# Rationalizability of square roots 

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#### Abstract

Feynman integral computations in theoretical high energy particle physics frequently involve square roots in the kinematic variables. Physicists often want to solve Feynman integrals in terms of multiple polylogarithms. One way to obtain a solution in terms of these functions is to rationalize all occurring square roots by a suitable variable change. In this paper, we give a rigorous definition of rationalizability for square roots of ratios of polynomials. We show that the problem of deciding whether a single square root is rationalizable can be reformulated in geometrical terms. Using this approach, we give easy criteria to decide rationalizability in most cases of square roots in one and two variables. We also give partial results and strategies to prove or disprove rationalizability of sets of square roots. We apply the results to many examples from actual computations in high energy particle physics.


Keywords: Feynman integrals, multiple polylogarithms, rationality problems.

## 1. Introduction

The measurements carried out at modern particle colliders require accurate theoretical predictions. To optimize the precision of these predictions, one has to solve Feynman integrals of increasing complexity. Using dimensional regularization, one writes a given Feynman integral as a Laurent series. For many Feynman integrals, each term of their Laurent expansion can be written as a linear combination of multiple polylogarithms. A representation in terms of these functions is favorable because their analytic structure and numerical implementation (cf. Vollinga and Weinzierl (2005); Bauer et al. (2000)) are well-understood. Multiple polylogarithms are iterated integrals with integration kernels like

$$
\omega_{j}=\frac{d x}{x-z_{j}}
$$

where the $z_{j}$ may depend on the kinematic variables but are independent of the integration variable $x$. The kernels that appear in the computation of Feynman integrals are, however, often more

[^0]complicated: they typically involve various square roots. For example, one encounters kernels like
$$
\frac{d x}{\sqrt{\left(x-z_{1}\right)\left(x-z_{2}\right)}}
$$

To still find a representation in terms of multiple polylogarithms, one usually tries to proceed as follows:

1. try to find a variable change that turns all square roots into rational functions;
2. use partial fractioning to obtain the desired integration kernels.

Changing variables to rationalize a given set of square roots has, therefore, been a crucial step in many particle physics computations, see Becchetti and Bonciani (2018); Broadhurst et al. (1993); Fleischer et al. (1999); Aglietti and Bonciani (2004); Gehrmann et al. (2018); Henn and Smirnov (2013); Lee and Pomeransky (2017); Bork and Onishchenko (2020); Abreu et al. (2019); Primo et al. (2019); Chaubey and Weinzierl (2019); Gehrmann et al. (2016). Especially for Feynman integrals in massless theories with dual conformal symmetry, momentum twistor variables turned out to be an excellent variable choice, cf. Bourjaily et al. (2018b). Applications can be found also in combinatorics, see for example Ablinger (2017, 2019). An algorithmic approach to the rationalization problem was brought from mathematics to the physics community in Besier et al. (2019) and recently automated with the RationalizeRoots software (cf. Besier et al. (2020b)), which is available for Maple (cf. Maple (2020)) and Mathematica (cf. Mathematica (2020)).

While these techniques often lead to a suitable variable change, there are still many practical examples where they do not apply. For these cases, two questions arise:

- Are the methods just not powerful enough to find a suitable variable change?
- Is it even possible to rationalize the given square roots? If not, can we prove it?

This paper is a first attempt to develop simple yet rigorous criteria that physicists can use to answer these questions. After giving a rigorous definition of rationalizability (Definition 3) that is compatible with the notion of "change of variables" used in physics, we show that the problem can be reduced to studying square roots of (squarefree) polynomials instead of square roots of rational expressions. This allows us to translate the problem into an arithmetic geometrical language and give some first general partial results. These are summarized in the following theorem.

Theorem 1. Let $k$ be any field and let $W=\sqrt{p / q}$ be a square root of a ratio of polynomials, i.e., $p, q \in k\left[X_{1}, \ldots, X_{n}\right]$ and $q$ non-zero. Then, the following statements hold:

1. There exists a squarefree polynomial $f \in k\left[X_{1}, \ldots, X_{n}\right]$ such that $W$ is rationalizable if and only if $\sqrt{f}$ is.
2. There exist two projective varieties over $k$ associated to $\sqrt{f}$, denoted by $\bar{V}$ (the associated hypersurface) and $\bar{S}$ (the associated double cover), such that the following statements are equivalent:
(a) $\sqrt{f}$ is rationalizable;
(b) $\overline{\bar{V}}$ is unirational;
(c) $\bar{S}$ is unirational.
3. Assume $k$ is algebraically closed and let d denote the degree of $f$. If $d=1,2$, or if $\bar{V}$ has a singular point of multiplicity $d-1$, then $\sqrt{f}$ is rationalizable.

By restricting to the physically relevant case $k=\mathbb{C}$, and by only considering square roots of polynomials in one or two variables, we can give more precise criteria.

Theorem 2. Let $f$ be a squarefree polynomial of degree $d>0$ over $\mathbb{C}$ and consider its square root $\sqrt{f}$. Let $\bar{S}$ denote the double cover associated to $\sqrt{f}$. Then, the following statements hold:

1. If $f$ is a polynomial in one variable, then $\sqrt{f}$ is rationalizable if and only if $d \leq 2$.
2. If $f$ is a polynomial in two variables and $\bar{S}$ has at most rational simple singularities, then $\sqrt{f}$ is rationalizable if and only if $d \leq 4$.

Theorems 1 and 2 summarize a number of statements that are proven in Sections 2 and 3, respectively. More precisely, we proceed as follows: in Subsection 2.1, we introduce the definition of (non-)rationalizability. The associated hypersurface and double cover are defined in Subsections 2.2 and 2.3. In these subsections, we also prove that the rationalizability of $\sqrt{f}$ is equivalent to the unirationality of the associated hypersurface and double cover, respectively. We prove the criteria for square roots of polynomials in one and two variables in Subsection 3.1 and 3.2. In Subsection 3.3, we deal with square roots of polynomials in more variables and give a criterion for square roots of homogeneous polynomials. Finally, in Subsection 3.4, we introduce the notion of rationalizability for sets of square roots and give a first partial condition that one can use to disprove it. We end the section with a discussion on how to prove non-rationalizability for several sets of square roots that are directly related to recent Feynman integral computations in theoretical high energy particle physics.

Throughout the paper, we deliberately try to keep the statements simple so that they are easy to apply in practice; most proofs are kept compact, preferring abstract but short arguments over arguments that might be more elementary but longer.

Acknowledgment 1. We want to thank Duco van Straten, Stefan Weinzierl, and Robert M. Schabinger for many fruitful conversations. Furthermore, we would like to thank Anne FrühbisKrüger for a detailed explanation of the Singular library classify2.lib. The second author was supported by the grant SFB/TRR 45 at the Johannes Gutenberg University in Mainz. We also thank the anonymous referees for useful comments and suggestions.

## 2. Foundations

### 2.1. Notion of rationalizability

Let $k$ be any field. Consider the polynomial ring $R:=k\left[X_{1}, \ldots, X_{n}\right]$, and denote its field of fractions by $Q:=k\left(X_{1}, \ldots, X_{n}\right)=\operatorname{Frac} R$. If we let $f$ and $g$ be two polynomials in $R$ with $g$ non-zero, then $f / g \in Q$. Fix an algebraic closure $\bar{Q}$ of $Q$, and consider the quantity $\sqrt{f / g} \in \bar{Q}$.

Definition 3. We call the square root $\sqrt{f / g}$ rationalizable if there is a homomorphism of $k$ algebras $\phi: Q \rightarrow Q$ such that $\phi(f / g)$ is a square in $Q$. Otherwise, we say that $\sqrt{f / g}$ is not rationalizable.

Remark 4. Since $\phi: Q \rightarrow Q$ is a homomorphism of $k$-algebras and since $Q$ is a field, $\phi$ is in particular a homomorphism of fields and preserves the zero element and the unit. This implies that $\phi$ is automatically non-constant and, in particular, injective.

Remark 5. Definition 3 is motivated by the fact that physicists are looking for a change of variables that turns $\mathrm{f} / \mathrm{g}$ into a square while preserving the number of variables, i.e., the number of newly introduced variables should be equal to the number of original variables. Such a change of variables is a non-constant homomorphism of $k$-algebras from $Q$ to $Q$ and vice versa.

Example 6. The square root $\sqrt{1-X^{2}}$ is rationalizable. If $k$ has characteristic 2, then

$$
1-X^{2}=X^{2}+1=(X+1)^{2}
$$

and hence $\sqrt{1-X^{2}}=X+1$ (the rationalizing map being the identity). If $\operatorname{char} k \neq 2$, consider $\phi: k(X) \rightarrow k(X)$ defined by

$$
X \mapsto \frac{2 X}{X^{2}+1}
$$

that is, the map that sends the polynomial $g(X)$ to $g\left(\frac{2 X}{X^{2}+1}\right)$. Then, we have

$$
\phi\left(1-X^{2}\right)=\left(\frac{X^{2}-1}{X^{2}+1}\right)^{2}
$$

A square root that is not rationalizable is, for example, $\sqrt{1-X^{3}}$. We will explain how to prove its non-rationalizability in Section 3 (cf. Corollary 30).

Remark 7. Let us stress that our notion of non-rationalizability only implies that there is no rational substitution that rationalizes the square root. For example, the substitution

$$
X \mapsto \sqrt[3]{-X^{2}+2 X}
$$

turns $\sqrt{1-X^{3}}$ into a square in $Q$. However, it does not define a homomorphism $Q \rightarrow Q$ since $\sqrt[3]{-X^{2}+2 X} \notin Q$. Physicists dealing with Feynman integrals will mainly be interested in the existence of rational substitutions since these do not introduce new square roots in other parts of their computation. Nevertheless, the existence of non-rational (called radical) parametrizations is studied and applied to other contexts in Sendra and Sevilla $(2011,2013)$ and Sendra et al. (2017).

Remark 8. If $\mathbb{A}_{k}^{n}$ denotes the affine space with $R$ as its coordinate ring, then homomorphisms of $k$-algebras $Q \rightarrow Q$ are in one-to-one correspondence with dominant rational maps $\mathbb{A}_{k}^{n} \rightarrow \mathbb{A}_{k}^{n}$ (Hartshorne, 1977, Theorem I.4.4). This correspondence is crucial since it will allow us to switch between the algebraic and the geometric point of view, see Proposition 18.

Definition 3 raises an obvious question: given a square root $\sqrt{f / g}$, can we determine whether or not it is rationalizable? To answer this question, we will use tools from algebraic geometry. In particular, we will relate the rationalizability of a square root to the unirationality of a certain variety associated to it. Before delving into this, let us prove the following elementary lemma.
Lemma 9. If $p$ and $q$ denote two non-zero elements of $Q$, then $\sqrt{p}$ is rationalizable if and only if $\sqrt{p q^{2}}$ is rationalizable.
Proof. Assume that $\sqrt{p}$ is rationalizable. Then, there exists a non-constant homomorphism of $k$-algebras $\phi: Q \rightarrow Q$ such that $\phi(p)=r^{2}$ with $r \in Q$. But this means that

$$
\phi\left(p q^{2}\right)=\phi(p) \phi(q)^{2}=r^{2} \phi(q)^{2}=(r \phi(q))^{2} \in Q .
$$

Hence, $\sqrt{p q^{2}}$ is rationalizable.
Conversely, assume that $\sqrt{p q^{2}}$ is rationalizable. Then, there is a non-constant homomorphism of $k$-algebras $\phi: Q \rightarrow Q$ such that $\phi\left(p q^{2}\right)$ is a square in $Q$. As $\phi$ is a morphism of
$k$-algebras, it follows that $\phi\left(p q^{2}\right)=\phi(p) \phi\left(q^{2}\right)=\phi(p) \phi(q)^{2}$ is a square in $Q$. As $\phi$ is a nonconstant morphism of $k$-algebras that are fields, it is injective, and so $\phi(q)$ is non-zero, and hence it follows that $\phi(p)=\phi\left(p q^{2}\right) / \phi(q)^{2}$ is a square. Hence, $\sqrt{p}$ is rationalizable.

Corollary 10. Let $p$ and $q$ be two polynomials in $R$ with $q$ non-zero and consider the fraction $p / q \in Q$. Then there exists a squarefree polynomial $f \in R$ such that $\sqrt{p / q}$ is rationalizable if and only if $\sqrt{f}$ is.

Proof. By Lemma 9, the square root $\sqrt{p / q}$ is rationalizable if and only if the square root $\sqrt{q^{2} \cdot p / q}=\sqrt{p q}$ is. Write $g:=p \cdot q$ and notice that $g \in R$. As $R$ is a unique factorization domain, we can write $g=f h^{2}$, with $f, h \in R$ and $f$ squarefree. Again by Lemma $9, \sqrt{g}$ is rationalizable if and only if $\sqrt{f}$ is, proving the statement.

Remark 11. Let us stress that ignoring squares in the argument is, in many cases, even mandatory for our criteria to be applicable. In other words, one often should ignore square factors in a square root argument: in Subsection 3.2, we will present rationalizability criteria that require the associated variety of the square root to have at most rational simple singularities. Simple singularities are, in particular, isolated singularities. The problem is that, if the square root argument contained a square, the associated varieties (see Definition 13 and 24) would have non-isolated singularities, i.e., a singular locus of positive dimension, and our criteria would not be applicable.

### 2.2. From a square root to a projective hypersurface

Recall that $R$ denotes the polynomial ring $k\left[X_{1}, \ldots, X_{n}\right]$, and $Q=\operatorname{Frac} R$ its field of fractions. We fixed an algebraic closure $\bar{Q}$ of $Q$. By Corollary 10, we can reduce our study to square roots of squarefree polynomials. Throughout this subsection, we use $f \in R$ to denote a non-constant squarefree polynomial of degree $d$ and consider the square root $\sqrt{f}$ in $\bar{Q}$.

Definition 12. Let $\mathbb{A}_{k}^{n+1}$ be the affine space over $k$ with coordinates $X_{1}, \ldots, X_{n}, W$. Let $V$ denote the hypersurface in $\mathbb{A}_{k}^{n+1}$ defined by the equation $W^{2}=f$, which is an affine variety of dimension $n$. We call $V$ the affine hypersurface associated to $\sqrt{f}$.

Definition 13. Let $f$ and $V$ be defined as above, and let $\mathbb{P}_{k}^{n+1}$ be the projective space over $k$ with coordinates $z, x_{1}, \ldots, x_{n}, w$, where affine and projective coordinates have the following relations: $X_{i}=x_{i} / z$ for $i=1, \ldots, n$ and $W=w / z$. We denote by $\bar{V}$ the projective closure of $V \subset \mathbb{A}_{k}^{n+1}$ in $\mathbb{P}_{k}^{n+1}$. We call $\bar{V}$ the hypersurface associated to $\sqrt{f}$. The defining equation of $\bar{V}$ is given by

$$
z^{d-2} w^{2}-z^{d} f\left(x_{1} / z, \ldots, x_{n} / z\right)=0
$$

where $d$ denotes the degree of $f$.
Example 14 (Elliptic curve). Consider the square root $\sqrt{X(X-1)(X-\lambda)}$ over $\mathbb{C}$, for some $\lambda \in \mathbb{C} \backslash\{0,1\}$. Its associated hypersurface is the projective cubic plane curve $\bar{V} \subset \mathbb{P}_{\mathbb{C}}^{2}$ defined by

$$
\bar{V}: z w^{2}-x(x-z)(x-\lambda z)=0
$$

Note that $\bar{V}$ is an elliptic curve in Legendre form.

The next step is to relate the rationalizability of a square root to the arithmetic of its associated hypersurface. More precisely, we will show that the rationalizability of a square root is equivalent to the unirationality of its associated hypersurface. We start by recalling the definitions of rationality and unirationality.
Definition 15. Let $Y$ be a variety defined over $k$, let $\bar{Y}$ be its projective closure, and denote their dimension by $N=\operatorname{dim} Y=\operatorname{dim} \bar{Y}$. We say that $Y$ is rational over $k$ if there is a birational map $\mathbb{P}^{N} \rightarrow \bar{Y}$. We say that $Y$ is unirational over $k$ if there is a rational dominant map $\mathbb{P}^{N} \rightarrow \bar{Y}$, i.e., a rational map $\mathbb{P}^{N} \rightarrow \bar{Y}$ with dense image.

Remark 16. One can also rephrase the notions of rationality and unirationality in algebraic terms, cf. (Hartshorne, 1977, Theorem I.4.4 and Corollary I.4.5): a variety $Y$ is rational if its function field $K(Y)$ is isomorphic to a pure transcendental extension field of $k$ of finite type; it is unirational if its function field can be embedded into a pure transcendental extension field of $k$ of finite type. The transcendental degree of the extension field equals the dimension of $Y$.

Remark 17. Notice that rationality always implies unirationality while the converse statement does, in general, not hold. There are, however, some special cases in which the two notions are indeed equivalent.

- Rationality and unirationality are always equivalent for varieties of dimension one defined over any field (Lüroth's theorem, cf. (Hartshorne, 1977, Example IV.2.5.5)).
- Over algebraically closed fields of characteristic 0 , the notions of rationality and unirationality are also equivalent for varieties of dimension two (Castelnuovo's theorem, cf. (Hartshorne, 1977, Remark V.6.2.1)). This is not true for varieties of higher dimension (see the counterexamples in Artin and Mumford (1972)).
- Over non-algebraically closed fields of characteristic 0 or algebraically closed fields of positive characteristic, things are more complicated: only the equivalence in the onedimensional case is proven to hold; in higher dimensions, the equivalence of unirationality and rationality is either disproved or still disputed, depending on the dimension and the base field.

Proposition 18. Let $f$ and $V$ be defined as above. Then, the square root $\sqrt{f}$ is rationalizable if and only if $V$ is unirational over $k$.
Proof. First assume that $\sqrt{f}$ is rationalizable. This means that there is a non-constant homomorphism of $k$-algebras $\phi: Q \rightarrow Q$ such that $\phi(f)=h^{2} \in Q$ for some $h \in Q$. For $i=1, \ldots, n$ let $\phi_{i}=\phi\left(X_{i}\right)$ denote the image of $X_{i}$ via $\phi$. The function field of $V$ is

$$
K(V)=\operatorname{Frac}\left(\frac{k\left[X_{1}, \ldots, X_{n}, W\right]}{\left(W^{2}-f\right)}\right)
$$

To show that $V$ is unirational, it suffices to show that $K(V)$ can be embedded into a pure transcendental extension field of $k$ of finite type (cf. Remark 16), e.g., embedded into $Q=k\left(X_{1}, \ldots, X_{n}\right)$. Consider the homomorphism of $k$-algebras $\Phi: K(V) \rightarrow Q$ defined by sending $X_{i}$ to $\phi_{i}$ for $i=1, \ldots, n$, and by sending $W$ to $h$. As $\phi(f)=h^{2}$, the map $\Phi$ is well-defined; as $\phi$ is nonconstant, $\Phi$ is non-constant and hence injective, proving the unirationality of $V$.

Conversely, assume that $V$ is unirational over $k$. As noted in Remark 16, this means that there is an injective homomorphism of $k$-algebras

$$
\Phi: K(V)=\operatorname{Frac}\left(\frac{k\left[X_{1}, \ldots, X_{n}, W\right]}{{ }_{6}^{\left(W^{2}-f\right)}}\right) \rightarrow Q .
$$

Consider the embedding $\iota: Q \rightarrow K(V)$ defined by sending the element $X_{i}$ to its equivalence class in $\operatorname{Frac}\left(\frac{k\left[X_{1}, \ldots, X_{n}, W\right]}{\left(W^{2}-f\right)}\right)$. Then $\iota(f)=W^{2}$ and let $g:=\Phi(W)$ be the image of $W$ via $\Phi$ in $Q$. The composition $\Phi \circ \iota: Q \rightarrow Q$ thus sends $f$ to $\Phi(\iota(f))=\Phi\left(W^{2}\right)=\Phi(W)^{2}=g^{2}$, showing that $\sqrt{f}$ is rationalizable.
Remark 19. The projective closure $\bar{Y}$ of a variety $Y$ is always birationally equivalent to the variety itself. Therefore, the unirationality of $\bar{Y}$ is equivalent to the unirationality of $Y$. This is reflected in Definition 15, where we define the (uni)rationality of any variety (affine or projective) only in terms of its projective closure. Thus, one can replace the variety $V$ by its projective closure $\bar{V}$ in the statement of Proposition 18.

An immediate consequence of Proposition 18 is that the rationalizability of a square root only depends on the number of variables that actually appear in the polynomial, and not the total number of variables of the ambient ring. This is shown in the following corollary.

Corollary 20. Let $f \in R$ be defined as before and assume that, after reordering the variables, there exists an $m<n$ such that $f \in R^{\prime}:=k\left[X_{1}, \ldots, X_{m}\right] \subset R$. Then the square root of $f$ viewed as polynomial in $R^{\prime}$ is rationalizable if and only if the square root of $f$ viewed as polynomial in $R$ is.
Proof. Let $\bar{V} \subset \mathbb{P}_{k}^{n+1}$ and $\overline{V^{\prime}} \subset \mathbb{P}_{k}^{m+1}$. Since the variables $X_{m+1}, \ldots, X_{n}$ do not appear in $f$, the hypersurface $\bar{V}$ is birationally equivalent to $\overline{V^{\prime}} \times \mathbb{P}_{k}^{n-m}$. So $\bar{V}$ is unirational if and only $\overline{V^{\prime}} \times$ $\mathbb{P}_{k}^{n-m}$ is, which in turn is unirational if and only if $\overline{V^{\prime}}$ is. The statement hence follows from Proposition 18.

Using Proposition 18, we can determine the (non-)rationalizability of a square root by studying the unirationality of its associated hypersurface. Doing this is, however, still a highly nontrivial task. While the unirationality of varieties is well-studied for one- and two-dimensional varieties over algebraically closed fields of characteristic 0 , the unirationality of varieties of higher dimension is largely not understood (cf. Remark 23). For this reason, in Section 3, we will assume $k=\mathbb{C}$ and focus on square roots in one or two variables. Nevertheless, there are also some partial results holding in any characteristic. For example, we have the following corollary.

Corollary 21. Assume $k$ to be algebraically closed. If $\bar{V}$ has a point of multiplicity $d-1$, then $\sqrt{f}$ is rationalizable. In particular, if $d \leq 2$, then $\sqrt{f}$ is rationalizable.

Proof. If $d>2$ then $\bar{V}$ is a variety of degree $d$. Assume $\bar{V}$ has a point $P$ of multiplicity $d-1$. Then the projection from $P$ gives a birational map $\mathbb{P}_{k}^{N} \rightarrow \bar{V}$. Therefore $\bar{V}$ is rational, hence unirational (cf. Remark 17) and so, by Proposition 18, $\sqrt{f}$ is rationalizable. For $d=1,2$, the variety $\bar{V}$ is of degree 2 and so any of its regular points is of multiplicity 1 . One can then project from any of these points and reason as above to prove the statement.

Remark 22. For a detailed discussion of Corollary 21 and a software package that returns an explicit rational parametrization of a degree-d hypersurface with a point of multiplicity $d-1$, see Besier et al. (2019) and Besier et al. (2020b).
Remark 23. For the interested reader, the literature on rationality of varieties in dimension 1 and 2 is vast and starts from the 19th century. For a modern account see, among others, Arbarello et al. (1985) and (Hartshorne, 1977, Chapter IV) for curves; (Hartshorne, 1977,

Chapters V), and (Kollár et al., 2004, Chapters 1-3) for surfaces. It is important to remark that the arithmetic of surfaces is much more complicated than the one of curves, and the mathematical tools needed to a comprehensive study of it go well beyond the scope of this article.

### 2.3. From a square root to a double cover

In addition to the associated hypersurface defined in the previous subsection, one can also associate another variety to a square root of a polynomial. We will see that these two varieties are not isomorphic in general but always birationally equivalent to each other. Hence, in view of our rationalizability study, the two approaches are equivalent. The approach described in this subsection is particularly convenient for a more geometrical analysis; the approach described in Subsection 2.2 is more suitable for a generalization and, a priori, requires less advanced geometrical tools. Both approaches have advantages and disadvantages in different contexts, and we will use both throughout this paper.

In this subsection, let $k$ be any field and $f \in R=k\left[X_{1}, \ldots, X_{n}\right]$ be a non-constant squarefree polynomial of degree $d$. Define $r:=\lceil d / 2\rceil$. Consider the square root $\sqrt{f}$. Let $\mathbb{A}_{k}^{n+1}$ be the affine space over $k$ with variables $X_{1}, \ldots, X_{n}, W$. Let $\mathbb{P}_{k}=\mathbb{P}_{k}(1, \ldots, 1, r)$ be the weighted projective space over $k$ with coordinates $s, y_{1}, \ldots, y_{n}, u$ of weights $1,1, \ldots, 1, r$, respectively. The relations between the affine and projective coordinates are $X_{i}=y_{i} / s$ for $i=1, \ldots, n$ and $W=u / s$.
Definition 24. We define the double cover associated to $\sqrt{f}$ to be the hypersurface in $\mathbb{P}_{k}$ given by

$$
\bar{S}: u^{2}-s^{2 r} f\left(y_{1} / s, \ldots, y_{n} / s\right)=0
$$

Remark 25. The associated affine hypersurface $V \subset \mathbb{A}_{k}^{n+1}$ (cf. Definition 12) has a natural structure of double cover of $\mathbb{A}_{k}^{n}$. If $S$ denotes $V$ viewed as a double cover, then $\bar{S}$ is the projective closure of $S$ in $\mathbb{P}_{k}$.

Proposition 26. Let $\bar{V}$ and $\bar{S}$ be the hypersurface and the double cover associated to $\sqrt{f}$. Then $\bar{V}$ and $\bar{S}$ are birationally equivalent.

Proof. Define the rational map $\Phi: \mathbb{P}_{k} \rightarrow \mathbb{P}_{k}^{n+1}$ via

$$
\Phi:\left(s, y_{1}, \ldots, y_{n}, u\right) \mapsto\left(s, y_{1}, \ldots, y_{n}, u / s^{r-1}\right)
$$

Then, $\Phi$ is well-defined over a Zariski open subset of $\mathbb{P}_{k}=\mathbb{P}_{k}(1,1, \ldots, 1, r)$, sends $\bar{S}$ to $\bar{V}$, and admits an inverse, namely the rational map $\Psi: \mathbb{P}_{k}^{n+1} \rightarrow \mathbb{P}_{k}$ defined by

$$
\Psi:\left(z, x_{1}, \ldots, x_{n}, w\right) \mapsto\left(z, x_{1}, \ldots, x_{n}, z^{r-1} w\right) .
$$

Hence, $\Phi$ is a birational map from $\bar{S}$ to $\bar{V}$.
Example 27. Take $k$ to be the field of complex numbers $\mathbb{C}$. In Example 14, we have seen that the associated hypersurface of the square root $\sqrt{X(X-1)(X-\lambda)}$ is the elliptic curve $\bar{V} \subset \mathbb{P}_{k}^{2}$ defined by

$$
\bar{V}: z w^{2}-x(x-z)(x-\lambda z)=0
$$

The double cover associated to the square root is the curve $\bar{S}$ in the weighted projective space $\mathbb{P}_{k}(1,1,2)$ with coordinates $s, y$, u of weights $1,1,2$, respectively, defined by the equation

$$
\bar{S}: u^{2}-s y(y-s)(y-\lambda s)=0
$$

Using the map $\bar{S} \rightarrow \mathbb{P}_{k}^{1}$ defined by $(s: y: u) \mapsto(s: y)$, one sees that $\bar{S}$ is a double cover of $\mathbb{P}_{k}^{1}$ ramified at four points: $(0: 1: 0),(1: 0: 0),(1: 1: 0),(1: \lambda: 0)$. Hence, $\bar{S}$ is an elliptic curve (by Hurwitz's theorem, cf. (Hartshorne, 1977, Corollary IV.2.4)). Finally, Proposition 26 shows that $\bar{S}$ and $\bar{V}$ are indeed birationally equivalent, although a priori they might look very different.

## 3. Rationalizability criteria

In the subsections that follow, we will always assume the base field to be $k=\mathbb{C}$ and use the shorthand notations $\mathbb{A}^{n}:=\mathbb{A}_{k}^{n}, \mathbb{P}^{n}:=\mathbb{P}_{k}^{n}$, and $\mathbb{P}:=\mathbb{P}_{k}$.

### 3.1. Square roots in one variable

Studying the rationalizability of square roots of polynomials in one variable is rather easy: it all boils down to computing the geometric genus of the curve associated to the square root. Until the end of this subsection, $f$ will always be a squarefree polynomial in $\mathbb{C}[X]$ of degree $d>0$. Let $\mathbb{A}^{2}$ be the affine plane with coordinates $X$ and $W$. Let $C$ denote the affine curve associated to the square root $\sqrt{f}$, and let $\mathbb{P}^{2}$ denote the projective plane with coordinates $z, x, w$ and relations $X=x / z, W=w / z$. We write $\bar{C}$ for the projective closure of $C$ in $\mathbb{P}^{2}$.
Theorem 28. The square root $\sqrt{f}$ is rationalizable if and only if $\bar{C}$ has geometric genus 0 .
Proof. This immediately follows from Remark 17 and Proposition 18, keeping in mind that a curve over $\mathbb{C}$ is rational if and only if it has geometric genus 0 (cf. Lüroth (1875); Clebsch (1865)). For a modern reference, see (Sendra et al., 2008, Theorems 4.11, 4.62).

Remark 29. After establishing the existence of a parametrization of the curve, the next natural question is about the possibility of explicitly providing it. Whenever a curve $C$ has geometric genus 0 , one can find a rational parametrization (cf. (Sendra et al., 2008, p. 133)).

In fact one can also decide the rationalizability of $\sqrt{f}$ just by looking at the degree of $f$, as shown by the following corollary. Note that, for this result to hold, it is crucial to assume that $f$ is a squarefree polynomial.

Corollary 30. The square root $\sqrt{f}$ is rationalizable if and only if $d \leq 2$.
Proof. Following Subsection 2.3, let $\bar{S}$ be the double cover associated to $\sqrt{f}$, i.e., the double cover of $\mathbb{P}^{1}$ ramified above the zeros of $f$ and, if $d$ is odd, over the point at infinity. Then the Riemann-Hurwitz formula (cf. (Hartshorne, 1977, Corollary IV.2.4)) tells us that

$$
2 g(\bar{S})-2=2\left(2 g\left(\mathbb{P}^{1}\right)-2\right)+\sum_{P \in \bar{S}}\left(e_{P}-1\right),
$$

where $e_{P}$ is the ramification index of $P \in \bar{S}$. Since $f$ is a separable polynomial of degree $d$, we have that

$$
\sum_{P \in \bar{S}}\left(e_{P}-1\right)= \begin{cases}d & \text { if } d \text { is even } \\ d+1 & \text { if } d \text { is odd } .\end{cases}
$$

As $g\left(\mathbb{P}^{1}\right)=0$, the formula yields

$$
g(\bar{S})=\left\{\begin{array}{cl}
(d-2) / 2 & \text { if d is even } \\
(d-1) / 2 & \text { if } d \text { is odd } \\
9 &
\end{array}\right.
$$

From this one clearly sees that $g(\bar{S})=0$ if and only if $d=1,2$. Then the statement follows from Theorem 28.

Remark 31. In practice, Corollary 10, and Corollary 30 allow us to almost immediately determine the rationalizability of a square root: assume we want to determine whether the square root $\sqrt{p / q}$ is rationalizable, with $p, q \in \mathbb{C}[X]$ any two non-zero polynomials and at least one of them non-constant. Consider the polynomial $h:=p \cdot q \in \mathbb{C}[X]$. Then $\sqrt{p / q}$ is rationalizable if and only if $h$ has at most two zeros with odd multiplicity.

### 3.2. Square roots in two variables

The criterion to decide whether the square root of a polynomial in one variable is rationalizable or not relies on the computation of the geometric genus of the associated curve. For surfaces, the situation is analogous with the role of the genus being played by the Kodaira dimension.

Remark 32. Recall that the Kodaira dimension of a projective variety $Y$ is an integer $\kappa=$ $\kappa(Y) \in\{-1,0,1, \ldots, \operatorname{dim} Y\}$. Please note that some sources use $-\infty$ instead of -1 . For the precise definition, we refer to (Hartshorne, 1977, Chapter 6).

The surfaces arising in our context are mostly not smooth. The theory of singular surfaces is extremely rich, and a detailed discussion of the topic goes well beyond the scope of this paper; for an overview, see Arnold et al. (1985). In this subsection, we will only deal with surfaces with mild isolated singularities, that is, surfaces with at most rational simple singularities (also called $D u V a l$ or $A D E$ singularities). For the definition and their properties, we refer the reader to (Arnold et al., 1985, Chapter 15). (We want to stress out that a "rational" simple singularity does not need to be defined over $\mathbb{Q}$; "rational" only means that by resolving it one gets an exceptional divisor birationally equivalent to $\mathbb{P}^{1}$.)

Throughout this subsection, $f$ will always denote a non-constant squarefree polynomial in $\mathbb{C}[X, Y]$ of degree $d$ (cf. Corollary 10 and Remark 11). Following Definition 13, we denote by $\bar{V}$ the hypersurface associated to $\sqrt{f(X, Y)}$, that is, the projective surface in $\mathbb{P}^{3}$ with coordinates $(z, x, y, w)$ defined by

$$
\bar{V}: z^{d-2} w^{2}-z^{d} f(x / z, y / z)=0
$$

Finally, recall that the notions of being rational and unirational are equivalent for surfaces, cf. Remark 17.

Lemma 33. Let $f, d$, and $\bar{V}$ be defined as above and assume $\bar{V}$ is smooth or has at most rational simple singularities. Then the following statements are equivalent.
(i) $\bar{V}$ has Kodaira dimension -1 .
(ii) the degree of $f$ is at most three, i.e., $d \leq 3$;
(iii) $\bar{V}$ is unirational;
(iv) $\sqrt{f}$ is rationalizable.

Proof. From Proposition 18, we know that $i i i) \Longleftrightarrow i v$ ). We are left to show that $i$ ), $i i$, and $i i i$ ) are equivalent. Therefore, we will show that $i) \Longrightarrow i i) \Longrightarrow i i i) \Longrightarrow i$ ).
i) $\Longrightarrow$ ii) Assume $\bar{V}$ has at most rational simple singularities and $\kappa(\bar{V})=-1$. Also, recall that $\bar{V}$ is a hypersurface in $\mathbb{P}^{3}$ of degree $d$. Since $\bar{V}$ has at most rational simple singularities, the canonical class is left unchanged after passing to the smooth model, and hence the Kodaira dimension of $\bar{V}$ is determined by its degree. Hypersurfaces in $\mathbb{P}^{3}$ of Kodaira dimension -1 have
degree $d=1,2,3$ (in fact $\kappa(\bar{V})=0$ for $d=4$ and $\kappa(\bar{V})=2$ for $d \geq 5$, cf. (Barth et al., 2004, Section VI.1)).
ii) $\Longrightarrow$ iii) If $d=1,2$, then the statement follows from Corollary 21 and Proposition 18. Assume $d=3$, then $\bar{V}$ is a cubic. If it is smooth, then it is rational (classic result by Clebsch, see Clebsch (1866) for the original paper, or (Hartshorne, 1977, Corollay V.4.7) for a more modern statement and proof). Rationality implies unirationality. If $\bar{V}$ is singular, by assumption the singularities must be rational simple and hence of multiplicity $2=d-1$. Then the unirationality of $\bar{V}$ follows again from Corollary 21 and Proposition 18.
iii) $\Longrightarrow i$ Assume $\bar{V}$ is unirational. As already noted, this is equivalent to saying that $\bar{V}$ is rational. Then by the Enriques-Kodaira classification of surfaces it follows that $\kappa(\bar{V})=-1$ (see also (Hartshorne, 1977, Theorem V.6.1)).
 $\bar{V}$ has a non-simple singular point at $(0: 0: 0: 1)$, so the result does not apply. If $d=1,2$ we already know (unconditionally) that $\sqrt{f}$ is rationalizable (cf. Proposition 18). The only interesting case is when $d=3$, as shown by the following corollary.
Corollary 35. Assume $d=3$. Then $\sqrt{f}$ is rationalizable if and only if $\bar{V}$ has no singular points of multiplicity 3.

Proof. If $\bar{V}$ has a singular point of multiplicity 3 , then it is the projective cone over a projective plane cubic curve (with the singular point being the vertex of the cone). Hence $\bar{V}$ is ruled and not (uni)rational and so, by Proposition 18, it follows that $\sqrt{f}$ is not rationalizable.

Conversely, assume that $\bar{V}$ has no singular points of multiplicity 3 . As $\bar{V}$ is a cubic, this means that it is either smooth or has singular points of multiplicity 2 . In the smooth case, the statement follows from Lemma 33; in the singular case, from Corollary 21.

Remark 36. The assumption in Lemma 33 for $\bar{V}$ to have at most rational simple singularities is strictly necessary, as shown by Corollary 35. We have seen that if $d=3$ and $\bar{V}$ has a point of multiplicity 3, then $\bar{V}$ is a cone over a cubic curve, which is ruled but not (uni)rational, providing a counterexample to $i i) \Longrightarrow$ iv). In Ishii and Nakayama (2004), one can find a ruled quartic surface having Kodaira dimension -1 while not being (uni)rational, hence a counterexample to $i) \Longrightarrow i i$ ). The implications $i i i) \Longleftrightarrow i v$ ), iii) $\Longrightarrow i$ ), and $i i) \Longrightarrow$ i) hold unconditionally.

The guaranteed existence of non-simple singular points on $\bar{V}$ (cf. Remark 36) prevents us from getting much information about the rationalizability of $\sqrt{f}$. However, we can also use the approach of Subsection 2.3, which turns out to be much more suitable for the case of square roots in two variables, as shown in the theorem below. Let $\bar{S}$ denote the double cover associated to $\sqrt{f}$ (cf. Definition 24).

Theorem 37. Assume that $\bar{S}$ has at most rational simple singularities. Then $\sqrt{f}$ is rationalizable if and only if $d \leq 4$.

Proof. From Remark 17 and Proposition 18, $\sqrt{f}$ is rationalizable if and only if $\bar{S}$ (or, equivalently, $\bar{V}$ ) is unirational.

So now assume $\bar{S}$ has at most rational simple singularities. Let $T$ be the desingularization of $\bar{S}$. By a result in Hironaka (1964), such a $T$ exists and is birationally equivalent to $\bar{S}$. This means that $\bar{S}$ is (uni)rational if and only $T$ is. As $\bar{S}$ has only rational simple singularities, the canonical
divisor of $T$ equals the canonical divisor of $\bar{S}$. Hence, we can use the degree of $f$ to compute the canonical divisor of $\bar{S}$ and hence the Kodaira dimension of $T$.

If $d=1,2$ then $\sqrt{f}$ is rationalizable by Proposition 21 .
If $d=3,4$, then $T$ (and hence $\bar{S}$ ) is birationally equivalent to a del Pezzo surface of degree 2 (cf. (Kollár, 1996, Theorem III.3.5), where the degree of a del Pezzo surface is defined to be the self-intersection of the canonical divisor of the surface; notice that it does not need to coincide with the degree of the defining polynomial). Del Pezzo surfaces are rational (cf. (Manin, 1986, Theorem IV.24.4)).

We are left to show that if $d>4$, then $\sqrt{f}$ is not rationalizable. In order to see this, we prove that $\bar{S}$ is not (uni)rational. Since $d>4$, we have that $T$ and, hence, $\bar{S}$ have Kodaira dimension greater than or equal to 0 (in fact, their Kodaira dimension is 0 if $d=5,6$ and 2 if $d \geq 7$, cf. (Barth et al., 2004, Section V.22)). As $\bar{S}$ and $\bar{V}$ are birationally equivalent, they have the same Kodaira dimension. Then Lemma 33 implies that $\sqrt{f}$ is not rationalizable, proving the statement.

Remark 38. At a first glance, Theorem 37 and Lemma 33 contradict each other, as the square root of a polynomial fof degree $d=4$ should be non-rationalizable, according to Lemma 33, but also rationalizable, according to Theorem 37. This contradiction does, however, not really exist: as already noted in Remark 36, the hypersurface $\bar{V}$ associated to $f$ always has a non-simple singular point and, therefore, one cannot apply the implication i) $\Longrightarrow$ ii) in Lemma 33 (cf. Remark 36) needed to conclude that $\sqrt{f}$ is not rationalizable.

Remark 39. In order to apply Theorem 37, one needs to study the singularities of $\bar{S}$. To simplify this task, we wrote a Magma (cf. Bosma et al. (1997)) function. For the source code of the function and a detailed explanation of how to apply it, see Besier and Festi (2020). Alternatively, one can use the classify2.lib library of Singular (cf. Decker et al. (2019)). Both Magma and Singular come with a free online calculator that one can use to perform the singularity classification.

Example 40. In Festi and van Straten (2019), the rationalizability of the square root

$$
\begin{equation*}
\sqrt{\frac{(X+Y)(1+X Y)}{X+Y-4 X Y+X^{2} Y+X Y^{2}}} \tag{1}
\end{equation*}
$$

coming from the Bhabha scattering (cf. Henn and Smirnov (2013)) is studied. Using Corollary 10, one immediately sees that this is equivalent to study the unirationality of the double cover $\bar{S}$ associated to the square root

$$
\sqrt{(X+Y)(1+X Y)\left(X+Y-4 X Y+X^{2} Y+X Y^{2}\right)}
$$

The surface $\bar{S}$ has only simple singularities as one can check either by hand or using our code, cf. Besier and Festi (2020). Therefore, from Theorem 37, it follows that the square root (1) is not rationalizable.

### 3.3. Square roots in three or more variables

In the previous subsections, we have seen that, if the argument of the square root is a polynomial in one or two variables, then we can often determine whether the square root is rationalizable or not by investigating the unirationality of the associated varieties.

Unfortunately, the situation becomes dramatically more complicated as the number of variables grows. Already in the case of three variables, the previous approaches will not work anymore as we lack easy criteria to assess the unirationality of threefolds. The same is true for varieties of even higher dimension.

In Subsection 2.2, we have already seen a partial result to deduce rationalizability in any number of variables (cf. Corollary 21). In this subsection, we give a result that can help practitioners in studying the rationalizability of a square root of a homogeneous polynomial. More precisely, we will show how to reduce the study of the square root of a homogeneous polynomial in $n$ variables to the study of a square root of a (non-homogeneous) polynomial in $n-1$ variables. This process is particularly interesting when $n=3$, as we can then use all the results of Subsection 3.2.

In what follows we will always assume that $f$ is a non-constant squarefree polynomial of degree $d$ in $R=k\left[X_{1}, \ldots, X_{n}\right]$; recall that we fixed $k=\mathbb{C}$. We use $\bar{V}$ to denote the hypersurface associated to $\sqrt{f}$ (see Definition 13).

Proposition 41. Let $f$ and $d$ be defined as above and let $F$ be the homogeneization of $f$ in $k\left[x_{0}, x_{1}, \ldots, x_{n}\right]$ with $X_{i}=x_{i} / x_{0}$ for $i=1, \ldots, n$, that $i$ s, $F=x_{0}^{d} f\left(x_{1} / x_{0}, \ldots, x_{n} / x_{0}\right)$.

The following holds:

1. if d is even, then $\sqrt{f}$ is rationalizable if and only if $\sqrt{F}$ is;
2. if $d$ is odd, then $\sqrt{f}$ is rationalizable if and only if $\sqrt{x_{0} F}$ is.

Proof. In what follows, let $Q^{\prime}$ be the field $k\left(x_{0}, x_{1}, \ldots, x_{n}\right)$ and recall $Q:=k\left(X_{1}, \ldots, X_{n}\right)$.

1. By assumption $d$ is even; write $d=2 r$. Assume $\sqrt{F}$ is rationalizable. Then there is a non-constant homomorphism of $k$-algebras $\Phi: Q^{\prime} \rightarrow Q^{\prime}$ such that

$$
\Phi(F)=F\left(\Phi_{0}, \ldots, \Phi_{n}\right)=H^{2}
$$

where $\Phi_{i}:=\Phi\left(x_{i}\right)$ and $H \in Q^{\prime}$. As $F=x_{0}^{d} f\left(x_{1} / x_{0}, \ldots, x_{n} / x_{0}\right)$ we have

$$
\begin{aligned}
H^{2}=\Phi(F) & =\Phi\left(x_{0}^{d} f\left(x_{1} / x_{0}, \ldots, x_{n} / x_{0}\right)\right) \\
& =\Phi\left(x_{0}\right)^{d} \Phi\left(f\left(x_{1} / x_{0}, \ldots, x_{n} / x_{0}\right)\right) \\
& =\Phi_{0}^{d} f\left(\Phi_{1} / \Phi_{0}, \ldots, \Phi_{n} / \Phi_{0}\right)
\end{aligned}
$$

from which it follows that

$$
f\left(\Phi_{1} / \Phi_{0}, \ldots, \Phi_{n} / \Phi_{0}\right)=\frac{H^{2}}{\Phi_{0}^{d}}=\left(\frac{H}{\Phi_{0}^{r}}\right)^{2}
$$

is a square in $Q^{\prime}$. Notice that, as $\Phi$ is a homomorphism of fields, it is injective and hence $\Phi_{0}$ is non-zero. Also, as $k$ is algebraically closed, it is infinite and hence there exists an element $c \in k$ such that the rational expressions $H\left(c, x_{1}, \ldots x_{n}\right)$ and $\Phi_{i}\left(c, x_{1}, \ldots, x_{n}\right)$, for $i=0,1, \ldots, n$, are well-defined and non-zero. Then the following homomorphism of $k$ algebras is well defined.

$$
\begin{aligned}
\phi: Q & \rightarrow Q \\
X_{i} & \mapsto \frac{\Phi_{i}\left(c, X_{1}, \ldots, X_{n}\right)}{\Phi_{0}\left(c, X_{1}, \ldots, X_{n}\right)} \text { for } i=1, \ldots, n
\end{aligned}
$$

It is easy to see that $\phi$ sends $f$ to a square, indeed

$$
\begin{aligned}
\phi(f) & =f\left(\frac{\Phi_{1}\left(c, X_{1}, \ldots, X_{n}\right)}{\Phi_{0}\left(c, X_{1}, \ldots, X_{n}\right)}, \ldots, \frac{\Phi_{n}\left(c, X_{1}, \ldots, X_{n}\right)}{\Phi_{0}\left(c, X_{1}, \ldots, X_{n}\right)}\right) \\
& =\left(f\left(\Phi_{1} / \Phi_{0}, \ldots, \Phi_{n} / \Phi_{0}\right)\right)\left(c, X_{1}, \ldots, X_{n}\right) \\
& =\frac{H^{2}}{\Phi_{0}^{d}}\left(c, X_{1}, \ldots, X_{n}\right) \\
& =\left(\frac{H\left(c, X_{1}, \ldots, X_{n}\right)}{\Phi_{0}^{r}\left(c, X_{1}, \ldots, X_{n}\right)}\right)^{2} .
\end{aligned}
$$

Hence, $\sqrt{f}$ is rationalizable.
Conversely, assume that $\sqrt{f}$ is rationalizable. Then, there is a non-constant homomorphism $\phi: Q \rightarrow Q$ of $k$-algebras such that

$$
\phi(f)=f\left(\phi_{1}, \ldots, \phi_{n}\right)=h^{2}
$$

for some $h \in Q$, where $\phi_{i}=\phi\left(X_{i}\right) \in Q$ for $i=1, \ldots, n$. All the $\phi_{i}$ 's can be expressed as a ratio of two polynomials; taking the least common multiple $\varphi_{0}$ of the denominators, we can write

$$
\phi_{i}=\frac{\varphi_{i}}{\varphi_{0}}
$$

for $i=1, \ldots, n$ (notice that $\varphi_{0}$ is fixed). Then, as above, we can find $n+1$ polynomials $\Phi_{0}, \Phi_{1}, \ldots, \Phi_{n} \in k\left[x_{0}, x_{1}, \ldots, x_{n}\right]$ (homogeneous and of the same degree) such that the following equalities hold in $Q^{\prime}$ :

$$
\frac{\varphi_{i}\left(x_{1} / x_{0}, \ldots, x_{n} / x_{0}\right)}{\varphi_{0}\left(x_{1} / x_{0}, \ldots, x_{n} / x_{0}\right)}=\frac{\Phi_{i}}{\Phi_{0}},
$$

for every $i=1, \ldots, n$. Define the homomorphism $\Phi$ as follows.

$$
\begin{aligned}
\Phi: Q^{\prime} & \rightarrow Q^{\prime} \\
x_{i} & \mapsto \Phi_{i} \text { for } i=0,1, \ldots, n
\end{aligned}
$$

Then $\Phi$ sends $F$ to a square, concluding the proof:

$$
\begin{aligned}
\Phi(F) & =\Phi\left(x_{0}^{d} f\left(x_{1} / x_{0}, \ldots, x_{n} / x_{0}\right)\right) \\
& =\Phi\left(x_{0}\right)^{d} \Phi\left(f\left(x_{1} / x_{0}, \ldots, x_{n} / x_{0}\right)\right) \\
& \left.=\Phi_{0}^{d} f\left(\Phi\left(x_{1} / x_{0}\right), \ldots, \Phi\left(x_{n} / x_{0}\right)\right)\right) \\
& =\Phi_{0}^{d} f\left(\Phi_{1} / \Phi_{0}, \ldots, \Phi_{n} / \Phi_{0}\right) \\
& =\Phi_{0}^{d} f\left(\frac{\varphi_{1}\left(x_{1} / x_{0}, \ldots, x_{n} / x_{0}\right)}{\varphi_{0}\left(x_{1} / x_{0}, \ldots, x_{n} / x_{0}\right)}, \ldots, \frac{\varphi_{n}\left(x_{1} / x_{0}, \ldots, x_{n} / x_{0}\right)}{\varphi_{0}\left(x_{1} / x_{0}, \ldots, x_{n} / x_{0}\right)}\right) \\
& =\Phi_{0}^{d}\left(f\left(\frac{\varphi_{1}}{\varphi_{0}}, \ldots, \frac{\varphi_{n}}{\varphi_{0}}\right)\right)\left(x_{1} / x_{0}, \ldots, x_{n} / x_{0}\right) \\
& =\Phi_{0}^{d}\left(f\left(\phi_{1}, \ldots, \phi_{n}\right)\right)\left(x_{1} / x_{0}, \ldots, x_{n} / x_{0}\right) \\
& =\Phi_{0}^{d} h\left(x_{1} / x_{0}, \ldots, x_{n} / x_{0}\right)^{2} \\
& =\left(\Phi_{0}^{r} h\left(x_{1} / x_{0}, \ldots, x_{n} / x_{0}\right)\right)^{2} . \\
& 14
\end{aligned}
$$

2. Assume that $d$ is odd and write $d=2 r-1$. Then $x_{0} F$ has degree $2 r$, and the proof goes as above.

Remark 42. Proposition 41 is particularly useful in the case of square roots in three variables. Indeed, if $n=3$ and $f$ happens to be homogeneous, then one can regard $f$ as the homogenization of a polynomial $g$ in two variables. Then $\sqrt{f}$ is rationalizable if and only if the hypersurface (or, equivalently, the double cover) associated to $\sqrt{g}$ (or to $\sqrt{x_{0} g}$, if $d$ is odd and $x_{0}$ is the homogenizing variable) is unirational. Subsequently, one can apply the methods of Subsection 3.2.

Example 43. With this example, we show that Proposition 41 can be helpful even with square roots in two variables. Consider the square root $\sqrt{F}$ with $F=X_{1}^{4}+X_{2}^{4}$ and let $\bar{S}$ be its associated double cover in $\mathbb{P}(1,1,1,2)$ with coordinates $s, y_{1}, y_{2}$, $u$, that is,

$$
\bar{S}: u^{2}=y_{1}^{4}+y_{2}^{4} .
$$

One can see that $\bar{S}$ has a non-simple (elliptic) singularity in (1:0:0:0) and therefore we cannot use Theorem 37 to conclude that $\sqrt{f}$ is rationalizable.

Nevertheless, $F$ can be seen as the homogenization of the polynomial $G=X^{4}+1$ and, by Proposition 41, $\sqrt{F}$ is rationalizable if and only if $\sqrt{G}$ is (as the degree of $G$ is 4 , even). As $G$ is a polynomial in one variable, we can then apply Theorem 28 to conlude that $\sqrt{G}$, and hence also $\sqrt{F}$, is not rationalizable.

Example 44. Fermat quartics give us also another interesting example. Consider the square root $\sqrt{F}$ with $F=X_{1}^{4}+X_{2}^{4}+X_{3}^{4}$. Notice that $F$ has degree $4>2$ so we cannot conclude right away from Corollary 21 that $\sqrt{F}$ is rationalizable. The associated hypersurface is $\bar{V}:=$ $\left\{z^{2} w^{2}-x_{1}^{4}-x_{2}^{4}-x_{3}^{4}=0\right\}$ and has two (non-simple) singular points, $(1: 0: 0: 0: 0)$ and (0:0:0:0:1), both of multiplicity 2. In particular, no triple points. So again, we cannot apply Corollary 21 to conclude that it is rationalizable.

Nevertheless, we notice that $F$ is homogeneous and it can be seen as the homogenization of $f=X^{4}+Y^{4}+1$ in $k\left[X_{1}, X_{2}, X_{3}\right]$ with $X=X_{1} / X_{3}$ and $Y=X_{2} / X_{3}$. Let $\bar{S}$ denote the double cover associated to $\sqrt{f}$. It is easy to see that $\bar{S}$ is smooth and so, in particular, has no non-simple singularities. From Theorem 37 it follows that $\sqrt{f}$ is rationalizable. Hence, by Proposition 41, so is $\sqrt{F}$.

### 3.4. Proving non-rationalizability of a set of square roots

In the context of Feynman integral computations, it is often not enough to study the rationalizability of a single square root. Instead, practitioners are usually interested in whether or not several different square roots can be rationalized simultaneously. However, we will see that the non-rationalizability of a set of square roots (also called alphabet) can often be deduced from the non-rationalizability of a single square root so that many of our previous methods can also be applied in this more general context.

As before, we fix $k=\mathbb{C}$, and write $R=k\left[X_{1}, \ldots, X_{n}\right]$ for the ring of polynomials and $Q=$ Frac $R$ for its field of fractions.

Definition 45. Let $f_{1}, \ldots, f_{r}$ be polynomials in $R$. An alphabet $\left\{\sqrt{f_{1}}, \ldots, \sqrt{f_{r}}\right\}$ is called rationalizable if there is a homomorphism of $k$-algebras $\phi: Q \rightarrow Q$ such that $\phi\left(f_{i}\right)=h_{i}^{2}$ for some $h_{i} \in Q$, where $i=1, \ldots, r$.

Remark 46. From the definition it immediately follows that if an alphabet $\mathcal{A}^{\prime}$ is non-rationalizable, then every alphabet $\mathcal{A} \supseteq \mathcal{A}^{\prime}$ containing $\mathcal{A}^{\prime}$ is also non-rationalizable.

This remark is particularly useful when $\mathcal{A}$ has a subset $\mathcal{A}^{\prime}$ containing only square roots of polynomials in fewer variables, that is, after possibly reordering the variables and the polynomials,

$$
\mathcal{A}^{\prime}=\left\{\sqrt{f_{1}}, \ldots, \sqrt{f_{s}}\right\}
$$

with $s<r$ and $f_{1}, \ldots, f_{s} \in k\left[X_{1}, \ldots, X_{m}\right] \subset R, m<n$. Then one can try to disprove the rationalizability of $\mathcal{A}$ by disproving the rationalizability of $\mathcal{A}^{\prime}$. The rationalizability of $\mathcal{A}^{\prime}$ as alphabet of square roots of polynomials in $R$ is equivalent to its rationalizability as square roots of polynomials in $k\left[X_{1}, \ldots, X_{m}\right]$ (Corollary 20). The latter task is easier because of the fewer variables involved and, in particular, if $m=1,2$, one can then apply the criteria given in Subsections 3.1 and 3.2.

Proposition 47. If the alphabet $\left\{\sqrt{f_{1}}, \ldots, \sqrt{f_{n}}\right\}$ is rationalizable then, for every non-empty subset $J \subseteq\{1, \ldots, n\}$, the square root

$$
\begin{equation*}
\sqrt{\prod_{j \in J} f_{j}} \tag{2}
\end{equation*}
$$

is rationalizable.
Proof. By definition, if $\left\{\sqrt{f_{1}}, \ldots, \sqrt{f_{n}}\right\}$ is rationalizable, then there exists a non-constant $k$-algebra homomorphism $\phi: Q \rightarrow Q$ such that, for $i=1, \ldots, n$, the map $\phi$ sends $f_{i}$ to $h_{i}^{2}$, for some $h_{i} \in Q$. In particular, $\phi\left(f_{j}\right)=h_{j}^{2}$ for every $j \in J$. Hence, $\phi\left(\prod_{j \in J} f_{j}\right)=\left(\prod_{j \in J} h_{j}\right)^{2}$, proving the statement.
Remark 48. The above straightforward proposition allows us to prove that a given alphabet is not rationalizable by showing that at least one square root of the form (2) is not rationalizable.

At the moment, we do not know whether the converse statement of Proposition 47 holds or not, as even if the product $\sqrt{\prod_{j \in J} f_{j}}$ is rationalizable for every $J \subseteq\{1, \ldots, n\}$, the rationalizing morphisms $\phi_{J}$ do not need to be a priori all equal. Nevertheless, one can often prove the rationalizability of a given alphabet by providing an explicit variable change that rationalizes all of its square roots simultaneously. To find such a variable change, one can try to apply the elementary strategy mentioned in Remark 52, which works for many practical examples.

Let us apply Proposition 47 to the alphabets of some recent Feynman integral computations.
Example 49. By Corollary 30, square roots of a squarefree polynomial in one variable of degree $d>2$ are not rationalizable. Such square roots occurred in many Feynman integral computations of the last decade, see Laporta and Remiddi (2005); Müller-Stach et al. (2012); Adams et al. (2013); Bloch and Vanhove (2015); Adams et al. (2014, 2015, 2016b); Sø gaard and Zhang (2015); Bloch et al. (2017); Remiddi and Tancredi (2016); Adams et al. (2016a); Bonciani et al. (2016a); von Manteuffel and Tancredi (2017); Adams and Weinzierl (2018a); Bogner et al. (2017); Ablinger et al. (2018); Remiddi and Tancredi (2017); Bourjaily et al. (2018a); Hidding and Moriello (2019); Broedel et al. (2018c,b,a); Adams and Weinzierl (2018b); Adams et al. (2018b,a). As an example, let us consider the following alphabet, which appears in perturbative corrections for Higgs production in Dulat (2018); Anastasiou et al. (2015):

$$
\mathcal{A}=\{\sqrt{X}, \sqrt{1+4 X}, \sqrt{X(X-4)}\} .
$$

These three square roots cannot be rationalized by a single rational variable change. To see this, define $f_{1}:=X, f_{2}:=1+4 X, f_{3}:=X(X-4)$, take $J:=\{2,3\}$, and consider the square root

$$
\begin{equation*}
\sqrt{\prod_{j \in J} f_{j}}=\sqrt{(1+4 X) X(X-4)} \tag{3}
\end{equation*}
$$

Note that the product $(1+4 X) X(X-4)$ is a squarefree polynomial of degree $3>2$. Therefore, by Corollary 30, the square root (3) is not rationalizable. Thus, by Proposition 47, we conclude that $\mathcal{A}$ is not rationalizable.

Example 50. Let us now consider the following set of square roots that is relevant for perturbative corrections to di-photon and di-jet hadro-production in Becchetti and Bonciani (2018):

$$
\begin{aligned}
\mathcal{A}= & \{\sqrt{X+1}, \sqrt{X-1}, \sqrt{Y+1}, \\
& \left.\sqrt{X+Y+1}, \sqrt{16 X+(4+Y)^{2}}\right\} .
\end{aligned}
$$

Write $f_{1}, \ldots, f_{5}$ for the polynomial arguments of the square roots in $\mathcal{A}$. To show that $\mathcal{A}$ is not rationalizable, consider the whole set of indices $J=\{1,2,3,4,5\}$ and define

$$
f:=\prod_{j \in J} f_{j}=(X+1)(X-1)(Y+1)(X+Y+1)\left(16 X+(4+Y)^{2}\right) .
$$

Let $\bar{S}$ be the associated double cover of $\sqrt{f}$ (cf. Definition 24). It is easy to check-for example by using our Magma function-that $\bar{S}$ has only rational simple singularities. Since $f$ has degree 6 , Theorem 37 tells us that $\sqrt{f}$ is not rationalizable. Hence, using Proposition 47, we can conclude that $\mathcal{A}$ cannot be rationalizable.

Example 51. It is important (and fair) to stress that the results presented in this paper are not always enough to get an answer. Consider the alphabet

$$
\begin{aligned}
& \mathcal{A}=\left\{\sqrt{X_{1}\left(X_{1}-4 X_{3}\right)}, \sqrt{-X_{1} X_{2}\left(4 X_{3}\left(X_{3}+X_{2}\right)-X_{1} X_{2}\right)},\right. \\
&\left.\sqrt{X_{1}\left(X_{2}^{2}\left(X_{1}-4 X_{3}\right)+X_{3} X_{1}\left(X_{3}-2 X_{2}\right)\right)}\right\}
\end{aligned}
$$

relevant for two-loop EW-QCD corrections to Drell-Yan scattering (cf. Heller et al. (2020); Besier et al. (2020a); Bonciani et al. (2016b)). Denote by $F_{1}, F_{2}, F_{3}$ the polynomial arguments of the square roots in $\mathcal{A}$. Proving non-rationalizability of $\mathcal{A}$ requires more than just the techniques presented in this paper. Notice that $F_{1}, F_{2}, F_{3}$ are all homogeneous. Therefore, we can view them as the homogenizations of three polynomials with respect to one of the three variables, for example $X_{3}$. Studying the rationalizability of $\mathcal{A}$ is, hence, equivalent to studying the rationalizability of

$$
\left\{\sqrt{f_{1}}, \sqrt{f_{2}}, \sqrt{f_{3}}\right\}
$$

where $f_{i}=f_{i}(X, Y)$ is the dehomogenization of $F_{i}$ with respect to $X_{3}$, that is,

$$
f_{i}(X, Y):=F_{i}(X, Y, 1)
$$

As $f_{1}, f_{2}$ and $f_{3}$ have degree 2,4 and 4 respectively, and since their associated double covers have at most rational simple singularities, one has that $\sqrt{f_{1}}, \sqrt{f_{2}}, \sqrt{f_{3}}$ are rationalizable when
considered individually (cf. Theorem 37). The products $f_{1} f_{2}$ and $f_{1} f_{3}$, after removing the square factors, also have degree 4 and associated double covers with at most rational simple singularities. Hence, their square roots are rationalizable. The product $f_{2} f_{3}$ has, after removing square factors, degree 6 but its associated double cover has (at least) two non-simple singularities, so we cannot conclude that $\sqrt{f_{2} f_{3}}$ is not rationalizable. (After further investigation it turns out that it is in fact rationalizable.) We are left with the square root of the product $f_{1} f_{2} f_{3}$. After removing the squares, the product has degree 8 but the associated double cover has some non-simple singularities as well. So again, we cannot use Theorem 37 to conclude that its square root is non-rationalizable.

Analogous computations and results are obtained if one dehomogenizes the polynomials $F_{1}, F_{2}, F_{3}$ with respect to $X_{1}$ or $X_{2}$. For this reason, using the results and techniques of the previous subsections, we cannot prove or disprove the rationalizability of the alphabet $\mathcal{A}$.

Only by using methods that are beyond the scope of this paper, one can see that the square root $\sqrt{f_{1} f_{2} f_{3}}$ is not rationalizable and hence that the alphabet $\mathcal{A}$ is not rationalizable, where the $f_{i}$ denote the above mentioned dehomogenizations with respect to $X_{3}$.

Remark 52. Finally, let us stress that, when trying to prove non-rationalizability in physics computations, it is crucial to pick the right starting point for the proof. To clarify this important subtlety, consider the following alphabet:

$$
\mathcal{A}:=\{\sqrt{X-1}, \sqrt{X-2}\} .
$$

To rationalize this set, we could proceed as follows:

1. try to rationalize the first square root;
2. if successful, plug the corresponding substitution into the second square root and try to rationalize the resulting square root;
3. if successful, compose both substitutions to obtain a single substitution that will rationalize both square roots.
(A more detailed discussion of this procedure can be found in Besier et al. (2020b, 2019).) We start out with the rationalization of the first square root, via the homomorphism $\phi: k(X) \rightarrow k(X)$ defined by $\phi: X \mapsto X^{4}+1$, hence $\sqrt{\phi(X-1)}=X^{2}$. Using $\phi$, the second square root becomes

$$
\begin{equation*}
\sqrt{X^{4}-1} \tag{4}
\end{equation*}
$$

giving us a non-rationalizable square root, cf. Theorem 28.
Therefore, one might be tempted to assume that the non-rationalizability of (4) implies nonrationalizability of $\mathcal{A}$. This assumption is, however, not true: consider the homomorphism $\psi: k(X) \rightarrow k(X)$ defined as $\psi: X \mapsto X^{2}+1$. Then one can easily see that $\psi(X-1)=X^{2}$ and $\psi(X-2)=X^{2}-1$. Notice that $X^{2}-1$ has degree 2 and so one can easily rationalize its square root (cf. Corollary 21). A suitable substitution is, for example, given by the homomorphism $\sigma: k(X) \rightarrow k(X)$ defined by $\sigma: X \mapsto \frac{2 X^{2}}{1-X^{2}}+1$. Finally, the composition

$$
\iota:=\sigma \circ \psi: k(X) \rightarrow k(X), X \mapsto \frac{2\left(X^{2}+1\right)^{2}}{1-\left(X^{2}+1\right)^{2}}+1
$$

rationalizes both square roots simultaneously, proving that $\mathcal{A}$ is rationalizable.
Besides illustrating a way to prove rationalizability for an alphabet, this example gives us the following important insight: proving non-rationalizability after some substitutions have already been made does, in general, not imply non-rationalizability of the original alphabet. For

Feynman integral computations, this means that one should always prove non-rationalizability as early as possible, i.e., as soon as the square roots arise in the computation.

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