a cone in $\mathbb{R}_{\geq 0}^{3g-3}$. Moreover, recall from Subsection 5.2 that if M_0 has trivalent $H(M_0)$ and is the limit at t_1 of some M in $\mathbb{FD}_H$, then a labelling compatible at t_1 has the property of embedding in $\mathbb{R}_{\geq 0}^{3g-3}$ the cones of two $M, \hat{M}$ in $\mathrm{Star}(M_0)$ in such a way that the intersection is the embedding of the cone of M_0. Thus, embed $\mathbb{FD}_H$ into $\mathbb{R}^{3g-3}$ by choosing labellings for the elements of $\mathbb{FD}_H$ such that for every M_0 that has trivalent $H(M_0)$ and is a limit at t_1 of some M in $\mathbb{FD}_H$, we have labellings compatible at t_1 for $\mathrm{Star}(M_0)$.

We now present the proof of Theorem 2. To realize a point y_1 in $\overline{C}_H$ we use Theorem 33 to choose an initial point y^* in C_H realizable by a tropical morphism, and draw a straight line in $\mathbb{R}_{\geq 0}^{3g-3}$ between y_1 and y^* to use as a deformation path. We then walk along this line, applying Proposition 75 as needed. To ensure that the hypotheses of Proposition 75 are met, we choose y^* to be in general position using the following standard argument.

Lemma 81. *Let $S = \bigcup_{L \in \mathcal{F}} L \subset \mathbb{R}^n$ be a countable union of affine linear spaces L of codimension at least 2, and y, y_1 be in $\mathbb{R}^n$. For every $\varepsilon > 0$ there is a point y^* in the radius-ε ball centred at y such that the half-open line segment $[y^*, y_1)$ does not intersect S.*

Proof. Let $\bar{S} = \bigcup_{L \in \mathcal{F}} \mathrm{aff\text{-}span}(y_1, L) \subset \mathbb{R}^n$. The codimension of $\mathrm{aff\text{-}span}(y_1, L)$ is at least 1; as $\mathcal{F}$ is countable, this implies that the ball $B(y, \epsilon)$ is not contained in $\bar{S}$. So choose y^* in $B(y, \varepsilon) \setminus \bar{S}$. $\qquad\square$

Proof of Theorem 2. Assume that $g = 2g'$. To realize a genus-g metric graph Γ, delete the dangling trees of Γ to obtain the metric graph $\widetilde{\Gamma}$, and let H be a trivalent combinatorial type such that for some $y_1 \in \overline{C}_H$ we have that $\widetilde{\Gamma} = (H, y_1)$. By Theorem 33 we have that $\mathbb{FD}_H$ is non-empty. Moreover it is a finite set. So embed $\mathbb{FD}_H$ into $\overline{C}_H$ via labellings compatible at t_1.

For a gluing datum M in $\mathbb{FD}_H$, denote by $\mathbb{L}_k M$ the set of affine spaces in $\mathbb{R}^{3g-3}$ associated to limits of M where at least k edges have been contracted. This means, images under A_M of sets in $\mathbb{R}^{|E(T)|}$ defined by fixing k coordinates to be zero; the members of $\mathbb{L}_k M$ have codimension at least k. Let

$$Z_k = \bigcup_{M \in \mathbb{FD}_H} \mathbb{L}_k M.$$

Since $\mathbb{FD}_H$ is finite, the set Z_k is closed in the topology that $\mathbb{R}^{3g-3}$ induces on $\overline{C}_H$. Choose M in $\mathbb{FD}_H$, and using Lemma 81 choose y^* in $C_M \subset \mathbb{R}_{\geq 0}^{3g-3}$ such that the half open line segment $\sigma = [y^*, y_1)$ does not intersect Z_2. Note that y^* is not in Z_1 since $C_M \cap Z_1 = \varnothing$ and $y^* \in C_M$. Thus, σ intersects with Z_1 finitely many times, and these intersections are not in Z_2. Now simply walk from y^* to y_1, always staying inside of the embedding of $\mathbb{FD}_H$ into $\mathbb{R}_{\geq 0}^{3g-3}$ by applying Proposition 75, if necessary, when arriving at one of the finitely many points $\sigma \cap Z_1$.

Now assume that g is odd. Choose a point x in Γ and attach to it a loop via a bridge. The lengths of the bridge and loop are not important, hence the construction depends on one parameter (the placement of x). The resulting graph $\Gamma_\mathrm{b} = (G_\mathrm{b}, \ell_\mathrm{b})$ has even genus $g + 1$. As g is odd, $\lceil g/2 \rceil + 1$ equals $\lceil (g+1)/2 \rceil + 1$. So there is a tropical morphism $\Phi \colon \Gamma_\mathrm{b} \to (T, \ell_T)$ of degree $\lceil g/2 \rceil + 1$. By Lemma 79, the bridge and loop added to Γ are above two edges of T, and no other point of Γ_b is above them. Make the desired morphism by deleting these two edges and everything above. $\qquad\square$

6. Constructions

We carry out the constructions necessary to prove deformation. This section is motivated by the following paramount observation. The gluing datum M_2 of Example 62 is the limit at t_1 of M_1, $M_3^{(a)}$, and $M_3^{(b)}$ with edge labellings λ_1, $\lambda_3^{(a)}$ and $\lambda_3^{(b)}$. There is an isomorphism from $M_3^{(a)}$ to $M_3^{(b)}$, but $\lambda_3^{(a)}$ does not induce $\lambda_3^{(b)}$; moreover $M_3^{(a)}$ and $M_3^{(b)}$ produce in total two non-isomorphic tropical morphisms realizing the same source $\widetilde{\Gamma}$. Let A_1, $A_2^{(a)}$, $A_3^{(b)}$ be the edge-length matrices of these gluing datums. They satisfy:

$$\det A_1 + \det A_3^{(a)} + \det A_3^{(b)} = 0.$$

We view this equation as a balancing condition, something ubiquitous in tropical geometry. We generalize this relation with Equation $(\star)$ in Subsection 6.2.

We construct several candidates to be in $\mathrm{Star}(M_0)$, for M_0 the limit at t_1 of some M in $\mathbb{FD}_g$. We use Equation $(\star)$ to prove that at least in one of these candidates the determinant of the edge-length matrix has the sign opposite to that of $\det A_M$; as needed in Lemma 78.

6.1. **Candidates to being full-dimensional.** Let M be in $\mathbb{FD}_g$ and M_0 be the limit at t_1. Recall that giving a construction for $\mathrm{Star}(M_0)$ is hard because having a full-rank edge-length matrix is a global condition. Instead, we pass to a larger class of gluing datums that contains $\mathrm{Star}(M_0)$. In return we get conditions that can be checked locally, namely they only depend on the gluing relations above t_1 and its ends u, v.

Definition 82. Let $g = 2g'$ and M be a genus-g gluing datum. We say that M is *possibly full-dimensional* if M is change-minimal, satisfies dangling-no-glue, no-return, and A_M is a $(3g-3) \times (3g-3)$-matrix.

We denote by $\mathrm{PStar}(M_0)$ the set of isomorphism classes of possibly full-dimensional gluing datums that contract to a limit isomorphic to M_0. By Lemma 74 we have that $\mathrm{PStar}(M_0)$ contains $\mathrm{Star}(M_0)$. We index the elements of $\mathrm{PStar}(M_0)$ by denoting them $M^{(q)}$, with $q = 1, 2$, and so on. We write $T^{(q)}$ for the base tree of $M^{(q)}$, $G^{(q)}$ for the graph that arises from $M^{(q)}$, $\varphi^{(q)}$ for $\varphi_{M^{(q)}}$, $r^{(q)}$ for $r_{\varphi^{(q)}}$, $m^{(q)}$ for $m_{\varphi^{(q)}}$, and $A^{(q)} = (a_{ij}^{(q)})$ for $A_{M^{(q)}}$. We also write T_0 for the base tree of M_0, G_0 for the graph that arises from M_0, φ_0 for φ_{M_0}, r_0 for r_{φ_0}, m_0 for m_{φ_0}, and $A_0 = (a_{ij})$ for the matrix we get by inserting the zero column at the beginning of A_{M_0}.

Recall that by Equation (C) on Page 32 we have that w_0 is not change-minimal; more precisely, $\mathrm{ch}\, w_0 + \mathrm{val}\, w_0 = 4$. Our strategy to construct the elements of $\mathrm{PStar}(M_0)$ is to regrow w_0 back to an edge in such a way that the resulting gluing datum is change-minimal and satisfies dangling-no-glue and no-return. This is possible because, as advertised, these conditions depend solely on the gluing relations above t_1 and its ends, u and v.

6.2. **A balancing condition.** On the other direction, note that since $A^{(q)}$ is a $(3g-3) \times (3g-3)$-matrix we have that $M^{(q)}$ in $\mathrm{PStar}(M_0)$ is full-dimensional if only if $\det A^{(q)} \neq 0$. So we need a relation for the determinants. This is where the balancing condition suggested by Example 62 comes in:

Lemma 83 (balancing condition)**.** *If M is in $\mathrm{Star}(M_0)$ and $H(M_0)$ is isomorphic to $H(M)$, then there exists a finite sequence of triples $(M^{(q)}, \lambda^{(q)}, K_q)$, indexed by q, where the $M^{(q)}$ are in $\mathrm{PStar}(M_0)$ with edge labellings $\lambda^{(q)}$ compatible at t_1 and the K_q are positive*

integers, such that every isomorphism class in $\mathrm{PStar}(M_0)$ *has a representative in the* $M^{(q)}$, *and the following equality holds:*

$$(\star) \qquad \sum_{q=1}^{r} K_q \det(A^{(q)}) = 0.$$

It is straightforward to see that the balancing condition of Lemma 83 implies Lemma 78.

Proof of Lemma 78. Since M is in $\mathrm{Star}(M_0)$, we have that M is in $\mathrm{PStar}(M_0)$. Then by the balancing condition we have some $\hat{M}$ in $\mathrm{PStar}(M_0)$ with $\mathrm{sgn}\det A_{\hat{M}} \neq \mathrm{sgn}\det A_M$ and $\det A_{\hat{M}} \neq 0$. The latter equation implies that $\hat{M}$ is in $\mathrm{Star}(M_0)$, as desired. □

We prove the balancing condition and construct $\mathrm{PStar}(M_0)$, simultaneously, with a case-by-case analysis. For verifying Equation $(\star)$ we rely on formulas that follow from Equations $(*)$ and $(**)$ on Page 33. To shorten, we write $c^{(q)}$ for $\det A^{(q)}$. Consider the adjugate matrix $\mathrm{adj}\, A^{(q)}$, with columns indexed by edges of $H(M^{(q)})$ and rows indexed by edges of $T^{(q)}$. Let $c_1, \ldots, c_{3g-3}$ be the first row of $\mathrm{adj}\, A^{(q)}$. Recall that the c_i depend only on M_0. For an edge h in $H(M^{(q)})$ or $H(M_0)$ we write c_h for $c_{\lambda_H^{(q)}(h)}$ or $c_{\lambda_{0,H}(h)}$, respectively. Recall that given a non-dangling e in $G^{(q)}$ or G_0 we denote by $h(e)$ the unique edge of $H(M^{(q)})$ or $H(M_0)$, respectively, that contains e. If e is dangling, we let $c_{h(e)}$ be zero. Fix a j in $[3g-3]$. We define:

$$\sigma_0(J, j) = \sum_{e \in J} \frac{c_{h(e)}}{|e|}, \ \text{ for } J \subseteq \varphi_{M_0}^{-1}(t_j),$$

$$\sigma^{(q)}(J, j) = \sum_{e \in J} \frac{c_{h(e)}}{|e|}, \ \text{ for } J \subseteq \varphi_{M^{(q)}}^{-1}(t_j).$$

For convenience we write $\sigma_0(j)$ for $\sigma_0(\varphi_{M_0}^{-1}(t_j), j)$ and $\sigma^{(q)}(j)$ for $\sigma^{(q)}(\varphi_{M^{(q)}}^{-1}(t_j), j)$.

Lemma 84. *The following equalities hold:*

$$\sigma^{(q)}(j) = c^{(q)} \ \text{ if } j = 1, \ \text{ otherwise } \sigma^{(q)}(j) = 0,$$

$$\sigma_0(J, j) = \sigma^{(q)}(J, j) \ \text{ for } j \neq 1 \ \text{ and any } J.$$

Proof. The first equality follows from Equations $(*)$ and $(**)$, and the fact that:

$$\sigma^{(q)}(J, j) = \sum_{e \in \varphi_{M^{(q)}}^{-1}(t_j)} \frac{c_{h(e)}}{|e|} = \sum_{i=1}^{3g-3} c_i a_{ij}^{(q)}.$$

Note that if $j \neq 1$ the edges in $\varphi_{M_0}^{-1}(t_j)$ correspond to those in $\varphi_{M^{(q)}}^{-1}(t_j)$, which together with the facts that the c_i depend only on M_0 and $a_{ij}^{(q)} = a_{ij}$ for $j \neq 1$, gives the second equality. □

Additionally, note that if J_1, J_2 are disjoint, then $\sigma_0(J_1, j) + \sigma_0(J_2, j) = \sigma_0(J_1 \cup J_2, j)$.

6.3. **Cases for constructing** $\mathrm{PStar}(M_0)$. The plan of action to construct $\mathrm{PStar}(M_0)$ has three steps:

(1) Finding the possible base trees that contract to T_0.
(2) Restricting the possibilities for the gluing relations $\sim_u^{(q)}$, $\sim_v^{(q)}$, $\sim_{t_1}^{(q)}$.

(3) Giving a construction of a gluing datum together with edge labellings compatible at t_1 for each of the possibilities found in Step (2).

We deal with Step (1) on Subsection 6.4. Step (2) is the most delicate one, as it proves that indeed we have representatives for all the elements of $\mathrm{PStar}(M_0)$. We are able to say some useful generalities in Subsections 6.5 and 6.6, but the definitive arguments come from assuming certain features of the gluing relations of M_0 above w_0 and edges in $E(w_0)$. Hence, we split the proof in cases. For Step (3) we refer the reader to the illustrations that accompany each case.

We have, by Equation (C) on Page 32, that $\mathrm{ch}\, w_0 + \mathrm{val}\, w_0 = 4$. Thus, $\mathrm{val}\, w_0$ is 1, 2, 3 or 4. In our setting the first option is not possible, because if $\mathrm{val}\, w_0$ were 1, then we would have $\mathrm{val}\, u = 1$ and $\mathrm{val}\, v = 2$ for all gluing datums in $\mathrm{PStar}(M_0)$, including the full-dimensional M. So Lemma 59 applied to M would imply that there is a loop above t_1, so the genus of $H(M_0)$ would drop upon contraction of t_1, a contradiction.

Let A_0 be a vertex above w_0. If $H(M_0)$ is isomorphic to $H(M)$, then $H(M_0)$ is trivalent. Hence, $\mathrm{nd\text{-}val}\, A_0$ is 0, 2, or 3. Also, since $\mathrm{ch}\, w_0 \le 2$ we have that $r_0(A_0)$ is 0, 1 or 2.

Our philosophy of division of cases is similar to the one for proving the local properties: the different cases for calculating $\mathrm{PStar}(M_0)$ are mainly determined by the values of $\mathrm{val}\, w_0$, $r_0(A_0)$, and $\mathrm{nd\text{-}val}\, A_0$. There are other minor case specific factors that come into play. This gives a total of 17 cases. See Figure 25 for a map that eases navigating through the 15 main cases. The diagram does not show 2 *auxiliary cases* contained in Case {aux-r0}, which we treat in Subsection 6.7.1.

6.4. **Trees contracting to T_0.** We describe the possibilities for $T^{(q)}$. Assume that the edges of T_0 incident to w_0 are labelled with $2, \ldots, \deg w_0 + 1$. To construct a tree that contracts to T_0 choose disjoint sets S, S' whose union is $\{2, \ldots, \deg w_0 + 1\}$. At most one of S, S' may be empty. Replace w_0 with an edge t_1 with ends u, v; the edges indexed by S are incident to u, and those indexed by S' are incident to v. We denote the resulting tree by T_S. For ease of notation we omit braces; for example we write $T_{2,3}$ for $T_{\{2,3\}}$, and $T_\varnothing$ for the tree corresponding to empty S. Without loss of generality we assume that $|S| \le |S'|$, and in case of equality that $2 \in S$. This gives a one-to-one correspondence between unordered partitions of $\{2, \ldots, \deg w_0 + 1\}$ into two parts and trees contracting to T_0. Since $M^{(q)}$ is change-minimal, $\mathrm{val}\, u \le 3$ and $\mathrm{val}\, v \le 3$, that is $|S|, |S'| \le 2$.

6.5. **The graph** $\mathrm{nd\text{-}} G^{(q)}_{A_0}$. Let $\mathrm{nd\text{-}} G^{(q)}_{A_0}$ be the subgraph of non-dangling elements of $G^{(q)}$ that contract to A_0. Since $g(H(M)) = g(H(M_0))$, the graph $\mathrm{nd\text{-}} G^{(q)}_{A_0}$ is a forest; moreover, by Lemma 67 it is a tree. By dangling-no-glue, determining the gluing relations $\sim_u^{(q)}, \sim_v^{(q)}, \sim_{t_1}^{(q)}$ is equivalent to determining $E(A)$ for each A in $V(\mathrm{nd\text{-}} G^{(q)}_{A_0})$ and the index $m^{(q)}(x)$ of each element x in $\mathrm{nd\text{-}} G^{(q)}_{A_0}$.

Lemma 85. *Let M be a gluing datum, M_0 the limit at t_1, and A_0 above w_0. If $g(H(M)) = g(H(M_0))$, then:*
$$\mathrm{nd\text{-}val}\, A_0 = \sum_{A \in V(\mathrm{nd\text{-}} G^{(q)}_{A_0})} (\mathrm{nd\text{-}val}\, A - 2) + 2.$$

FIGURE 25. Logical flow of cases to regrow w_0.

Proof. Since nd-$G_{A_0}^{(q)}$ is a tree, $|E(\text{nd-}G_{A_0}^{(q)})| = |V(\text{nd-}G_{A_0}^{(q)})| - 1$. So we have that:

$$\text{nd-val}\,A_0 = \sum_{A \in V(\text{nd-}G_{A_0}^{(q)})} \text{nd-val}\,A - 2|E(\text{nd-}G_{A_0}^{(q)})|$$

$$= \sum_{A \in V(\text{nd-}G_{A_0}^{(q)})} \text{nd-val}\,A - 2(|V(\text{nd-}G_{A_0}^{(q)})| - 1)$$

$$= \sum_{A \in V(\text{nd-}G_{A_0}^{(q)})} (\text{nd-val}\,A - 2) + 2. \qquad \square$$

Let A be in $V(\text{nd-}G_{A_0}^{(q)})$. By Lemma 85 if nd-val $A_0 = 2$, then nd-val $A = 2$. If nd-val $A_0 = 3$, then either nd-val $A = 2$ or nd-val $A = 3$; moreover, the latter case occurs exactly once. So when nd-val A is 3 we denote by $A^{(q)}$ the unique vertex of $G^{(q)}$ with non-dangling valency 3 that contracts to A_0. We make two further observations, one on the edge set and one on the vertex set of nd-$G_{A_0}^{(q)}$:

Lemma 86. *Let r be the number of vertices of $G^{(q)}$ that contract to A_0 and belong either to Case (r1-nd3) or Case (r2-nd2) of Proposition 57 (local properties). Then $|E(\text{nd-}G_{A_0}^{(q)})| \le r + 1 \le r_0(A_0) + 1$.*

Proof. Let A be in $V(\text{nd-}G_{A_0}^{(q)})$. By the local properties and the fact that all the edges of nd-$G_{A_0}^{(q)}$ are above t_1, we have that $\text{val}_{\text{nd-}G_{A_0}^{(q)}} A$ is either 1 or 2. Thus, nd-$G_{A_0}^{(q)}$ is in fact a path, so the inner vertices have valency 2. Note that valency 2 implies that A belongs either to Case (r1-nd3) or Case (r2-nd2), so the result follows. $\qquad\square$

Lemma 87. *If $r_0(A_0) \leq 1$, then the vertices of* nd-$G_{A_0}^{(q)}$ *are the ends above u, v of the edges $e^{(q)}$, where $e^{(q)}$ corresponds to e in* nd-$E(A_0)$.

Proof. Let A be in $V(\text{nd-}G_{A_0}^{(q)})$. Since $r_0(A_0) \leq 1$, we have that $r^{(q)}(A) \leq 1$. Thus, by the local properties A is not above a leaf. So A satisfies the no-return condition, which means that not all of the edges in nd-$E(A)$ are above t_1, so at least one of them corresponds to one of the edges in nd-$E(A_0)$. $\qquad\square$

6.6. **Combinatorial type.** We can obtain $H(M_0)$ from $H(M)$ by contracting a set of edges. If the number of edges contracted is zero, then $H(M_0)$ and $H(M)$ are isomorphic. This is equivalent to having a trivalent $H(M_0)$. This is the setting of Lemma 83 (balancing condition). Thus, we assume for the rest of this section that $H(M_0)$ and $H(M)$ are isomorphic. Since the edge labelling on $\text{PStar}(M_0)$ is compatible at t_1, the identification is canonical.

The case where $H(M_0)$ and $H(M)$ are not isomorphic is of interest as well, and we deal with it in Part II of this series of articles. It is by studying the $\text{PStar}(M_0)$ of M_0 with non-trivalent $H(M_0)$ that we show how it is possibly to walk between cones C_H with different combinatorial types H. We use this to prove that the space $\Phi^d \mathcal{M}_g^{\text{trop}}$ we alluded to in the sneak peak part of Subsection 2.7 is connected.

6.7. **Case-work.** Let A_0 be above w_0. Let e in $G^{(q)}$ contract to A_0. Let A_u, A_v be the ends of e, above u, v respectively. If e is dangling, then it does not contribute to $\sigma^{(q)}(1)$. So, by Lemma 65 (dangling in the limit) we can assume that A_0 is non-dangling. We denote by J_r the set of those e contracting to A_0 such that $r^{(q)}(A_u) + r^{(q)}(A_v) = r$. By the discussion in Subsection 6.3 we have that $r_0(A_0) \leq 2$, hence:

$$\det A^{(q)} = \sigma^{(q)}(1) = \sigma^{(q)}(J_0, 1) + \sigma^{(q)}(J_1, 1) + \sigma^{(q)}(J_2, 1).$$

In each of the following cases we calculate $\sigma^{(q)}(J_0, 1)$, $\sigma^{(q)}(J_1, 1)$, and $\sigma^{(q)}(J_2, 1)$ separately. To avoid having several levels of subindices, we write $c(e)$ for $c_{h(e)}$.

6.7.1. *Case {aux-r0}.* We begin by showing that $T^{(q)} = T_{S^{(q)}}$ determines nd-$G_{A_0}^{(q)}$ when $r_0(A_0)$ is 0. This is an auxiliary case for calculating $\sigma^{(q)}(J_0, 1)$. Since $\text{val}\, w_0 \geq 2$, by Lemma 55 (r1 implies no-return) we have that A_0 satisfies the no-return condition. Thus, if nd-val $A_0 = 2$, then φ_0 is injective on nd-$E(A_0)$. If nd-val $A_0 = 3$, suppose that φ_0 is not injective on nd-$E(A_0)$, so $\sum_{e \in \text{nd-}E(A_0)} |e| \leq 2|A_0|$. Hence, $r_0(A_0) = \text{nd-val}\, A_0 - 2 + 2|A_0| - \sum_{e \in \text{nd-}E(A_0)} |e| \geq 1$, a contradiction.

- Case {aux-r0-nd2}: Assume that nd-val A_0 is 2. Let e_α, e_β be the edges in nd-$E(A_0)$, above t_α and t_β respectively. By Lemma 87 the vertices of nd-$G_{A_0}^{(q)}$ are the ends of $e_\alpha^{(q)}$ and $e_\beta^{(q)}$ above u or v. If $\{\alpha, \beta\} \subset S^{(q)}$, the ends of $e_\alpha^{(q)}$, $e_\beta^{(q)}$ are above u; since $G_{A_0}^{(q)}$ is connected, they equal one vertex A in $G^{(q)}$, which equals A_0 as subsets of $[d]$ by Lemma 67. Similarly if $\{\alpha, \beta\} \subset S'^{(q)}$. Otherwise, one end A_u is above u, the other

end A_v above v, so they are distinct. By connectivity of nd-$G_{A_0}^{(q)}$ there is one edge e' joining A_u, A_v. By Lemma 85 and since $r_0(A_0) = 0$, the vertices A_u and A_v belong to Case (r0-nd2) of the local properties. So, as subsets of $[d]$, the classes $e_\alpha^{(q)}$, A_u, e', A_v, $e_\beta^{(q)}$ are equal; and $h(e_\alpha^{(q)}) = h(e') = h(e_\beta^{(q)})$.

- Case {aux-r0-nd3}: Assume that nd-val A_0 is 3. Let $e_\alpha, e_\beta, e_\gamma$ be the edges in nd-$E(A_0)$, above t_α, t_β, t_γ, respectively. Since $|\varphi_0(\text{nd-}E(A_0))| = 3$ and $\max(|S^{(q)}|, |S'^{(q)}|) \leq 2$, both intersections $S^{(q)} \cap \varphi_0(\text{nd-}E(A_0))$ and $S'^{(q)} \cap \varphi_0(\text{nd-}E(A_0))$ are non-empty. One of these intersections is a singleton. Assume without loss of generality that the singleton is $\{\alpha\}$. By Lemma 86 there is at most one edge in nd-$G_{A_0}^{(q)}$, therefore at most two vertices. By Lemma 87 the vertices of nd-$G_{A_0}^{(q)}$ are the end A_2 of $e_\alpha^{(q)}$ above u, and the vertex A_3 that is the end of both $e_\beta^{(q)}$ and $e_\gamma^{(q)}$ above v. So A_2 and A_3 are distinct, joined by a non-dangling edge e', and nd-val $A_2 = 2$, nd-val $A_3 = 3$. By the Case (r0-nd2) of the local properties, as subsets of $[d]$, we get that $e_\alpha^{(q)}$, A_2, and e' are equal, and $h(e_\alpha^{(q)}) = h(e')$. Note that they are also a subset of A_3. Finally Lemma 67 implies that $A_0 = A_3 \cup A_2 = A_3$.

6.7.2. *Case {w4}.* Assume that val w_0 is 4. Then ch $w_0 = 0$ by Equation (C) on Page 32, so $r_0(A_0) = 0$. Thus, the two Cases {aux-r0-nd2} and {aux-r0-nd3} settle entirely this case. Namely, the elements of PStar(M_0) are $M^{(1)}$, $M^{(2)}$ and $M^{(3)}$, determined by $T_{2,3}$, $T_{2,4}$, and $T_{2,5}$, respectively.

- Case {w4-r0-nd2}: Let nd-$E(A_0) = \{e_\alpha, e_\beta\}$ with $\varphi_0(e_\alpha) = t_\alpha$, $\varphi_0(e_\beta) = t_\beta$. Applying Case {aux-r0-nd2} we get the local parts shown in Figure 26, and a formula.

M_0 $\qquad$ $T_{\alpha,\beta}$ $\qquad$ $T_{\alpha,\gamma}$ $\qquad$ $T_{\alpha,\delta}$

FIGURE 26

(w4-nd2) $$\sum_{q=1}^{3} \sum_{e \in E(\text{nd-}G_{A_0}^{(q)})} \frac{c(e)}{|e|} = \frac{c(e_\alpha)}{|e_\alpha|} + \frac{c(e_\beta)}{|e_\beta|}.$$

- Case {w4-r0-nd3}: Let nd-$E(A_0) = \{e_\alpha, e_\beta, e_\gamma\}$ with $\varphi_0(e_\alpha) = t_\alpha$, $\varphi_0(e_\beta) = t_\beta$, $\varphi_0(e_\gamma) = t_\gamma$. Applying Case {aux-r0-nd3} we get the local parts shown in Figure 27, and a formula.

FIGURE 27

(w4-nd3)
$$\sum_{q=1}^{3} \sum_{e \in E(\text{nd-}G_{A_0}^{(q)})} \frac{c(e)}{|e|} = \frac{c(e_\alpha)}{|e_\alpha|} + \frac{c(e_\beta)}{|e_\beta|} + \frac{c(e_\gamma)}{|e_\gamma|}.$$

- We verify Equation ($\star$) for Case $\{w4\}$ by putting together Lemma 84 and Equations (w4-nd2), (w4-nd3):

(1)
$$c^{(1)} + c^{(2)} + c^{(3)} = \sum_{q=1}^{3} \sigma^{(q)}(1) = \sum_{q=1}^{3} \sum_{A_0 \in \varphi_0^{-1}(A)} \sum_{e \in E(\text{nd-}G_{A_0}^{(q)})} \frac{c(e)}{|e|}$$
$$= \sum_{j=2}^{5} \sigma^{(q)}(j) = 0.$$

6.7.3. *Case $\{w3\}$.* Assume that $\text{val}\, w_0$ is 3. Then $\text{ch}\, w_0 = 1$ by Equation (C). There is exactly one vertex of G_0 with r_0-value 1 above w_0, the others have r_0-value 0. By Subsection 6.4, the possible base trees contracting to T_0 are T_2, T_3 and T_4; all with $\text{val}\, u = 2$, $\text{val}\, v = 3$. Let $\{\alpha\} = S^{(q)}$. We analyse the cases where $r_0(A_0)$ is either 0 or 1.

- Case $\{w3\text{-r0}\}$: Assume that $r_0(A_0) = 0$, and let A_u in $G_{A_0}^{(q)}$ be non-dangling above u. Then $r^{(q)}(A_u) = 0$. Since $\text{val}\, u = 2$, only Case (r0-nd2) of the local properties is possible, so nd-val $A = 2$ and as a subset of $[d]$ the class A_u is equal to both of the elements in nd-$E(A_u)$. Since any non-dangling edge in $(\varphi^{(q)})^{-1}(t_\alpha)$ has a non-dangling end above u, we conclude that:
$$\sigma^{(q)}(J_0, 1) = \sigma^{(q)}(J_0, \alpha) = \sigma_0(J_0, \alpha).$$

- Case $\{w3\text{-r1}\}$: Since H_0 is trivalent either nd-val A_0 is 3 or 2.
- Case $\{w3\text{-r1-nd3}\}$: Assume that nd-val A_0 is 3. By the no-return condition $\varphi_0(\text{nd-}E(A_0))$ has at least two elements. So let e_2, e_3, e_4 be the edges in nd-$E(A_0)$, labelled such that $|e_4| = \max(|e_2|, |e_3|, |e_4|)$, $\varphi_0(e_3) = t_3$ and $\varphi_0(e_4) = t_4$. Let $k_i = |e_i|$. Lemma 54 (nd. r_φ formula) gives $r_0(A_0) = 1 = 3 + 2|A_0| - 2 - (k_2 + k_3 + k_4)$. That is, $2|A_0| = k_2 + k_3 + k_4$. If e_2 were above t_4, then $k_2 + k_4 \leq |A_0|$. Substituting in $k_3 = 2|A_0| - (k_2 + k_4)$ gives $k_3 \geq |A_0| \geq k_2 + k_4 > k_4$, a contradiction. So e_2 is either above t_2 or above t_3.
- Case $\{w3\text{-r1-nd3-t2}\}$: Assume that e_2 is above t_2. Let $\{\alpha, \beta, \gamma\} = \{2, 3, 4\}$. Recall from Subsection 6.5 that $A^{(q)}$ is the unique vertex of $G^{(q)}$ with nd-val $A^{(q)} = 3$ that contracts to A_0. We say that $M^{(q)}$ is in Position I or II if $A^{(q)}$ is above u or v, respectively. We prove that $T^{(q)}$ and the position of $A^{(q)}$ determine $M^{(q)}$.

   Position I: Assume that $A^{(q)}$ is above u. Since $\text{val}\, u = 2$ and nd-val $A^{(q)} = 3$ we have that $A^{(q)}$ belongs to Case (r1-nd3) of the local properties (assuming Case (r0-nd3)

contradicts that $\mathrm{val}\, u = 2$). So there are exactly two edges e', e'' of nd-$E(A^{(q)})$ above t_1; moreover $|e_\alpha| = |A^{(q)}| = |e'| + |e''|$. By Lemma 86 these are the two edges of nd-$G_{A_0}^{(q)}$. Hence, nd-$G_{A_0}^{(q)}$ has three vertices. By Lemma 87 these are: $A^{(q)}$ incident to $e_\alpha^{(q)}$; and the two ends of $e_\beta^{(q)}$, $e_\gamma^{(q)}$ above v, with non-dangling valency 2 and $r^{(q)}$-value equal to 0. By the Case (r0-nd2) of the local properties we conclude that as subsets of $[d]$, the edges e', $e_\beta^{(q)}$ and their common end are equal, and $h(e') = h(e_\beta^{(q)})$; similarly with e'' and $e_\gamma^{(q)}$. Thus, $|e_\alpha| = |e_\beta| + |e_\gamma|$ and $|A^{(q)}| = |A_0|$, so this construction is possible only when $\alpha = 4$ and $k_4 = |A_0|$.

Position II: Assume that $A^{(q)}$ is above v. Since $\mathrm{val}\, v = 3$ and nd-val $A^{(q)} = 3$ we have that $A^{(q)}$ belongs to Case (r0-nd3) of the local properties (assuming Case (r1-nd3) contradicts that $\mathrm{val}\, v = 3$). So there is exactly one edge e' of nd-$E(A^{(q)})$ above t_1. Let A' be the other end of e', which by Lemma 85 has non-dangling valency 2. Lemma 86 implies that the two vertices of nd-$G_{A_0}^{(q)}$ are $A^{(q)}$ and A'. Hence, by Lemma 87 A' is incident to $e_\alpha^{(q)}$; and $h(e_\alpha^{(q)}) = h(e')$. Since $r_0(A_0) = 1$, we have that A' belongs to Case (r1-nd2) of the local properties and this gives two possible cases for the values of $|A'|$, $|e'|$ and $|A^{(q)}|$:

Position II.a: $|A'| = |e'|$ and $|A'| = |e_\alpha^{(q)}| + 1$. In this case, as subsets of $[d]$, we get that $e_\alpha^{(q)} \subset A'$, that A' and e' are equal, that $e' \subset A^{(q)}$, and therefore, by Lemma 67, that $A_0 = A^{(q)} \cup A' = A^{(q)}$. We also get that $|A_0| \geq |A'| > k_\alpha$.

Position II.b: $|A'| = |e'| + 1$ and $|A'| = |e_\alpha^{(q)}|$. As subsets of $[d]$ we have that $e' \subset A' \cap A^{(q)}$, so $A' \setminus A^{(q)}$ is either empty or a singleton. The former would imply that for $i \in A' \setminus e'$, the classes e' and $\overline{(t_1, i)}_{M^{(q)}}$ are distinct edges of $G^{(q)}$ above t_1 with equal ends, which gives a cycle over t_1, a contradiction. So $|A_0| = |A^{(q)}| + |A'| - |A^{(q)} \cap A'| = |A^{(q)}| + 1$. The ends of $e_\beta^{(q)}$ and $e_\gamma^{(q)}$ are above v, so $|A^{(q)}| \geq k_\beta, k_\gamma$. Hence, $|A_0| > \max(k_\beta, k_\gamma)$, so $k_\alpha \geq 2$.

Since Position I implies $|A_0| = k_4$, we treat two cases: $|A_0| = k_4$ and $|A_0| > k_4$.

- Case {w3-r1-nd3-t2-(a=k4)}: Assume that $|A_0| = k_4$. Then $k_2 + k_3 = |A_0|$, so $k_4 > k_2, k_3$. Let $\alpha = 4$, so $|A_0| = k_4$, which precludes Position II.a. Position I and Position II.b give $M^{(1)}$ and $M^{(2)}$, respectively. Let $\alpha = 2$, which precludes Position I. Then it is not true that $|A_0| > \max(k_\beta, k_\gamma) = k_4$, which precludes Position II.b as well. Position II.a gives $M^{(3)}$. An analogous argument with $\alpha = 3$ gives $M^{(4)}$. See Figure 28.

$$\frac{c(e_4)}{k_2 + k_3} + \sigma_0(J_0, 4) = 0$$

$$\frac{c(e_2)}{k_2} + \sigma_0(J_0, 2) = 0$$

$$\frac{c(e_3)}{k_3} + \sigma_0(J_0, 3) = 0$$

$$|e'| = k_2 \qquad |e''| = k_3$$

$$\sigma^{(q)}(J_0, 1) = \sigma_0(J_0, 4)$$

$$c^{(1)} = \frac{c(e_2)}{k_2} + \frac{c(e_3)}{k_3} + \sigma^{(q)}(J_0, 1)$$

$|e'| = k_4 - 1 = k_2 + k_3 - 1$ $|e'| = k_2 + 1$ $|e'| = k_3 + 1$

$\sigma^{(2)}(J_0, 1) = \sigma_0(J_0, 4)$ $\sigma^{(3)}(J_0, 1) = \sigma_0(J_0, 2)$ $\sigma^{(4)}(J_0, 1) = \sigma_0(J_0, 3)$

$c^{(2)} = \dfrac{c(e_4)}{k_2 + k_3 - 1} + \sigma^{(2)}(J_0, 1)$ $c^{(3)} = \dfrac{c(e_2)}{k_2 + 1} + \sigma^{(3)}(J_0, 1)$ $c^{(4)} = \dfrac{c(e_3)}{k_3 + 1} + \sigma^{(4)}(J_0, 1)$

FIGURE 28

To verify Equation $(\star)$ we compute:

$$(2) \qquad c^{(1)} + (k_2 + k_3 - 1)c^{(2)} + (k_2 + 1)c^{(3)} + (k_3 + 1)c^{(4)}$$

$$= \left(\tfrac{(k_2+1)}{k_2} c(e_2) + (k_2 + 1)\sigma_0(J_0, 2) \right) + \left(\tfrac{(k_3+1)}{k_3} c(e_3) + (k_3 + 1)\sigma_0(J_0, 3) \right)$$

$$+ (c(e_4) + (k_2 + k_3)\sigma_0(J_0, 4)) = 0 + 0 + 0 = 0.$$

- Case {w3-r1-nd3-t2-(a>k4)}: Assume that $|A_0| > k_4$, which precludes Position I. If $\min(k_2, k_3, k_4)$ were 1, then $k_2 + k_3 + k_4 \leq 2|A_0| - 1$, contradicting $0 = 2|A| - k_2 - k_3 - k_4$. Hence, $\min(k_2, k_3, k_4) \geq 2$, so both cases of Position II are possible. It turns out that they balance each other for each possible α. The case $\alpha = 4$ is shown in Figure 29.

M_0 $M^{(1)}$ $M^{(2)}$

                  $|e'| = k_4 - 1$ $|e'| = k_4 + 1$

                  $\sigma^{(1)}(J_0, 1) = \sigma_0(J_0, 4)$ $\sigma^{(2)}(J_0, 1) = \sigma_0(J_0, 4)$

$\dfrac{c(e_4)}{k_4} + \sigma_0(J_0, 4) = 0$ $c^{(1)} = \dfrac{c(e_4)}{k_4 - 1} + \sigma^{(1)}(J_0, 1)$ $c^{(2)} = \dfrac{c(e_4)}{k_4 + 1} + \sigma^{(2)}(J_0, 1)$

FIGURE 29

We conclude that:

$$(k_4 - 1)c^{(1)} + (k_4 + 1)c^{(2)} = 2c(e_4) + 2k_4\sigma_0(J_0, 4) = 0.$$

Analogous constructions exist for $\alpha = 2$, which gives $M^{(3)}$, $M^{(4)}$; and $\alpha = 3$, which gives $M^{(5)}$, $M^{(6)}$. We verify Equation ($\star$):

$$(3) \qquad \Big((k_4 - 1)c^{(1)} + (k_4 + 1)c^{(2)}\Big) + \Big((k_2 - 1)c^{(3)} + (k_2 + 1)c^{(4)}\Big) +$$
$$\Big((k_3 - 1)c^{(5)} + (k_3 + 1)c^{(6)}\Big) = 0 + 0 + 0 = 0.$$

- Case {w3-r1-nd3-t3}: Assume that e_2 is above t_3. Then $k_2 + k_3 \leq |A_0|$. Recall that $2|A_0| = k_2 + k_3 + k_4$ and $|A_0| \geq k_i$, so $k_4 = |A_0|$, $k_2 + k_3 = |A_0|$ and $|A_0| > k_2, k_3$. We argue that $T^{(q)}$ determines nd-$G_{A_0}^{(q)}$. If α were 2, then by Lemma 87 all the vertices of nd-$G_{A_0}^{(q)}$ would be above v, so nd-$G_{A_0}^{(q)}$ would have a single vertex with $r^{(q)}$-value 1 above trivalent v, contradicting that $M^{(q)}$ is change-minimal. For $\alpha = 3, 4$, we get that nd-$G_{A_0}^{(q)}$ has vertices above u and v. The vertices above v belong to the Case (r0-nd2) of the local properties, since Case (r0-nd3) would produce a non-dangling element above t_2. So if $\alpha = 3$, we get $M^{(1)}$ where $e_2^{(q)}$, $e_3^{(q)}$ have ends above v, and by Case (r0-nd2) these determine the two edges e', e'' of nd-$G_{A_0}^{(q)}$; similarly, if $\alpha = 4$ we get $M^{(2)}$ where $e_4^{(q)}$ has an end above v and this determines the edge e' of nd-$G_{A_0}^{(q)}$. So the vertex of nd-$G_{A_0}^{(q)}$ above u is $A^{(q)}$, and it belongs to Case (r1-nd3). So $A^{(q)}$ equals e_4, equals A_0 as subsets of $[d]$. This gives $M^{(1)}$ and $M^{(2)}$ for $\alpha = 3, 4$, respectively. See Figure 30.

$$\frac{c(e_2)}{k_2} + \frac{c(e_3)}{k_3} + \sigma_0(J_0, 3) = 0$$
$$\frac{c(e_4)}{k_2 + k_3} + \sigma_0(J_0, 4) = 0$$

$M^{(1)}$

$|e'| = k_4 = k_2 + k_3$
$\sigma^{(1)}(J_0, 1) = \sigma_0(J_0, 3)$
$c^{(1)} = \frac{c(e_4)}{k_2 + k_3} + \sigma^{(1)}(J_0, 1)$

$M^{(2)}$

$|e'| = k_2 \qquad |e''| = k_3$
$\sigma^{(2)}(J_0, 1) = \sigma_0(J_0, 4)$
$c^{(2)} = \frac{c(e_2)}{k_2} + \frac{c(e_3)}{k_3}$
$\qquad + \sigma^{(2)}(J_0, 1)$

FIGURE 30

We verify Equation ($\star$):

$$(4) \qquad c^{(1)} + c^{(2)} =$$
$$\left(\tfrac{1}{k_2 + k_3} c(e_4) + \sigma_0(J_0, 4)\right) + \left(\tfrac{1}{k_2} c(e_2) + \tfrac{1}{k_3} c(e_3) + \sigma_0(J_0, 3)\right) = 0 + 0 = 0.$$

- Case {w3-r1-nd2}: Assume that nd-val A_0 is 2. Let e, e' be in nd-$E(A_0)$. They are in the same edge h of H_0. Assume that $|e| \leq |e'|$. By no-return we may assume that e and e' are above t_3 and t_4 respectively. Lemma 54 (nd. r_φ formula) gives $r_\varphi(A_0) = 1 = 2|A_0| - |e'| - |e|$. Thus, if $k = |e|$, then $|A_0| = |e'| = k + 1$. From here, follow a reasoning analogous to case {w3-nd3-t2} to show that $T^{(q)}$ determines $M^{(q)}$, and that the vertices

of nd-$G^{(q)}_{A_0}$ above v belong to Case (r0-nd2) of the local properties, and the ones above u to Case (r1-nd2). See Figure 31.

$$M_0$$

$$\frac{c_h}{k} + \sigma_0(J_0, 3) = 0$$

$$\frac{c_h}{k+1} + \sigma_0(J_0, 4) = 0$$

$$M^{(1)}$$

$$\sigma^{(1)}(J_0, 1) = \sigma_0(J_0, 3)$$

$$c^{(1)} = \frac{c_h}{k+1} + \sigma^{(1)}(J_0, 1)$$

$$M^{(2)}$$

$$\sigma^{(2)}(J_0, 1) = \sigma_0(J_0, 4)$$

$$c^{(2)} = \frac{c_h}{k} + \sigma^{(2)}(J_0, 1)$$

FIGURE 31

We verify Equation $(\star)$:

(5) $$c^{(1)} + c^{(2)} = \left(\tfrac{1}{k}c_h + \sigma_0(J_0, 3)\right) + \left(\tfrac{1}{k+1}c_h + \sigma_0(J_0, 4)\right) = 0 + 0 = 0.$$

6.7.4. *Case {w2}.* Assume that $\operatorname{val} w_0$ is 2. Then $r_0(A_0) = \operatorname{val} A_0 - 2$. So if $r_0(A_0) = 0$, then $\operatorname{val} A_0 = 2$. Hence, $\sigma_0(J_0, 2) = \sigma_0(J_0, 3)$, and we denote this quantity by s. Also, $\operatorname{ch} w_0 = 2$ by Equation (C). The trees contracting to T_0 are $T_\varnothing$ with $\operatorname{val} u = 1$, $\operatorname{val} v = 3$; and T_2 with $\operatorname{val} u = \operatorname{val} v = 2$. If $T^{(q)} = T_\varnothing$, then Remark 51 implies that $\sigma^{(q)}(J_0, 1) = \sigma^{(q)}(J_1, 1) = 0$, so $\det A^{(q)} = \sigma^{(q)}(J_2, 1)$. If $T^{(q)} = T_2$, then a reasoning analogous to Case {w3-r0} gives

$$\sigma^{(q)}(J_0, 1) = \sigma^{(q)}(J_0, 2) = s,$$
$$\sigma^{(q)}(J_0, 1) = \sigma^{(q)}(J_0, 3) = s.$$

So we are left with the cases $r_0(A_0) = 2$ and $r_0(A_0) = 1$.

- Case {w2-r2}: Assume that $r_0(A_0) = 2$. Then $\operatorname{val} A_0 = 4$. Since A_0 satisfies no-return, let e_2, e_3 be non-dangling edges in $E(A_0)$, above t_2, t_3, respectively. Let e_1, e_4 be the remaining two edges of $E(A_0)$. We may assume without loss of generality that e_1 is above t_2. By the refinement property:

$$\sum_{\substack{e \in E(A_0) \\ \varphi_0(e) = t_2}} |e| = \sum_{\substack{e \in E(A_0) \\ \varphi_0(e) = t_3}} |e| = |A_0|.$$

So in particular $\sum_{i=1}^{4} k_i = 2|A_0|$. We explore how $T^{(q)}$ affects nd-$G^{(q)}_{A_0}$:

  Base I: assume that $T^{(q)}$ is $T_\varnothing$. Then, by Remark 51 (change-minimal leaves) we have that nd-$G^{(q)}_{A_0}$ has a vertex A_u above u with $|A_u| = 2$, $r^{(q)}(A_u) = 2$, and two incident edges e', e'' with $|e'| = |e''| = 1$. These are all the edges of nd-$G^{(q)}_{A_0}$ by Lemma 86 and the fact that $r_0(A_0) = 2$. Thus, A_u and the ends A', A'' of e', e'', respectively, are the vertices of nd-$G^{(q)}_{A_0}$. We may assume that $e^{(q)}_2$ and A' are incident. There are two possibilities for $E(A')$ and $E(A'')$. Either $e^{(q)}_3$ is incident to A' or A''. We call the first case Base I.a. Since $r^{(q)}(A') = r^{(q)}(A'') = 0$, we have that $k_2 = |A'| = k_3$, $k_1 = |A''| = k_4$. We call the second case Base I.b; it implies that k_2 equals the cardinality of the edge above t_3 incident to A'. So either $k_2 = k_4$ and $k_3 = k_1$; or $k_2 = k_1$ and

$k_3 = k_4$. In particular, here the two edges of $E(A_0)$ are above t_2, and the other two above t_3. See Case {w2-r2-nd3-M-11} for figures of both possibilities.

Base II: assume that $T^{(q)}$ is T_2. Then the vertices of nd-$G_{A_0}^{(q)}$ belong to Cases (r0-nd2), (r1-nd2), or (r1-nd3) of the local properties. By Lemma 85 at most one vertex of nd-$G_{A_0}^{(q)}$ belongs to Case (r1-nd3). So by Lemma 86 we get that nd-$G_{A_0}^{(q)}$ has either one edge e'; or two edges e', e''. We call them Base II.1, Base II.2, respectively.

- Case {w2-r2-nd3}: Assume that nd-val A_0 is 3. We may assume without loss of generality that e_4 is dangling. So $k_4 = 1$ and $2|A_0| - 1 = \sum_{i=1}^{3} k_i$. If e_4 is above t_3, then $k_3 + 1 = |A_0| = k_1 + k_2$. If e_4 is above t_2, then $k_3 = |A_0| = k_1 + k_2 + 1$. We call these two possibilities Cardinality M and Cardinality P.

For Base II we have that exactly one vertex of nd-$G_{A_0}^{(q)}$ belongs to Case (r1-nd3), so the other vertex of nd-$G_{A_0}^{(q)}$ with $r^{(q)}$-value 1 belongs to Case (r1-nd2).

In Base II.1, Lemma 87 implies that the ends of $e_1^{(q)}$, $e_2^{(q)}$ above u are the same vertex of nd-$G_{A_0}^{(q)}$; by no-return this vertex has non-dangling valency 3, so it is $A^{(q)}$. By Case (r1-nd3) we have that $e_1 \cup e_2$, $A^{(q)}$, and e' are equal as subsets of $[d]$; therefore, $|e'| = |A^{(q)}| = k_1 + k_2$. Hence, the other end A' of e' (which is above v, incident to $e_3^{(q)}$) belongs to Case (r1-nd2). Either $|e'| = |A'| = k_3 + 1$, or $|e'| = |A'| - 1 = k_3 - 1$. The first possibility implies $k_1 + k_2 = k_3 + 1$, namely Cardinality M; the second possibility implies $k_1 + k_2 = k_3 - 1$, namely Cardinality P. We call these possibilities Base II.1.M and Base II.1.P, respectively.

In Base II.2, nd-$G_{A_0}^{(q)}$ has three vertices. By Lemma 87 these are the ends A', A'' of $e_1^{(q)}$, $e_2^{(q)}$, respectively, above u, and the end B of $e_3^{(q)}$ above v. By no-return we may assume that e', e'' are incident to A', A'', respectively. Since these are all the edges of nd-$G_{A_0}^{(q)}$, nd-val $A' = $ nd-val $A'' = 2$, and nd-val $B = 3$. So B is $A^{(q)}$ and $h(e_1^{(q)}) = h(e')$, $h(e_2^{(q)}) = h(e'')$. So Case (r1-nd3) gives that $k_3 = |A^{(q)}| = |e'| + |e''|$. One of A', A'' belongs to Case (r1-nd2) and the other to Case (r0-nd2). If A' belongs to Case (r1-nd2) then either $|e'| = |A'| - 1 = k_1 - 1$, so $k_1 > 1$ and $(k_1 - 1) + k_2 = k_3$, that is Cardinality M; or $|e'| = |A'| + 1 = k_1 + 1$, which gives $(k_1 + 1) + k_2 = k_3$, that is Cardinality P. We call these possibilities Base II.2.1.M and Base II.2.1.P, respectively. Analogously, if A'' belongs to Case (r1-nd2) we get Base II.2.2.M and Base II.2.2.P, where the former gives $k_2 > 1$.

- Case {w2-r2-nd3-M}: Assume that e_4 is above t_3, namely Cardinality M. The possibilities are Base I.a, Base I.b, Base II.1.M, Base II.2.1.M, and Base II.2.2.M.
- Case {w2-r2-nd3-M-11}: Assume that $k_1 = 1$ and $k_2 = 1$. Base II.2.1.M, Base II.2.2.M are precluded since $k_1 \not> 1$, $k_2 \not> 1$, respectively. Base I.a, Base I.b, Base II.1.M determine $M^{(1)}$, $M^{(2)}$ and $M^{(3)}$, respectively. See Figure 32.

$$c(e_1) + c(e_2) + s = 0$$
$$c(e_1) + s = 0$$

$M^{(1)}$

$M^{(2)}$

$M^{(3)}$

$\sigma^{(1)}(J_0, 1) = \sigma^{(1)}(J_1, 1) = 0$ $\sigma^{(2)}(J_0, 1) = \sigma^{(2)}(J_1, 1) = 0$ $|e| = 2$

$c^{(1)} = 2c(e_1)$ $c^{(2)} = 2c(e_2)$ $\sigma^{(3)}(J_0, 1) = s$

$c^{(3)} = \dfrac{c(e_3)}{2} + s$

FIGURE 32

We verify Equation $(\star)$:

(6) $c^{(1)} + c^{(2)} + 4c^{(3)} = 2(c(e_1) + c(e_2) + s) + 2(c(e_3) + s) = 2 \cdot 0 + 2 \cdot 0 = 0.$

- Case {w2-r2-nd3-M-1k}: Assume that $k_1 = 1$ and $k_2 \geq 2$. Let $k = k_2$, so $|A_0| = k + 1$ and $|e_3| = k$. Base I.b and Base II.2.1.M are precluded since $|e_2| \neq |e_4|$ and $k_1 \not> 1$, respectively. Base I.a, Base II.2.2.M, Base II.1.M determine $M^{(1)}$, $M^{(2)}$ and $M^{(3)}$, respectively. See Figure 33.

M_0

$$c(e_1) + \frac{c(e_2)}{k} + s = 0$$

$$\frac{c(e_3)}{k} + s = 0$$

$M^{(1)}$

$M^{(2)}$

$M^{(3)}$

$\sigma^{(1)}(J_0, 1) = \sigma^{(1)}(J_1, 1) = 0$ $|e'| = 1 \quad |e''| = k - 1$ $|e'| = k + 1$

$c^{(1)} = 2c(e_1)$ $\sigma^{(2)}(J_0, 1) = s$ $\sigma^{(3)}(J_0, 1) = s$

$c^{(2)} = c(e_1) + \dfrac{c(e_2)}{k - 1} + s$ $c^{(3)} = \dfrac{c(e_3)}{k + 1} + s$

FIGURE 33

We verify Equation ($\star$):

(7)
$$c^{(1)} + 2(k-1)c^{(2)} + 2(k+1)c^{(3)} =$$
$$2(kc(e_1) + c(e_2) + ks) + (c(e_3) + ks) = 2 \cdot 0 + 0 = 0.$$

- Case {w2-r2-nd3-M-kk}: Assume that $k_1 \geq 2$ and $k_2 \geq 2$. Then $k_3 > k_1, k_2$, so Base I is precluded since $|e_2| \neq |e_3|$ nor $|e_2| \neq |e_4|$. Base II.2.1.M, Base II.2.2.M, and Base II.1.M determine $M^{(1)}$, $M^{(2)}$ and $M^{(3)}$, respectively. See Figure 34.

$$\frac{c(e_1)}{k_1} + \frac{c(e_2)}{k_2} + s = 0$$

$$\frac{c(e_3)}{k_1 + k_2 - 1} + s = 0$$

$|e'| = k_1 - 1 \quad |e''| = k_2$
$\sigma^{(1)}(J_0, 1) = s$
$c^{(1)} = \dfrac{c(e_1)}{k_1 - 1} + \dfrac{c(e_2)}{k_2} + s$

$|e'| = k_1 \quad |e''| = k_2 - 1$
$\sigma^{(2)}(J_0, 1) = s$
$c^{(2)} = \dfrac{c(e_1)}{k_1} + \dfrac{c(e_2)}{k_2 - 1} + s$

$|e'| = k_1 + k_2$
$\sigma^{(3)}(J_0, 1) = s$
$c^{(3)} = \dfrac{c(e_3)}{k_1 + k_2} + s$

FIGURE 34

We verify Equation ($\star$):

(8) $(k_1 - 1)c^{(1)} + (k_2 - 1)c^{(2)} + (k_1 + k_2)c^{(3)} =$
$$\left(\frac{k_1 + k_2 - 1}{k_1} c(e_1) + \frac{k_1 + k_2 - 1}{k_2} c(e_2) + (k_1 + k_2 - 1)s \right) + (c(e_3) + (k_1 + k_2 - 1)s) = 0.$$

- Case {w2-r2-nd3-P}: Assume that e_4 is above t_2, namely Cardinality P. Base I is precluded. Base II.2.1.P, Base II.2.2.P, Base II.1.P determine $M^{(1)}$, $M^{(2)}$ and $M^{(3)}$, respectively. See Figure 35.

$$\frac{c(e_1)}{k_1} + \frac{c(e_2)}{k_2} + s = 0$$

$$\frac{c(e_3)}{k_1 + k_2 + 1} + s = 0$$

$|e'| = k_1 + 1 \quad |e''| = k_2$ $\qquad$ $|e'| = k_1 \quad |e''| = k_2 + 1$ $\qquad$ $|e'| = k_1 + k_2$

$\sigma^{(q)}(J_0, 1) = s$ $\qquad$ $\sigma^{(2)}(J_0, 1) = s$ $\qquad$ $\sigma^{(3)}(J_0, 1) = s$

$c^{(1)} = \dfrac{c(e_1)}{k_1 + 1} + \dfrac{c(e_2)}{k_2} + s$ $\quad$ $c^{(2)} = \dfrac{c(e_1)}{k_1} + \dfrac{c(e_2)}{k_2 + 1} + s$ $\quad$ $c^{(3)} = \dfrac{c(e_3)}{k_1 + k_2} + s$

FIGURE 35

We verify Equation $(\star)$:

(9) $\quad (k_1 + 1)c^{(1)} + (k_2 + 1)c^{(2)} + (k_1 + k_2)c^{(3)} =$

$\quad (k_1 + k_2 + 1)(\frac{1}{k_1}c(e_1) + \frac{1}{k_2}c(e_2) + s) + (k_1 + k_2 + 1)(\frac{1}{k_1 + k_2 + 1}c(e_3) + s) = 0 + 0 = 0.$

- Case {w2-r2-nd2}: Assume that nd-val A_0 is 2. Then both e_1, e_4 are dangling. Let h be equal to $h(e_2)$ equal to $h(e_3)$.
- Case {w2-r2-nd2-M}: Recall that e_1 is above t_2. Assume that e_4 is above t_3. Then $k_2 + 1 = |A_0| = k_3 + 1$. Since $|e_2| = |e_3|$, the columns corresponding to t_2 and t_3 in A_0 are equal. So M_0 is not full-rank, contradicting Lemma 70.
- Case {w2-r2-nd2-P}: Assume that e_4 is above t_2. Then $k_3 = |A_0| = k_2 + 2$, and Base I is precluded. In Base II all vertices of nd-$G_{A_0}^{(q)}$ belong to Case (r1-nd2). Thus, nd-$G_{A_0}^{(q)}$ has a single edge $|e'|$. The Case (r1-nd2) on the end of e_3 above v implies that $|e'| = k_3 - 1 = k_2 + 1$. This is the only possibility, shown in Figure 36.

M_0 $\qquad\qquad$ $M^{(1)}$

$s = \sigma_0(J_0, 2) = \sigma_0(J_0, 3)$ $\qquad$ $|e'| = k_2 + 1$

$\dfrac{c_h}{k_2} + s = 0, \quad \dfrac{c_h}{k_2 + 2} + s = 0$ $\qquad$ $\sigma^{(1)}(J_0, 1) = s$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad c^{(1)} = \dfrac{c_h}{k_2 + 1} + s$

FIGURE 36

From $c_h + k_2 s = 0$ and $c_h + (k_2 + 2)s = 0$ it follows that $c^{(1)} = c_h + (k_2 + 1)s = 0$, hence $M^{(1)}$ is not full-dimensional.

- Case {w2-r1}: Assume that $r_0(A_0) = 1$. Then val $A_0 = 3$. Since A_0 satisfies no-return, let e_2, e_3 be non-dangling edges in $E(A_0)$, above t_2, t_3, respectively. Let e_1 be the remaining edge of $E(A_0)$. We may assume without loss of generality that e_1 is above t_2. By the refinement property $k_1 + k_2 = |A_0| = k_3$. Let $\tilde{A}$ be the unique vertex of nd-$G_{A_0}^{(q)}$ with $r^{(q)}(\tilde{A}) = 1$. If $\tilde{A}$ is above u (resp. v), then the vertices of nd-$G_{A_0}^{(q)}$ above v (resp. u) belong to the Case (r0-nd2) of the local properties (Case (r0-nd3) would contradict that v (resp. u) is divalent). These facts and Lemma 87 determine the classes.
- Case {w2-r1-nd3}: Assume that nd-val A_0 is 3. We get two elements in PStar(M_0), shown in Figure 37.

local part around A_0 $\qquad$ $\varphi^{(1)}(\tilde{A}) = u$ $\qquad\qquad$ $\varphi^{(2)}(\tilde{A}) = v$

$\sigma_0(J_{A_0}, 2) = \dfrac{c(e_1)}{k_1} + \dfrac{c(e_2)}{k_2}$ $\quad |e'| = k_3$ $\qquad\qquad |e'| = k_1 \quad |e''| = k_2$

$\qquad\qquad\qquad\qquad\qquad \sigma^{(1)}(J_{A_0}, 1) = \dfrac{c(e_3)}{k_1 + k_2}$ $\quad \sigma^{(2)}(J_{A_0}, 1) = \dfrac{c(e_1)}{k_1} + \dfrac{c(e_2)}{k_2}$

$\sigma_0(J_{A_0}, 3) = \dfrac{c(e_3)}{k_1 + k_2}$

FIGURE 37

Thus,

$$\sigma^{(1)}(J_{A_0}, 1) + \sigma^{(2)}(J_{A_0}, 1) = \sigma_0(J_{A_0}, 2) + \sigma_0(J_{A_0}, 3).$$

- Case {w2-r1-nd2}: Assume that nd-val A_0 is 2. We may assume that e_1 is dangling. So $k_3 = k_2 + 1$. Let $h = h(e_2) = h(e_3)$. We get two elements in PStar(M_0), shown in Figure 38.

local part around A_0 $\qquad$ $\varphi^{(1)}(\tilde{A}) = u$ $\qquad$ $\varphi^{(2)}(\tilde{A}) = v$

$$\sigma_0(J_{A_0}, 2) = \frac{c_h}{k_1} \qquad |e'| = k_2 + 1 \qquad |e'| = k_2$$

$$\sigma_0(J_{A_0}, 3) = \frac{c_h}{k_1 + 1} \qquad \sigma^{(1)}(J_{A_0}, 1) = \frac{c_h}{k_2 + 1} \qquad \sigma^{(2)}(J_{A_0}, 1) = \frac{c_h}{k_2}$$

FIGURE 38

Thus,
$$\sigma^{(1)}(J_{A_0}, 1) + \sigma^{(2)}(J_{A_0}, 1) = \sigma_0(J_{A_0}, 2) + \sigma_0(J_{A_0}, 3).$$

- Proof of $(\star)$ for case $\{\text{w2-r1}\}$: There is another vertex B_0 above w_0 with $r_0(B_0) = 1$. The previous analysis holds for B_0, with notation entirely analogous. Note that in $M^{(q)}$, $\varphi^{(q)}(\tilde{A}) \neq \varphi^{(q)}(\tilde{B})$ because $\operatorname{ch} u = \operatorname{ch} v = 1$. So $\varphi^{(q)}(\tilde{A})$ determines the gluing datum, and it still holds that
$$\sigma^{(1)}(J_{B_0}, 1) + \sigma^{(2)}(J_{B_0}, 1) = \sigma_0(J_{B_0}, 2) + \sigma_0(J_{B_0}, 3).$$

This gives the following calculation, which verifies Equation $(\star)$:

$$c^{(1)} = \sigma^{(1)}(J_{A_0}, 1) + \sigma^{(1)}(J_{B_0}, 1) + s, \qquad c^{(2)} = \sigma^{(2)}(J_{A_0}, 1) + \sigma^{(2)}(J_{B_0}, 1) + s,$$

$$(10) \qquad\qquad c^{(1)} + c^{(2)} = \sigma_0(2) + \sigma_0(3) = 0 + 0 = 0.$$

Conclusion. It is remarkable how diverse the arguments of Section 6 are. By no means do we stand in front of a construction that has been repeated with subtle variations. The richness and diversity of the behaviour of possibly full-dimensional gluing datums defied many attempts of further consolidation into fewer cases. The end result is exhaustive, so Lemma 83 is verified. This finishes the proof of Theorem 2. The method is effective as well; see [Dra] for code.

Preview part II. To summarize the past sneak peeks, let $\varphi^{(1)}$, $\varphi^{(2)}$ be DT-morphisms that share a limit φ_0. The cone C_{φ_0} may be viewed as a common face to the cones $C_{\varphi^{(1)}}$ and $C_{\varphi^{(2)}}$. Hence, it is natural to glue at C_{φ_0} and mod out by isomorphisms. The possibilities for C_{φ_0} include the cases that we skipped where the source of φ_0 is not trivalent. The moduli space $\Phi^d \mathcal{M}_g^{\mathrm{trop}}$ that emerges by gluing all genus-g full-dimensional cones, as mentioned in Subsection 2.7, is the main object in Part II of this series. We study the connectivity of this space and its projection to $\mathcal{M}_g^{\mathrm{trop}}$ to get a stronger version of Theorem 2, in the sense that we obtain a count of realizations. Also, the result becomes independent from the work in [CD18].

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