# On Bézout Inequalities for non-homogeneous Polynomial Ideals* 

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#### Abstract

We introduce a "workable" notion of degree for non-homogeneous polynomial ideals and formulate and prove ideal theoretic Bézout Inequalities for the sum of two ideals in terms of this notion of degree and the degree of generators. We compute probabilistically the degree of an equidimensional ideal.


Keywords. Non-homogeneous polynomial ideal, ideal degree, idealistic Bézout Inequality.

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## Introduction

Motivated by the aim to formulate and prove an idealistic version of Bézout's Theorem (see [18]) and by applications to transcendence theory (see [15, 4]), the notion of degree of

[^0]homogeneous polynomial ideals became intensively studied. In general this work relied on the notion of degree of homogeneous polynomial ideals based on the Hilbert polynomial (see [14] for a different view).

In this paper we propose an alternative and self-contained approach for non-homogeneous polynomial ideals leading to specific results which are not simple consequences of their homogeneous counterparts.

We introduce a "workable" notion of degree for non-homogeneous polynomial ideals and formulate and prove ideal theoretic Bézout Inequalities for the sum of two ideals in terms of this notion of degree. However it turns out, that, due to the presence of embedded primes, a Bézout Inequality in completely intrinsic terms (depending only on the degrees of the two given ideals) is unfeasable. Hence in some place the degrees of generators of at least one of the ideals comes into play and our main Bézout Inequality will be of this mixed type.

We finish the paper with a probabilistic algorithm which computes the degree of an equidimensional ideal given by generators.

## Organization of the paper

The first three sections are devoted to the development of the tools we are going to use in the sequel. The technical highlight is Proposition 19 in the third section which anticipates in some sense the "correctness" of our ideal theoretic notion of degree (see Theorem 21 and Definition (22) and the main result of the paper, namely the mixed type Bézout Inequality, Theorem 29, in the fourth section.

These results become combined in the fifth section with techniques going back to Masser and Wüstholz [15] in order to estimate the degree of a polynomial ideal in terms of the degrees of generators (Theorem (32).

Finally the sixth section contains a probabilistic complexity result concerning the computation of the degree of an equidimensional ideal.

## 1 Notions and Notations

Let $K$ be an algebraically closed field, $\vec{X}=\left(X_{1}, \ldots, X_{n}\right)$ with $X_{1}, \ldots, X_{n}$ indeterminates over $K$ and $K[\vec{X}]$ the ring of $n$-variate polynomials with coefficients in $K$. The affine space $K^{n}$ with the Zariski topology is denoted by $\mathbb{A}^{n}$. For any ideal $\mathfrak{a}$ of $K[\vec{X}]$, we denote by $V(\mathfrak{a})$ the set of its common zeros in $\mathbb{A}^{n}$ and by $A:=K[\vec{X}] / \mathfrak{a}$ the associated factor ring.

We shall use freely standard notions and notations from Commutative Algebra and affine Algebraic Geometry. These can be found for example in [1, 16, [17].

Less standard is the notion of degree of closed (affine) subvarieties of $\mathbb{A}^{n}$ we are going to use.

For an irreducible closed subvariety $V$ of $\mathbb{A}^{n}$ we define the degree $\operatorname{deg} V$ of $V$ as the maximum number of points that can arise when we intersect $V$ with an affine linear subspace $E$ of $\mathbb{A}^{n}$ such that $V \cap E$ is finite (observe that it is a nontrivial fact that $\operatorname{deg} V<\infty$ holds). The degree $\operatorname{deg} V$ of an arbitrary closed subvariety $V$ of $\mathbb{A}^{n}$ is the sum of the degrees of the irreducible components of $V$.

It is a remarkable fact that for this notion of degree that for closed subvarieties $V$ and
$W$ an intrinsic Bézout Inequality holds: $\operatorname{deg}(V \cap W) \leq \operatorname{deg}(V) \cdot \operatorname{deg}(W)$ (with intrinsic we mean that the degree of $V \cap W$ is estimated in terms of the degrees of $V$ and $W$ only). For more details we refer to [10, 6, 18].

## 2 Secant and regular sequences

Let $\mathfrak{a} \subset K[\vec{X}]$ be an arbitrary ideal of dimension $m$ (i.e. the Krull dimension of the ring $A$ equals $m$ ) such that all its isolated primes have dimension $m$ (or, equivalently, the variety $V(\mathfrak{a})$ is equidimensional of dimension $m$ ). Under this condition we say that the ideal $\mathfrak{a}$ is equidimensional of dimension $m$.

Definition 1 Let $\vec{f}:=f_{1}, \ldots, f_{m^{\prime}}, m^{\prime} \leq m$, be a sequence of polynomials of $K[\vec{X}]$. Then $\vec{f}$ is called a secant sequence for $\mathfrak{a}$ of length $m^{\prime}$ if for any index $j, 1 \leq j \leq m^{\prime}$, the dimension of the ideal $\mathfrak{a}+\left(f_{1}, \ldots, f_{j}\right)$ is $m-j$. Moreover, $\vec{f}$ is called a regular sequence of length $m^{\prime}$ with respect to $\mathfrak{a}$ if $\mathfrak{a}+\left(f_{1}, \ldots, f_{m^{\prime}}\right)$ is a proper ideal and if for any index $j$, $1 \leq j \leq m^{\prime}$ the polynomial $f_{j}$ is not a zero-divisor modulo the ideal $\mathfrak{a}+\left(f_{1}, \ldots, f_{j-1}\right)$. If this is the case the residue classes $f_{1}+\mathfrak{a}, \ldots, f_{m^{\prime}}+\mathfrak{a}$ are said to form a regular sequence in $A$.

Remark 2 In the sequel we shall only consider secant sequences of maximal length $m$.
Observe that for a secant sequence $\vec{f}:=f_{1}, \ldots, f_{m}$ for $\mathfrak{a}$, all the isolated components of the ideal $\mathfrak{a}+\left(f_{1}\right)$ have dimension $m-1$ and $f_{2}, \ldots, f_{m}$ is a secant sequence for $\mathfrak{a}+\left(f_{1}\right)$. More generally, for $1 \leq j<m$ the polynomials $f_{j+1}, \ldots, f_{m}$ form a secant sequence for $\mathfrak{a}+\left(f_{1}, \ldots, f_{j}\right)$.

Any regular sequence of maximal length $m$ with respect to $\mathfrak{a}$ constitutes a secant sequence for $\mathfrak{a}$, because any member of the regular sequence drops the Krull dimension by one at each step, up to reach dimension 0 .

Proposition 3 Let $\vec{f}:=f_{1}, \ldots, f_{m}$ be a secant sequence for the equidimensional ideal $\mathfrak{a}$ of dimension $m$. Let us consider the regular morphism:

$$
\begin{align*}
\vec{f}: V(\mathfrak{a}) & \longrightarrow \mathbb{A}^{m} \\
x & \longmapsto \vec{f}(x):=\left(f_{1}(x), \ldots, f_{m}(x)\right) . \tag{1}
\end{align*}
$$

Then $\vec{f}(V(\mathfrak{a}))$ is Zariski dense in $\mathbb{A}^{m}$. Said otherwise, $\vec{f}: V(\mathfrak{a}) \rightarrow \mathbb{A}^{m}$ is dominant.
Proof. The proof is an easy consequence of the Theorem of the Dimension of the Fibers (see for instance [17, $\S 6.3$ Theorem 7]). Since the sequence $\vec{f}$ is secant, the fiber $\vec{f}^{-1}(0)$ is a zero-dimensional algebraic set and, therefore, there exists an irreducible component of $V(\mathfrak{a})$ which intersects this fiber. If $\mathfrak{p}$ is the corresponding minimal associated prime of $\mathfrak{a}$, the Theorem of the Dimension of the Fibers applied to the restriction $\left.\vec{f}\right|_{V(\mathfrak{p})}: V(\mathfrak{p}) \longrightarrow \mathbb{A}^{m}$ implies that $\left.\vec{f}\right|_{V(\mathfrak{p})}$ is dominant. Thus $\vec{f}(V(\mathfrak{p}))$ is Zariski dense in $\mathbb{A}^{m}$ and hence also
$\vec{f}(V(\mathfrak{a}))$.
From Proposition 3 we deduce immediately the following statement:
Corollary 4 With the same notations as in Proposition 圆, the following is a monomorphism of $K$-algebras:

$$
\begin{aligned}
& \vec{f}^{*}: K\left[Y_{1}, \ldots, Y_{m}\right] \longrightarrow A=K[\vec{X}] / \mathfrak{a}, \\
& Y_{i} \longmapsto \\
& f_{i}+\mathfrak{a}, \quad 1 \leq i \leq m .
\end{aligned}
$$

In particular, the residual classes $\left\{f_{1}+\mathfrak{a}, \ldots, f_{m}+\mathfrak{a}\right\}$ are algebraically independent over $K$.

Corollary 5 Let $\vec{f}$ be a secant sequence for the equidimensional ideal $\mathfrak{a}$ of dimension $m$. Then there exists a non-empty Zariski open subset $U$ of $\mathbb{A}^{m}$, such that for all $\vec{a}:=$ $\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in U \cap K^{m}$ the sequence $\vec{f}-\vec{a}:=f_{1}-\alpha_{1}, \ldots, f_{m}-\alpha_{m}$ is a secant sequence for $\mathfrak{a}$.

Proof. Fix an index $j, 1 \leq j \leq m$ and let $\vec{f}_{j}: V(\mathfrak{a}) \rightarrow \mathbb{A}^{j}$ be the polynomial map $\left(f_{1}(x), \ldots, f_{j}(x)\right)$. As in the proof of Proposition 3, we deduce that there exists at least one irreducible component of $V(\mathfrak{a})$ such that the restriction of $\vec{f}_{j}$ is dominant and its typical fiber has dimension $m-j$. On the other hand, if $\mathfrak{p}$ is an isolated associated prime of $\mathfrak{a}$ such that the typical fiber of the restriction of $\vec{f}_{j}$ to $V(\mathfrak{p})$ has not dimension $m-j$, then the Zariski closure $\overrightarrow{\vec{f}_{j}(V(\mathfrak{p}))}$ is properly contained in $\mathbb{A}^{j}$. Therefore, the set of points $\vec{y} \in \mathbb{A}^{j}$ such that the fiber $\vec{f}_{j}^{-1}(\vec{y})$ is $m-j$ dimensional contains a nonempty Zariski open set $U_{j} \subset \mathbb{A}^{j}$. One verifies immediately that $U:=\bigcap_{j=1}^{m}\left(U_{j} \times \mathbb{A}^{m-j}\right) \subset \mathbb{A}^{m}$ satisfies the statement of the corollary.

Remark 6 There exists in $\left(\mathbb{A}^{n}\right)^{m}$ a nonempty Zariski open set $\mathcal{O}$ of linear forms of $K[\vec{X}]$ such that each element $\vec{f}=f_{1}, \ldots, f_{m}$ of $\mathcal{O}$ constitutes a secant family for $\mathfrak{a}$.

Proof. The statement is a consequence of Noether's Normalization Lemma as in 10 , Lemma 1] applied to the equidimensional variety $V(\mathfrak{a})$.

We need in the sequel the following technical lemma concerning Zariski dense subsets:
Lemma 7 Let be given positive integers $n_{1}, \ldots, n_{s}$ and a subset $U$ of $\mathbb{A}^{n_{1}} \times \cdots \times \mathbb{A}^{n_{s}}$. For each $i=1, \ldots, s$ denote by $\pi_{i}$ the canonical projection of the product space onto $\mathbb{A}^{n_{i}}$. Assume that the set $U$ satisfies the following conditions:
i) $\pi_{1}(U)$ is Zariski dense in $\mathbb{A}^{n_{1}}$.
ii) For each $i=1, \ldots, s-1$ and each $a:=\left(a_{1}, \ldots, a_{i}\right) \in\left(\pi_{1} \times \ldots \times \pi_{i}\right)(U)$ the set

$$
\pi_{i+1}\left(\left(\{a\} \times \mathbb{A}^{n_{i+1}} \times \cdots \times \mathbb{A}^{n_{s}}\right) \cap U\right) \subset \mathbb{A}^{n_{i+1}}
$$

is Zariski dense in $\mathbb{A}^{n_{i+1}}$.

Then, the set $U$ is Zariski dense in $\mathbb{A}^{n_{1}} \times \cdots \times \mathbb{A}^{n_{s}}$.
Proof. By induction on $s$. For $s=1$ there is nothing to prove because condition $i$ ) implies already the conclusion (the second condition is vacuous). Suppose now $s>1$. For each $i=1, \ldots, s$ denote by $T_{i}$ the $n_{i}$-tuple of coordinates of $\mathbb{A}^{n_{i}}$ and suppose that a polynomial $F\left(T_{1}, \ldots, T_{s}\right)$ vanishes on $U$. Consider $F$ as polynomial in the variables $T_{s}$ :

$$
F=\sum_{\alpha \in \mathbb{N}_{0}^{n_{s}}} f_{\alpha}\left(T_{1}, \ldots, T_{s-1}\right) T_{s}^{\alpha} .
$$

For an arbitrary point $a:=\left(a_{1}, \ldots, a_{s-1}\right) \in\left(\pi_{1} \times \ldots \times \pi_{s-1}\right)(U)$ the $n_{s}-$ variate polynomial $F\left(a, T_{s}\right)$ vanishes at any point $b \in \mathbb{A}^{n_{s}}$ with $(a, b) \in U$. In other words, the polynomial $F\left(a, T_{s}\right)$ vanishes on the set $\pi_{s}\left(\left(\{a\} \times \mathbb{A}^{n_{s}}\right) \cap U\right)$, which is Zariski dense in $\mathbb{A}^{n_{s}}$ by condition ii) for $i=s-1$. Hence the coefficients $f_{\alpha}(a)$ are zero for all subindexes $\alpha$ and for all $a \in\left(\pi_{1} \times \ldots \times \pi_{s-1}\right)(U)$.

We consider now the set $U^{\prime} \subset \mathbb{A}^{n_{1}} \times \cdots \times \mathbb{A}^{n_{s-1}}$ defined as

$$
U^{\prime}:=\left(\pi_{1} \times \ldots \times \pi_{s-1}\right)(U) .
$$

It is easy to see that $U^{\prime}$ satisfies conditions $i$ ) and $i i$ ). Hence, by induction hypothesis, the set $U^{\prime}$ is Zariski dense and therefore each polynomial $f_{\alpha}$ must be identically zero. Hence $F=0$, which implies that $U$ is Zariski dense in $\mathbb{A}^{n_{1}} \times \cdots \times \mathbb{A}^{n_{s}}$.

Corollary 8 There exists in $\left(\mathbb{A}^{n+1}\right)^{m}$ a nonempty Zariski open set $\mathcal{O}$ of polynomials of degree one such that each element $\vec{f}=f_{1}, \ldots, f_{m}$ of $\mathcal{O}$ constitutes a secant sequence for $\mathfrak{a}$.

Proof. The condition to form a secant sequence is expressible in first order logic and hence constructible. From Corollary 5, Remark 6 and Lemma 7 we deduce that the sequences $\vec{f}=f_{1}, \ldots, f_{m}$ of degree one polynomials which are secant sequences for $\mathfrak{a}$ constitutes a Zariski dense subset of $\left(\mathbb{A}^{n+1}\right)^{m}$. Since this set is also constructible the corollary follows.

We are now going to analyze the ubiquity of regular sequences with respect to an equidimensional polynomial ideal $\mathfrak{a} \subset K[\vec{X}]$ of positive dimension. We start with the following simple observation.

Remark 9 Let $\mathfrak{p}$ be a prime ideal of the polynomial ring $K[\vec{X}]$ of positive dimension. Then the set

$$
T:=\left\{\left(a_{1}, \ldots, a_{n}\right) \in K^{n} \mid \text { there exists } \lambda \in K \text { such that } a_{1} X_{1}+\cdots+a_{n} X_{n}+\lambda \in \mathfrak{p}\right\}
$$

is a proper linear subspace of $K^{n}$.
Proof. The set $T$ is clearly a linear subspace because $\mathfrak{p}$ is an ideal. If $T=K^{n}$ the elements of the canonical basis of $K^{n}$ belong to $T$. Hence, there exist scalars $\lambda_{1}, \ldots, \lambda_{n}$ such that $X_{1}+\lambda_{1}, \ldots, X_{n}+\lambda_{n}$ belong to $\mathfrak{p}$. Thus the ideal $\mathfrak{p}$ is maximal. This contradicts the assumption that $\mathfrak{p}$ is of positive dimension.

From Remark 9 and Lemma 7 we deduce now the following result about the density of degree one regular sequences with respect to an equidimensional polynomial ideal.

Proposition 10 Let $\mathfrak{a} \subset K[\vec{X}]$ be an equidimensional ideal of dimension $m>0$. Then there exists a Zariski dense subset $U$ of $\left(\mathbb{A}^{n+1}\right)^{m}$ such that for all $\left(\overrightarrow{a_{1}}, \ldots, \overrightarrow{a_{m}}\right) \in U$ with $\overrightarrow{a_{i}}:=\left(a_{1 i}, \ldots, a_{n i}, a_{(n+1) i}\right), 1 \leq i \leq m$, the polynomials $\ell_{1}, \ldots, \ell_{m}$ of degree 1 defined by

$$
\ell_{i}:=a_{1 i} X_{1}+\cdots+a_{n i} X_{n}+a_{(n+1) i}
$$

form a regular sequence with respect to the ideal $\mathfrak{a}$.
Proof. We start by the construction of a suitable regular generic polynomial of degree 1. Let $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{t}$ be the associated primes of $\mathfrak{a}$ of positive dimension and let $\mathfrak{p}_{t+1}, \ldots, \mathfrak{p}_{r}$ those associated primes which are maximal ideals. Observe $1 \leq t$ since the ideal $\mathfrak{a}$ is of positive dimension. For each $j=1, \ldots, t$ let $T_{j}$ be the proper linear subspace of $\mathbb{A}^{n}$ associated to $\mathfrak{p}_{j}$ following Remark 6. Thus $U_{1}:=\mathbb{A}^{n} \backslash \bigcup_{j=1}^{t} T_{j}$ is a constructible Zariski dense subset of $\mathbb{A}^{n}$. Observe that for $1 \leq j \leq t$ and any (homogeneous) linear form $\ell$ whose coefficients belong to $U_{1}$, the constructible set $\ell\left(V\left(\mathfrak{p}_{j}\right)\right)$ is Zariski dense in $\mathbb{A}^{1}$. Thus, the intersection $U_{\ell}:=\bigcap_{j=1}^{s}-\ell\left(V\left(\mathfrak{p}_{j}\right)\right)$ is constructible and Zariski dense too. Hence $\mathcal{U}:=\left\{(\ell, u) ; \ell \in U_{1}, u \in U_{\ell}\right\}$ is a constructible subset of $\mathbb{A}^{n+1}$ which is Zariski dense following Lemma 7

Now, for each maximal ideal $\mathfrak{p}_{j}, t<j \leq r$, associated to $\mathfrak{a}$ we consider $W_{j} \subset \mathbb{A}^{n+1}$, the $n$-dimensional linear subspace of the polynomials of degree one contained in $\mathfrak{p}_{j}$. Since $\mathcal{U}$ is Zariski dense and constructible in $\mathbb{A}^{n+1}$, we conclude that

$$
\mathcal{U}_{1}:=\mathcal{U} \backslash \bigcup_{j=t+1}^{r} W_{j}
$$

is constructible and Zariski dense in $\mathbb{A}^{n+1}$.
Now we consider an arbitrary polynomial $\ell_{1}:=a_{1} X_{1}+\cdots a_{n} X_{n}+a_{n+1}$ with $\left(a_{1}, \ldots, a_{n+1}\right) \in$ $\mathcal{U}_{1}$. Then, $\ell_{1} \notin \mathfrak{p}_{j}$ for $t<j \leq r$ since the vector $\left(a_{1}, \ldots, a_{n+1}\right)$ does not belong to $\cup_{j=t+1}^{r} W_{j}$. On the other hand, we have $\ell_{1} \notin \mathfrak{p}_{j}$ for all $1 \leq j \leq t$ because the homogeneous part of $\ell_{1}$ is in $U_{1}$.

In other words, for $\ell_{1} \in \mathcal{U}_{1}$ we infer that $\ell_{1}$ does not belong to any associated prime of $\mathfrak{a}$. Thus $\ell_{1}$ is not a zero divisor mod $\mathfrak{a}$.

In order to prove that $\ell_{1}$ is a regular element it suffices to show that $\ell_{1}$ is not a unity modulo $\mathfrak{a}$. Otherwise $\ell_{1}$ must be also a unity $\bmod \mathfrak{p}_{1}$, i.e. $\ell_{1}(p) \neq 0$ for all $p \in V\left(\mathfrak{p}_{1}\right)$ and then $-a_{n+1}$ is not in the image $\ell\left(V\left(\mathfrak{p}_{1}\right)\right)$ which contradicts the fact that $a_{n+1} \in U_{\ell}$ (recall that $1 \leq t$ holds).

Summarizing, we have shown that there exists a (constructible) Zariski dense subset $\mathcal{U}_{1}$ of $\mathbb{A}^{n+1}$ such that any polynomial of degree one with coefficients in $\mathcal{U}_{1}$ is regular $\bmod$ $\mathfrak{a}$.

Now the corollary follows by an inductive argument based on Lemma 7 take any $\ell_{1}$ with coefficients in $\mathcal{U}_{1}$. Then by Krull's Principal Ideal Theorem, the ideal $\mathfrak{a}_{1}:=\mathfrak{a}+\left(\ell_{1}\right)$ is equidimensional of dimension $m-1$. If $m=1$ there is nothing to prove.

Assume $m>1$. We build a constructible Zariski dense subset $\mathcal{U}_{2, \ell_{1}}$ of $\mathbb{A}^{n+1}$ (depending on $\ell_{1}$ ) such that for any $\ell_{2}$ with coefficients in this set, the pair $\ell_{1}, \ell_{2}$ is a regular sequence of length 2 for $\mathfrak{a}$. By repeating this argument for each $\ell_{1}$, Lemma 7 ensures the existence of a Zariski dense subset $\mathcal{U}_{2}$ of $\mathbb{A}^{n+1} \times \mathbb{A}^{n+1}$ representing the coefficients of regular sequences
of length 2 for $\mathfrak{a}$, formed by polynomials of degree 1 . The corollary follows now inductively after $m$ steps.

## 3 Generic fiber vs. special fibers

Throughout this section let notations and assumptions be the same as in the previous section. Let us now consider a minimal primary decomposition of the equidimensional ideal $\mathfrak{a}$, of dimension $m$, namely

$$
\mathfrak{a}=\mathfrak{q}_{1} \cap \cdots \cap \mathfrak{q}_{r}
$$

where each ideal $\mathfrak{q}_{i}$ is $\mathfrak{p}_{i}$-primary. Assume $\vec{f}$ is a secant sequence with respect to the ideal $\mathfrak{a}$. Let us denote by $K[\vec{f}]$ the image of $K\left[Y_{1}, \ldots, Y_{m}\right]$ in $A=K\left[X_{1}, \ldots, X_{n}\right] / \mathfrak{a}$.

Let us recall the polynomial mapping $\vec{f}: V(\mathfrak{a}) \longrightarrow \mathbb{A}^{m}$ and let $\vec{f}_{i}$ be its restriction to each of the components $V\left(\mathfrak{p}_{i}\right)$ :

$$
\overrightarrow{f_{i}}:=\left.\vec{f}\right|_{V\left(\mathfrak{p}_{i}\right)}: V\left(\mathfrak{p}_{i}\right) \longrightarrow \mathbb{A}^{m}, \text { where } 1 \leq i \leq r
$$

Taking into account Proposition 3 we may assume without loss of generality that there exists an index $1 \leq s \leq r$ such that $\vec{f}_{i}$ is dominant if and only if $1 \leq i \leq s$.

Lemma 11 With these notations the following statements hold:
i) For any $i, s+1 \leq i \leq r$, there exists a non-zero polynomial $Q_{i} \in K\left[Y_{1}, \ldots, Y_{m}\right]$ such that $Q_{i}\left(f_{1}, \ldots, f_{m}\right)$ belongs to $\mathfrak{q}_{i}$.
ii) For any $i, 1 \leq i \leq s$, and for any $j, 1 \leq j \leq n$, there is a polynomial $Q_{i j} \in$ $K\left[Y_{1}, \ldots, Y_{m}\right][T]$, of positive degree in the variable $T$ such that $Q_{i j}\left(f_{1}, \ldots, f_{m}\right)\left(X_{j}\right)$ belongs to $\mathfrak{q}_{i}$.

Proof. For the proof of statement $i$, observe that for any $i, s+1 \leq i \leq r$, the image $\vec{f}\left(V\left(\mathfrak{p}_{i}\right)\right)$ is contained in a proper hypersurface of $\mathbb{A}^{m}$. Taking $G_{i} \in K\left[Y_{1}, \ldots, Y_{m}\right]$ the minimal equation of this hypersurface, we conclude that $G_{i}\left(f_{1}, \ldots, f_{m}\right)$ belongs to $\mathfrak{p}_{i}$. Since the radical of $\mathfrak{q}_{i}$ is $\mathfrak{p}_{i}$, there exists some power $Q_{i}:=G_{i}^{k_{i}}$ such that $Q_{i}\left(f_{1}, \ldots, f_{m}\right)$ belongs to $\mathfrak{q}_{i}$, as wanted.

For the proof of statement $i i)$, let us write by $\mathfrak{p}:=\mathfrak{p}_{i}$ where $1 \leq i \leq s$. As $\vec{f}_{i}(V(\mathfrak{p}))$ is Zariski dense in $\mathbb{A}^{m}$, and $\mathfrak{a}$ is $m$-equidimensional we have a monomorphism of finitely generated $K$-algebras of the same Krull dimension:

$$
\overrightarrow{f_{i}^{*}}: K\left[Y_{1}, \ldots, Y_{m}\right] \hookrightarrow K\left[X_{1}, \ldots, X_{n}\right] / \mathfrak{p}
$$

with $\vec{f}_{i}^{*}\left(Y_{k}\right)=f_{k}+\mathfrak{p}, 1 \leq k \leq m$. Thus we may consider $K[\vec{f}]$ as a subalgebra of $K\left[X_{1}, \ldots, X_{n}\right] / \mathfrak{p}$. Let $S:=K[\vec{f}] \backslash\{0\}$, Then $K(\vec{f}):=S^{-1} K[\vec{f}]$ is the quotient field of $K[\vec{f}]$ and we have an extension of domains of the same Krull dimension:

$$
K(\vec{f}) \hookrightarrow S^{-1}\left(K\left[X_{1}, \ldots, X_{n}\right] / \mathfrak{p}\right)
$$

which is therefore a finite field extension. Then, for any $1 \leq j \leq n$ the residue class $X_{j}+\mathfrak{p}$ is algebraic over $K(\vec{f})$. In particular, there is a polynomial $H_{j} \in K\left[Y_{1}, \ldots, Y_{m}\right][T]$
(depending on $\mathfrak{p}$ ), of positive degree in $T$ such that $H_{j}\left(f_{1}, \ldots, f_{m}\right)\left(X_{j}\right)$ belongs to $\mathfrak{p}$. As $\mathfrak{p}=\mathfrak{p}_{i}$, let us write $G_{i j}:=H_{j}$. Since $\mathfrak{p}_{i}$ is the radical of $\mathfrak{q}_{i}$, there is some power $Q_{i j}:=G_{i j}^{k_{i j}}$ such that $Q_{i j}\left(f_{1}, \ldots, f_{m}\right)\left(X_{j}\right)$ belongs to $\mathfrak{q}_{i}$ as wanted.

For the next statement observe that Corollary 4 implies that for any nonzero polynomial $q \in K\left[Y_{1}, \ldots, Y_{m}\right]$ we have $q(\vec{f})=q\left(f_{1}, \ldots, f_{m}\right) \neq 0$. Thus $K[\vec{f}] \backslash\{0\}$ forms a multiplicative closed set of $A$ which we denote in the sequel by $S(\vec{f})$. Observe that $K(\vec{f}):=S(\vec{f})^{-1} K[\vec{f}]$ is the quotient field of $K[\vec{f}]$.

Proposition 12 There is a nonzero polynomial $q \in K\left[Y_{1}, \ldots, Y_{m}\right]$ such that for $q(\vec{f})=$ $q\left(f_{1}, \ldots, f_{m}\right) \in S(\vec{f})$ the localizations of $K[\vec{f}]$ and $A$ by $q(\vec{f})$ define an integral ring extension:

$$
K[\vec{f}]_{q(\vec{f})} \hookrightarrow A_{q(\vec{f})}
$$

In particular $A_{q(\vec{f})}$ is a finite $K[\vec{f}]_{q(\vec{f})}$-module.
Proof. We use the notations of Lemma 11, Let us consider the polynomial $q \in K\left[Y_{1}, \ldots, Y_{m}\right]$ given by

$$
q:=\left(\prod_{i=1}^{s} \prod_{j=1}^{n} a_{i j}\left(Y_{1}, \ldots, Y_{m}\right)\right) \times\left(\prod_{i=s+1}^{r} Q_{i}\left(Y_{1}, \ldots, Y_{m}\right)\right) \in K\left[Y_{1}, \ldots, Y_{m}\right]
$$

where for $1 \leq i \leq s$ and $1 \leq j \leq n$ the polynomial $a_{i j}$ is the non-zero leading coefficient of the polynomial $Q_{i j} \in K\left[Y_{1}, \ldots, Y_{m}\right][T]$ which satisfies the condition $a_{i j}\left(f_{1}, \ldots, f_{m}\right) \notin \mathfrak{p}_{i}$ (see the proof of Lemma 11). As $q$ is a non-zero polynomial we have $q(\vec{f})=q\left(f_{1}, \ldots, f_{m}\right) \notin$ $\mathfrak{a}$. Thus we obtain a $K$-algebra extension

$$
K[\vec{f}]_{q(\vec{f})} \hookrightarrow A_{q(\vec{f})} .
$$

As $q(\vec{f}) \in \mathfrak{q}_{i}$ for $s+1 \leq i \leq r$, the minimal primary decomposition of the ideal (0) in $A_{q(\vec{f})}$ is given by:

$$
(0)=\mathfrak{q}_{1}^{e} \cap \cdots \cap \mathfrak{q}_{s}^{e}
$$

where $\mathfrak{q}_{i}^{e}$ is the extension of $\mathfrak{q}_{i}$ to $A_{q(\vec{f})}$. In particular, the ring extension above is integral since for every $1 \leq j \leq n$ the residue class $X_{j}+\mathfrak{a}$ satisfies the algebraic dependence equation given by $\prod_{i=1}^{s} Q_{i j}$ described in Lemma 11, In particular, as $A_{q(\vec{f})}$ is a finitely generated $K[\vec{f}]_{q(\vec{f})}$-algebra and as every residue class $X_{j}+\mathfrak{a}, 1 \leq j \leq n$ is integral over $K[\vec{f}]_{q(\vec{f})}$, the algebra $A_{q(\vec{f})}$ is a finite $K[\vec{f}]_{q(\vec{f})}$-module.

We deduce from Proposition 12 the following statement.
Corollary 13 The localized ring $S(\vec{f})^{-1} A$ is a $K(\vec{f})$-algebra of finite dimension.

Proposition 14 There is a nonzero polynomial $p \in K\left[Y_{1}, \ldots, Y_{m}\right]$ such that $p(f):=$ $p\left(f_{1}, \ldots, f_{m}\right) \in S(\vec{f})$ has the following property. The localizations by $p(f)$ define an integral ring extension:

$$
K[\vec{f}]_{p(\vec{f})} \hookrightarrow A_{p(\vec{f})}
$$

and $A_{p(\vec{f})}$ is a free $K[\vec{f}]_{p(\vec{f})}$-module of finite rank. Moreover, its rank satisfies the condition

$$
\operatorname{rank}_{K[\vec{f}]_{p(\vec{f})}}\left(A_{p(\vec{f})}\right)=\operatorname{dim}_{K(\vec{f})} S(\vec{f})^{-1} A .
$$

Proof. Let $q \in K\left[Y_{1}, \ldots, Y_{m}\right]$ be the non-zero polynomial of Proposition 12. Then, we have

- a finite subset $\beta$ of $A$, such that $\beta$ is a basis of $S(\vec{f})^{-1} A$ as $K(\vec{f})$-vector space.
- a finite subset $\mathscr{M}$ of $A$, such that $\mathscr{M}$ is a system of generators of the $K[\vec{f}]_{q(\vec{f})}$-module $A_{q(\vec{f})}$.

As $\beta$ is a basis of $S(\vec{f})^{-1} A$ as $K(\vec{f})$-vector space, there is some nonzero polynomial $h \in K\left[Y_{1}, \ldots, Y_{m}\right]$ such that $h(\vec{f}):=h\left(f_{1}, \ldots, f_{m}\right) \neq 0$ in $A$ and such that all elements in $\mathscr{M}$ are linear combinations of the elements in the basis $\beta$ with coefficients in $K[\vec{f}]_{h(\vec{f})}$. Then, $p:=q h \in K\left[Y_{1}, \ldots, Y_{m}\right]$ is a non-zero polynomial such that

$$
K[\vec{f}]_{p(\vec{f})} \hookrightarrow A_{p(\vec{f})}
$$

is an integral ring extension and such that $A_{p(\vec{f})}$ is a free $K[\vec{f}]_{p(\vec{f})}$-module of finite rank. As rank is stable under localizations, we conclude

$$
\operatorname{rank}_{K[\vec{f}]_{p(\vec{f})}}\left(A_{p(\vec{f})}\right)=\operatorname{dim}_{K(\vec{f})} S^{-1} A .
$$

Lemma 15 Suppose that $\vec{f}$ is a regular sequence. Then the inequality $\operatorname{dim}_{K} A /(\vec{f}) \leq$ $\operatorname{dim}_{K(\vec{f})} S(\vec{f})^{-1} A$ holds.

Proof. By induction on the Krull dimension $m$ of $A$.
If $m=0$ there is nothing to prove. Suppose $m \geq 1$ and denote by $B:=A /\left(f_{1}\right)$. Since $\vec{f}$ is a secant sequence, the ring $B$ has Krull dimension $m-1$ and $\vec{f}^{\prime}:=f_{2}, \ldots, f_{m}$ is a regular sequence for the ideal $\mathfrak{a}+\left(f_{1}\right)$. Therefore, if $T$ denotes the multiplicative subset $K\left[\vec{f}^{\prime}\right] \backslash\{0\}$ of $B$ and $L^{\prime}:=K\left(\overrightarrow{f^{\prime}}\right)$ we have by induction hypothesis the inequality $\operatorname{dim}_{K} B /\left(\overrightarrow{f^{\prime}}\right) \leq \operatorname{dim}_{L^{\prime}} T^{-1} B$.

Since $B /\left(\overrightarrow{f^{\prime}}\right)=A /(\vec{f})$ it suffices to show the inequality $\operatorname{dim}_{L^{\prime}} T^{-1} B \leq \operatorname{dim}_{K(\vec{f})} S^{-1} A$.
Let $g_{1}, \ldots, g_{s}$ be polynomials in $K[\vec{X}]$ whose classes form a basis of $T^{-1} B$ over $L^{\prime}$. We consider the images of $g_{1}, \ldots, g_{s}$ in the ring $S(\vec{f})^{-1} A$. We show that these elements are linearly independent over $K(\vec{f})$. Suppose on the contrary that they are $K(\vec{f})$-linearly dependent. Cleaning denominators we may suppose that there exists a non trivial linear combination $\sum_{i=1}^{s} p_{i} g_{i}$ which belongs to the ideal $\mathfrak{a}$ where the $p_{i}$ 's are polynomials in
$K[\vec{f}]$. Since the polynomials $f_{1}, \ldots, f_{m}$ are algebraically independent modulo $\mathfrak{a}, f_{1} \in K[\vec{f}]$ may be viewed as an irreducible element in a factorial domain. Moreover, since $f_{1}$ is not zero-divisor modulo $\mathfrak{a}$ we may suppose without loss of generality that $f_{1}$ is not a common factor of all polynomials $p_{1}, \ldots, p_{s}$. Taking the class of each polynomial $p_{i}$ modulo the ideal $\left(f_{1}\right) \subset K[\vec{f}]$ we conclude that the polynomials $g_{i}$ are not linearly independent in $T^{-1} B$, contradiction.

Lemma 15 fails for secant sequences:
Example 16 Consider the one dimensional ideal $\mathfrak{a}=\left(X_{1}^{3}, X_{1}^{2} X_{2}\right) \subset K\left[X_{1}, X_{2}\right]$ whose primary decomposition is $\mathfrak{a}=\left(X_{1}^{2}\right) \cap\left(X_{1}^{3}, X_{2}\right)$. The polynomial $X_{2}$ is a secant sequence for $\mathfrak{a}$ and $A /\left(X_{2}\right)=K\left[X_{1}, X_{2}\right] /\left(X_{1}^{3}, X_{2}\right) \simeq K\left[X_{1}\right] /\left(X_{1}^{3}\right)$. Therefore $\operatorname{dim}_{K} A /\left(X_{2}\right)=3$.
On the other hand, if $S:=K\left[X_{2}\right] \backslash\{0\}$ and $L:=K\left(X_{2}\right)$, we have

$$
S^{-1} A \simeq S^{-1} K\left[X_{1}, X_{2}\right] /\left(X_{1}^{3}, X_{1}^{2}\right) \simeq K\left(X_{2}\right)\left[X_{1}\right] /\left(X_{1}^{2}\right)
$$

(since $X_{2}$ is invertible in $S^{-1} A$ ). But then $\operatorname{dim}_{L} S^{-1} A=2$.
Observe that $X_{2}$ is a secant family for $\mathfrak{a}$ but it is a zero divisor in $A$ since $X_{2} X_{1}^{2} \in \mathfrak{a}$ and $X_{1}^{2} \notin \mathfrak{a}$.

Typically we have equality in Lemma 15
Proposition 17 Let $\vec{f}$ be a secant sequence for $\mathfrak{a}$, as before. Then there exists a nonempty Zariski open subset $\mathcal{O}$ of $\mathbb{A}^{m}$ such that for any $\vec{a} \in \mathcal{O}$ the equality $\operatorname{dim}_{K} A /(\vec{f}-\vec{a})=$ $\operatorname{dim}_{K(\vec{f})} S(\vec{f})^{-1} A$ holds.

Proof. By Proposition 14, there exists a non-zero polynomial $p \in K\left[Y_{1}, \ldots, Y_{m}\right]$ such that the localizations by $p(\vec{f})$ define an integral ring extension

$$
K[\vec{f}]_{p(\vec{f})} \hookrightarrow A_{p(\vec{f})}
$$

and $A_{p(\vec{f})}$ is a free $K[\vec{f}]_{p(\vec{f})}$-module of finite rank

$$
N:=\operatorname{rank}_{K[\vec{f}]_{p(\vec{f})}}\left(A_{p(\vec{f})}\right)=\operatorname{dim}_{K(\vec{f})} S(\vec{f})^{-1} A
$$

Let $H:=\{p \neq 0\} \subset \mathbb{A}^{m}$ and let $U$ be the nonempty Zariski open subset of $\mathbb{A}^{m}$ of Corollary 5. Then, $\mathcal{O}:=H \cap U$ is also a non-empty Zariski open subset of $\mathbb{A}^{m}$. Let $\vec{a} \in \mathcal{O}$ be a point in this open set and let us denote by $\mathfrak{m}_{\vec{a}}$ the ideal in $K[\vec{f}]$ generated by the sequence $f_{1}-a_{1}, \ldots, f_{m}-a_{m}$ in $K[\vec{f}]$. As $K[\vec{f}]$ is a polynomial ring, $\mathfrak{m}_{\vec{a}}$ is a maximal ideal in $K[\vec{f}]$. Let us denote by $\mathfrak{m}_{\vec{a}} A$ its extension to $A$. Moreover, as $p(\vec{a}) \neq 0$, we have $p(\vec{f}) \notin \mathfrak{m}_{\vec{a}}$. In particular, the extension $\mathfrak{m}_{\vec{a}}^{(1)}:=\mathfrak{m}_{\vec{a}} K[\vec{f}]_{p(\vec{f})}$ is also a maximal ideal in $K[\vec{f}]_{p(\vec{f})}$. Let us consider the submodule $\mathfrak{m}_{\vec{a}}^{(1)} A_{p(\vec{f})}$ of $A_{p(\vec{f})}$ as $K[\vec{f}]_{p(\vec{f})}$-module. Note that $\left(A / \mathfrak{m}_{\vec{a}} A\right)_{p(\vec{f})}$ is isomorphic as $K[\vec{f}]_{p(\vec{f})}$-module to

$$
\left(K[\vec{f}]_{p(\vec{f})}\right)^{N} / \mathfrak{m}_{\vec{a}}^{(1)}\left(K[\vec{f}]_{p(\vec{f})}\right)^{N}
$$

Thus, we conclude that as $K[\vec{f}]_{p(\vec{f})}$-module, $\left(A / \mathfrak{m}_{\vec{a}} A\right)_{p(\vec{f})}$ is isomorphic to

$$
\left(K[\vec{f}]_{p(\vec{f})} / \mathfrak{m}_{\vec{a}}^{(1)}\right)^{N} \cong\left(K[\vec{f}] / \mathfrak{m}_{\vec{a}}\right)^{N} \cong K^{N} .
$$

Hence $\operatorname{dim}_{K} A /(\vec{f}-\vec{a})=N=\operatorname{dim}_{K(\vec{f})} S(\vec{f})^{-1} A$ for all $\vec{a} \in \mathcal{O}$.
Corollary 18 There exists a Zariski dense subset $\mathcal{U}$ of $\left(\mathbb{A}^{n+1}\right)^{m}$ such that for any sequence $\vec{f}=f_{1}, \ldots, f_{m}$ of degree one polynomials of $\mathcal{U}$ the sequence $\vec{f}$ is secant and the equality $\operatorname{dim}_{K(\vec{f})} S(\vec{f})^{-1} A=\operatorname{dim}_{K} A /(\vec{f})$ holds.

Proof. Combine Corollary 8 with Proposition 17 and Lemma 7
Let $1 \leq q \leq r$ be the number of isolated primes of the ideal $\mathfrak{a}$. Without loss of generality we may assume that these are $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{q}$. Then for each $1 \leq j \leq q$ the ring $A_{\mathfrak{p}_{j}}$ is local and Artinian with maximal ideal $\left(\mathfrak{p}_{j} / \mathfrak{a}\right)_{\mathfrak{p}_{j}}$. In the sequel we denote by $\ell_{j}$ the length of $A_{\mathfrak{p}_{j}}$.

As in Lemma 11 we may assume without loss of generality that there exists an index $1 \leq s \leq q$ such that $\overrightarrow{f_{i}}=\left.\vec{f}\right|_{V\left(\mathfrak{p}_{i}\right)}: V\left(\mathfrak{p}_{i}\right) \rightarrow \mathbb{A}^{m}$ is dominant if and only if $1 \leq i \leq s$.

We say that the ring extension $K(\vec{f}) \subseteq S(\vec{f})^{-1} A$ is separable if for every $1 \leq i \leq s$ the field extension $K(\vec{f}) \subseteq S(\vec{f})^{-1} A /\left(\mathfrak{p}_{i} / \mathfrak{a}\right)$ is separable (compare Corollary (13)).

With these notions and notations we may formulate the following result.

## Proposition 19

i) Assume that the ring extension $K(\vec{f}) \subseteq S(\vec{f})^{-1} A$ is separable. Then we have

$$
\begin{aligned}
& \operatorname{dim}_{K(\vec{f})} S(\vec{f})^{-1} A \leq\left(\sum_{i=1}^{s} \operatorname{deg} V\left(\mathfrak{p}_{i}\right) \ell_{i}\right) \prod_{k=1}^{m} \operatorname{deg}\left(f_{k}\right) \leq \\
& \leq\left(\sum_{j=1}^{q} \operatorname{deg} V\left(\mathfrak{p}_{j}\right) \ell_{j}\right) \prod_{k=1}^{m} \operatorname{deg}\left(f_{k}\right) .
\end{aligned}
$$

ii) There exists a nonempty Zariski open subset $\mathcal{O}$ of $\left(\mathbb{A}^{n+1}\right)^{m}$ such that for any sequence $\vec{f}=f_{1}, \ldots, f_{m}$ of degree one polynomials of $\mathcal{O}$, the sequence $\vec{f}$ is secant for $\mathfrak{a}$ and such that

$$
\operatorname{dim}_{K(\vec{f})} S(\vec{f})^{-1} A=\sum_{j=1}^{q} \operatorname{deg} V\left(\mathfrak{p}_{j}\right) \ell_{j}
$$

holds.
Proof. We are going to show statement $i$ ). Let us abbreviate $L:=K(\vec{f}), S:=S(\vec{f})$ and $R:=S(\vec{f})^{-1} A=S^{-1} A$.

By virtue of Lemma $11 i)$ the localizations $\mathfrak{m}_{i}:=S^{-1}\left(\mathfrak{p}_{i} / \mathfrak{a}\right), 1 \leq i \leq s$, are exactly the maximal ideals of $R$. By Corollary 13 the $L$-algebra $R$ is finite dimensional and
therefore Artinian. From the Chinese Remainder Theorem we deduce now $R=\bigoplus_{i=1}^{s} R_{\mathfrak{m}_{i}}$. Let $1 \leq i \leq s$ and observe that the rings $R_{\mathfrak{m}_{i}}$ and $A_{\mathfrak{p}_{i}}$ are isomorphic.

Therefore $\ell_{i}$ is also the length of the Artinian local ring $R_{\mathfrak{m}_{i}}$. This implies $\operatorname{dim}_{L} R_{m_{i}}=$ $\left[R / \mathfrak{m}_{i}: L\right] \ell_{i}$.

Putting all this together we obtain

$$
\begin{equation*}
\operatorname{dim}_{L} S^{-1} A=\operatorname{dim}_{L} R=\sum_{i=1}^{s}\left[R / \mathfrak{m}_{i}: L\right] \ell_{i} \tag{2}
\end{equation*}
$$

Fix again $1 \leq i \leq s$. We are now going to analyze in geometric terms the quantity $\left[R / \mathfrak{m}_{i}: L\right]$.

Observe that the dominating morphism $\vec{f}_{i}=\left.\vec{f}\right|_{V\left(\mathfrak{p}_{i}\right)}: V\left(\mathfrak{p}_{i}\right) \rightarrow \mathbb{A}^{m}$ induces canonical field isomorphisms

$$
R / \mathfrak{m}_{i} \cong S^{-1}\left(A /\left(\mathfrak{p}_{\mathfrak{i}} / \mathfrak{a}\right)\right) \cong K\left(V\left(\mathfrak{p}_{i}\right)\right)
$$

where $K\left(V\left(\mathfrak{p}_{i}\right)\right)$ denotes the fraction field of $V\left(\mathfrak{p}_{i}\right)$. Since by assumption the field extension $K(\vec{f}) \subset S^{-1}\left(A /\left(\mathfrak{p}_{\mathfrak{i}} / \mathfrak{a}\right)\right)$ is separable, we see that $\overrightarrow{f_{i}}$ is generically unramified. In particular, we are in conditions to apply [17, Theorem 4, Chapter II, §6.3] to conclude that there exists a Zariski open subset $W$ of $\mathbb{A}^{m}$ such that for any point $\vec{a} \in W$ the fiber $\vec{f}_{i}^{-1}(\vec{a})$ is unramified and of cardinality $\left[R / \mathfrak{m}_{i}: L\right]$.

Choose $\vec{a} \in W$. From the Bézout Inequality [10] we deduce now

$$
\left[R / \mathfrak{m}_{i}: L\right]=\sharp \vec{f}_{i}^{-1}(\vec{a}) \leq \operatorname{deg} V\left(\mathfrak{p}_{i}\right) \prod_{k=1}^{m} \operatorname{deg}\left(f_{k}\right) .
$$

From (2) we infer finally the statement $i$ ) of the proposition, namely

$$
\begin{gathered}
\operatorname{dim}_{K(\vec{f})} S(\vec{f})^{-1} A=\operatorname{dim}_{L} S^{-1} A=\sum_{i=1}^{s} \sharp \vec{f}_{i}^{-1}(\vec{a}) \ell_{i} \leq \\
\leq\left(\sum_{i=1}^{s} \operatorname{deg} V\left(\mathfrak{p}_{i}\right) \ell_{i}\right) \prod_{k=1}^{m} \operatorname{deg}\left(f_{k}\right) \leq\left(\sum_{j=1}^{q} \operatorname{deg} V\left(\mathfrak{p}_{j}\right) \ell_{j}\right) \prod_{k=1}^{m} \operatorname{deg}\left(f_{k}\right)
\end{gathered}
$$

We are now going to prove statement $i i$ ) of the proposition.
Following Corollary 8 and [10, Lemma 1] we may choose a nonempty Zariski open subset of $\left(\mathbb{A}^{n+1}\right)^{m}$ such that for any sequence $\vec{f}=f_{1}, \ldots, f_{m}$ of degree one polynomials of $\mathcal{O}$ the following conditions are satisfied:

- $\vec{f}$ is a secant sequence for $\mathfrak{a}$;
- for $1 \leq j \leq q$ the morphisms $\vec{f}_{j}:=\left.\vec{f}\right|_{V\left(\mathfrak{p}_{j}\right)}: V\left(\mathfrak{p}_{j}\right) \rightarrow \mathbb{A}^{m}$ are dominant, generically unramified and of degree $\operatorname{deg} V\left(\mathfrak{p}_{j}\right)$.

Thus, in particular, any sequence $\vec{f}=f_{1}, \ldots, f_{m}$ of $\mathcal{O}$ fulfills the assumption of the statement $i$ ) of the proposition.

From the proof of statement $i$ ) we deduce:

$$
\operatorname{dim}_{K(\vec{f})} S(\vec{f})^{-1} A=\sum_{j=1}^{q} \sharp \vec{f}_{j}^{-1}(\vec{a}) \ell_{j}
$$

for a suitable, generically chosen point $\vec{a} \in \mathbb{A}^{m}$. Since $f_{j}$ is generically unramified of degree $\operatorname{deg} V\left(\mathfrak{p}_{j}\right)$ we may choose $\vec{a}$ such that $\sharp \vec{f}_{j}^{-1}(\vec{a})=\operatorname{deg} V\left(\mathfrak{p}_{j}\right)$ holds for $1 \leq j \leq q$. This implies the statement $i i)$ of the proposition, namely $\operatorname{dim}_{K(\vec{f})} S(\vec{f})^{-1} A=\sum_{j=1}^{q} \operatorname{deg} V\left(\mathfrak{p}_{j}\right) \ell_{j}$.

## 4 Notion of the degree of an ideal of non-homogeneous polynomials

As at the end of the last section let be given an equidimensional ideal $\mathfrak{a}$ of $K[\vec{X}]$ of dimension $m$ with isolated primes $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{q}$. Let $A:=K[\vec{X}] / \mathfrak{a}$, recall that for $1 \leq j \leq q$ the ring $A_{\mathfrak{p}_{j}}$ is local and Artinian and let $\ell_{j}$ be the length of $A_{\mathfrak{p}_{j}}$.

We shall need the following technical result.
Lemma 20 There exists a nonempty Zariski open subset $\mathcal{U}$ of $\left(\mathbb{A}^{n+1}\right)^{m}$ such that for any sequence $\vec{f}=f_{1}, \ldots, f_{m}$ of degree one polynomials of $\mathcal{U}$, the sequence $\vec{f}$ is secant for $\mathfrak{a}$ and $\operatorname{dim}_{K} A /(\vec{f})$ is constant, independently from $\vec{f}$.

Proof. Let $T_{i j}, 1 \leq i \leq m, 0 \leq j \leq n$ be new indeterminates, $\vec{T}=\left(T_{i j}\right)_{\substack{1 \leq i \leq m \\ 0 \leq j \leq n}}$ and let $F_{i}:=\sum_{j=1}^{m} T_{i j} X_{j}+T_{i 0}, 1 \leq i \leq m$. Fix any monomial order of $\vec{X}$ and let $g_{1}, \ldots g_{s} \in K[\vec{X}]$ be a set of generators of $\mathfrak{a}$. Observe that the ideal $\left(g_{1}, \ldots g_{s}, F_{1}, \ldots, F_{m}\right)$ of $K(\vec{T})[\vec{X}]$ is zero-dimensional.

Consider an arbitrary Gröbner basis computation $\beta$ of this ideal.
The leading coefficients occurring in $\beta$ form a finite set of nonzero rational functions of $K(\vec{T})$. Hence, there exists a nonempty Zariski open subset $\mathcal{U}$ of $\mathbb{A}^{(n+1) m}$ where none of the numerators and denominators of these rational functions vanishes.

Let $\vec{f}=f_{1}, \ldots, f_{m}$ be a sequence of degree one polynomials of $\mathcal{U}$. Then $\beta$ may be specialized to a Gröbner basis computation of $\left(g_{1}, \ldots g_{s}, f_{1}, \ldots, f_{m}\right)$ in $K[\vec{X}]$ which yields the stair of $\beta$.

In view of Corollary图we may assume without loss of generality that for every sequence $\vec{f}=f_{1}, \ldots, f_{m}$ of degree one polynomials of $\mathcal{U}$ the sequence $\vec{f}$ is secant for $\mathfrak{a}$.

Therefore everything is well defined and $\operatorname{dim}_{K} A /(\vec{f})$ is finite and constant on $\mathcal{U}$.
Theorem 21 There exists a nonempty Zariski open subset $\mathcal{O}$ of $\left(\mathbb{A}^{n+1}\right)^{m}$ such that for any sequence $\vec{f}=f_{1}, \ldots, f_{m}$ of degree one polynomials of $\mathcal{O}$ the sequence $\vec{f}$ is secant for $\mathfrak{a}$ and $\operatorname{dim}_{K} A /(\vec{f})=\sum_{j=1}^{q} \operatorname{deg} V\left(\mathfrak{p}_{j}\right) \ell_{j}$ holds.

Proof. Combining Proposition $19 i^{\text {) }}$ with Lemma 20 we find a nonempty Zariski open subset $\mathcal{O}$ of $\left(\mathbb{A}^{n+1}\right)^{m}$ such that for any sequence $\vec{f}=f_{1}, \ldots, f_{m}$ of degree one polynomials of $\mathcal{O}$ the sequence $\vec{f}$ is secant for $\mathfrak{a}$ such that $\operatorname{dim}_{K} A /(\vec{f})$ is constant and $\operatorname{dim}_{K(\vec{f})} S^{-1}(\vec{f}) A=$ $\sum_{j=1}^{q} \operatorname{deg} V\left(\mathfrak{p}_{j}\right) \ell_{j}$ holds.

From Corollary 18 we conclude that there exists a sequence $\overrightarrow{f_{0}}=f_{1}^{0}, \ldots, f_{m}^{0}$ of degree one polynomials of $\mathcal{O}$ satisfying the equality

$$
\operatorname{dim}_{K\left(\vec{f}_{0}\right)} S^{-1}\left(\vec{f}_{0}\right) A=\operatorname{dim}_{K} A /\left(\vec{f}_{0}\right)
$$

This implies

$$
\operatorname{dim}_{K} A /\left(\vec{f}_{0}\right)=\sum_{j=1}^{q} \operatorname{deg} V\left(\mathfrak{p}_{j}\right) \ell_{j}
$$

Hence, for an arbitrary sequence $\vec{f}=f_{1}, \ldots, f_{m}$ of degree one polynomials belonging to $\mathcal{O}$ the sequence $\vec{f}$ is secant for $\mathfrak{a}$ and it holds

$$
\operatorname{dim}_{K} A /(\vec{f})=\operatorname{dim}_{K} A /\left(\vec{f}_{0}\right)=\sum_{j=1}^{q} \operatorname{deg} V\left(\mathfrak{p}_{j}\right) \ell_{j}
$$

We may simplify the somewhat complicated formulation of Theorem 21 saying that a generic sequence $\vec{f}=f_{1}, \ldots, f_{m}$ of degree one polynomials is secant for $\mathfrak{a}$ and $\operatorname{dim}_{K} A /(\vec{f})=$ $\sum_{j=1}^{q} \operatorname{deg} V\left(\mathfrak{p}_{j}\right) \ell_{j}$ holds. In this sense the word generic refers always to the existence of a nonempty Zariski open set which not always is made explicit.

Using this terminology we may define the degree of the equidimensional ideal $\mathfrak{a}$ in two different ways as follows.

Definition 22 The degree $\operatorname{deg}(\mathfrak{a})$ of the equidimensional ideal $\mathfrak{a}$ may be equivalently defined as
i) $\operatorname{deg}(\mathfrak{a}):=\operatorname{dim}_{K} A /(\vec{f})$ for a generic sequence $\vec{f}=f_{1}, \ldots, f_{m}$ of degree one polynomials of $K[\vec{X}]$
or
ii) $\operatorname{deg}(\mathfrak{a}):=\sum_{j=1}^{q} \operatorname{deg} V\left(\mathfrak{p}_{j}\right) \ell_{j}$, where $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{q}$ are the isolated primes of $\mathfrak{a}$ and $\ell_{1}, \ldots, \ell_{q}$ are the lengths of the Artinian local rings $A_{\mathfrak{p}_{1}}, \ldots, A_{\mathfrak{p}_{q}}$.

The formulation ii) for the degree of $\mathfrak{a}$ was introduced in [2] for homogeneous ideals whereas the formulation i) seems new for non-homogeneous ideals and represents our "workable" notion of degree (see Section (6)).

The next statement is a straightforward consequence of this definition:
Proposition 23 Let $\mathfrak{a}$ and $\mathfrak{b}$ two equidimensional ideals in the ring $K[\vec{X}]$ of the same dimension $m$. If $\mathfrak{a} \subseteq \mathfrak{b}$ then $\operatorname{deg}(\mathfrak{b}) \leq \operatorname{deg}(\mathfrak{a})$.

Proof. Since the ideals have the same dimension we can take the same generic degree one polynomials $\vec{f}$ for both ideals. Thus $\mathfrak{a}+(\vec{f}) \subseteq \mathfrak{b}+(\vec{f})$ and the proposition follows.

If the ideal $\mathfrak{a}$ is generated by a single polynomial its degree agrees with the total degree of the polynomial which generates it.

Proposition 24 Let $\mathfrak{a}$ be the ideal generated by a non constant polynomial $g$. The $\operatorname{deg}(\mathfrak{a})=\operatorname{deg}(g)$.

Proof. It suffices to observe that after a generic linear change of coordinates, the degree of the polynomial $g$ does not change if $n-1$ variables are generically specialized and it agrees with the total degree of $g$.

In the case of general polynomial ideals, as customary, we extend our notion of degree as follows.

Definition 25 Let $I \subset K[\vec{X}]$ be an arbitrary proper polynomial ideal with isolated primary components $\mathfrak{q}_{1}, \ldots, \mathfrak{q}_{t}$. We define:

$$
\operatorname{deg}(I):=\sum_{h=1}^{t} \operatorname{deg}\left(\mathfrak{q}_{h}\right) .
$$

Remark 26 For radical polynomial ideals Definitions 22 and 25 coincide with the usual notions of geometric degree of (equidimensional or arbitrary) algebraic closed subvarieties of affine spaces following [10].

### 4.1 On the Bézout Inequality

In view of the Bézout Inequality [10, 6, 18] for affine varieties one might expect that for arbitrary ideals $I, J \subset K[\vec{X}]$ the following estimation holds:

$$
\operatorname{deg}(I+J) \leq \operatorname{deg}(I) \cdot \operatorname{deg}(J)
$$

That this may become wrong shows the following example.
Example 27 Consider the one-dimensional ideal $I=\left(X_{1}^{3}, X_{1}^{2} X_{2}\right) \subset K\left[X_{1}, X_{2}\right]$ whose primary decomposition is $I=\left(X_{1}^{2}\right) \cap\left(X_{1}^{3}, X_{2}\right)$. The primary isolated component is $\left(X_{1}^{2}\right)$ while $\left(X_{1}^{3}, X_{2}\right)$ is the embedded one. By Definition 22 we have $\operatorname{deg}(I)=\operatorname{deg}\left(\left(X_{1}^{2}\right)\right)=2$, since for any generic linear polynomial $f_{1}:=a X_{1}+b X_{2}+c$ the ring $K[\vec{X}] /\left(X_{1}^{2}, f_{1}\right)$ has the $K$-basis $\left\{1, X_{1}\right\}$.
Take the ideal $J:=\left(X_{2}\right) \subset K[\vec{X}]$, which has degree one, and consider the degree of the sum $I+J$. We have

$$
I+J=\left(X_{1}^{3}, X_{1}^{2} X_{2}, X_{2}\right)=\left(X_{1}^{3}, X_{2}\right)
$$

which is a 0 -dimensional $\left(X_{1}, X_{2}\right)$-primary ideal. Clearly $\operatorname{deg}(I+J)=3$.
But on the other hand we have $\operatorname{deg}(I) \cdot \operatorname{deg}(J)=2.1=2<3$.
The following example illustrates that there is no chance to obtain an intrinsic Bézout Inequality (which depends only on the degrees of the ideals but not their generators).

Example 28 Let $k \in \mathbb{N}$ and $I=\left(X_{1}^{k}, X_{1} X_{2}\right) \subset K\left[X_{1}, X_{2}\right]$. It is easy to see that the primary decomposition of $I$ is: $I=\left(X_{1}\right) \cap\left(X_{1}^{k}, X_{2}\right)$ and then $\operatorname{deg}(I)=\operatorname{deg}\left(\left(X_{1}\right)\right)=1$. By adding the ideal ( $X_{2}$ ) we have

$$
\operatorname{deg}\left(I+\left(X_{2}\right)\right)=\operatorname{deg}\left(X_{1}^{k}, X_{1} X_{2}, X_{2}\right)=\operatorname{deg}\left(X_{1}^{k}, X_{2}\right)=
$$

$$
=\operatorname{length}\left(K\left[X_{1}, X_{2}\right] /\left(X_{1}^{k}, X_{2}\right)\right)=k .
$$

Observe that the degree of the sum of the ideals depends on the degree of the generators of $I$ but not on the degree of $I$.

Nevertheless, Proposition 19 implies the following Bézout-type Inequality for equidimensional ideals.

Theorem 29 Let $K$ be of characteristic zero and let $\mathfrak{a} \subset K[\vec{X}]$ be an equidimensional ideal of dimension $m>0$. Let $f_{1}, \ldots, f_{k}$ be a regular sequence, not necessarily maximal, for $\mathfrak{a}$. Then the inequality

$$
\operatorname{deg}\left(\mathfrak{a}+\left(f_{1}, \ldots, f_{k}\right)\right) \leq \operatorname{deg}(\mathfrak{a}) \prod_{i=1}^{k} \operatorname{deg}\left(f_{i}\right)
$$

holds.
Proof. Since $\mathfrak{a}$ is assumed equidimensional and $f_{1}, \ldots, f_{k}$ is a regular sequence, Krull's Principal Ideal Theorem implies that ideal $\mathfrak{b}:=\mathfrak{a}+\left(f_{1}, \ldots, f_{k}\right)$ is also equidimensional. Combining Proposition 10 with Theorem 21 we see that there exists a regular sequence $f_{k+1}, \ldots, f_{m}$ of degree one polynomials such that $\operatorname{deg}(\mathfrak{b})=\operatorname{dim}_{K} K[\vec{X}] / \mathfrak{b}+\left(f_{k+1}, \ldots, f_{m}\right)$ holds.

Let $S:=K\left[f_{1}, \ldots, f_{k}, f_{k+1}, \ldots, f_{m}\right] \backslash\{0\}$ and $L:=K\left(f_{1}, \ldots, f_{k}, f_{k+1}, \ldots, f_{m}\right)$, then Lemma 15 states the inequality

$$
\operatorname{deg}(\mathfrak{b})=\operatorname{dim}_{K} A /\left(f_{1}, \ldots, f_{k}, f_{k+1}, \ldots, f_{m}\right) \leq \operatorname{dim}_{L} S^{-1}(A) .
$$

On the other hand, taking into account that the characteristic of $K$ is zero, from Proposition (19 i) we deduce

$$
\operatorname{dim}_{L} S^{-1}(A) \leq\left(\sum_{j=1}^{q} \operatorname{deg}\left(V\left(\mathfrak{p}_{j}\right)\right) \ell_{j}\right) \prod_{i=1}^{m} \operatorname{deg}\left(f_{i}\right)=\operatorname{deg}(\mathfrak{a}) \prod_{i=1}^{k} \operatorname{deg}\left(f_{i}\right) .
$$

This implies the theorem.

## 5 A Masser-Wüstholz type degree bound for non-homogeneous polynomial ideals

The constructions in this section are inspired by [15.
Let $\mathfrak{a}$ be an arbitrary non-zero proper ideal of the polynomial ring $R:=K[\vec{X}]$ where $K$ is an algebraically closed field of characteristic zero and $\vec{X}:=\left(X_{1}, \ldots, X_{n}\right)$ is a set of variables. Denote by $r:=h t(\mathfrak{a})$ the height of the ideal $\mathfrak{a}$.

From a primary decomposition of $\mathfrak{a}$ we obtain a decomposition of $\mathfrak{a}$ as follows:

$$
\mathfrak{a}=\mathfrak{Q}_{r} \cap \mathfrak{Q}_{r+1} \cap \cdots \cap \mathfrak{Q}_{n} \cap \mathfrak{I},
$$

where, for each $j=r, \ldots, n$ the ideal $\mathfrak{Q}_{j}$ is the intersection of all isolated primary components of $\mathfrak{a}$ having height $j$, or the whole ring $R$ otherwise. The ideal $\mathfrak{I}$ is the intersection of the embedded primary components.

For any $j=r, \ldots, n$ such that $\mathfrak{Q}_{j} \neq R$, let $\mathfrak{Q}_{j}=\bigcap_{i=1}^{s_{j}} \mathfrak{q}_{j i}$ be its primary decomposition. Observe that $\mathfrak{Q}_{j}$ is unmixed of height $j$.

### 5.1 A family of suitable multiplicative sets related to the ideal $\mathfrak{a}$

In this section we introduce suitable simple multiplicative sets such that the respective localizations detect each equidimensional component of the ideal $\mathfrak{a}$ (see Proposition 30 below).

With the previous notations, for each pair $(k, \ell)$ such that $\mathfrak{Q}_{k} \neq R$ and $1 \leq \ell \leq s_{k}$, there exists a point $z_{k \ell} \in \mathbb{A}^{n}$ lying in the variety $V\left(\mathfrak{q}_{k \ell}\right)$ but outside the union of the remaining irreducible components $V\left(\mathfrak{q}_{j i}\right)$, with $j \neq k$ or $i \neq \ell$ if $j=k$, and the immerse variety $V(\mathfrak{I})$. In purely idealistic terms, there exists a maximal ideal $\mathfrak{m}_{k \ell}$ such that $\mathfrak{q}_{k \ell} \subseteq \mathfrak{m}_{k \ell}$ but $\mathfrak{q}_{j i} \nsubseteq \mathfrak{m}_{k \ell}$ for all the other pairs $(j, i)$ and $\mathfrak{I} \nsubseteq \mathfrak{m}_{k \ell}$.

For each index $j=r, \ldots, n$ such that $\mathfrak{a}$ has isolated components of height $j$ we introduce the multiplicative set

$$
S_{j}:=R \backslash \bigcup_{i=1}^{s_{j}} \mathfrak{m}_{j i}
$$

If there is no isolated component of $\mathfrak{a}$ with height $j$ we define $S_{j}:=R \backslash\{0\}$.
From $\mathfrak{q}_{k \ell} \subseteq \mathfrak{m}_{k \ell}$ we infer $\mathfrak{q}_{k \ell} \cap S_{k}=\emptyset$. On the other hand, for $j \neq k$, we have $\mathfrak{q}_{k \ell} \cap S_{j} \neq \emptyset$. If $S_{j}=R \backslash\{0\}$ this is obvious. If $S_{j} \neq R \backslash\{0\}$ the assumption $\mathfrak{q}_{k \ell} \cap S_{j}=\emptyset$ implies the inclusion $\mathfrak{q}_{k \ell} \subset \bigcup_{i=1}^{s_{j}} \mathfrak{m}_{j i}$. By [1, Proposition 1.11], there exists then an index $i$ with $\mathfrak{q}_{k \ell} \subset \mathfrak{m}_{j i}$, in contradiction with the choice of the maximal ideals and $j \neq k$. A similar argument shows $\mathfrak{I} \cap S_{j} \neq \emptyset$ for all $j=r, \ldots n$. Namely, if $\mathfrak{I}$ is disjoint from $S_{j}$ then $\mathfrak{I}$ must be included in suitable maximal ideal $\mathfrak{m}_{j i}$, which again contradicts the choice of the maximal ideals.

With these considerations we have
Proposition 30 For any index $k=r, \ldots, n$ the equality

$$
S_{k}^{-1}(\mathfrak{a})=S_{k}^{-1}\left(\mathfrak{Q}_{k}\right)
$$

holds in the fraction ring $S_{k}^{-1} R$.
In particular, if $\mathfrak{Q}_{k} \neq R$, the ideal $S_{k}^{-1}(\mathfrak{a})$ is unmixed of height $k$ (or equivalently unmixed of dimension $n-k)$.

Proof. By the previous arguments we have

$$
\begin{gathered}
S_{k}^{-1}(\mathfrak{a})=S_{k}^{-1}\left(\mathfrak{Q}_{r}\right) \cap S_{k}^{-1}\left(\mathfrak{Q}_{r+1}\right) \cap \cdots \cap S_{k}^{-1}\left(\mathfrak{Q}_{n}\right) \cap S_{k}^{-1}(\mathfrak{I})= \\
\quad=\bigcap_{j i} S_{k}^{-1}\left(\mathfrak{q}_{j i}\right) \cap S_{k}^{-1}(\mathfrak{I})=\bigcap_{k i} S_{k}^{-1}\left(\mathfrak{q}_{k i}\right)=S_{k}^{-1}\left(\mathfrak{Q}_{k}\right) .
\end{gathered}
$$

Thus, in case $\mathfrak{Q}_{k} \neq R$, the ideal $S_{k}^{-1}(\mathfrak{a})$ is unmixed of height $k$ because $\mathfrak{Q}_{k}$ has this property.

### 5.2 A suitable local regular sequence contained in $\mathfrak{a}$

Let $\vec{g}:=g_{1}, \ldots, g_{s}$ be a system of generators of $\mathfrak{a}$ with degrees $D_{1} \geq D_{2} \geq \cdots \geq D_{s}$, respectively. Since $\mathfrak{a}$ is assumed generated by $s$ many polynomials, Krull's Principal Ideal Theorem (see [1, Corollary 11.16]) implies that in the primary decomposition of $\mathfrak{a}$ only unmixed components $\mathfrak{Q}_{k}$ with $k \leq s$ may appear.

Lemma 31 Fix an index $k=r, \ldots, n$ with $\mathfrak{Q}_{k} \neq R$. Then, there exist polynomials $p_{1}, \ldots, p_{k} \in \mathfrak{a}$ such that for all $j, 1 \leq j \leq k$, the following conditions are satisfied:
i) The polynomial $p_{j}$ is a generic linear combination of the polynomials $g_{j}, \ldots, g_{s}$ (in particular, $\left.\operatorname{deg}\left(p_{j}\right)=\operatorname{deg}\left(g_{j}\right)=D_{j}\right)$.
ii) $p_{1}, \ldots, p_{j}$ is a regular sequence in the localized ring $S_{k}^{-1}(R)$.
iii) If $\mathfrak{a}_{j}:=\left(p_{1}, \ldots, p_{j}\right)$ and $\mathfrak{a}_{j}^{*}:=S_{k}^{-1}\left(\mathfrak{a}_{j}\right) \cap R$, the ideal $\mathfrak{a}_{j}^{*}$ is an unmixed ideal of height $j$ in $R$.
iv) The inequality $\operatorname{deg}\left(\mathfrak{a}_{j}^{*}\right) \leq \operatorname{deg}\left(\mathfrak{a}_{j-1}^{*}\right) D_{j}$ holds.

## Proof.

The proof runs by induction on $j$, where $1 \leq j \leq k$.
For $j=1$ let $p_{1}$ be a generic linear combination of the generators $g_{1}, \ldots, g_{s}$. Clearly $p_{1}$ satisfies the conditions $i$,,$\left.\left.i\right), i i i\right)$ for $j=1$ because $p_{1}$ is not invertible in $S_{k}^{-1}(R)$ since it belongs to the unmixed $(n-k)$-dimensional ideal $S_{k}^{-1}(\mathfrak{a})$ (see Proposition 30). Remark that, even if condition $i v$ ) is vacuous for $j=1$, the inequality $\operatorname{deg}\left(\mathfrak{a}_{1}^{*}\right) \leq \operatorname{deg}\left(p_{1}\right)$ holds (see Propositions (23) and (24).

Suppose now that the lemma holds for $1 \leq j<k$ and let $p_{1}, \ldots, p_{j}$ be polynomials verifying conditions $i$ ) $-i v$ ). Let

$$
S_{k}^{-1}\left(\mathfrak{a}_{j}\right)=\bigcap_{i=1}^{q} \mathfrak{h}_{i}
$$

be a primary decomposition of the ideal $S_{k}^{-1}\left(\mathfrak{a}_{j}\right)$ in the ring $S_{k}^{-1} R$. Remark that all the primary components in this decomposition are isolated and ( $n-j$ )-dimensional because $S_{k}^{-1} R$ is a Cohen-Macaulay ring and $S_{k}^{-1}\left(\mathfrak{a}_{j}\right)$ is generated by the regular sequence $p_{1}, \ldots, p_{j}$.

For each $i=1, \ldots, q$ consider the linear subspace

$$
T_{i}:=\left\{\mu \in K^{s-j-1} \mid \sum_{t \geq j+1} \mu_{t} g_{t} \in \sqrt{\mathfrak{h}_{i}}\right\},
$$

where $\sqrt{\mathfrak{h}_{i}}$ denotes the radical of $\mathfrak{h}_{i}$. If for some $i, T_{i}$ is the whole space $K^{s-j-1}$, then $g_{j+1}, \ldots, g_{s} \in \sqrt{\mathfrak{h}}$. Then, since $p_{j} \in \mathfrak{h}_{i}$ and $p_{j}$ is a generic linear combination of $g_{j}, g_{j+1}, \ldots, g_{s}$ we infer that $g_{j}$ also belongs to $\sqrt{\mathfrak{h}_{i}}$ and by repeating this argument we
conclude that $S_{k}^{-1}(\mathfrak{a}) \subset \sqrt{\mathfrak{h}_{i}}$. But then, by Proposition 30, we have the inequality of heights $k \leq j$, which contradicts the choice of $j$.

Therefore, any $T_{i}$ is a proper linear subspace of $K^{s-j-1}$, and in particular, a generic vector $\mu \in K^{s-j-1}$ verifies $\mu \notin \bigcup_{i} T_{i}$ and so, the associated polynomial $p_{j+1}:=\sum_{t \geq j+1} \mu_{t} g_{t}$ is not a zero divisor modulo the ideal $S_{k}^{-1}\left(\mathfrak{a}_{j}\right)$. Moreover, $p_{j+1}$ is not invertible modulo $S_{k}^{-1}\left(\mathfrak{a}_{j}\right)$ because of the inclusion $S_{k}^{-1}\left(\left(p_{j+1}\right)+\mathfrak{a}_{j}\right) \subseteq S_{k}^{-1}(\mathfrak{a})$ and Proposition 30. Hence, $p_{1}, \ldots, p_{j+1}$ is a regular sequence in $S_{k}^{-1} R$ and conditions $i$ ) and $i i$ ) are satisfied for $j+1$.

Condition $i i i)$ is a consequence of $i i)$ by Macaulay's Theorem applied to the CohenMacaulay ring $S_{k}^{-1} R$ and the well-known fact that the contraction to $R$ of a primary ideal in $S_{k}^{-1} R$ remains primary of same dimension.

We finish the proof showing condition $i v$ ): Since $p_{1}, \ldots, p_{j}$ is a regular sequence in $S_{k}^{-1} R$, the ideals $S_{k}^{-1}\left(\mathfrak{a}_{j}\right)$ and $\mathfrak{a}_{j}^{*}$ are both equidimensional of dimension $n-j$. On the other hand, as $p_{j+1}$ is regular with respect to $S_{k}^{-1}\left(\mathfrak{a}_{j}\right)$, then it is also regular with respect to $S_{k}^{-1}\left(\mathfrak{a}_{j}\right) \cap R=\mathfrak{a}_{j}^{*}$. In particular, $\mathfrak{a}_{j}^{*}+\left(p_{j+1}\right)$ is equidimensional of dimension $n-j-1$.

Since $\mathfrak{a}_{j}^{*} \subseteq \mathfrak{a}_{j+1}^{*}$ and $p_{j+1} \in \mathfrak{a}_{j+1}^{*}$ holds, we have the inclusion $\mathfrak{a}_{j}^{*}+\left(p_{j+1}\right) \subseteq \mathfrak{a}_{j+1}^{*}$ and both ideals are equidimensional of dimension $n-j-1$. Thus, Proposition 23 implies

$$
\operatorname{deg}\left(\mathfrak{a}_{j+1}^{*}\right) \leq \operatorname{deg}\left(\mathfrak{a}_{j}^{*}+\left(p_{j+1}\right)\right)
$$

Finally, by Theorem 29 applied to the ideal $\mathfrak{a}_{j}^{*}$ and the polynomial $p_{j+1}$, we have the inequalities

$$
\operatorname{deg}\left(\mathfrak{a}_{j+1}^{*}\right) \leq \operatorname{deg}\left(\mathfrak{a}_{j}^{*}+\left(p_{j+1}\right)\right) \leq \operatorname{deg}\left(\mathfrak{a}_{j}^{*}\right) \operatorname{deg}\left(p_{j+1}\right)=\operatorname{deg}\left(\mathfrak{a}_{j}^{*}\right) D_{j+1}
$$

and the lemma is proved.

### 5.3 Bézout Inequality in Masser-Wüstholz style

Lemma 31 implies the following Bézout Inequality which appears in [15] in the projective case with the usual notion of degree for homogeneous ideals:

Theorem 32 Let $K$ be an algebraically closed field of characteristic zero, $\vec{X}:=\left(X_{1}, \ldots, X_{n}\right)$ variables over $K$ and $\mathfrak{a} \subset R:=K[\vec{X}]$ a non-zero and proper polynomial ideal generated by polynomials $g_{1}, \ldots, g_{s}$ of total degrees $D_{1} \geq \cdots \geq D_{s}$, respectively. Fix an index $k$, $1 \leq k \leq n$, such that in the primary decomposition of $\mathfrak{a}$ there exist isolated components of height $k$ and denote by $\mathfrak{Q}_{k}$ the intersection of this components. Then:
i) The inequality $\operatorname{deg}\left(\mathfrak{Q}_{k}\right) \leq D_{1} \ldots D_{k}$ holds.
ii) The inequality $\operatorname{deg}(\mathfrak{a}) \leq \sum_{k \in \mathcal{C}} D_{1} \ldots D_{k}$, holds, where $\mathcal{C}$ is the set of those $k$ such that the ideal $\mathfrak{a}$ has isolated primary components of height $k$.

Proof. Following the previous lemma, consider the polynomials $p_{1}, \ldots, p_{k}$ defining a regular sequence in the localized ring $S_{k}^{-1}(R)$. Since $\mathfrak{a}_{k}:=\left(p_{1}, \ldots, p_{k}\right) \subseteq \mathfrak{a}$, we have $S_{k}^{-1}\left(\mathfrak{a}_{k}\right) \subseteq S_{k}^{-1}(\mathfrak{a})=S_{k}^{-1}\left(\mathfrak{Q}_{k}\right)$ (Proposition 30). Hence, $\mathfrak{a}_{k}^{*}:=S_{k}^{-1}\left(\mathfrak{a}_{k}\right) \cap R \subseteq \mathfrak{Q}_{k}$, and both ideals are unmixed of height $k$ (and in particular, equidimensional of dimension $n-k$ ). From Proposition 23 we conclude $\operatorname{deg}\left(\mathfrak{Q}_{k}\right) \leq \operatorname{deg}\left(\mathfrak{a}_{k}^{*}\right)$.

By iteration of Condition $i v$ ) in Lemma 31we obtain $\operatorname{deg}\left(\mathfrak{a}_{k}^{*}\right) \leq D_{1} \ldots D_{k}$ which implies inequality $i$ ).

The assertion $i i$ ) is immediate from $i$ ) and the definition of degree of arbitrary ideals (Definition 25).

## 6 Computing the degree of an equidimensional polynomial ideal

Let $\mathfrak{a}$ be an equidimensional ideal of the polynomial ring $\mathbb{Q}[\vec{X}]$ in $n$ variables $\vec{X}:=$ $\left(X_{1}, \ldots, X_{n}\right)$. Assume that the dimension $m$ and a system of generators $g_{1}, \ldots, g_{s} \in \mathbb{Z}[\vec{X}]$ of the ideal $\mathfrak{a}$ are known. Let $d$ and $\sigma$ be upper bounds for the degrees of $g_{1}, \ldots, g_{s}$ and the bit-sizes of their coefficients.

The goal of this section is to discuss the complexity character of the problem of computing $\operatorname{deg}(\mathfrak{a})$.

Of course, one could compute (uniformly and deterministically) a primary decomposition of the ideal $\mathfrak{a}$ (see [8]) and determine $\operatorname{deg}(\mathfrak{a})$ by means of Definition $22 i i$ ). This would involve a computational cost which is doubly exponential in $n$.

We present here a probabilistic approach which is more efficient and discuss then whether its complexity can be improved.

Let $T_{i j}, 1 \leq i \leq m, 0 \leq j \leq n$ be new indeterminates, $\vec{T}=\left(T_{i j}\right)_{\substack{1 \leq \leq \leq m \\ 0 \leq j \leq n}}$ and let for $1 \leq i \leq m f_{i}:=\sum_{j=1}^{n} T_{i j} X_{j}+T_{i 0}$. Let $\vec{f}:=f_{1}, \ldots, f_{m}$ and fix a monomial order for $\vec{X}$.

Let $\mathfrak{b}$ the ideal generated by $\mathfrak{a}$. From the proof of Lemma 20 and Theorem 21 we deduce

$$
\operatorname{deg}(\mathfrak{a})=\operatorname{dim}_{\mathbb{Q}(\vec{T})} \mathbb{Q}(\vec{T})[\vec{X}] / \mathfrak{b}
$$

Since $\mathfrak{b}$ is an ideal of dimension zero in $\mathbb{Q}(\vec{T})[\vec{X}]$ we have just to compute a Gröbner basis of $\mathfrak{b}$ from the generators $g_{1}, \ldots, g_{s}$ and $f_{1}, \ldots, f_{m}$.

Following [5, Theorem 3.3] this can be done using $\left(s d^{n^{2}}\right)^{O(1)}$ arithmetic operations in $\mathbb{Q}(\vec{T})$.

Applying [11, Theorem 4.4] we obtain a non-uniform deterministic or uniform probabilistic algorithm which computes $\operatorname{deg}(\mathfrak{a})$ by means of $\left(s d^{n^{2}}\right)^{O(1)}$ arithmetic operations in $\mathbb{Q}$.

The non-uniform deterministic version of the algorithm is based on a hitting sequence of integers having bit-size $\left(s d^{n^{2}}\right)^{O(1)}$. This sequence has to be chosen probabilistically in the uniform complexity model. The whole algorithm requires therefore $\left(\sigma s d^{n^{2}}\right)^{O(1)}$ bit operations.

Putting everything together we obtain the following complexity statement.
Theorem 33 There exists a uniform probabilistic algorithm implementable on a Turing Machine with advice which computes $\operatorname{deg}(\mathfrak{a})$ in time $\left(\sigma s d^{n^{2}}\right)^{O(1)}$

This result raises two questions. What is the uniform deterministic complexity of computing $\operatorname{deg}(\mathfrak{a})$ ? This question seems to be out of reach with the actual techniques.

The other question asks whether $\operatorname{deg}(\mathfrak{a})$ can be computed probabilistically using $\left(\sigma s d^{n}\right)^{O(1)}$ bit operations.

This question can be answered positively if it is possible to guess probabilistically generic degree one polynomials $f_{1}, \ldots, f_{m}$ of $\mathbb{Z}[\vec{X}]$ of bit size $\left(\sigma s d^{n}\right)^{O(1)}$.

In this case the probabilistic algorithm of Lakshman [13], techniques of 9], a suitable arithmetic Bézout Inequality (see e.g. [3, Théorème 2] or [12, §1.2.4]) and efficient factorization of univariate polynomials over $\mathbb{Z}$ (see for instance [7, Corollary 16.25]) can be combined to obtain the desired complexity result. We do not go into the (lengthy) details of this approach.

## Conclusion

We introduced a suitable notion of degree for non-homogeneous polynomial ideals and proved extrinsic Bézout Inequalities for this notion. We argued that an intrinsic Bézout Inequality for the sum of two ideals is unfeasable. We exhibit a probabilistic algorithm of single exponential complexity which computes the degree of an equidimensional ideal.

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