# FINITE GROUP CHARACTERS ON FREE RESOLUTIONS 

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#### Abstract

Under reasonable assumptions, a group action on a module extends to the minimal free resolutions of the module. Explicit descriptions of these actions can lead to a better understanding of free resolutions by providing, for example, convenient expressions for their differentials or alternative characterizations of their Betti numbers. This article introduces an algorithm for computing characters of finite groups acting on minimal free resolutions of finitely generated graded modules over polynomial rings.


## 1. Introduction

Let $\mathbb{k}$ be a field and let $R$ be a polynomial ring over $\mathbb{k}$ with a positive $\mathbb{Z}^{r}$-grading ${ }^{1}$ in the sense of [KR05, Definition 4.2.4]. The Betti numbers of a finitely generated graded $R$-module $M$ are the integers

$$
\beta_{i, j}(M)=\operatorname{dim}_{\mathbb{k}} \operatorname{Tor}_{i}^{R}(M, \mathbb{k})_{j},
$$

for $i \in \mathbb{N}$ with $i \leqslant \operatorname{dim} R$ and $j \in \mathbb{Z}^{r}$. It is well understood that Betti numbers encode many interesting algebraic and geometric properties of $M$ (see [Eis95, Eis05]).

Let $G$ be a finite group. Suppose there is a $\mathbb{k}$-linear action of $G$ on both $R$ and $M$ that preserves degrees and is compatible with the $R$-module structures. If $\mathbb{k}$ has characteristic zero or positive characteristic not dividing the order of $G$ (the non modular case), then $G$ is linearly reductive (see [Lan02, XVIII, Theorem 1.2]). In this case, the $G$-action extends to a minimal free resolution of $M$ in an essentially unique way (see [Gal15, §1]). There has been significant interest in naturally occurring examples of minimal free resolutions with such group actions [ZGS14, ELSW18, GGW18, BDRH ${ }^{+}$19, Gal20, BdAG ${ }^{+}$20, Mur20, SY20, MR20, Rai21].

Understanding group actions on minimal free resolutions in terms of representation theory can provide useful information about the structure of the resolution or offer interesting combinatorial interpretations of the Betti numbers. The goal is to describe the $\mathbb{k}$-vector space $\operatorname{Tor}_{i}^{R}(M, \mathbb{k})_{j}$ as a representation of $G$. In the non modular case, this is completely equivalent to describing the character of $G$ on $\operatorname{Tor}_{i}^{R}(M, \mathbb{k})_{j}$, i.e., the function $G \rightarrow \mathbb{k}$ that returns the trace of the action of an element $g \in G$ on $\operatorname{Tor}_{i}^{R}(M, \mathbb{k})_{j}$ (see [Ser77, Ch. 2]). We propose the following:

Definition. The $(i, j)$-th Betti character of $G$ on $M$, denoted $\beta_{i, j}^{G}(M)$, is the character of $G$ on $\operatorname{Tor}_{i}^{R}(M, \mathbb{k})_{j}$.

[^0]Note that the Betti character $\beta_{i, j}^{G}(M)$ evaluated at the identity element of $G$ is the usual Betti number $\beta_{i, j}(M)$.

The goal of this work is to describe an algorithm (Algorithm 1) to compute Betti characters of finite groups in the non modular case. This is similar in spirit to our earlier work [Gal16, Gal15], which dealt with the case of resolutions with an action of a semisimple Lie group in characteristic zero. Our algorithm requires access to the following data:

- a minimal free resolution of $M$;
- matrices for the action of $G$ on the generators of $R$;
- matrices for the action of $G$ on the generators of $M$.

While free resolutions are generally quite difficult to construct, the assumption is that a minimal free resolution of $M$ can be computed with the aid of software. Since characters are class functions (i.e., constant on conjugacy classes), it is enough to provide matrices for the action of a set of representatives of the conjugacy classes of $G$ on $R$ and $M$. Since $G$ typically acts trivially on the multiplicative identity of $R$, when $M$ is a quotient of $R$ by an ideal, the action on the generator of $M$ is trivial. Therefore, in most scenarios of interest, the requirements for running the algorithm are minimal. The algorithm uses the action matrices to construct a new minimal free resolution of $M$, then produces a map of complexes between the given resolution and the new one (see Proposition 2.1). The trace of the individual maps in the map of complexes are the entries of the desired Betti characters (see Theorem 2.2). While most currently available results describing finite group actions on resolutions are limited to certain classes of modules (e.g., monomial ideals) or certain groups (mainly symmetric groups), the proposed algorithm bears no such restriction; as such, the author hopes the algorithm will prove useful in exploring further examples of finite group actions on free resolutions.

This paper is organized as follows: Section 2 contains the theoretical background, Section 4 describes the algorithm, and Section 3 illustrates the results with a few examples.

## 2. Theory

With the same notation as in Section 1, let $\left(F_{\bullet}, d_{\bullet}\right)$ be a minimal free resolution of $M$ as an $R$-module and recall that the differentials $d_{i}: F_{i} \rightarrow F_{i-1}$ are $G$-equivariant (i.e., they commute with the action of the group; see [Gal16, Proposition 2.4.9] or [Gal15, § 1]). Computing Tor along this resolution gives

$$
\operatorname{Tor}_{i}^{R}(M, \mathbb{k})_{j} \cong\left(F_{i} \otimes_{R} \mathbb{k}\right)_{j}
$$

where $\left(F_{i} \otimes_{R} \mathbb{k}\right)_{j}$ is the graded component of degree $j$ of the graded $\mathbb{k}$-vector space $F_{i} \otimes_{R} \mathbb{k}$. For $g \in G$, denote $\phi_{i}^{g}$ the $\mathbb{k}$-linear map which operates as multiplication by $g$ on $F_{i} \otimes_{R} \mathbb{k}$. The value of the Betti character $\beta_{i, j}^{G}(M)$ at $g$ is the trace of $\phi_{i}^{g}$ restricted to degree $j$ (in both domain and codomain). Note that multiplication by $g$ on the modules $F_{i}$ is not $R$ linear, which makes it inconvenient to work with directly. We proceed to repackage the same information using maps of complexes.

For each $i \geqslant 0$, fix an $R$-basis $\mathcal{E}_{i}$ of $F_{i}$. Observe that the set $\left\{e \otimes 1 \mid e \in \mathcal{E}_{i}\right\}$ is a $\mathbb{k}$-basis of $F_{i} \otimes_{R} \mathbb{k}$. For $g \in G$, let $\psi_{i}^{g}: F_{i} \rightarrow F_{i}$ be the map of $R$-modules defined by $\psi_{i}^{g}(e)=g e$ for all $e \in \mathcal{E}_{i}$, and extended by $R$-linearity. Note that $\psi_{i}^{g}$ is invertible and $\left(\psi_{i}^{g}\right)^{-1}=\psi_{i}^{g^{-1}}$. For every $e \in \mathcal{E}_{i}$, we have $\psi_{i}^{g}(e)=g e$ and $\phi_{i}^{g}(e \otimes 1)=(g e) \otimes 1$, so $\psi_{i}^{g}$ and $\phi_{i}^{g}$ output $\mathbb{k}$-linear combinations with the same coefficients relative to the bases $\mathcal{E}_{i}$ and $\left\{e \otimes 1 \mid e \in \mathcal{E}_{i}\right\}$
respectively. Now the value of the Betti character $\beta_{i, j}^{G}(M)$ at $g$ is the trace of $\psi_{i}^{g}$ restricted to degree $j$, and $\psi_{i}^{g}$ is an $R$-linear map.

In addition to being $R$-linear, the maps $\psi_{i}^{g}$ can be collected into a map of complexes of $R$-modules. For this purpose, we define for every element $g \in G$ a map of $R$-modules $d_{i}^{g}: F_{i} \rightarrow F_{i-1}$ by setting $d_{i}^{g}=\psi_{i-1}^{g} d_{i}\left(\psi_{i}^{g}\right)^{-1}$.

Proposition 2.1. The following are true for every $g \in G$.
(1) The collection $\left(F_{\bullet}, d_{\bullet}^{g}\right)$ is a minimal free resolution of $M$.
(2) The collection $\psi_{\bullet}^{g}:\left(F_{\bullet}, d_{\bullet}\right) \rightarrow\left(F_{\bullet}, d_{\bullet}^{g}\right)$ is a map of complexes.
(3) If $\left[a_{u v}\right]$ is the matrix of $d_{i}$ relative to the bases $\mathcal{E}_{i}$ and $\mathcal{E}_{i-1}$, then the matrix of $d_{i}^{g}$ relative to the same bases is $\left[g^{-1} a_{u v}\right]$.

Proof. Fix an arbitrary element $g \in G$ throughout the proof.
(1) The definition of $d_{i}^{g}$ immediately implies that the diagrams

commute. Commutativity of the left diagram implies $\psi_{i}^{g}\left(\operatorname{ker} d_{i}\right) \subseteq \operatorname{ker} d_{i}^{g}$. Commutativity of the right diagram implies $\left(\psi_{i}^{g}\right)^{-1}\left(\operatorname{ker} d_{i}^{g}\right) \subseteq \operatorname{ker} d_{i}$, hence $\operatorname{ker} d_{i}^{g} \subseteq \psi_{i}^{g}\left(\operatorname{ker} d_{i}\right)$. Hence we have the equality $\psi_{i}^{g}\left(\operatorname{ker} d_{i}\right)=\operatorname{ker} d_{i}^{g}$. A similar reasoning shows $\psi_{i-1}^{g}\left(\operatorname{im} d_{i}\right)=\operatorname{im} d_{i}^{g}$. Since $\left(F_{\bullet}, d_{\bullet}\right)$ is a free resolution, there is an equality $\operatorname{ker} d_{i}=\operatorname{im} d_{i+1}$ for every integer $i \geqslant 1$. Applying the morphism $\psi_{i}^{g}$ and combining with the previous equalities, we deduce $\operatorname{ker} d_{i}^{g}=\operatorname{im} d_{i+1}^{g}$, thus showing $\left(F_{\bullet}, d_{\mathbf{\bullet}}^{g}\right)$ is also a free resolution.

Let $d_{0}: F_{0} \rightarrow M$ be the surjection that augments $\left(F_{\bullet}, d_{\bullet}\right)$ to a free resolution of the $R$-module $M$, so that ker $d_{0}=\operatorname{im} d_{1}$. Define the function $d_{0}^{g}: F_{0} \rightarrow M$ by setting $d_{0}^{g}=$ $d_{0}\left(\psi_{0}^{g}\right)^{-1}$, so that $d_{0}^{g} \psi_{0}^{g}=d_{0}$. It follows immediately from the definition that $d_{0}^{g}$ is an epimorphism of $R$-modules. Moreover, the same kind of argument used in the previous paragraph shows that $\psi_{0}^{g}\left(\operatorname{ker} d_{0}\right)=\operatorname{ker} d_{0}^{g}$. Combining equalities as before, we conclude that $d_{0}^{g}$ augments $\left(F_{\bullet}, d_{\bullet}^{g}\right)$ to a free resolution of $M$.

It remains to prove that $\left(F_{\bullet}, d_{\bullet}^{g}\right)$ is minimal, i.e., for every integer $i \geqslant 1$, $\operatorname{im} d_{i}^{g} \subseteq \mathfrak{m} F_{i-1}$ where $\mathfrak{m}$ denotes the maximal ideal generated by the variables in $R$. We have previously established that $\psi_{i-1}^{g}\left(\operatorname{im} d_{i}\right)=\operatorname{im} d_{i}^{g}$, and we know $\operatorname{im} d_{i} \subseteq \mathfrak{m} F_{i-1}$ because $\left(F_{\bullet}, d_{\bullet}\right)$ is minimal. Combining this with the fact that $\psi_{i-1}^{g}$ is an isomorphism of $R$-modules, we have

$$
\operatorname{im} d_{i}^{g}=\psi_{i-1}^{g}\left(\operatorname{im} d_{i}\right) \subseteq \psi_{i-1}^{g}\left(\mathfrak{m} F_{i-1}\right)=\mathfrak{m} \psi_{i-1}^{g}\left(F_{i-1}\right)=\mathfrak{m} F_{i-1},
$$

as desired.
(2) This statement is the commutativity of the left diagram in part (1) of the proof.
(3) Let $\mathcal{E}_{i}=\left\{f_{1}, \ldots, f_{n}\right\}$ and $\mathcal{E}_{i-1}=\left\{e_{1}, \ldots, e_{m}\right\}$. The entry in row $u$, column $v$ in the matrix of $d_{i}^{g}$ relative to $\mathcal{E}_{i}$ and $\mathcal{E}_{i-1}$ is the coefficient of $e_{u}$ in the expression of $d_{i}^{g}\left(f_{v}\right)$ as a linear combination of $e_{1}, \ldots, e_{m}$. Since $d_{i}$ is $G$-equivariant, we have

$$
d_{i}^{g}\left(f_{v}\right)=\psi_{i-1}^{g} d_{i}\left(\psi_{i}^{g}\right)^{-1}\left(f_{v}\right)=\psi_{i-1}^{g} d_{i}\left(g^{-1} f_{v}\right)=\psi_{i-1}^{g}\left(g^{-1} d_{i}\left(f_{v}\right)\right) .
$$

If the matrix of $d_{i}$ relative to $\mathcal{E}_{i}$ and $\mathcal{E}_{i-1}$ is $\left[a_{u v}\right]$, then we get

$$
d_{i}^{g}\left(f_{v}\right)=\psi_{i-1}^{g}\left(g^{-1} \sum_{i=1}^{m} a_{u v} e_{u}\right)=\psi_{i-1}^{g}\left(\sum_{i=1}^{m}\left(g^{-1} a_{u v}\right)\left(g^{-1} e_{u}\right)\right),
$$

because the $G$-action is compatible with the $R$-module operations. Finally, by definition of $\psi_{i-1}^{g}$, we obtain

$$
d_{i}^{g}\left(f_{v}\right)=\sum_{i=1}^{m}\left(g^{-1} a_{u v}\right) \psi_{i-1}^{g}\left(g^{-1} e_{u}\right)=\sum_{i=1}^{m}\left(g^{-1} a_{u v}\right) \psi_{i-1}^{g} \psi_{i-1}^{g^{-1}}\left(e_{u}\right)=\sum_{i=1}^{m}\left(g^{-1} a_{u v}\right) e_{u} .
$$

Therefore the coefficient of $e_{u}$ in the expression of $d_{i}^{g}\left(f_{v}\right)$ as a linear combination of $e_{1}, \ldots, e_{m}$ is $g^{-1} a_{u v}$.

We come to our main result, which, combined with Proposition 2.1, gives a practical way to compute $\beta_{i, j}^{G}(M)(g)$, the value of the $(i, j)$-th Betti character of $G$ on $M$ at an element $g \in G$. As observed earlier, $\beta_{i, j}^{G}(M)(g)$ is equal to the trace of $\psi_{i}^{g}$ restricted to degree $j$. The problem is to determine matrices for the maps $\psi_{i}^{g}$. As we will show, this is not entirely necessary. In fact, it suffices to find maps $\hat{\psi}_{i}^{g}$ that agree with $\psi_{i}^{g}$ in degree $j$.

Fix a free $R$-module $F_{i}$ with basis $\mathcal{E}_{i}$, and let $F_{i, j}$ be the free direct summand of $F_{i}$ generated by the elements of $\mathcal{E}_{i}$ of degree $j$. Let $\pi_{i, j}: F_{i} \rightarrow F_{i, j}$ be the natural projection, which does not depend on the chosen bases. Observe that $\pi_{i, j}$ is $G$-equivariant because the action of $G$ sends an element of $\mathcal{E}_{i}$ of degree $j$ to a $\mathbb{k}$-linear combination of elements of $\mathcal{E}_{i}$ of degree $j$.
Theorem 2.2. For every $g \in G$, if $\hat{\psi}_{\mathbf{\bullet}}^{g}:\left(F_{\bullet}, d_{\bullet}\right) \rightarrow\left(F_{\bullet}, d_{\bullet}^{g}\right)$ is a map of complexes with $\hat{\psi}_{0}^{g}=\psi_{0}^{g}$, then $\pi_{i, j} \hat{\psi}_{i}^{g}(e)=g e=\psi_{i}^{g}(e)$ for every $e \in \mathcal{E}_{i}$ of degree $j$. In particular, $\beta_{i, j}^{G}(M)(g)$ is equal to the trace of $\hat{\psi}_{i}^{g}$ restricted to degree $j$.

Proof. By Proposition 2.1, $\left(F_{\bullet}, d_{\bullet}\right)$ and ( $F_{\bullet}, d_{\bullet}^{g}$ ) are both minimal free resolutions of $M$. Moreover, $\hat{\psi}_{\bullet}^{g}$ and $\psi_{\bullet}^{g}$ are both maps of complexes lifting the $R$-automorphism of $M$ induced by $\hat{\psi}_{0}^{g}=\psi_{0}^{g}$. By the Comparison Theorem (see [Wei94, Theorem 2.2.6]), $\hat{\psi}_{\bullet}^{g}$ and $\psi_{\bullet}^{g}$ are homotopic, and therefore they induce the same map in homology (see [Wei94, Lemma 1.4.5]). It follows that, for every $i \geqslant 0, \operatorname{im}\left(\hat{\psi}_{i}^{g}-\psi_{i}^{g}\right) \subseteq \mathfrak{m} F_{i}$.

Consider an element $e \in \mathcal{E}_{i}$ of degree $j$. We have $\hat{\psi}_{i}^{g}(e)=\psi_{i}^{g}(e)+f=g e+f$ for some $f \in \mathfrak{m} F_{i}$. Note that both $\hat{\psi}_{i}^{g}(e)$ and ge have degree $j$ because $\hat{\psi}_{i}^{g}$ and the group action preserve degrees. It follows that $f$ also has degree $j$. Now apply the projection $\pi_{i, j}$ to obtain $\pi_{i, j} \hat{\psi}_{i}^{g}(e)=g e+\pi_{i, j}(f)$. Since $f \in \mathfrak{m} F_{i}$, we have that $\pi_{i, j}(f) \in \mathfrak{m} F_{i, j}$. However, since $F_{i, j}$ is a free module generated in degree $j$, the only element of degree $j$ in $\mathfrak{m} F_{i, j}$ is zero; hence, $\pi_{i, j}(f)=0$. We conclude that $\pi_{i, j} \hat{\psi}_{i}^{g}(e)=g e=\psi_{i}^{g}(e)$, as desired. The equality between $\beta_{i, j}^{G}(M)$ and the trace of $\hat{\psi}_{i}^{g}$ restricted to degree $j$ is an immediate consequence.
Remark 2.3. After evaluation at representatives of all conjugacy classes of $G$, the character $\beta_{i, j}^{G}(M)$ can be decomposed into irreducible characters using the character table of $G$ (see for example [Ser77, Ch. 2]). This allows further insight into the structure of $\operatorname{Tor}_{i}^{R}(M, \mathbb{k})_{j}$ as a representation of $G$.

## 3. Examples

Let $R=\mathbb{Q}\left[x_{1}, x_{2}, x_{3}, x_{4}\right]$ and consider the monomial ideal

$$
I=\left\langle x_{1} x_{2}, x_{1} x_{3}, x_{2} x_{3}, x_{1} x_{4}, x_{2} x_{4}, x_{3} x_{4}\right\rangle .
$$

We regard $I$ as an $R$-module and compute a minimal free resolution of $I$ with the help of the software Macaulay2 [GS]. The resolution $\left(F_{\bullet}, d_{\bullet}\right)$ has the following form:

$$
R(-2)^{6} \stackrel{d_{1}}{\leftrightarrows} R(-3)^{8} \stackrel{d_{2}}{\longleftarrow} R(-4)^{3}
$$

with

$$
d_{1}=\left[\begin{array}{cccccccc}
-x_{3} & 0 & -x_{4} & 0 & 0 & 0 & 0 & 0 \\
x_{2} & -x_{2} & 0 & 0 & -x_{4} & 0 & 0 & 0 \\
0 & x_{1} & 0 & 0 & 0 & 0 & -x_{4} & 0 \\
0 & 0 & x_{2} & -x_{2} & x_{3} & -x_{3} & 0 & 0 \\
0 & 0 & 0 & x_{1} & 0 & 0 & x_{3} & -x_{3} \\
0 & 0 & 0 & 0 & 0 & x_{1} & 0 & x_{2}
\end{array}\right]
$$

and

$$
d_{2}=\left[\begin{array}{ccc}
x_{4} & x_{4} & 0 \\
0 & x_{4} & 0 \\
-x_{3} & -x_{3} & 0 \\
0 & -x_{3} & x_{3} \\
x_{2} & 0 & 0 \\
0 & 0 & -x_{2} \\
0 & x_{1} & 0 \\
0 & 0 & x_{1}
\end{array}\right] .
$$

The symmetric group $\mathfrak{S}_{4}$ acts on $R$ by permuting variables, i.e., $\sigma x_{i}=x_{\sigma(i)}$ for every $\sigma \in \mathfrak{S}_{4}$. The ideal $I$ is stable under this action, hence $\mathfrak{S}_{4}$ acts on the minimal free resolution above. We are going to determine the Betti characters $\beta_{i, j}^{\mathfrak{G}_{4}}(I)$.

Recall that conjugacy classes in $\mathfrak{S}_{4}$ are determined by cycle type [Sag01, § 1.1], and therefore correspond to partitions of 4 . We choose the following permutations to represent the conjugacy classes in $\mathfrak{S}_{4}$ : in cycle notation, (1234), (123), (12)(34), (12), and id $\mathfrak{S}_{4}$. It is easy to describe the action of $G$ on the $\mathbb{Q}$-vector space spanned by the generators of $I$ as this is a permutation representation. This action determines the matrices of the maps $\psi_{0}^{\sigma}: R(-2)^{6} \rightarrow R(-2)^{6}$ for $\sigma \in \mathfrak{S}_{4}$. We have:

$$
\begin{aligned}
\psi_{0}^{(1234)}=\left[\begin{array}{llllll}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{array}\right], \quad \psi_{0}^{(123)}=\left[\begin{array}{llllll}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}\right], \\
\psi_{0}^{(12)(34)}=\left[\begin{array}{llllll}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right], \quad \psi_{0}^{(12)}=\left[\begin{array}{llllll}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right] .
\end{aligned}
$$

From the trace of these matrices, we can determine the Betti character

|  | $(1234)$ | $(123)$ | $(12)(34)$ | $(12)$ | $\mathrm{id}_{\mathfrak{S}_{4}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\beta_{0,2}^{\mathfrak{S}_{4}}(I)$ | 0 | 0 | 2 | 2 | 6 |

where the last value is the trace of the identity map on the module $F_{0}$, which is the same as the rank of $F_{0}$.

We illustrate the computation of the other Betti characters by focusing on the element (1234). The resolution $\left(F_{\bullet}, d_{\bullet}^{(1234)}\right)$ has the following form:

$$
R(-2)^{6} \stackrel{d_{1}^{(1234)}}{\leftrightarrows} R(-3)^{8} \stackrel{d_{2}^{(1234)}}{\leftrightarrows} R(-4)^{3}
$$

with

$$
d_{1}^{(1234)}=\left[\begin{array}{cccccccc}
-x_{2} & 0 & -x_{3} & 0 & 0 & 0 & 0 & 0 \\
x_{1} & -x_{1} & 0 & 0 & -x_{3} & 0 & 0 & 0 \\
0 & x_{4} & 0 & 0 & 0 & 0 & -x_{3} & 0 \\
0 & 0 & x_{1} & -x_{1} & x_{2} & -x_{2} & 0 & 0 \\
0 & 0 & 0 & x_{4} & 0 & 0 & x_{2} & -x_{2} \\
0 & 0 & 0 & 0 & 0 & x_{4} & 0 & x_{1}
\end{array}\right]
$$

and

$$
d_{2}^{(1234)}=\left[\begin{array}{ccc}
x_{3} & x_{3} & 0 \\
0 & x_{3} & 0 \\
-x_{2} & -x_{2} & 0 \\
0 & -x_{2} & x_{2} \\
x_{1} & 0 & 0 \\
0 & 0 & -x_{1} \\
0 & x_{4} & 0 \\
0 & 0 & x_{4}
\end{array}\right]
$$

By Proposition 2.1 (3), the matrices of the differentials $d_{i}^{(1234)}$ are obtained by applying the permutation $(1234)^{-1}=(1432)$ to the matrices of the differentials $d_{i}$.

Next we compute $\hat{\psi}_{1}^{(1234)}$ by solving the equation $\psi_{0}^{(1234)} d_{1}=d_{1}^{(1234)} \hat{\psi}_{1}^{(1234)}$. In this example, the solution can be found by direct inspection although in general it would require solving a system of linear equations. The result is

$$
\hat{\psi}_{1}^{(1234)}=\left[\begin{array}{cccccccc}
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

Note that this is an invertible matrix with entries in $\mathbb{Q}$ whose domain and codomain are both generated in degree 3 . It follows that this is actually the matrix of $\psi_{1}^{(1234)}$, the map defined to act like (1234) on the given basis of $F_{1}$.

Similarly, we compute $\hat{\psi}_{2}^{(1234)}$ by solving the equation $\hat{\psi}_{1}^{(1234)} d_{2}=d_{2}^{(1234)} \hat{\psi}_{2}^{(1234)}$. The result is

$$
\hat{\psi}_{2}^{(1234)}=\left[\begin{array}{ccc}
0 & -1 & 1 \\
1 & 1 & 0 \\
0 & 1 & 0
\end{array}\right]
$$

Repeating the process for the representatives of the other conjugacy classes, we find a complete description of the Betti characters, which we summarize in the table below.

|  | $(1234)$ | $(123)$ | $(12)(34)$ | $(12)$ | $\mathrm{id}_{\mathfrak{S}_{4}}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $\beta_{0,2}^{\mathfrak{S}_{4}}(I)$ | 0 | 0 | 2 | 2 | 6 |
| $\beta_{1,3}^{\mathfrak{E}_{4}}(I)$ | 0 | -1 | 0 | 0 | 8 |
| $\beta_{2,4}^{\mathfrak{E}_{4}}(I)$ | 1 | 0 | -1 | -1 | 3 |

The reader may verify the Betti characters found here are the same as the characters of the representations found in [Gal20, § 4]. We also mention that $I$ is a symmetric shifted ideal and its equivariant resolution can be found as described in $\left[\mathrm{BdAG}^{+}\right.$20, Theorem 6.2, Example 6.3].

For completeness, we include matrices of the maps $\hat{\psi}_{\bullet}^{\sigma}$ for the representatives of the other conjugacy classes (except the identity). These matrices were found using our implementation of Algorithm 1 in Macaulay2.

$$
\begin{aligned}
& \hat{\psi}_{1}^{(123)}=\left[\begin{array}{ccccccc}
-1 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right], \\
& \hat{\psi}_{1}^{(12)(34)}=\left[\begin{array}{cccccccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right], \\
& \hat{\psi}_{1}^{(12)}=\left[\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right], \\
& \hat{\psi}_{2}^{(123)}=\left[\begin{array}{ccc}
0 & 1 & 0 \\
-1 & -1 & 0 \\
0 & -1 & 1
\end{array}\right], \\
& \hat{\psi}_{2}^{(12)(34)}=\left[\begin{array}{ccc}
0 & -1 & 1 \\
-1 & 0 & -1 \\
0 & 0 & -1
\end{array}\right],
\end{aligned}
$$

## 4. Algorithm

We generalize the ideas of the example from Section 3 and formalize them into Algorithm 1 , which we present in pseudocode below. We refer again to the notations introduced in Sections 1 and 2.

We start by describing some auxiliary functions. The implementation will depend on the system but should be fairly straightforward.
degrees $(A)$ : Takes as input the matrix $A$ of a map between graded free $R$-modules. It returns a list of degrees $\left\{\operatorname{deg}\left(e_{1}\right), \ldots, \operatorname{deg}\left(e_{n}\right)\right\}$, where $\left\{e_{1}, \ldots, e_{n}\right\}$ is the basis used for the domain of $A$.
$\operatorname{Trace}(A, j)$ : Takes as input the matrix $A$ of a map between two copies of the same graded free $R$-module and a degree $j$. It returns the trace of the submatrix of $A$ formed by taking only rows and columns corresponding to basis elements of degree $j$.
action $(g, A)$ : Takes as input an element $g \in G$ and the matrix $A=\left[a_{u v}\right]$ of a map between graded free $R$-modules. It returns the matrix $\left[g a_{u v}\right]$ obtained by acting with $g$ on each entry of $A$.
factor $(A, B)$ : Takes as input two matrices $A$ and $B$ corresponding to maps between graded free $R$-modules. The maps should have the same codomain. Also $A$ and $B$ should be expressed relative to the same basis of their common codomain. It returns a matrix $C$ with homogeneous entries in $R$ such that $A=B C$.

Remark 4.1. In Macaulay2, the function action can be implemented as a substitution that sends each variable $x_{i}$ in the ring $R=\mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$ to the element $g x_{i}$, which can be expressed as a linear combination of the variables. Similarly, the function factor is already implemented in Macaulay2 via the operator //. We also point out that Macaulay2 has the ability to compute extensions of module maps to maps of complexes via the function extend; this can be used to obtain the map of complexes $\hat{\psi}_{\bullet}^{g}$ starting from a matrix of the map $\psi_{0}^{g}$ for a given $g \in G$.
Remark 4.2. As observed in Section 3, the use of factor in the context of Algorithm 1 requires solving a system of linear equations. In systems where a matrix factorization function is implemented (such as Macaulay2), this could rely on Gröbner bases calculations. Moreover, the result of such a matrix factorization is, in principle, not unique. However, Theorem 2.2 guarantees any map of complexes $\hat{\psi}_{\bullet}^{g}$ that extends $\psi_{0}^{g}$ can be used to determine the value of the Betti character $\beta_{i, j}^{G}(M)$ at $g$. Therefore how factorization is actually implemented should not be a source of concern.

Given a set $\left\{g_{1}, \ldots, g_{r}\right\}$ of representatives of the conjugacy classes of $G$, the character $\chi$ of a finite dimensional representation of $G$ can be encoded into the list $\left\{\chi\left(g_{1}\right), \ldots, \chi\left(g_{r}\right)\right\}$. This data type will be used by our algorithm to return Betti characters. Moreover, the collection of all nonzero Betti characters can be assembled into a hash table with keys $(i, j)$ and corresponding values $\beta_{i, j}^{G}(M)$, where $i$ is a homological degree and $j$ is a degree.

The computation of $d_{i}^{g_{k}}$ in line 11 follows Proposition 2.1 (3). The construction of $\hat{\psi}_{i}^{g_{k}}$ via a matrix factorization in line 12 ensures that $\hat{\psi}_{\bullet}^{g_{k}}$ is a map of complexes extending $\psi_{0}^{g_{k}}$. The computation of the values of the Betti characters in line 14 is justified by Theorem 2.2.

Remark 4.3. Many interesting equivariant resolutions arise as minimal free resolutions of quotient rings $R / I$ where $I$ is a $G$-stable ideal, i.e., $G I \subseteq I$. This is the case of the example in Section 3. Note that a minimal free resolution $\left(F_{\bullet}, d_{\bullet}\right)$ of $M=R / I$ has $F_{0}=R$ with a trivial action of $G$ on $\operatorname{Tor}_{0}^{R}(M, \mathbb{k}) \cong F_{0} \otimes_{R} \mathbb{k} \cong \mathbb{k}$. In this setting, the matrices $P_{1}, \ldots, P_{r}$ in the input of Algorithm 1 are 1 by 1 identity matrices, hence they can be omitted from the input. This implies that the description of the group action on the resolution is entirely determined by group action on the ring, which is computed by the function Action.

Remark 4.4. Algorithm 1 assumes the action of the group $G$ on the minimal free resolution $\left(F_{\bullet}, d_{\bullet}\right)$ can be explicitly described in homological degree zero by matrices $\psi_{0}^{g}$ for certain $g \in G$. The action in higher homological degrees is then computed by recursive matrix factorizations. More explicitly, if the matrix $\psi_{i-1}^{g}$ is known, the matrix $\psi_{i}^{g}$ can be computed by a matrix factorization via the following diagram.


Note that $\psi_{i-1}^{g}$ can also be computed from $\psi_{i}^{g}$ via a matrix factorization. Therefore it is possible to extend the algorithm so that the starting point for computing the action on a resolution can be given in arbitrary homological degree rather than degree zero.

Remark 4.5. In the setting of Section 1, the graded components of $M$ are also finite dimensional representations of $G$, and the reader may be interested in finding their characters. This is a much easier task than computing Betti characters. For completeness, we briefly outline a strategy for computing characters of graded components.

Consider a (minimal) presentation $d_{1}: F_{1} \rightarrow F_{0}$ of $M$. Fix a basis of $F_{0}$ and a module term ordering on the terms of $F_{0}$ relative to this basis. Let $j$ be a degree such that the graded component $M_{j}$ is nonzero, and let $\mathcal{B}=\left\{b_{1}, \ldots, b_{s}\right\}$ be the set of terms of degree $j$ in $F_{0}$ not contained in the leading term module of im $d_{1}$. By Macaulay's Basis Theorem

```
Algorithm 1 Betti characters
Input: - \(D=\left\{D_{1}, \ldots, D_{n}\right\}\), a list with \(D_{i}\) the matrix of the differential \(d_{i}\) relative to
    the bases \(\mathcal{E}_{i}\) and \(\mathcal{E}_{i-1}\)
    - \(P=\left\{P_{1}, \ldots, P_{r}\right\}\), a list with \(P_{k}\) the matrix of \(\psi_{0}^{g_{k}}\) for a set \(\left\{g_{1}, \ldots, g_{r}\right\}\) of repre-
        sentatives of the conjugacy classes of \(G\)
Output: \(B\), a hash table with keys \((i, j)\) and values \(B_{i, j}=\beta_{i, j}^{G}(M)\) the nonzero Betti
    characters of \(G\) on \(M\) represented as lists
    function BettiCharacters \((D, P)\)
    \(B \leftarrow\} \quad \triangleright\) initialize empty hash table
    for \(k \leftarrow 1, \ldots, r\) do \(\quad \triangleright\) loop for homological degree 0
        \(Q_{k} \leftarrow P_{k} \quad \triangleright\) matrix of \(\psi_{0}^{g_{k}}\)
        for all \(j \in \operatorname{DEGREes}\left(Q_{k}\right)\) do
            \(B_{0, j} \leftarrow B_{0, j} \cup\left\{\operatorname{TracE}\left(Q_{k}, j\right)\right\} \quad \triangleright\) append to (possibly empty) list
        end for
        end for
        for \(i \leftarrow 1, \ldots, n\) do \(\quad \triangleright\) loop for homological degrees \(>0\)
        for \(k \leftarrow 1, \ldots, r\) do
            \(C_{k} \leftarrow \operatorname{Action}\left(g_{k}^{-1}, D_{i}\right) \quad \triangleright\) matrix of \(d_{i}^{g_{k}}\)
            \(Q_{k} \leftarrow \operatorname{FACTOR}\left(Q_{k} D_{i}, C_{k}\right) \quad \triangleright\) matrix of \(\hat{\psi}_{i}^{g_{k}}\)
            for all \(j \in \operatorname{DEGREes}\left(Q_{k}\right)\) do
                    \(B_{i, j} \leftarrow B_{i, j} \cup\left\{\operatorname{TrACE}\left(Q_{k}, j\right)\right\} \quad \triangleright\) append to (possibly empty) list
                end for
                end for
        end for
        return \(B\)
    end function
```

[KR00, Theorem 1.5.7], the residue classes $\bar{b}_{1}, \ldots, \bar{b}_{s}$ modulo im $d_{1}$ form a $\mathbb{k}$-basis of $M_{j}$. For $g \in G$, we may write

$$
g \bar{b}_{v}=\sum_{u=1}^{s} a_{u v} \bar{b}_{u}
$$

for some $a_{u v} \in \mathbb{k}$. The character of $G$ on $M_{j}$ evaluated at $g$ is the trace of the matrix $\left[a_{u v}\right]$. It is worth noting that the input of Algorithm 1 provides all the information needed to compute these characters (and more).

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    ${ }^{1}$ This assumption includes many multigradings of interest in algebra and geometry. If preferred, the reader may safely assume to be working with a standard grading.

