# Conjugacy classes and centralisers in wreath products 

Dominik Bernhardt, Alice C. Niemeyer,<br>Friedrich Rober, Lucas Wollenhaupt

November 29, 2021


#### Abstract

In analogy to the disjoint cycle decomposition in permutation groups, Ore and Specht define a decomposition of elements of the full monomial group and exploit this to describe conjugacy classes and centralisers of elements in the full monomial group. We generalise their results to wreath products whose base group need not be finite and whose top group acts faithfully on a finite set. We parameterise conjugacy classes and centralisers of elements in such wreath products explicitly. For finite wreath products, our approach yields efficient algorithms for finding conjugating elements, conjugacy classes, and centralisers.


## 1 Introduction

Wreath product constructions feature prominently in the theory of permutation groups, see for example [1,8] and references therein. A general approach to working with permutation groups is first to apply a reduction to primitive groups and subsequently to study primitive groups. Many problems for primitive groups are solved with the help of the O'Nan-Scott Theorem, see for example [4,10], in which most classes are defined via wreath products.
To our knowledge, Specht was the first to describe the conjugacy classes of the full monomial group, namely $K 2_{\Gamma} \operatorname{Sym}(\Gamma)$ for a finite set $\Gamma$, see [14, Sätze II, III, IV]. Specht and Ore define a wreath cycle decomposition for elements of full monomial groups in analogy to the disjoint cycle decomposition of elements of the symmetric group. Moreover, Ore gives criteria when two elements are conjugate in the group and determines the centraliser of an element, see [11, Theorem 8].
In order to make Specht's and Ore's theory more widely known, we have decided to restate their results in the modern language of wreath products. Moreover, we have extended the results to the more general setting of arbitrary wreath products $K{ }_{\Gamma} H$. We no longer require $H$ to be the full symmetric group on $\Gamma$, however, we assume that $H$ acts faithfully on $\Gamma$. For a detailed description of the groups we investigate, see Hypothesis A.

We describe the wreath cycle decomposition and define a territory decomposition for an element of $K \imath_{\Gamma} H$, see Definition 23, which generalises the type of a wreath product element defined by Specht [14, (8b)]. Viewing elements in wreath products in a disjoint wreath cycle decomposition quickly becomes very intuitive and highlights much of the underlying structure. For example, it is just as easy to read off the order of an element in disjoint wreath cycle decomposition in a wreath product as it is to read off the order of an element from its disjoint cycle decomposition in the symmetric group. Additionally, it is a very efficient way of representing elements of wreath products on a computer.
The main aim of this paper is to prove several results that can be summarised as follows:

Theorem 1. Two elements $w=(f, h)$ and $v=(e, g)$ of $K{l_{\Gamma}} \operatorname{Sym}(\Gamma)$ are conjugate in $K{l_{\Gamma}}_{\Gamma}$ if and only if there exists an element $t \in H$ that conjugates $h$ to $g$ and maps the territory decomposition of $w$ to that of $v$. Moreover, there exists an explicit, computable bijection from an iterated cartesian product into the conjugacy classes of $K{ }_{\Gamma} H$.

Theorem 2. There exists an explicit, computable bijection from an iterated cartesian product into the centraliser of an element of $K{ }_{{ }_{\Gamma}} \operatorname{Sym}(\Gamma)$ in $K{ }_{{ }_{\Gamma}} H$. Moreover, this centraliser is an extension of two groups.

The statements of our main theorems facilitate efficient computation of centralisers, conjugacy classes of elements and conjugacy testing on a computer. This approach has been implemented in the GAP-package WPE [12] by the third author. For example, one is now able to test elements for conjugacy and to compute conjugating elements in groups as large as $S_{25} \backslash S_{100}$ in a few seconds. For further computational results, see Section 6 , Cannon and Holt describe algorithms in [3] to compute centralisers and conjugacy classes of elements and perform conjugacy testing in a finite group with trivial soluble radical. They embed the given group into a direct product of certain wreath products and solve these tasks for each direct factor. Hulpke, see [6], presents an algorithm to compute the conjugacy classes in finite permutation groups in which he considers a more general situation of subdirect products of the base group. Compared to both [3] and [6], our methods are further reaching for wreath products as we exploit the underlying wreath cycle decomposition. We translate the explicit descriptions in Theorems 1 and 2 into very efficient algorithms in practice, see Section 6. We hope that our methods could be used to improve the algorithms of Cannon and Holt when treating the wreath products that occur as direct factors.

### 1.1 The structure of this paper

In Section 2 in Theorem 9 we restate, in the modern language of wreath products, Ore's decomposition of a wreath product element generalising the decomposition of a permutation into disjoint cycles. In Section 3 we first define the territory decomposition of a wreath product element, see Definition 23 , Next, we give a criterion to decide whether two elements of $K{ }_{{ }_{\Gamma}} \operatorname{Sym}(\Gamma)$ are conjugate in $K{ }_{\Gamma} H$ and, if they are, construct a conjugating element, see Theorem 27, Theorem 35 in Section 4 parameterises the conjugacy classes of
$K{ }_{\Gamma} H$ explicitly. Theorem 1 immediately follows from Theorems 27] and 35. Theorem 47 parameterises the centraliser of a wreath product element in $K{\imath_{\Gamma}}_{\Gamma}$ explicitly. Together with Theorem 48 this immediately implies Theorem 2, As some of the results may seem quite technical on first reading, we have illustrated them with several examples. The final section gives evidence of the computational power of our results.

## 2 Wreath Cycle Decompositions in Wreath Products

The aim of this section is to generalise the concept of a disjoint cycle decomposition of permutations to arbitrary elements of wreath products. The main statement of this section is Theorem 9 in which we give an explicit decomposition of wreath product elements as a product of disjoint wreath cycles. This result is well known, see for example Ore [11] or Kerber et. al. [7, Section 4.2]. We adapt this theorem to our notation and give an explicit constructive formula for the decomposition of a wreath product element. For a permutation $h \in \operatorname{Sym}(\Gamma)$, we denote by $\operatorname{supp}_{\Gamma}(h)$ its support on $\Gamma$, i.e. the set of points in $\Gamma$ moved by $h$. The set of fixed points of a permutation $h$ is denoted by fix ${ }_{\Gamma}(h)$. If the choice of $\Gamma$ is clear from the context we omit it. For the entire paper, we fix the following setting:

Hypothesis A. Let $K$ be a not necessarily finite group, $\Gamma$ a finite set and $H \leq \operatorname{Sym}(\Gamma)$. Further, set $W:=K \imath_{\Gamma} H:=K^{\Gamma} \rtimes H=\left\{(f, h): f \in K^{\Gamma}, h \in H\right\}$ and denote by $\left.S:=K\right\rangle_{\Gamma} \operatorname{Sym}(\Gamma)$ the full monomial group (on $K$ with respect to $\Gamma$ ). We denote the set of functions from $\Gamma$ to $K$ by $K^{\Gamma}$ and apply functions from the right, i.e. we write $[\gamma] f$ for the image of $\gamma \in \Gamma$ under $f \in K^{\Gamma}$. Accordingly, all groups act from the right.

We start by extending the concepts of support and cycles in permutation groups to arbitrary wreath products. Wreath cycles were already introduced by Ore [11] who called them monomial cycles.

Definition 3. Let $w:=(f, h), v:=(e, g) \in K{ }_{{ }_{\Gamma}} H$.

1. The element $h \in H$ is called the top component of $w=(f, h)$ and the element $f \in K^{\Gamma}$ is called the base component of $w$.
2. We define $\operatorname{terr}_{\Gamma}(w):=\operatorname{supp}_{\Gamma}(h) \cup\left\{\gamma \in \Gamma:[\gamma] f \neq 1_{K}\right\}$. We call terr${ }_{\Gamma}(w)$ the territory of $w$. If the choice of $\Gamma$ is clear from context or of no importance, we write terr $(w)$ instead.
3. We call $w=(f, h)$ a wreath cycle if either $h$ induces the identity on $\Gamma$ and $|\operatorname{terr}(w)|=1$ or $h$ induces a single non-trivial cycle in its action on $\Gamma$ and $\operatorname{terr}(w)=\operatorname{supp}(h)$.
4. We say $w$ and $v$ are disjoint if $\operatorname{terr}(w) \cap \operatorname{terr}(v)=\varnothing$.

If $\Gamma=\{1, \ldots, n\} \subseteq \mathbb{Z}_{>0}$, we denote an element $w:=(f, h) \in W$ as $(f, h)=\left(f_{1}, \ldots, f_{n} ; h\right)$, where for $i \in \Gamma$ the element $f_{i}:=[i] f$ is called the $i$-th base component of $w$. For $v:=(e, g)=$ $\left(e_{1}, \ldots, e_{n} ; g\right) \in W$, multiplication is given by

$$
w \cdot v=\left(f \cdot e^{h^{-1}}, h \cdot g\right)=\left(f_{1} \cdot e_{1^{h}}, \ldots, f_{n} \cdot e_{n^{h}} ; h \cdot g\right)
$$

and hence inverses are given by

$$
w^{-1}=\left(\left(f^{-1}\right)^{h}, h^{-1}\right)=\left(\left(f_{1^{h^{-1}}}\right)^{-1}, \ldots,\left(f_{n^{h^{-1}}}\right)^{-1} ; h^{-1}\right) .
$$

Example 4. Let $K:=\operatorname{Sym}(\{1, \ldots, 4\}), \Gamma:=\{1, \ldots, 8\}$ and $S:=K \imath \operatorname{Sym}(\Gamma)$. Throughout this paper the element
with $w=(f, h) \in S$ is used in all examples. The territory of $w$ is

$$
\operatorname{terr}(w)=\{1,2,3,4,5,6\} \cup\{1,2,4,5,7\}=\{1,2,3,4,5,6,7\}
$$

In Lemma 6 we prove that elements with disjoint territories commute, which can be viewed of as an extension of the fact that permutations with disjoint support commute. Now consider the following two elements of $S$ :

$$
\begin{aligned}
& v:=\left(\begin{array}{ccccccccc}
1 & 2 \\
(0), & (), & 3 & 4 & 5 & { }^{6} & (), & (), & (), \\
1 & (1,2), & 8 & \text { (top } & () & () \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text { top }
\end{array}\right), \\
& u:=(\quad(1,2)(3,4), \quad(3,4), \quad(), \quad(), \quad(), \quad(), \quad(), \quad() ;(1,2)) \text {. }
\end{aligned}
$$

These elements are examples of the two different types of wreath cycles in part 3 of Definition 3. For $v$, the top component acts trivially on $\Gamma$ and $|\operatorname{terr}(v)|=|\{7\}|=1$. For $u$, we observe that the top component of $u$ induces the single cycle (1,2) on $\Gamma$ and $\operatorname{terr}(u)=$ $\{1,2\}=\operatorname{supp}((1,2))$.

It is sometimes useful to write the territory of a wreath product element as a disjoint union of the support of the top component and the fixed points of the top component that are contained in the territory of the wreath product element. This will be used throughout this paper in several proofs.

Remark 5. Let $w:=(f, h) \in K{\ell_{\Gamma}} H$. Then $\operatorname{terr}(w)=\operatorname{supp}(h) \cup(\operatorname{fix}(h) \cap \operatorname{terr}(w))$.
The next lemma shows that disjoint wreath product elements commute.
Lemma 6. Let $w=(f, h)$ and $v=(e, g)$ be two disjoint elements of $K{ }_{\Gamma} H$. Then $w$ and $v$ commute.

Proof. Let $w=(f, h)$ and $v=(e, g)$ be disjoint. We first show

$$
f e=e f, \quad e^{h}=e \text { and } f^{g}=f .
$$

As $g$ and $h$ are disjoint, it follows that $[\gamma] f=1_{K}$ or $[\gamma] e=1_{K}$ and thus we have $[\gamma](f \cdot e)=$ $[\gamma](e \cdot f)$ for any $\gamma \in \Gamma$. Further, for all $\gamma \in \operatorname{supp}(h)$ we have $\gamma, \gamma^{h} \notin \operatorname{terr}(v)$ and therefore
$[\gamma] e^{h^{-1}}=\left[\gamma^{h}\right] e=1_{K}=[\gamma] e$. Additionally, if $\gamma \notin \operatorname{supp}(h)$ we have $[\gamma] e^{h^{-1}}=\left[\gamma^{h}\right] e=[\gamma] e$. This, together with a similar argument for $f^{g^{-1}}$, shows that

$$
f e^{h^{-1}}=f e=e f=e f^{g^{-1}}
$$

Furthermore, $h$ and $g$ are disjoint permutations of $\Gamma$ and therefore commute. Thus

$$
(f, h)(e, g)=\left(f e^{h^{-1}}, h g\right)=\left(e f^{g^{-1}}, g h\right)=(e, g)(f, h)
$$

Analogously to a disjoint cycle decomposition for permutations we define a wreath cycle decomposition of a wreath product element into disjoint wreath cycles. Just as the individual cycles in a disjoint cycle decomposition of a permutation need not be elements of the group, the wreath cycles in the disjoint wreath cycle decomposition need not be elements of the wreath product.

Definition 7. A wreath cycle decomposition for a wreath product element $w \in W=K{ }_{\Gamma} H$ is a decomposition of $w$ as $w=\prod_{i=1}^{\ell} w_{i}$ where the $w_{i} \in S=K{ }_{l_{\Gamma}} \operatorname{Sym}(\Gamma)$ are pairwise disjoint wreath cycles for all $i=1, \ldots, \ell$.

Our next aim is to give a formula to compute a wreath cycle decomposition and to show that it is unique up to ordering of the factors. Throughout this paper, the following function simplifies the definition of certain elements. In particular, it is used in the construction of a disjoint wreath cycle decomposition.

Definition 8. Let $\Omega \subseteq \Gamma$. For a map $f: \Gamma \rightarrow K$ we define

$$
\left.f\right|_{\Omega} ^{\Gamma}: \Gamma \rightarrow K, \gamma \mapsto \begin{cases}{[\gamma] f,} & \text { if } \gamma \in \Omega \\ 1_{K}, & \text { else } .\end{cases}
$$

For simplicity, we set $\left.f\right|_{\gamma} ^{\Gamma}:=\left.f\right|_{\{\gamma\}} ^{\Gamma}$ for $\gamma \in \Gamma$.
Note that the function $\left.f\right|_{\Omega} ^{\Gamma}$ agrees with $f$ on all of $\Omega$ and maps the elements of $\Gamma \backslash \Omega$ to $1_{K}$.
Theorem 9. Every element of $K\rangle_{\Gamma} H$ can be written as a finite product of disjoint wreath cycles in $S$. This decomposition is unique up to ordering of the factors.

Proof. Let $w:=(f, h) \in K{\imath_{\Gamma} H \text {. We prove the first statement of this theorem by giving an }}$ explicit wreath cycle decomposition of $w$.
Let $h=h_{1} \cdots h_{\ell}$ be the disjoint cycle decomposition of $h$ in $\operatorname{Sym}(\Gamma)$. We claim that the following is the desired decomposition of $w$ :

$$
w=\prod_{i=1}^{\ell}\left(\left.f\right|_{\operatorname{supp}\left(h_{i}\right)} ^{\Gamma}, h_{i}\right) \quad \cdot \prod_{\gamma \in \operatorname{fix}(h) \cap \operatorname{terr}(w)}\left(\left.f\right|_{\gamma} ^{\Gamma}, 1_{H}\right) .
$$

Note that all factors are disjoint wreath cycles because their territories are pairwise disjoint. We now show equality by using Lemma6, i.e. the fact that disjoint wreath product
elements commute:

$$
\begin{align*}
& \prod_{i=1}^{\ell}\left(\left.f\right|_{\operatorname{supp}\left(h_{i}\right)} ^{\Gamma}, h_{i}\right) \cdot \prod_{\gamma \in \operatorname{fix}(h) \cap \operatorname{terr}(w)}\left(\left.f\right|_{\gamma} ^{\Gamma}, 1_{H}\right) \\
= & \left(\prod_{i=1}^{\ell}\left(\left.f\right|_{\operatorname{supp}\left(h_{i}\right)} ^{\Gamma}, 1_{H}\right) \cdot\left(1_{K^{\Gamma}}, h_{i}\right)\right)_{\gamma \in \operatorname{fix}(h) \cap \operatorname{terr}(w)}\left(\left.f\right|_{\gamma} ^{\Gamma}, 1_{H}\right)  \tag{1}\\
= & \left(\prod_{i=1}^{\ell}\left(\left.f\right|_{\operatorname{supp}\left(h_{i}\right)} ^{\Gamma}, 1_{H}\right) \cdot \prod_{\gamma \in \operatorname{fix}(h) \text { กterr }(w)}\left(\left.f\right|_{\gamma} ^{\Gamma}, 1_{H}\right)\right) \cdot \prod_{i=1}^{\ell}\left(1_{K^{\Gamma}}, h_{i}\right) \\
= & \left(f, 1_{H}\right) \cdot\left(1_{K^{\Gamma}}, h\right)=(f, h)=w .
\end{align*}
$$

To prove uniqueness, suppose $w=\prod_{i=1}^{m}\left(e_{i}, g_{i}\right)$ is another disjoint wreath cycle decomposition of $w$. Thus $h=g_{1} \cdots g_{m}$ is a disjoint cycle decomposition of $h$ and we may assume $g_{1}, \ldots, g_{k} \neq 1_{H}$ and $g_{k+1}, \ldots, g_{m}=1_{H}$ for some $1 \leq k \leq m$. As the $g_{i}$ are pairwise disjoint and due to the uniqueness of the cycle decomposition in $\operatorname{Sym}(\Gamma)$ we have $k=\ell$ and there exists a permutation $\sigma \in \operatorname{Sym}(\ell)$ such that $h_{i}=g_{i^{\sigma}}$ for all $i \in\{1, \ldots, \ell\}$. Hence, without loss of generality we may assume $h_{i}=g_{i}$ for all $i \in\{1, \ldots, \ell\}$. By a calculation along the lines of computation (1) above, we obtain $\prod_{i=1}^{m} e_{i}=f$. For a given $\gamma \in \operatorname{terr}(w)$ with $[\gamma] f \neq 1_{K}$, there exists a unique $j \in\{1, \ldots, m\}$ such that $[\gamma] e_{j}=[\gamma] f$.
First assume $\gamma \in \operatorname{supp}\left(h_{i}\right)$ for some $i \in\{1, \ldots, \ell\}$. Note

$$
\gamma \in \operatorname{terr}\left(\left(e_{j}, g_{j}\right)\right)=\operatorname{supp}\left(g_{j}\right) \text { and } \gamma \in \operatorname{supp}\left(h_{i}\right)=\operatorname{terr}\left(\left(\left.f\right|_{\operatorname{supp}\left(h_{i}\right)} ^{\Gamma}, h_{i}\right)\right)
$$

and since $h=g_{1} \cdots g_{\ell}=h_{1} \cdots h_{\ell}$ are two cycle decompositions of the same permutation $h$ we must have $i=j$. Since the ( $e_{j}, g_{j}$ ) are pairwise disjoint wreath cycles for all $j=1, \ldots, m$ and $f=e_{1} \cdots e_{m}$ we obtain $e_{i}=\left.f\right|_{\operatorname{supp}\left(h_{i}\right)} ^{\Gamma}$ for all $i \in\{1, \ldots, \ell\}$. Now we assume $\gamma \in \operatorname{fix}(h) \cap \operatorname{terr}(w)$. As $\gamma \in \operatorname{fix}(h)$, we have $j>\ell$. As $g_{j}=1_{H}$ and $\left|\operatorname{terr}\left(\left(e_{j}, g_{j}\right)\right)\right|=$ 1 we have $e_{j}=\left.f\right|_{\gamma} ^{\Gamma}$.

Example 10. Let $K:=\operatorname{Sym}(\{1, \ldots, 4\}), \Gamma:=\{1, \ldots, 8\}$ and $S:=K$ 亿 $\operatorname{Sym}(\Gamma)$. We give an example of a wreath cycle decomposition by considering the element $w=(f, h) \in S$ from Example 4 :

$$
w:=\left((1,2)(3,4),(3,4), \quad(),(1,2),(1,2,3), \quad \begin{array}{cccccc}
1 & (), & (1,2), & () ; & (1,2)(3,4)(5,6)) .
\end{array}\right.
$$

We obtain the decomposition by applying the steps of the proof of Theorem 9 to $w$. We know $\operatorname{terr}(w)=\{1,2,3,4,5,6,7\}$ and $\operatorname{supp}(h)=\{1,2,3,4,5,6\}$. A disjoint cycle decomposition of $h$ is $h=h_{1} \cdot h_{2} \cdot h_{3}$ with $h_{1}=(1,2), h_{2}=(3,4)$ and $h_{3}=(5,6)$. Next we compute base components in $K^{\Gamma}$ for each cycle in the disjoint cycle decomposition of $h$, which are given by $f_{1}:=\left.f\right|_{\operatorname{supp}\left(h_{1}\right)} ^{\Gamma}, f_{2}:=\left.f\right|_{\operatorname{supp}\left(h_{2}\right)} ^{\Gamma}$ and $f_{3}:=\left.f\right|_{\operatorname{supp}\left(h_{3}\right)} ^{\Gamma}$ As an example, we consider

$$
f_{1}=\left.f\right|_{\operatorname{supp}\left(h_{1}\right)} ^{\Gamma}: \Gamma \rightarrow K, \gamma \mapsto \begin{cases}{[\gamma] f,} & \text { if } \gamma \in\{1,2\} \\ (), & \text { else }\end{cases}
$$

and compute some images under this map explicitly. The image of 1 under $f_{1}$ is [1] $f_{1}=$ [1] $f=(1,2)(3,4)$ and $[3] f_{1}=()$. Note $\operatorname{fix}(h) \cap \operatorname{terr}(w)=\{7\}$, so the only remaining point for which we need to compute a base component in $K^{\Gamma}$ is the point 7 which is given by $f_{4}:=\left.f\right|_{\{7\}} ^{\Gamma}$. We set $h_{4}=()$ and $w_{i}=\left(f_{i}, h_{i}\right)$ for $1 \leq i \leq 4$. This yields a wreath cycle decomposition of $w$ into the product of the following four wreath cycles in $S$ :


## 3 Solving the conjugacy problem in wreath products

Recall the setting from Hypothesis A. The main result of this section, Theorem[27, gives a solution to the conjugacy problem for wreath product elements: Given two wreath product elements $w$ and $v$ of $S$, decide whether they are conjugate in $W$ and give a conjugating element if it exists. Theorem 1 is an immediate consequence.
The wreath cycle decomposition plays a crucial role in the solution of the conjugacy problem. It turns out that it suffices to define many concepts only for wreath cycles and then apply them to every cycle in the wreath cycle decomposition. We first define a property of a wreath cycle which is invariant under conjugation by elements of $S$. For this we need the following function.

Definition 11. Define $\mathscr{C}(W):=\{w \in W: w$ is a wreath cycle $\}$ as the set of all wreath cycles in $W$ and the Yade-map by

$$
[-,-] \text { Yade }: \Gamma \times \mathscr{C}(W) \rightarrow K,(\gamma,(f, h)) \mapsto \prod_{i=0}^{|h|-1}[\gamma]\left(f^{h^{-i}}\right)=\prod_{i=0}^{|h|-1}\left[\gamma^{h^{i}}\right] f
$$

Given a wreath cycle $w=(f, h)$ and $\gamma \in \Gamma$ we call $[\gamma, w]$ Yade the Yade of $w$ in $\gamma$.
Yade stands for Yet another determinant since, after choosing a suitable matrix representation of $K{ }_{\Gamma} \operatorname{Sym}(\Gamma)$, the Yade-map can be interpreted as a matrix determinant whence Ore called it a determinant in [11, p. 19]. James and Kerber also introduce this map in [7, Section 4.3] and call it a cycle product.
Observe that $[\gamma, w]$ Yade $=1_{K}$ for all $\gamma \notin \operatorname{terr}(w)$ for a wreath cycle $w \in S$. In computations involving the Yade-map, it is therefore enough to consider the restriction $[-, w]$ Yade $\left.\right|_{\text {terr }(w)}$. Suppose that $w$ is a wreath cycle. The next lemma shows that [ $\alpha, w$ ]Yade and [ $\beta, w$ ]Yade are $K$-conjugate whenever $\alpha$ and $\beta$ are contained in the territory of $w$. We also give a conjugating element explicitly.
 conjugate in $K$ for every $\alpha, \beta \in \operatorname{terr}(w)$. For some $j \in\{0, \ldots,|h|-1\}$ with $\alpha^{h^{j}}=\beta$, we have

$$
([\alpha, w] \text { Yade })^{y}=[\beta, w] \text { Yade, } \quad \text { where } y=\prod_{i=0}^{j-1}\left[\alpha^{h^{i}}\right] f .
$$

Proof. As $w$ is a wreath cycle we prove this statement by distinguishing two cases according to part 3 of Definition 3, Either $h=1_{H}$ and terr $(w)$ is a singleton or $h$ induces a single non-trivial cycle on $\Gamma$ and $\operatorname{terr}(w)=\operatorname{supp}(h)$.
First, consider the case of $h=1_{H}$. Then $\alpha=\beta$ since terr $(w)$ is a singleton and the claim becomes trivial. Now consider the case of $h$ being a non-trivial cycle. Let $\alpha, \beta \in \operatorname{supp}(h)$. Then we have

$$
a:=[\alpha, w] \text { Yade }=\prod_{i=0}^{|h|-1}[\alpha] f^{h^{-i}}=\prod_{i=0}^{|h|-1}\left[\alpha^{h^{i}}\right] f
$$

and

$$
b:=[\beta, w] \text { Yade }=\prod_{i=0}^{|h|-1}[\beta] f^{h^{-i}}=\prod_{i=0}^{|h|-1}\left[\beta^{h^{i}}\right] f .
$$

Since $h$ is a cycle there is a $j \in\{0, \ldots,|h|-1\}$ such that $\alpha^{h^{j}}=\beta$. A straight forward computation shows that for $y:=\prod_{i=0}^{j-1}\left[\alpha^{h^{i}}\right] f$ we have $b=a^{y}$ thus $a$ and $b$ are conjugate in $K$ and the claim follows.

The previous lemma justifies the following definition.
Definition 13. Let $w=(f, h) \in S$ be a wreath cycle and let $\alpha \in \operatorname{terr}(w)$ be fixed. We call the conjugacy class $([\alpha, w] \text { Yade })^{K}$ of $[\alpha, w]$ Yade in $K$ the Yade-class of $w$.

The following example shows how to compute the Yade-map.
Example 14. Let $K:=\operatorname{Sym}(\{1, \ldots, 4\}), \Gamma:=\{1, \ldots, 8\}$ and $S:=K$ 亿 $\operatorname{Sym}(\Gamma)$. Consider the wreath cycles from Example 4 .

$$
\begin{aligned}
& v:=\left(\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text { top } \\
(), & (), & (), & (), & (), & (), & (1,2), & () ; & () \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text { top }
\end{array}\right), \\
& u:=(e, g)=(\quad(1,2)(3,4), \quad(3,4), \quad(), \quad(), \quad(), \quad(), \quad(), \quad() ; \quad(1,2)) .
\end{aligned}
$$

We compute the Yade of $u$ in 1 :

$$
[1, u] \text { Yade }=\prod_{i=0}^{|g|-1}\left[1^{g^{i}}\right] e=[1] e \cdot[2] e=(1,2)(3,4) \cdot(3,4)=(1,2)
$$

Then

$$
\begin{aligned}
& {[-, v] \text { Yade }:=\left(\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
(), & (), & (), & (), & (), & (), & (1,2), & () \\
& 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline-, u] \text { Yade }:=\left(\begin{array}{llllll} 
& (1,2), & (1,2), & (), & (), & (), \\
(), & (), & ()
\end{array}\right) .
\end{array} .\right.}
\end{aligned}
$$

The following corollary shows how to compute the order of a wreath cycle as a product of the order of its top component and of an element in $K$. This and the following lemma are stated in Ore, see [11, Theorem 4]. Note that by $|w|$ we denote the order of $w \in W$.

Corollary 15. Let $w=(f, h) \in S$ be a wreath cycle. Then for any $\gamma \in \operatorname{terr}(w)$ we have

$$
|w|=\mid[-, w] \text { Yade }|\cdot| h|=|[\gamma, w] \text { Yade }|\cdot| h \mid
$$

where $\mid[-, w]$ Yade $\mid$ is the order of $[-, w]$ Yade as an element of $K^{\Gamma}$.
Proof. By the definition of the Yade-map and the multiplication rule in wreath products we have $w^{|h|}=\left([-, w]\right.$ Yade, $\left.1_{H}\right)$, thus $|w|=\mid[-, w]$ Yade $|\cdot| h \mid$. It remains to show that for any $\gamma \in \operatorname{terr}(w)$ we have $\mid[-, w]$ Yade $|=|[\gamma, w]$ Yade $\mid$. This follows immediately by the pointwise multiplication in $K^{\Gamma}$ and the fact that the non-trivial images of $[-, w]$ Yade are conjugate in $K$ by Lemma 12 and thus all have the same order in $K$.

We can generalize this observation to arbitrary wreath product elements. This can be seen as an analogue to computing orders of permutations given in disjoint cycle decomposition. Recall that LCM denotes the least common multiple.

Lemma 16. Let $w=w_{1} \cdots w_{\ell} \in W$ be a wreath cycle decomposition of $w$ with $w_{i}=\left(f_{i}, h_{i}\right) \in$ $S$. Then

$$
|w|=\operatorname{LCM}\left(\left|w_{1}\right|, \ldots,\left|w_{\ell}\right|\right)=\operatorname{LCM}\left(\mid\left[-, w_{1}\right] \text { Yade }|\cdot| h_{1}|, \ldots,|\left[-, w_{\ell}\right] \text { Yade }|\cdot| h_{\ell} \mid\right) .
$$

Proof. The elements $w_{i}$ and $w_{j}$ commute pairwise for all $1 \leq i, j \leq \ell$ and therefore $|w|=$ $\operatorname{LCM}\left(\left|w_{1}\right|, \ldots,\left|w_{\ell}\right|\right)$. The result follows by Corollary 15 ,

We now turn our attention to the conjugacy problem in wreath products and first consider the case of wreath cycles. It turns out that a generalization of the length of a cycle, which we call the load, will be very useful. Recall that by Lemma 12 we know that [ $\alpha, w$ ] Yade and $[\beta, w]$ Yade are conjugate in $K$ for all $\alpha, \beta \in \operatorname{terr}(w)$.

Definition 17. Let $w=(f, h) \in W$ be a wreath cycle and let $\alpha \in \operatorname{terr}(w)$. We define the load of $w$ as the tuple

$$
\operatorname{ld}(w):=\left(([\alpha, w] \text { Yade })^{K},|h|\right) .
$$

We now prove that the load of a wreath cycle is invariant under conjugation in $S$.
Lemma 18. Let $w=(f, h) \in S$ be a wreath cycle. For every $a=(s, t) \in S$ the conjugate $a^{-1} w a=w^{a}$ is a wreath cycle such that for each $\gamma \in \Gamma$ we have

$$
\left[\gamma, w^{a}\right] \text { Yade }=\left[\gamma^{t^{-1}}\right] s^{-1} \cdot\left[\gamma^{t^{-1}}, w\right] \text { Yade } \cdot\left[\gamma^{t^{-1}}\right] s
$$

In particular, $\operatorname{ld}(w)=\operatorname{ld}\left(w^{a}\right)$.

Proof. Observe that

$$
w^{a}=(f, h)^{(s, t)}=\left(\left(s^{-1}\right)^{t} f^{t} s^{h^{-1} t}, t^{-1} h t\right) .
$$

We first show that $w^{a}$ is a wreath cycle. For this, first consider the case of $h=1_{H}$. Then

$$
w^{a}=\left(\left(f^{s}\right)^{t}, 1_{H}\right) .
$$

Thus the top component of $w^{a}$ is trivial. Since for every $\gamma \in \Gamma$ we have

$$
[\gamma]\left(f^{s}\right)^{t}=\left[\gamma^{t^{-1}}\right] s^{-1} \cdot\left[\gamma^{t^{-1}}\right] f \cdot\left[\gamma^{t^{-1}}\right] s
$$

and as $w$ is a wreath cycle with trivial top component, $w$ has exactly one non-trivial base component and we obtain $\left|\operatorname{terr}\left(w^{a}\right)\right|=1$.
Now let us consider $h \neq 1_{H}$. The top component $h$ is a non-trivial cycle and hence $h^{t}$ is a non-trivial cycle. Further for any $\gamma \notin \operatorname{supp}\left(h^{t}\right)$ we know $\gamma^{t^{-1}} \notin \operatorname{supp}(h)$ and $\gamma^{t^{-1} h}=\gamma^{t^{-1}}$.
Thus we have

$$
[\gamma]\left(\left(s^{-1}\right)^{t} f^{t} s^{h^{-1} t}\right)=\left[\gamma^{t^{-1}}\right] s^{-1} \cdot\left[\gamma^{t^{-1}}\right] f \cdot\left[\gamma^{t^{-1}}\right] s=1_{K}
$$

and thus terr $\left(w^{a}\right)=\operatorname{supp}\left(h^{t}\right)$. The claimed identity for $\left[\gamma, w^{a}\right]$ Yade follows from the following calculation:

$$
\begin{aligned}
{\left[\gamma, w^{a}\right] \text { Yade } } & =\prod_{i=0}^{\left|h^{t}\right|-1}[\gamma]\left(\left(s^{-1}\right)^{t} f^{t} s^{h^{-1}} t\right)^{\left(h^{t}\right)^{-i}}=\prod_{i=0}^{|h|-1}[\gamma]\left(s^{-1} f s^{h^{-1}}\right)^{t t^{-1} h^{-i} t} \\
& =\prod_{i=0}^{|h|-1}\left[\gamma^{t^{-1}}\right]\left(\left(s^{-1}\right)^{h^{-i}} f^{h^{-i}} s^{h^{-i-1}}\right)=\left[\gamma^{t^{-1}}\right] s^{-1} \cdot\left[\gamma^{t^{-1}}, w\right] \text { Yade } \cdot\left[\gamma^{t^{-1}}\right] s .
\end{aligned}
$$

One can now derive the following corollary relating the territories of conjugate wreath cycles.

Corollary 19. Let $w=(f, h) \in S$ be a wreath cycle and $a=(s, t) \in S$. Then

$$
\operatorname{terr}\left(w^{a}\right)=\operatorname{terr}(w)^{t}
$$

Proof. The claim follows from the proof of Lemma 18 if $h=1_{H}$. If $h \neq 1_{H}$, the claim follows because $w$ and $w^{a}$ are wreath cycles and hence

$$
\operatorname{terr}\left(w^{a}\right)=\operatorname{supp}\left(h^{t}\right)=\operatorname{supp}(h)^{t}=\operatorname{terr}(w)^{t} .
$$

Next we show the converse of Lemma 18 , i.e. that two wreath cycles with the same load are always conjugate in $S$. In order to do so, we need the following lemma, which is [11, Theorem 2] (we repeat the proof as we require the constructed elements).

Lemma 20. Let $K$ be a group, $\ell \in \mathbb{Z}_{>0}, a_{0}, a_{1}, \ldots, a_{\ell-1}, b_{0}, b_{1}, \ldots, b_{\ell-1} \in K$ and set

$$
a:=a_{0} a_{1} \cdots a_{\ell-1} \text { and } b:=b_{0} b_{1} \cdots b_{\ell-1}
$$

Then there exist $c_{0}, \ldots, c_{\ell} \in K$ where $c_{0}=c_{\ell}$ such that $b_{i}=c_{i}^{-1} a_{i} c_{i+1}$ for every $i \in\{0, \ldots, \ell-$ 1 \} if and only if $a$ and $b$ are conjugate in $K$.

Proof. Let $a$ and $b$ be conjugate in $K$ and $c_{0} \in K$ such that $a^{c_{0}}=b$, in particular $\left(a_{0} a_{1} \cdots a_{\ell-1}\right)^{c_{0}}=$ $b_{0} b_{1} \cdots b_{\ell-1}$. Define $c_{i}:=a_{i-1}^{-1} a_{i-2}^{-1} \cdots a_{0}^{-1} c_{0} b_{0} b_{1} \cdots b_{i-1}$, then a short calculation shows that $c_{i}$ has the desired property. Conversely, if such $c_{0}, \ldots, c_{\ell} \in K$ exist, we have $\left(a_{0} a_{1} \cdots a_{\ell-1}\right)^{c_{0}}=$ $b_{0} b_{1} \cdots b_{\ell-1}$ and thus $a^{c_{0}}=b$.

This enables us to show the following lemma, which is [11, Theorem 6] restated in our notation.

Lemma 21. Let $w, v \in S$ be wreath cycles. Then $w$ and $v$ are conjugate in $S$ if and only if they have the same load.

Proof. The if direction is Lemma 18, Now consider the wreath cycles $w=(f, h), v=(e, g) \in$ $S$ with the same load $\left(k^{K}, j\right)$. We construct an element $a=(s, t) \in S$ with $w^{a}=v$.
First assume $j=1$. Then $|\operatorname{terr}(w)|=|\operatorname{terr}(v)|=1$ and for some $\gamma_{0} \in \operatorname{terr}(w)$ there exists a $t \in \operatorname{Sym}(\Gamma)$ with $\gamma_{0}^{t} \in \operatorname{terr}(v)$. As the Yade-classes of $w$ and $v$ agree there exists a $c \in K$ with

$$
c^{-1} \cdot\left[\gamma_{0}\right] f \cdot c=c^{-1} \cdot\left[\gamma_{0}, w\right] \text { Yade } \cdot c=\left[\gamma_{0}^{t}, v\right] \text { Yade }=\left[\gamma_{0}^{t}\right] e .
$$

Using

$$
s: \Gamma \rightarrow K, \gamma \mapsto \begin{cases}c, & \text { if } \gamma=\gamma_{0} \\ 1_{K}, & \text { else }\end{cases}
$$

we conclude $w^{(s, t)}=v$.
Now assume $j>1$. As the order of $h$ equals that of $g$ there exists a $t \in \operatorname{Sym}(\Gamma)$ with $h^{t}=g$. We continue by constructing the base component $s$ of a conjugating element. Fix $\gamma_{0} \in \operatorname{supp}(h)$. As

$$
\left[\gamma_{0}, w\right] \text { Yade }=\prod_{i=0}^{j-1}\left[\gamma_{0}\right] f^{h^{-i}} \text { and }\left[\gamma_{0}^{t}, v\right] \text { Yade }=\prod_{i=0}^{j-1}\left[\gamma_{0}^{t}\right] e^{g^{-i}} \in K
$$

are conjugate in $K$, by Lemma 20 there exist $c_{0}, \ldots, c_{j} \in K$ where $c_{0}=c_{j}$ such that

$$
c_{i}^{-1} \cdot\left[\gamma_{0}\right] f^{h^{-i}} \cdot c_{i+1}=\left[\gamma_{0}^{t}\right] e^{g^{-i}}
$$

Now define

$$
s: \Gamma \rightarrow K, \gamma \mapsto \begin{cases}1_{K}, & \text { if } \gamma \notin \operatorname{terr}(w) \\ c_{i}, & \text { if } \gamma=\gamma_{0}^{h^{i}} \text { for } 0 \leq i<j\end{cases}
$$

We now show that $a=(s, t)$ has the desired property. First note

$$
\operatorname{terr}(v)=\operatorname{supp}(g)=\operatorname{supp}(h)^{t}=\operatorname{terr}(w)^{t}
$$

and

$$
w^{a}=(f, h)^{(s, t)}=\left(\left(s^{-1}\right)^{t} \cdot f^{t} \cdot s^{h^{-1} \cdot t}, t^{-1} \cdot h \cdot t\right)=\left(\left(s^{-1}\right)^{t} \cdot f^{t} \cdot s^{h^{-1} \cdot t}, g\right) .
$$

Next we prove that the base component of $w^{a}$ equals that of $v$. Let $\gamma \in \Gamma$ and observe

$$
\begin{equation*}
[\gamma]\left(\left(s^{-1}\right)^{t} \cdot f^{t} \cdot s^{h^{-1} \cdot t}\right)=\left[\gamma^{t^{-1}}\right] s^{-1} \cdot\left[\gamma^{t^{-1}}\right] f \cdot\left[\gamma^{t^{-1} \cdot h}\right] s \tag{2}
\end{equation*}
$$

We now distinguish two cases for $\gamma \in \Gamma$ :

- $\gamma \notin \operatorname{terr}(v)$ : Using Corollary 19 we obtain $\operatorname{terr}\left(w^{a}\right)=\operatorname{terr}(w)^{t}=\operatorname{terr}(v)$ and hence

$$
[\gamma]\left(\left(s^{-1}\right)^{t} \cdot f^{t} s^{h^{-1} \cdot t}\right)=1_{K}=[\gamma] e
$$

- $\gamma \in \operatorname{terr}(v)=\operatorname{supp}(g)$ : Then $\gamma^{t^{-1}} \in \operatorname{supp}(h)$ and there exists an $i \in\{0, \ldots, j-1\}$ with $\gamma^{t^{-1}}=\gamma_{0}^{h^{i}}$. By Equation 2

$$
\begin{aligned}
{[\gamma]\left(\left(s^{-1}\right)^{t} \cdot f^{t} \cdot s^{h^{-1} t}\right) } & =\left[\gamma^{t^{-1}}\right] s^{-1} \cdot\left[\gamma^{t^{-1}}\right] f \cdot\left[\gamma^{t^{-1} \cdot h}\right] s=c_{i}^{-1} \cdot\left[\gamma_{0}^{h^{i}}\right] f \cdot c_{i+1} \\
& =\left[\gamma_{0}^{t}\right] e^{g^{-i}}=\left[\gamma_{0}^{t \cdot g^{i}}\right] e=\left[\gamma_{0}^{t \cdot\left(t^{-1} \cdot h \cdot t\right)^{i}}\right] e=\left[\gamma_{0}^{h^{i} \cdot t}\right] e=[\gamma] e
\end{aligned}
$$

which concludes the proof.
We generalise the above results to arbitrary wreath product elements. For this, we need to introduce a few additional concepts.

Definition 22. Let $w=(f, h)=w_{1} \cdots w_{\ell} \in W$ be an arbitrary wreath product element in disjoint wreath cycle decomposition, i.e. the $w_{i} \in S$ are disjoint wreath cycles. Define $\mathscr{C}(w):=\left\{w_{1}, \ldots, w_{\ell}\right\}$ as the set of all wreath cycles in a disjoint wreath cycle decomposition of $w$ and

$$
\mathscr{L}(w):=\{\operatorname{ld}(z): z \in \mathscr{C}(w)\} \text { and for } L \in \mathscr{L}(w) \text { set } \mathscr{C}(L, w):=\{z \in \mathscr{C}(w): \operatorname{ld}(z)=L\}
$$

Further, set $\mathscr{C}^{*}(w):=\left\{z=(e, g) \in \mathscr{C}(w): g \neq 1_{H}\right\}$ as the set of all wreath cycles of $w$ with non-trivial top component.

Note that the above sets are well-defined as the cycles in a wreath cycle decomposition are unique up to permutation. Moreover, one can now write

$$
\begin{equation*}
w=w_{1} \cdots w_{\ell}=\prod_{L \in \mathscr{L}(w) z \in \mathscr{C}(L, w)} z \tag{3}
\end{equation*}
$$

The disjoint cycle decomposition of a permutation induces a partition of the underlying set by considering the support of each cycle. We extend this concept and decompose the territory of each wreath product element in accordance with the decomposition given in Equation 3 .

Definition 23. Let $w \in W$ and define the territory decomposition of $w$ as

$$
\mathscr{P}(w)=\underset{L \in \mathscr{L}(w)}{\times}\{\operatorname{terr}(z): z \in \mathscr{C}(L, w)\} .
$$

Remark 24. Suppose that $K$ has finitely many conjugacy classes $k_{1}^{K}, \ldots, k_{r}^{K}$ and let $w \in W$. Suppose that $A=\times_{L \in \mathscr{L}(w)} A_{L}$ is a set indexed by $\mathscr{L}(w)$. In our examples and in analogy to the example by Specht in [14] we illustrate $A$ as an $r \times|\Gamma|$-matrix $M$ via $M_{i, j}=A_{L}$, if $L=\left(k_{i}^{K}, j\right) \in L(w)$ and $M_{i, j}=\varnothing$ else.

Example 25. Let $K:=\operatorname{Sym}(\{1, \ldots, 4\}), \Gamma:=\{1, \ldots, 8\}$ and $S:=K \imath \operatorname{Sym}(\Gamma)$. We give an example for $\mathscr{P}(w)$ by considering the wreath cycle decomposition of the element $w=(f, h) \in S$ from Example 10, For this we choose

$$
\mathscr{R}(K):=\left\{k_{1}:=(), k_{2}:=(1,2), k_{3}:=(1,2)(3,4), k_{4}:=(1,2,3), k_{5}:=(1,2,3,4)\right\}
$$

as a set of representatives for the conjugacy classes of $K$ and set $r:=|\mathscr{R}(K)|=5$. Recall the wreath cycle decomposition of $w$ given by the following four wreath cycles:

$$
\begin{aligned}
& w_{1}=\left(\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text { top } \\
(1,2)(3,4), & (3,4), & (), & (), & (), & (), & (), & () ; & (1,2)
\end{array}\right), \\
& w_{2}=\left(\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text { top } \\
(), & (), & (), & (1,2), & (), & (), & (), & () ; & (3,4)
\end{array}\right) \text {, } \\
& w_{3}=\left(\quad \begin{array}{lllllllll}
1 & 2 \\
(), & (), & (), & (), & (1,2,3), & (), & (), & () ; & (5,6)
\end{array}\right), \\
& w_{4}=\left(\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text { top } \\
(), & (), & (), & (), & (), & (), & (1,2), & () ; & ()
\end{array}\right) \text {. }
\end{aligned}
$$

The Yade classes for each of the $w_{i}$ are

$$
\begin{aligned}
& {\left[1, w_{1}\right] \text { Yade }=(1,2)(3,4) \cdot(3,4)=(1,2) \in k_{2}^{K},} \\
& {\left[3, w_{2}\right] \text { Yade }=() \cdot(1,2)=(1,2) \in k_{2}^{K},} \\
& {\left[5, w_{3}\right] \text { Yade }=(1,2,3) \cdot()=(1,2,3) \in k_{4}^{K},} \\
& {\left[7, w_{4}\right] \text { Yade }=(1,2) \in k_{2}^{K} .}
\end{aligned}
$$

We write $\mathscr{P}(w)$ in matrix notation, where according to Remark 24 the entry in position $(i, j)$ is $\cup_{z \in \mathscr{C}(w)}\left\{\operatorname{terr}(z): \operatorname{ld}(z)=\left(k_{i}^{K}, j\right)\right\}$. We omit the additional set braces for each entry of $\mathscr{P}(w)$ and write $\cdot$ if an entry of $\mathscr{P}(w)$ is the empty set, so

$$
\mathscr{P}(w)=\left[\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\{7\} & \{1,2\},\{3,4\} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
k_{1} \\
k_{2} \\
k_{3} \\
\cdot & \{5,6\} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
k_{4} \\
\bullet_{4}
\end{array}\right.
$$

Corollary 26. Let $w=(f, h) \in W$ and $a=(s, t) \in W$. Then $\mathscr{P}\left(w^{a}\right)=\mathscr{P}(w)^{t}$ and in particular $\operatorname{terr}\left(w^{a}\right)=\operatorname{terr}(w)^{t}$.

Proof. By Lemma 18 we have $\operatorname{ld}(z)=\operatorname{ld}\left(z^{a}\right)$ for all $z \in \mathscr{C}(w)$ and thus $\mathscr{C}\left(L, w^{a}\right)=\mathscr{C}(L, w)^{a}$ for all $L \in \mathscr{L}(w)$. Therefore,

$$
\begin{aligned}
\mathscr{P}\left(w^{a}\right) & =\underset{L \in \mathscr{L}\left(w^{a}\right)}{\times}\left\{\operatorname{terr}(z): z \in \mathscr{C}\left(L, w^{a}\right)\right\}=\underset{L \in \mathscr{L}(w)}{\times}\left\{\operatorname{terr}\left(z^{a}\right): z \in \mathscr{C}(L, w)\right\} \\
& =\underset{L \in \mathscr{L}(w)}{\times}\left\{\operatorname{terr}(z)^{t}: z \in \mathscr{C}(L, w)\right\}=\mathscr{P}(w)^{t}
\end{aligned}
$$

by Corollary 19 .

The following theorem, which is an explicit version of Theorem 1 , gives us a way to test whether two elements of the wreath product $S$ are conjugate in $W$ after having computed their wreath cycle decomposition. The proof of this theorem is constructive as it shows how to construct a conjugating element if it exists and therefore solves the conjugacy problem.

Theorem 27. Two elements $w=(f, h), v=(e, g) \in S$ are conjugate in $W$ if and only if there exists a $t \in H$ such that $h^{t}=g$ and $\mathscr{P}(w)^{t}=\mathscr{P}(v)$.

Proof. We first show the if direction: If there exists $a=(s, t) \in W$ such that $w^{a}=v$, then $h^{t}=g$ and by Corollary 26 the claim follows.
We now show the only-if direction, so assume the existence of a $t \in H$ with $\mathscr{P}(w)^{t}=\mathscr{P}(v)$ and $h^{t}=g$, hence $\mathscr{L}(w)=\mathscr{L}(v)$. As we can identify each cycle with its territory, $t$ induces a unique bijection $\sigma: \mathscr{C}(w) \xrightarrow{1: 1} \mathscr{C}(v)$ such that $\sigma$ maps $\mathscr{C}(L, w)$ to $\mathscr{C}(L, v)$ bijectively for all $L \in \mathscr{L}(w)$ and $\operatorname{terr}(z)^{t}=\operatorname{terr}([z] \sigma)$ for all $z \in \mathscr{C}(L, w)$. Moreover, $t$ conjugates the top component of $z \in \mathscr{C}(L, w)$ to the top component of $[z] \sigma$ as $h^{t}=g$. Additionally, $\operatorname{ld}(z)=$ $\operatorname{ld}([z] \sigma)$ for all $z \in \mathscr{C}(L, w)$.
We now construct a base component $s \in K^{\Gamma}$ such that $a=(s, t) \in W$ satisfies $w^{a}=v$. As in the proof of Lemma 21, for each $z \in \mathscr{C}(L, w)$, we can find an $s_{z} \in K^{\Gamma}$ with $\left(s_{z}, t\right)^{-1} \cdot z \cdot\left(s_{z}, t\right)=$ $[z] \sigma$. Define

$$
s:=\prod_{z \in \mathscr{C}(w)} s_{z}
$$

as the product of all these maps for all disjoint wreath cycles in a decomposition of $w$. By construction $[\gamma] s=[\gamma] s_{z}$ if $\gamma \in \operatorname{terr}(z)$ since $[\gamma] s_{y}=1_{K}$ for all $y \neq z$. Thus

$$
w^{a}=\prod_{z \in \mathscr{C}(w)} z^{(s, t)}=\prod_{z \in \mathscr{C}(w)}\left(s_{z}, t\right)^{-1} \cdot z \cdot\left(s_{z}, t\right)=\prod_{z \in \mathscr{C}(w)}[z] \sigma=v,
$$

which concludes the proof.
There are several ways of finding a conjugating element. One could formulate this as a single backtrack problem seeking an element $t \in H$ which simultaneously conjugates $h$ to $g$ and maps $\mathscr{P}(w)$ to $\mathscr{P}(v)$. Alternatively, one could first find an element $t \in H$ with $h=g^{t}$, then compute $C_{H}(h)$ and check if $\mathscr{P}(v)^{t}$ is in the orbit of $\mathscr{P}(w)$ under $C_{H}(h)$. If not, $w$ and $v$ are not conjugate. Backtrack may also be helpful in the second approach. The implementation by the third author in [12] uses the second approach. For many search problems in permutation groups, partition backtrack is still the state of the art algorithm. For an exposition on the backtrack strategy frequently used, namely partition backtrack, see for example [9].
The special case of $H=\operatorname{Sym}(\Gamma)$ is already described in [7] Theorem 4.2.8]. We obtain the same result as a corollary to the above theorem.

Corollary 28. Two elements $w, v \in S$ are conjugate in $S$ if and only if $\mathscr{L}(w)=\mathscr{L}(v)$ and $|\mathscr{C}(L, w)|=|\mathscr{C}(L, v)|$ for all $L \in \mathscr{L}(w)$.

Applying the above theory, we give an example for conjugacy testing in wreath products.

Example 29. We use the notation from Example 25 and highlight wreath cycle decompositions by colouring points of $\Gamma$. We computed a wreath cycle decomposition of $w$ in Example 10 and $\mathscr{P}(w)$ in Example 25 using the same colours. Consider the elements $w=(f, h), v=(e, g) \in S:$

$$
\begin{aligned}
& w:=\left(\begin{array}{ccccccc}
1 & (1,2)(3,4),(3,4),(), & (1,2), & (1,2,3),(),(1,2), & () ; & (1,2)(3,4)(5,6)),
\end{array}\right. \\
& v:=(\quad(3,4), \quad(), \quad(),(1,2,3),(1,2), \quad(), \quad(), \quad(3,4) ;(1,2)(3,4)(5,6)) .
\end{aligned}
$$

The top components of $w$ and $v$ are equal and we have

Observe $\mathscr{L}(v)=\mathscr{L}(w)$ and $|\mathscr{C}(L, v)|=|\mathscr{C}(L, w)|$ for all $L \in \mathscr{L}(v)$ and thus $v$ and $w$ are conjugate in $S=K$ 亿 $\operatorname{Sym}(\Gamma)$ by Corollary 28 ,
Now let us consider three different choices for the top group $H_{i}(1 \leq i \leq 3)$ for the wreath product $W_{i}:=K \iota H_{i}$ and decide whether $w$ and $v$ are conjugate in $W$. For this we need to check if $\mathscr{P}(v) \in \mathscr{P}(w)^{C_{H}(h)}$. We use three different top groups:

$$
\begin{aligned}
H_{1}:=\langle(1,2)(3,4),(1,2,3,4),(5,6),(7,8)\rangle & \cong D_{8} \times C_{2} \times C_{2}, \\
H_{2}:=\langle(1,2)(3,4)(5,6),(3,5)(4,6)(7,8)\rangle & \cong C_{2} \times C_{2}, \\
H_{3}:= & \langle(1,2)(3,4)(5,6),(7,8)\rangle
\end{aligned}
$$

Since the submatrix induced by the columns from 3 to 8 in $\mathscr{P}(w)$ consists of empty entries, we condense the notation and write $\cdot$ for a matrix of appropriate dimension with empty
entries. We have

$$
\begin{aligned}
& \mathscr{P}(w)^{C_{H_{3}}(h)}=\mathscr{P}(w)^{C_{H_{1}}(h)} .
\end{aligned}
$$

Hence $w$ and $v$ are not conjugate in $W_{1}$ and $W_{3}$, but are conjugate in $W_{2}$. Now let us construct an element $a=(s, t) \in W_{2}$ with $w^{a}=v$. First we compute an element $t \in C_{H_{2}}(h)$ with $\mathscr{P}(w)^{t}=\mathscr{P}(v)$, for example $t=(3,5)(4,6)(7,8) \in C_{H_{2}}(h)$. Using the above colouring to encode the wreath cycles, we write $w$ and $v$ in a disjoint wreath cycle decomposition as in Equation 3, where $w_{i, j, \ell}$ denotes the $\ell$-th wreath cycle of load $\left(k_{i}^{K}, j\right)$ :

$$
\begin{aligned}
w & =w_{2,1,1} \cdot w_{2,2,1} \cdot w_{2,2,2} \cdot w_{4,2,1}, \\
v & =v_{2,1,1} \cdot v_{2,2,1} \cdot v_{2,2,2} \cdot v_{4,2,1} .
\end{aligned}
$$

Suppose $w_{i, j, \ell}=\left(f_{i, j, \ell}, h_{i, j, \ell}\right)$ and $v_{i, j, \ell}=\left(e_{i, j, \ell}, h_{i, j, \ell}\right)$. Next, compute the bijection $\sigma: \mathscr{C}(w) \xrightarrow{1: 1}$ $\mathscr{C}(v)$ recording the mapping induced by $a$, i.e. $z^{a}=[z] \sigma$ for every $z \in \mathscr{C}(w)$, where the base component of $a=(s, t)$ is yet to be constructed. As $\sigma$ only depends on the top component of $a$, it is already determined by $\left[w_{i, j, \ell}\right] \sigma=v_{i, j, \ell}$. Now it remains to construct the base component $s \in K^{\Gamma}$ of the conjugating element $a$. For this we construct elements $s_{i, j, \ell} \in K^{\Gamma}$, such that $w_{i, j, \ell}^{\left(s_{i, j, \ell}, t\right)}=v_{i, j, \ell}$ as in the proof of Theorem 27. We demonstrate this for the wreath cycle $w_{2,2,1}$ which takes the place of the cycle $z$ in the proof. First we compute two Yades:

$$
\begin{aligned}
& {\left[1, w_{2,2,1}\right] \text { Yade }=(1,2)(3,4) \cdot(3,4)=(1,2),} \\
& {\left[1^{t}, u_{2,2,1}\right] \text { Yade }=(3,4) \cdot()=(3,4) .}
\end{aligned}
$$

Next, note that $x:=(1,3)(2,4) \in K$ conjugates $(1,2)$ to $(3,4)$. Then compute the following elements of $K$ used in Lemma 20,

$$
\begin{aligned}
& c_{0}:=x=(1,3)(2,4) \\
& c_{1}:=[1] f_{2,2,1}{ }^{-1} \cdot c_{0} \cdot\left[1^{t}\right] e_{2,2,1}=((1,2)(3,4))^{-1} \cdot(1,3)(2,4) \cdot(3,4)=(1,3,2,4) .
\end{aligned}
$$

We proceed to define the element $s_{2,2,1}$ as

$$
s_{2,2,1}: \Gamma \rightarrow K, \gamma \mapsto \begin{cases}1_{K}, & \text { if } \gamma \notin \operatorname{terr}\left(w_{2,2,1}\right), \\ c_{i}, & \text { if } \gamma=1^{h_{2,2,1}^{i}} \text { for } 0 \leq i \leq 1 .\end{cases}
$$

Hence we have

$$
s_{2,2,1}=\left(\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
(1,3)(2,4), & (1,3,2,4), & (), & (), & (), & (), & (), & ()
\end{array}\right) .
$$

Analogously we compute $s_{2,1,1}, s_{2,2,1}$ and $s_{4,2,1}$. This yields

$$
\begin{aligned}
& \begin{array}{llllllllll}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text { top }
\end{array} \\
& a=((1,3)(2,4),(1,3,2,4),(),(1,2),(),(1,3,2),(1,3)(2,4),() ;(3,5)(4,6)(7,8)) .
\end{aligned}
$$

## 4 Conjugacy classes in wreath products

Recall the setting from Hypothesis A. In this section, we parameterise the $W$-conjugacy classes $w^{W}$ of arbitrary elements $w \in S$. This is achieved via defining bijections between certain iterated cartesian products and $w^{W}$. The notation is chosen to reflect the way we construct these conjugacy classes. When using a cartesian product $A \times B$ to parameterise a set, one should view this as first choosing an element in $A$ and then in $B$.
The conjugacy class sizes and the number of conjugacy classes in the full monomial group $S$ are already known, see for example James and Kerber [7, 4.2.9, 4.2.10].
The maps we define to parameterise the $W$-conjugacy classes of elements of $S$ are constructed in such a way that they can be implemented directly in computer algebra systems such as GAP[5] or MAGMA[2]. For instance, the third author implemented computation of $W$-conjugacy classes in this way in the GAP package WPE, see [12].
We extend the notation $\mathscr{C}(W)$ from Definition 11 naturally to refer to wreath cycles of $W$ with a fixed top component $h$.

Definition 30. For $h \in H$ denote the set of all wreath cycles of $W$ with top component $h$ by

$$
\mathscr{C}(W, h):=\{(f, h) \in W:(f, h) \text { is a wreath cycle }\} .
$$

Let $1_{H} \neq h \in H$ be a single cycle and $\gamma_{0} \in \operatorname{supp}(h)$. Given an $x \in K$, we construct all wreath cycles $w=(f, h) \in K\rangle_{\Gamma} H$ with top component $h$ and $\left[\gamma_{0}, w\right]$ Yade $=x$. The image of the map [ $\left.h, \gamma_{0}, x,-\right] \mathscr{B}$ defined in the following lemma yields the base components of the desired wreath cycles.

Lemma 31. Let $1_{H} \neq h \in H$ be a cycle, $\gamma_{0} \in \operatorname{supp}(h)$ and $x \in K$. Define

$$
\left[h, \gamma_{0}, x,-\right] \mathscr{B}: K^{\operatorname{supp}(h) \backslash\left\{\gamma_{0}\right\}} \hookrightarrow K^{\Gamma}, d \mapsto\left[h, \gamma_{0}, x, d\right] \mathscr{B}
$$

where

$$
\left[h, \gamma_{0}, x, d\right] \mathscr{B}=\gamma \mapsto \begin{cases}x \cdot \prod_{i=1}^{|h|-1}\left[\gamma^{h^{|| |-i}}\right] d^{-1}, & \text { if } \gamma=\gamma_{0} \\ {[\gamma] d,} & \text { if } \gamma \in \operatorname{supp}(h) \backslash\left\{\gamma_{0}\right\} \\ 1_{K}, & \text { if } \gamma \in \Gamma \backslash \operatorname{supp}(h)\end{cases}
$$

Then the following statements hold:

1. $\left[h, \gamma_{0}, x,-\right] \mathscr{B}$ is an injection.
2. $w:=\left(\left[h, \gamma_{0}, x, d\right] \mathscr{B}, h\right) \in \mathscr{C}(W, h)$, i.e. $w$ is a wreath cycle of $W$ with top component $h$.
3. The map $\left[h, \gamma_{0}, x,-\right] \mathscr{B}$ induces a bijection

$$
\varphi: K^{\operatorname{supp}(h) \backslash\left\{\gamma_{0}\right\}} \xrightarrow{1: 1}\left\{w \in \mathscr{C}(W, h):\left[\gamma_{0}, w\right] \text { Yade }=x\right\}, d \mapsto\left(\left[h, \gamma_{0}, x, d\right] \mathscr{B}, h\right) .
$$

4. If $K$ is finite, we have

$$
\mid\left\{w \in \mathscr{C}(W, h):\left[\gamma_{0}, w\right] \text { Yade }=x\right\}\left|=|K|^{|h|-1}\right.
$$

Proof. First note that $\left[h, \gamma_{0}, x,-\right] \mathscr{B}$ is injective since we embed $d$ in $\operatorname{supp}(h) \backslash\left\{\gamma_{0}\right\}$. Next we show that $\varphi$ is well-defined. Let $d \in K^{\operatorname{supp}(h) \backslash\left\{\gamma_{0}\right\}}$, define $f:=\left[h, \gamma_{0}, x, d\right] \mathscr{B}$ and $w:=(f, h)$. Then $\operatorname{terr}(w)=\operatorname{supp}(h)$ and

$$
\left[\gamma_{0}, w\right] \text { Yade }=\prod_{i=0}^{|h|-1}\left[\gamma_{0}^{h^{i}}\right] f=x \cdot \prod_{i=1}^{|h|-1}\left[\gamma_{0}^{h^{|h|-i}}\right] d^{-1} \cdot \prod_{i=1}^{|h|-1}\left[\gamma_{0}^{h^{i}}\right] d=x .
$$

This shows $\operatorname{im}(\varphi) \subseteq\left\{w \in \mathscr{C}(W, h):\left[\gamma_{0}, w\right]\right.$ Yade $\left.=x\right\}$ and clearly $\varphi$ is injective. Now let $w=(f, h) \in\left\{w \in \mathscr{C}(W, h):\left[\gamma_{0}, w\right]\right.$ Yade $\left.=x\right\}$. We construct an element $d \in K^{\operatorname{supp}(h) \backslash\left\{\gamma_{0}\right\}}$ with $\left[h, \gamma_{0}, x, d\right] \mathscr{B}=f$. Define $d: \operatorname{supp}(h) \backslash\left\{\gamma_{0}\right\} \rightarrow K, \gamma \mapsto[\gamma] f$ and $e:=\left[h, \gamma_{0}, x, d\right] \mathscr{B}$. For any $\gamma \in \Gamma \backslash\left\{\gamma_{0}\right\}$ we have $[\gamma] e=[\gamma] f$. Now suppose $\gamma=\gamma_{0}$. Then

$$
\left[\gamma_{0}\right] e=\left[\gamma_{0}, w\right] \text { Yade } \cdot \prod_{i=1}^{|h|-1}\left[\gamma_{0}^{h^{|h|-i}}\right] f^{-1}=\left[\gamma_{0}\right] f
$$

In particular $e=f$ and thus part 3 follows immediately.
Example 32. Let $K:=\operatorname{Sym}(\{1, \ldots, 4\}), \Gamma:=\{1, \ldots, 8\}$ and $S:=K \imath \operatorname{Sym}(\Gamma)$. We compute the image of $d$ under $\left[h, \gamma_{0}, x,-\right] \mathscr{B}$, where

$$
h:=(1,4), \quad \gamma_{0}:=1, \quad x:=(3,4) \quad \text { and } \quad d:=((1,2,3,4)) .
$$

Let $e:=\left[h, \gamma_{0}, x, d\right] \mathscr{B}$. For all points $\gamma \in \Gamma \backslash \operatorname{supp}(h)=\{2,3,5,6,7,8\}$, we obtain $[\gamma] e=1_{K}$. Next we compute the images of the points of $\operatorname{supp}(h)=\{1,4\}$ under $e$ : [1] $=x \cdot\left[1^{h}\right] d^{-1}=$ $(3,4) \cdot(1,4,3,2)=(1,4,2)$ and $[4] e=[4] d=(1,2,3,4)$. Then

$$
\left[h, \gamma_{0}, x, d\right] \mathscr{B}=\left(\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
(1,4,2), & (), & (), & (1,2,3,4), & (), & (), & (), \\
()
\end{array}\right) .
$$

In particular,

$$
\left(\left[h, \gamma_{0}, x, d\right] \mathscr{B}, h\right)=((1,4,2),(),(),(1,2,3,4),(),(),(),() ;(1,4))
$$

is contained in $\left\{w \in \mathscr{C}(W, h):\left[\gamma_{0}, w\right]\right.$ Yade $\left.=x\right\}$.
Using the above lemma we can derive the proportion of wreath cycles with non-trivial top-component whose Yade in a given point lies in a given subset $P \subseteq K$.

Corollary 33. Let $K$ be finite, $1 \neq h \in H$ a single cycle, $\gamma \in \operatorname{supp}(h)$ and $P \subseteq K$. Then

$$
\frac{\mid\{w \in \mathscr{C}(W, h):[\gamma, w] \text { Yade } \in P\} \mid}{|\mathscr{C}(W, h)|}=\frac{|P|}{|K|} .
$$

Proof.

$$
\begin{aligned}
\mid\{w \in \mathscr{C}(W, h):[\gamma, w] \text { Yade } \in P\} \mid & =\mid \bigcup_{x \in P}\{w \in \mathscr{C}(W, h):[\gamma, w] \text { Yade }=x\} \mid \\
& =\sum_{x \in P} \mid\{w \in \mathscr{C}(W, h):[\gamma, w] \text { Yade }=x\}\left|=|K|^{|h|-1}\right| P \mid .
\end{aligned}
$$

The result follows as $|\mathscr{C}(W, h)|=|K|^{|h|}$.
We now turn towards parameterising conjugacy classes of arbitrary wreath product elements.
We commence our investigation with the $W$-conjugacy class of a single wreath cycle in $S$. Note that if the top component of a wreath cycle is the identity we require an embedding of $\Gamma \times K$ into the base component, which we achieve via the map $\mathscr{E}$ defined in the following lemma, replacing the map $\mathscr{B}$ from Lemma 31.

Lemma 34. Let $w=(f, h) \in S$ be a wreath cycle and $\gamma_{0} \in \operatorname{terr}(w)$. Then, for $h \neq 1_{H}$, the map

$$
h^{H} \times\left(\left[\gamma_{0}, w\right] \text { Yade }\right)^{K} \times K^{\operatorname{terr}(w) \backslash\left\{\gamma_{0}\right\}} \xrightarrow{1: 1} w^{W},\left(h^{t}, x, d\right) \mapsto\left(\left[h^{t}, \gamma_{0}^{t}, x, d^{t}\right] \mathscr{B}, h^{t}\right)
$$

and for $h=1_{H}$, the map

$$
\gamma_{0}^{H} \times\left(\left[\gamma_{0}, w\right] \text { Yade }\right)^{K} \xrightarrow{1: 1} w^{W},\left(\gamma_{0}^{t}, x\right) \mapsto\left(\left[\gamma_{0}^{t}, x\right] \mathscr{E}, 1_{H}\right)
$$

are bijections into the conjugacy class of $w$ in $W$, where $\mathscr{B}$ is as in Lemma 31] and

$$
\mathscr{E}: \Gamma \times K \hookrightarrow K^{\Gamma},\left(\gamma^{\prime}, x\right) \mapsto\left[\gamma^{\prime}, x\right] \mathscr{E}
$$

with

$$
\left[\gamma^{\prime}, x\right] \mathscr{E}: \Gamma \rightarrow K, \gamma \mapsto \begin{cases}x, & \text { if } \gamma=\gamma^{\prime} \\ 1_{K}, & \text { if } \gamma \neq \gamma^{\prime}\end{cases}
$$

is an injection.

Proof. Let $v=(e, g) \in S$. Using Theorem 27, $w$ is conjugate to $v$ in $W$ if and only if $v$ is a wreath cycle, there exists a $t \in H$ with $h^{t}=g$ and $\mathscr{P}(w)^{t}=\mathscr{P}(v)$. As $w$ and $v$ are wreath cycles and the load of a wreath cycle is invariant under conjugation we know that $\mathscr{P}(w)^{t}=\mathscr{P}(v)$ holds if and only if $\left(\left[\gamma_{0}^{t}, v\right] \text { Yade }\right)^{K}=\left(\left[\gamma_{0}, w\right] \text { Yade }\right)^{K}$.
We now parameterise the different elements $v \in S$ one can construct with these properties. Suppose first $h \neq 1_{H}$. Then we need to choose an element $g=h^{t} \in h^{H}$ as a possible top component for $v$ and an element $x \in\left(\left[\gamma_{0}, w\right] \text { Yade }\right)^{K}$ as the Yade of $v$ at the point $\gamma_{0}^{t}$. By Lemma 31, the possible elements $v$ with the above requirements are parameterised by the following bijection

$$
K^{\operatorname{terr}(w) \backslash\left\{\gamma_{0}\right\}} \xrightarrow{1: 1}\left\{v \in \mathscr{C}(S, g):\left[\gamma_{0}^{t}, v\right] \text { Yade }=x\right\}, d \mapsto\left(\left[h^{t}, \gamma_{0}^{t}, x, d^{t}\right] \mathscr{B}, h^{t}\right),
$$

observing $K^{\operatorname{terr}(w) \backslash\left\{\gamma_{0}\right\}} \xrightarrow{1: 1} K^{\operatorname{supp}\left(h^{t}\right) \backslash\left\{\gamma_{0}^{t}\right\}}, d \mapsto d^{t}$ is a bijection.
Now let $h=1_{H}$. Then the top component of $v$ must equal $1_{H}$. Since $v$ is a wreath cycle, we must have $|\operatorname{terr}(v)|=1$. For the territory of $v$, we need to choose $\gamma_{0}^{t} \in \gamma_{0}^{H}$ and $x=$ $\left[\gamma_{0}^{t}\right] e=\left[\gamma_{0}^{t}, v\right]$ Yade $\in\left(\left[\gamma_{0}, w\right] \text { Yade }\right)^{K}$. These choices fix the element $v$, which must equal $\left(\left[\gamma_{0}^{t}, x\right] \mathscr{E}, 1_{H}\right)$.

We now parameterise the $W$-conjugacy class of arbitrary wreath product elements $w \in S$. Recall the definition of $\mathscr{C}(w)$ and $\mathscr{C}^{*}(w)$ from Definition 22,

Theorem 35. Let $w=(f, h) \in S$ be an arbitrary wreath product element and for each $z \in \mathscr{C}(w)$ choose $\gamma_{z} \in \operatorname{terr}(z)$. Fix a transversal $\left\{t_{1}, \ldots, t_{m}\right\}$ of the right cosets of $C_{H}(h)$ in $H$. Then the $W$-conjugacy class of $w$ is parameterised by the following bijection

$$
\begin{aligned}
& h^{H} \times \mathscr{P}(w)^{C_{H}(h)} \times \underset{z \in \mathscr{C}(w)}{\times}\left(\left[\gamma_{z}, z\right] \text { Yade }\right)^{K} \times \underset{z \in \mathscr{C}^{*}(w)}{\times} K^{\operatorname{supp}\left(h_{z}\right) \backslash\left\{\gamma_{z}\right\}} \xrightarrow{1: 1} w^{W} \\
& \left(h^{t_{a}}, \mathscr{P}(w)^{c}, x, d\right) \mapsto \prod_{z \in \mathscr{C}(w) \backslash \mathscr{C} *}{ }^{*}(w) \\
& \left(\left[\gamma_{z}^{b_{a}}, x_{z}\right] \mathscr{E}, 1_{H}\right) \cdot \prod_{z \in \mathscr{C}^{*}(w)}\left(\left[h_{z}^{b_{a}}, \gamma_{z}^{b_{a}}, x_{z}, d_{z}^{b_{a}}\right] \mathscr{B}, h_{z}^{b_{a}}\right),
\end{aligned}
$$

where $b_{a}:=c \cdot t_{a}$.
Proof. Let $v=(e, g) \in W$. By Theorem 18, $w$ and $v$ are conjugate in $W$ if and only if there exists a $t \in H$ with $g=h^{t}$ and $\mathscr{P}(v)=\mathscr{P}(w)^{t}$. We first claim that

$$
h^{H} \times \mathscr{P}(w)^{C_{H}(h)} \xrightarrow{1: 1}\left\{\left(h^{t}, \mathscr{P}(w)^{t}\right): t \in H\right\},\left(h^{t_{a}}, \mathscr{P}(w)^{c}\right) \mapsto\left(h^{c t_{a}}, \mathscr{P}(w)^{c t_{a}}\right)
$$

is a bijection and first show injectivity. Fix $1 \leq a, b \leq m$ and $c, c^{\prime} \in C_{H}(h)$ and assume $\left(h^{c t_{a}}, \mathscr{P}(w)^{c t_{a}}\right)=\left(h^{c^{\prime} t_{b}}, \mathscr{P}(w)^{c^{\prime} t_{b}}\right)$. Then $t_{a}=t_{b}$ as they are representatives of right cosets of $C_{H}(h)$ in $H$ and hence $\mathscr{P}(w)^{c}=\mathscr{P}(w)^{c^{\prime}}$. Surjectivity follows as, for an arbitrary $t \in H$, there exists $1 \leq a \leq m$ and $c \in C_{H}(h)$ with $t=c t_{a}$.
Now fix $\left(h^{t}, \mathscr{P}(w)^{t}\right)$ for some $t \in H$. In order to parameterise all elements $x$ of $W$ with top component $h^{t}$ and territory decomposition $\mathscr{P}(w)^{t}$, we consider a wreath cycle decomposition for each such element $x=x_{1} \cdots x_{\ell} \in W$. Note that for each wreath cycle $x_{i} \in S$ in such a decomposition, its load, top component and territory are fixed by our hypothesis. Thus for each $x_{i}$, we only need to consider its base component. By using the maps $\mathscr{E}$ and $\mathscr{B}$ one proceeds as in Lemma 34 .

Example 36. We use the notation from Example 29 and highlight wreath cycle decompositions by colouring points of $\Gamma$. We computed a wreath cycle decomposition of $w$ in Example 10 and $\mathscr{P}(w)$ in Example 25 using the same colours. Consider the element $w=(f, h) \in S$

$$
\left.\left.w:=\left(\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array}\right) \stackrel{\text { top }}{(1,2)(3,4),}(3,4),(),(1,2),(1,2,3), \stackrel{( }{( }\right),(1,2), \stackrel{( }{(1)} ;(1,2)(3,4)(5,6)\right)
$$

with

$$
\mathscr{P}(w)=\left[\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\{7\} & \{1,2\},\{3,4\} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & k_{1} \\
k_{2} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & k_{3} \\
\cdot & \{5,6\} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
k_{4} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right] \begin{aligned}
& k_{5}
\end{aligned}
$$

Using this colouring to encode the wreath cycles, we write $w$ in a disjoint wreath cycle decomposition as in Equation3, where $w_{i, j, \ell}$ denotes the $\ell$-th wreath cycle of load $\left(k_{i}^{K}, j\right)$ :

$$
w=w_{2,1,1} \cdot w_{2,2,1} \cdot w_{2,2,2} \cdot w_{4,2,1}
$$

Further let $w_{i, j, k}=\left(f_{i, j, k}, h_{i, j, k}\right)$ and fix points in the territory of each wreath cycle:

$$
\gamma_{2,1,1}:=7, \quad \gamma_{2,2,1}:=1, \quad \gamma_{2,2,2}:=3, \quad \gamma_{4,2,1}:=5 .
$$

Recall the three different top groups:

$$
\begin{array}{rll}
H_{1}:=\langle(1,2)(3,4),(1,2,3,4),(5,6),(7,8)\rangle & \cong D_{8} \times C_{2} \times C_{2}, \\
H_{2}:= & \langle(1,2)(3,4)(5,6),(3,5)(4,6)(7,8)\rangle & \cong C_{2} \times C_{2} \\
H_{3}:= & \langle(1,2)(3,4)(5,6),(7,8)\rangle & \cong C_{2} \times C_{2}
\end{array}
$$

Using the computations from Example 29 we compute the cardinality of the conjugacy class $w^{W_{i}}$, where $W_{i}:=K \backslash H_{i}$ for $1 \leq i \leq 3$. First note that two factors of the cartesian product occurring in the source of the bijection defined in Theorem 35 do not depend on the chosen top group, namely

$$
\begin{aligned}
& \mid\left(\left[7, w_{2,1,1}\right] \text { Yade }\right)^{K} \times\left(\left[1, w_{2,2,1}\right] \text { Yade }\right)^{K} \times\left(\left[3, w_{2,2,2}\right] \text { Yade }\right)^{K} \times\left(\left[5, w_{4,2,1}\right] \text { Yade }\right)^{K} \mid \\
& =\left|k_{2}^{K} \times k_{2}^{K} \times k_{2}^{K} \times k_{4}^{K}\right|=6^{3} \cdot 8=1,728
\end{aligned}
$$

and

$$
\left|K^{\{1,2\} \backslash\{1\}} \times K^{\{3,4\} \backslash\{3\}} \times K^{\{5,6\} \backslash\{5\}}\right|=24^{3}=13,824 .
$$

We have

$$
\begin{aligned}
\left|w^{W_{1}}\right| & =\left|h^{H_{1}}\right| \cdot\left|\mathscr{P}^{C_{H_{1}}(h)}\right| \cdot 1,728 \cdot 13,824=2 \cdot 2 \cdot 1,728 \cdot 13,824=95,551,488, \\
\left|w^{W_{2}}\right| & =\left|h^{H_{2}}\right| \cdot\left|\mathscr{P}^{C_{H_{2}}(h)}\right| \cdot 1,728 \cdot 13,824=1 \cdot 2 \cdot 1,728 \cdot 13,824=47,775,744, \\
\left|w^{W_{3}}\right| & =\left|w^{W_{2}}\right| .
\end{aligned}
$$

Let $\Phi$ be the bijection from Theorem 35 for the wreath product $W_{1}$, where we fix the transversal $\left\{t_{1}:=(), t_{2}:=(2,4)\right\}$ of the right cosets of $C_{H_{1}}(h)$ in $H_{1}$. We choose an element from the domain of $\Phi$ using the same notation as in the above Theorem:

$$
\begin{aligned}
& t_{a}:=t_{2}, \quad c:=(7,8) \in C_{H_{1}}(h),
\end{aligned}
$$

We compute the image of $\left(h^{t_{a}}, c, x, d\right)$ under $\Phi$ by computing each non-trivial factor in the
product separately:

$$
\begin{aligned}
& u_{2,1,1}=\left(\quad\left[\gamma_{2,1,1} c t_{a}, x_{2,1,1}\right] \mathscr{E}, \quad\right. \text { () ) } \\
& =\left(\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text { top } \\
(), & (), & (), & (), & (), & (), & (),(2,4) ; & ()
\end{array}\right) \text {, } \\
& u_{2,2,1}=\left(\quad\left[h_{2,2,1}{ }^{c t_{a}}, \gamma_{2,2,1}{ }^{c t_{a}}, x_{2,2,1}, d_{2,2,1}{ }^{c t_{a}}\right] \mathscr{B}, \quad h_{2,2,1}{ }^{c t_{a}}\right) \\
& =\left(\begin{array}{cccccccc}
1 & 2 & 3 \\
(1,4,2), & (), & (), & (1,2,3,4), & (), & (), & (), & () ; \\
(1,4)
\end{array}\right), \\
& u_{2,2,2}=\left(\quad\left[h_{2,2,2}{ }^{c t_{a}}, \gamma_{2,2,2}{ }^{c t_{a}}, x_{2,2,2}, d_{2,2,2}{ }^{c t_{a}}\right] \mathscr{B}, \quad h_{2,2,2}{ }^{c t_{a}}\right) \\
& =\left(\begin{array}{cccccccc}
1 & 2 \\
(), & (1,4,3),(3,4), & (), & (), & (), & (), & () ; & (2,3)
\end{array}\right), \\
& u_{4,2,1}=\left(\quad\left[{h_{4,2,1}}^{c t_{a}}, \gamma_{4,2,1}{ }^{c t_{a}}, x_{4,2,1}, d_{4,2,1}{ }^{c t_{a}}\right] \mathscr{B}, \quad \quad h_{4,2,1}{ }^{c t_{a}}\right) \\
& =\left(\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
(), & (), & (), & (), & (1,4,2),(1,2,3),(), & () ; & (5,6)
\end{array}\right) .
\end{aligned}
$$

Then $u:=(e, g)=u_{2,1,1} \cdot u_{2,2,1} \cdot u_{2,2,2} \cdot u_{4,2,1} \in w^{W_{1}}$ is a wreath cycle decomposition of $u=\left[h^{t_{a}}, c, x, d\right] \Phi$, where $u_{i, j, \ell}$ denotes the $\ell$-th wreath cycle of load $\left(k_{i}^{K}, j\right)$. Note that by construction we have $h^{c t_{a}}=g$ and $\mathscr{P}(w)^{c t_{a}}=\mathscr{P}(u)$.

In the following theorem we construct representatives of all conjugacy classes of elements of $W$. For this, we need to consider orbits under the natural action of $C_{H}(h)$ on $\{\mathscr{P}(w)$ : $w=(f, h) \in S\}$ for a fixed $h \in H$. Note that as $K$ need not be finite, the index set $I$ in the theorem below need not be finite either.

Theorem 37. Let $h \in H$ and define $\Omega:=\{\mathscr{P}(w): w=(f, h) \in S\}$ as the set of all territory decompositions of elements of $S$ with fixed top component $h$. Fix a set of representatives $\left\{\mathscr{P}\left(w_{i}\right): i \in I\right\}$ of the orbits under the action of $C_{H}(h)$ on $\Omega$, where $I$ is some index set. Denote by $\gamma_{z}$ the minimum of $\operatorname{terr}(z)$ with respect to a fixed total ordering on $\Gamma$. Then the map

$$
\varphi_{h}:\left\{\mathscr{P}\left(w_{i}\right): i \in I\right\} \hookrightarrow W, \mathscr{P}(w) \mapsto\left(\prod_{z \in \mathscr{C}(w)}\left(\left[\gamma_{z},\left[\gamma_{z}, z\right] \text { Yade }\right] \mathscr{E}, 1_{H}\right)\right) \cdot\left(1_{K^{\Gamma}}, h\right)
$$

is an injective map and $\operatorname{im}\left(\varphi_{h}\right)$ consists of a system of representatives of conjugacy classes of elements of $W$ whose top component is conjugate to $h$ in $H$.
In particular, a system of representatives of $W$-conjugacy classes is given by

$$
\mathscr{R}(W)=\bigcup_{x \in \mathscr{R}(H)} \operatorname{im}\left(\varphi_{x}\right),
$$

where the union ranges over a system of representatives $\mathscr{R}(H)$ of $H$-conjugacy classes.
Proof. Note that for $i \in I$ the elements $w_{i}$ and $v_{i}:=\left[\mathscr{P}\left(w_{i}\right)\right] \varphi_{h}$ are conjugate in $W$ by Theorem 27, since $\mathscr{P}\left(w_{i}\right)=\mathscr{P}\left(v_{i}\right)$ and the top components of $w_{i}$ and $v_{i}$ are identical. We first prove that elements of $\operatorname{im}\left(\varphi_{h}\right)$ for fixed $h \in H$ are non-conjugate. Let $i \neq j \in I$. By Theorem 27] $w_{i}$ and $w_{j}$ are conjugate in $W$ if and only if there exists a $t \in H$ with $h^{t}=h$ and $\mathscr{P}\left(w_{i}\right)^{t}=\mathscr{P}\left(w_{j}\right)$, so $t \in C_{H}(h)$. As $\mathscr{P}\left(w_{i}\right)$ and $\mathscr{P}\left(w_{j}\right)$ are in different centraliser orbits we conclude that $w_{i}$ and $w_{j}$ are not conjugate in $W$.
Let $\left(f, h^{t}\right) \in W$ for some $t \in H$. We now show that $\left(f, h^{t}\right)$ is conjugate to an element in $\operatorname{im}\left(\varphi_{h}\right)$. Note $\left(f, h^{t}\right)^{\left(1_{K^{\Gamma}}, t^{-1}\right)}=(e, h)$ for a suitable $e \in K^{\Gamma}$, so it suffices to show that $(e, h)$ is conjugate to an element of $\operatorname{im}\left(\varphi_{h}\right)$. There exists an $i \in I$ and a $c \in C_{H}(h)$ with $\mathscr{P}((e, h))=$ $\mathscr{P}\left(w_{i}\right)^{c}$, hence $\left[\mathscr{P}\left(w_{i}\right)\right] \varphi_{h}$ is conjugate to $(e, h)$ in $W$.

Example 38. Recall the groups $H_{i}$ and $W_{i}$ from Example 36. The number of conjugacy classes of $H_{i}$ for $i=1,2,3$ is

$$
\left|\mathscr{R}\left(H_{1}\right)\right|=20,\left|\mathscr{R}\left(H_{2}\right)\right|=4,\left|\mathscr{R}\left(H_{3}\right)\right|=4
$$

and using the above theorem we compute

$$
\left|\mathscr{R}\left(W_{1}\right)\right|=92000,\left|\mathscr{R}\left(W_{2}\right)\right|=103000 \text { and }\left|\mathscr{R}\left(W_{3}\right)\right|=160000 .
$$

As an example we demonstrate how this is done for $W_{2}$. We first choose a set $\mathscr{R}\left(H_{2}\right)$ of representatives of the $H_{2}$-conjugacy classes as

$$
\left\{x_{1}:=(), x_{2}:=(3,5)(4,6)(7,8), x_{3}:=(1,2)(3,4)(5,6), x_{4}:=(1,2)(3,6)(4,5)(7,8)\right\} .
$$

For each element $x_{i} \in \mathscr{R}\left(H_{2}\right)$, we compute the images under $\varphi_{x_{i}}$ :

$$
\left|\operatorname{im}\left(\varphi_{x_{1}}\right)\right|=99375,\left|\operatorname{im}\left(\varphi_{x_{2}}\right)\right|=1625,\left|\operatorname{im}\left(\varphi_{x_{3}}\right)\right|=1625,\left|\operatorname{im}\left(\varphi_{x_{4}}\right)\right|=375 .
$$

In particular, these computations show that there are 99375 conjugacy classes in $W_{2}$ whose elements have trivial top component.

## 5 Centralisers in wreath products

Recall the setting from Hypothesis A. We first introduce the notion of sparse wreath cycles. These are wreath cycles with at most one non-trivial base component. It turns out that every wreath cycle is conjugate in $K^{\Gamma} \times\left\langle 1_{H}\right\rangle$ to a sparse wreath cycle and we show that one can conjugate every wreath product element into a sparse wreath cycle decomposition, see Corollary 40. We use this to parameterise the $W$-centraliser of a product of sparse wreath cycles which, after conjugation, then parameterises the $W$-centraliser of an arbitrary wreath product element. The structure of $C_{S}(w)$ for the full monomial group $S=K\rangle_{\Gamma} \operatorname{Sym}(\Gamma)$ is described in [11, Theorem 8].

Definition 39. Let $w=(f, h) \in W$ be a wreath cycle. We call $w$ a sparse wreath cycle if there exists a $\gamma_{0} \in \Gamma$ such that $[\gamma] f=1_{K}$ for all $\gamma \in \Gamma \backslash\left\{\gamma_{0}\right\}$.

The concept of sparse wreath cycles is described in Ore [11, Theorem 7].
Note that in a disjoint wreath cycle decomposition of an element $w \in W$ in Theorem 9 we have

$$
w=(f, h)=\prod_{i=1}^{\ell}\left(\left.f\right|_{\operatorname{supp}\left(h_{i}\right)} ^{\Gamma}, h_{i}\right) \quad \cdot \prod_{\gamma \in \operatorname{fix}(h) \cap \operatorname{terr}(w)}\left(\left.f\right|_{\gamma} ^{\Gamma}, 1_{H}\right) \in W
$$

and the factors $\left(\left.f\right|_{\gamma} ^{\Gamma}, 1_{H}\right)$ are sparse wreath cycles for all $\gamma \in \operatorname{fix}(h) \cap \operatorname{terr}(w)$.
The following corollary shows that every wreath cycle is conjugate to a sparse wreath cycle and that one can write a $K^{\Gamma} \rtimes\left\langle 1_{H}\right\rangle$-conjugate of any wreath product element as a product of sparse wreath cycles. The following is a corollary of Theorem 27.

Corollary 40. Let $w=w_{1} \cdots w_{\ell} \in S$ be a disjoint wreath cycle decomposition of $w$. Then there exists an $a \in K^{\Gamma} \rtimes\left\langle 1_{H}\right\rangle$ such that $w^{a}=w_{1}^{a} \cdots w_{\ell}^{a}$ and $w_{i}^{a}$ is a sparse wreath cycle for all $1 \leq i \leq \ell$. This is called a sparse wreath cycle decomposition of $w^{a}$.

Proof. For $1 \leq i \leq \ell$ let $w_{i}=\left(f_{i}, h_{i}\right) \in S$, choose $\gamma_{i} \in \operatorname{terr}\left(w_{i}\right)$ and define

$$
e_{i}: \Gamma \rightarrow K, \gamma \mapsto \begin{cases}{\left[\gamma_{i}, w_{i}\right] \text { Yade, }} & \text { if } \gamma=\gamma_{i} \\ 1_{K}, & \text { else } .\end{cases}
$$

Now, for $1 \leq i \leq \ell$ set $v_{i}:=\left(e_{i}, h_{i}\right)$ and $v:=v_{1} \cdots v_{\ell}$. Choosing $t=1_{H}$, we obtain $h^{t}=h$ and $\mathscr{P}(w)^{t}=\mathscr{P}(v)$ and the existence of an $a \in W$ with $w^{a}=v$ and top component $t$ follows by the proof of Theorem 27

We now turn towards centralisers of elements of $W$. It is well known that for a single cycle $h \in \operatorname{Sym}(\Gamma)$, the centraliser of $h$ in $\operatorname{Sym}(\Gamma)$ is given by $C_{\operatorname{Sym}(\Gamma)}(h) \simeq\langle h\rangle \times \operatorname{Sym}(\Gamma \backslash \operatorname{supp}(h))$. The goal of this section is to give an explicit parametrisation of the $W$-centraliser of an arbitrary wreath product element $w \in W$ by an iterated cartesian product.
We first observe a relation between the top component of elements of $C_{W}(w)$ and the stabiliser of the territory decomposition of $w$.

Lemma 41. Let $w=(f, h) \in S$ and $a=(s, t) \in C_{W}(w)$. Then $t \in \operatorname{Stab}_{C_{\mathrm{Sym}(\mathrm{T})}(h)}(\mathscr{P}(w))$.
Proof. Let $a=(s, t) \in C_{W}(w)$. Then $\mathscr{P}(w)=\mathscr{P}\left(w^{a}\right)=\mathscr{P}(w)^{t}$ and therefore $t \in \operatorname{Stab}_{\operatorname{Sym}(\Gamma)}(\mathscr{P}(w))$ by Corollary 26. It remains to show $t \in C_{\text {Sym( } \Gamma)}(h)$. Observe

$$
(f, h)=w=w^{a}=\left(\left(s^{-1}\right)^{t} \cdot f^{t} \cdot s^{h^{-1} \cdot t}, t^{-1} \cdot h \cdot t\right)
$$

which implies $h=h^{t}$ and the claim follows.
To describe $C_{W}(w)$ explicitly, Lemma 41]suggests to investigate the structure of $\operatorname{Stab}_{C_{\mathrm{Sym}(\mathrm{T})}(h)}(\mathscr{P}(w))$ further. First, we restate the group theoretic structure of $C_{\operatorname{Sym}(\Gamma)}(h)$ for $h \in \operatorname{Sym}(\Gamma)$.

Lemma 42 ([13, Lemma 6.1.8]). Let $h \in \operatorname{Sym}(\Gamma)$ and $\left\{\mathscr{O}_{1}, \ldots, \mathscr{O}_{k}\right\}$ be a system of representatives of equivalence classes of orbits of $\langle h\rangle$ on $\Gamma$, where two orbits are equivalent if and only if they have the same cardinality. Then

$$
C_{\operatorname{Sym}(\Gamma)}(h) \simeq \stackrel{k}{x} C_{i=1}{\operatorname{Sym}\left(\mathscr{O}_{i}\right)}\left(\langle h\rangle^{\mathscr{O}_{i}}\right)\left\langle\operatorname{Sym}\left(\mid \overline{\mathscr{O}_{i} \mid}\right)\right.
$$

where for all $i=1, \ldots, k$ we denote by $\langle h\rangle^{\mathscr{C}_{i}}$ the group $\langle h\rangle$ induces on the orbit $\mathscr{O}_{i}$ and $\overline{\mathscr{O}_{i}}$ denotes the equivalence class of the representative $\mathscr{O}_{i}$.

As our goal is to describe the centraliser of a wreath product element explicitly, we require concrete isomorphisms. Our next aim is to give a constructive version of the above lemma. For this, we start by investigating what structure the preimage of $\operatorname{Sym}\left(\left|\overline{\mathscr{O}_{i}}\right|\right)$ must have under such an isomorphism.
The following definition constructs an element in $\operatorname{Sym}(\Gamma)$ induced by a permutation of a set of pairwise disjoint cycles in $\operatorname{Sym}(\Gamma)$ of the same order.

Definition 43. Let $I$ be a finite set and, for $i \in I$, let $h_{i} \in \operatorname{Sym}(\Gamma)$ be pairwise disjoint cycles of the same order and fix $\gamma_{i} \in \operatorname{supp}\left(h_{i}\right)$. Define the map $\Psi$ via

$$
\begin{aligned}
& {\left[\left\{\left(h_{i}, \gamma_{i}\right): i \in I\right\},-\right] \Psi: \operatorname{Sym}(I) \hookrightarrow \operatorname{Sym}(\Gamma),} \\
& \sigma \mapsto\left(\gamma \mapsto\left\{\begin{array}{ll}
{\left[\gamma_{[i] \sigma}\right] h_{[i] \sigma}^{j},} & \text { if } \gamma=\left[\gamma_{i}\right] h_{i}^{j} \text { for some } i \in I, j \in \mathbb{Z}_{>0} \\
\gamma, & \text { else }
\end{array}\right) .\right.
\end{aligned}
$$

The proof of the following lemma is straightforward.
Lemma 44. The map $\Psi$ in the definition above is a monomorphism.
The next lemma is a constructive version of Lemma 42 by using the map $\Psi$ introduced in Definition 43 to describe $C_{\operatorname{Sym}(\Gamma)}(h)$ for $h=h_{1} \cdots h_{\ell} \in \operatorname{Sym}(\Gamma)$ in a fixed disjoint cycle decomposition. We consider a partition of $\Gamma$ whose parts consist of the union of the supports of cycles $h_{i}$ of equal order. This partition corresponds to the equivalence classes $\mathscr{O}_{1}, \ldots, \mathscr{O}_{k}$ from Lemma 42, A necessary condition for an element of $\operatorname{Sym}(\Gamma)$ to centralise $h$ is that it stabilises this partition.

Lemma 45. Let $h=h_{1} \cdots h_{\ell} \in \operatorname{Sym}(\Gamma)$ be in disjoint cycle decomposition and define $\mathscr{O}(h):=$ $\left\{\left|h_{i}\right|: 1 \leq i \leq \ell\right\}$ and for $\mathfrak{o} \in \mathscr{O}(h)$ define $\mathscr{C}(\mathfrak{o}, h):=\left\{h_{i}:\left|h_{i}\right|=\mathfrak{o}, 1 \leq i \leq \ell\right\}$. For each $\mathfrak{o} \in \mathscr{O}(h)$ and each $z \in \mathscr{C}(\mathfrak{o}, h)$, choose $\gamma_{z} \in \operatorname{supp}(z)$ and define $\Psi_{\mathfrak{o}}: \operatorname{Sym}(\mathscr{C}(\mathfrak{o}, h)) \hookrightarrow \operatorname{Sym}(\Gamma), \sigma \mapsto$ $\left[\left(\left(z, \gamma_{z}\right): z \in \mathscr{C}(\mathfrak{o}, h)\right), \sigma\right] \Psi$ as in Definition 43, Then the elements of $C_{\mathrm{Sym}(\Gamma)}(h)$ are parameterised by the following group isomorphism

$$
\begin{array}{r}
\underset{\mathfrak{o} \in \mathscr{O}(h)}{\times}\left(\langle ( 1 , \ldots , \mathfrak { o } ) \rangle \left\langle\mathscr{C}(\mathfrak{o}, h) \operatorname{Sym}(\mathscr{C}(\mathfrak{o}, h)) \times \operatorname{Sym}(\Gamma \backslash \operatorname{supp}(h)) \stackrel{\sim}{\rightarrow} C_{\operatorname{Sym}(\Gamma)}(h),\right.\right. \\
\left(\left(\left((1, \ldots, \mathfrak{o})^{e_{z}}\right)_{z \in \mathscr{C}(\mathfrak{o}, h)}, \sigma_{\mathfrak{o}}\right)_{\mathfrak{o} \in \mathscr{O}(h)}, \pi_{0}\right) \mapsto\left(\prod_{\mathfrak{o} \in \mathscr{O}(h)}\left(\prod_{z \in \mathscr{C}(\mathfrak{o}, h)} h_{z}^{e_{z}}\right) \cdot\left[\sigma_{\mathfrak{o}}\right] \Psi_{\mathfrak{o}}\right) \cdot \pi_{0},
\end{array}
$$

where for $z \in \mathscr{C}(\mathfrak{o}, h)$, the integer $e_{z} \in\{0, \ldots, \mathfrak{o}-1\}$.
As announced after Lemma 41, we give an explicit bijection from an iterated cartesian product into $\operatorname{Stab}_{C_{\text {Sym(T) }}(h)}(\mathscr{P}(w))$ which is a crucial step towards the parametrisation of $C_{W}(w)$. We proceed in a similar way as in Lemma 45, where we consider the partition on $\Gamma$ induced by the orders of the disjoint cycles of $h$. We now translate these concepts from permutation groups to wreath products. Recall the decomposition of a wreath product
element $w$ into disjoint wreath cycles $w=(f, h)=w_{1} \cdots w_{\ell}$ from Theorem 9 By Lemma 21 for each $a=(s, t) \in C_{W}(w)$ and each $w_{i} \in \mathscr{C}(w)$ we have $w_{i}^{a} \in \mathscr{C}(w)$ and $\operatorname{ld}\left(w_{i}^{a}\right)=\operatorname{ld}\left(w_{i}\right)$. By Lemma41the top element $t \in \operatorname{Sym}(\Gamma)$ must centralise $h$ and stabilise $\mathscr{P}(w)$. This territory decomposition can be viewed as a refinement of the partition of $\Gamma$ above.

Lemma 46. Let $w=(f, h) \in S$. For each $z \in \mathscr{C}(w)$ choose $\gamma_{z} \in \operatorname{terr}(z)$. For all $L \in \mathscr{L}(w)$, define

$$
\Psi_{L}: \operatorname{Sym}(\mathscr{C}(L, w)) \hookrightarrow \operatorname{Sym}(\Gamma), \sigma \mapsto\left[\left\{\left(h_{z}, \gamma_{z}\right): z=\left(f_{z}, h_{z}\right) \in \mathscr{C}(L, w)\right\}, \sigma\right] \Psi
$$

where $\Psi$ is as in Definition 43. Then the elements of $\operatorname{Stab}_{C_{\text {Sym( })}(h)}(\mathscr{P}(w))$ are parameterised by the following group isomorphism

$$
\begin{aligned}
& \underset{L \in \mathscr{L}(w)}{\times}\left(\left\langle h_{L}\right\rangle \mathscr{C}_{(L, w)} \operatorname{Sym}(\mathscr{C}(L, w)) \times \operatorname{Sym}(\Gamma \backslash \operatorname{terr}(w)) \xrightarrow{\sim} \operatorname{Stab}_{\operatorname{Stsm}_{\operatorname{Sy)}(T)}(h)}(\mathscr{P}(w)),\right. \\
& \quad\left(\left(\left(h_{L}^{e_{z}}\right)_{z \in \mathscr{C}(L, w)}, \sigma_{L}\right)_{L \in \mathscr{L}(w)}, \pi_{0}\right) \mapsto\left(\prod_{L \in \mathscr{L}(w)}\left(\prod_{z \in \mathscr{C}(L, w)} h_{z}^{e_{z}}\right) \cdot\left[\sigma_{L}\right] \Psi_{L}\right) \cdot \pi_{0},
\end{aligned}
$$

where for $L=\left(k^{K}, j\right) \in \mathscr{L}(w)$ we define $h_{L}:=(1, \ldots, j)$ and for each $z \in \mathscr{C}(L, w)$ the integer $e_{z} \in\left\{0, \ldots,\left|h_{L}\right|-1\right\}$.

As in Corollary 40, every wreath product element $w^{\prime} \in W$ is conjugate to a wreath product element $w \in W$ in sparse disjoint wreath cycle decomposition. Therefore, the next theorem is only stated for elements in sparse, disjoint wreath cycle decomposition as $C_{W}\left(w^{\prime}\right)=$ $C_{W}\left(w^{a}\right)=C_{W}(w)^{a}$ for a suitable $a \in W$. This element $a$ can be constructed explicitly by Corollary 40. We generalise Ore's result [11, Theorem 8] for the full monomial group to the case $W=K{\imath_{\Gamma}} H$, where $H$ need not be the full symmetric group and $K$ need not be finite. We remark that in the parametrisation of the elements of $C_{W}(w)$ the stabiliser $\operatorname{Stab}_{C_{W}(w)}(\mathscr{P}(w))$ arises. This is due to Lemma 41,
We first introduce some notation. Let $w=(f, h)=\prod_{i=1}^{\ell} w_{i} \in W$ be in sparse, disjoint wreath cycle decomposition as in Corollary 40, For each cycle $z \in \mathscr{C}(w)$, fix a point $\gamma_{z} \in \operatorname{terr}(z)$ such that $[\gamma] f=1_{K}$ for all $\gamma \in \operatorname{terr}(z) \backslash\left\{\gamma_{z}\right\}$. Moreover, for each load $L \in \mathscr{L}(w)$, choose one representative cycle $z_{L} \in \mathscr{C}(L, w)$ and fix $\gamma_{L}:=\gamma_{z_{L}}$. For any other $z \in \mathscr{C}(L, w)$, fix elements $x_{z} \in K$ with $\left[\gamma_{z}\right] f=x_{z}^{-1} \cdot\left[\gamma_{L}\right] f \cdot x_{z}$. In particular, for every element $\gamma \in \operatorname{terr}(w)$ and any non-negative integer $e$ there exists a unique $z=\left(f_{z}, h_{z}\right) \in \mathscr{C}(w)$ and $0 \leq j<\left|h_{z}\right|$ such that $\gamma=\gamma_{z}^{h^{j-e}}$.

Theorem 47. Let $w=(f, h)=\prod_{i=1}^{\ell} w_{i} \in W$ be in sparse, disjoint wreath cycle decomposition. Then the elements of $C_{w}(w)$ can be parameterised by the following bijection $\Phi$ defined below:

$$
\begin{aligned}
& \Phi:\left(\underset{L \in \mathscr{L}(w)}{\times} C_{K}\left(\left[\gamma_{L}\right] f\right)^{\{\operatorname{terr}(z): z \in \mathscr{C}(L, w)\}} \times K^{\Gamma \backslash \operatorname{terr}(w)}\right) \times \operatorname{Stab}_{C_{H}(h)}(\mathscr{P}(w)) \xrightarrow{1: 1} C_{W}(w), \\
& (c, t)=\left(\left(\left(c_{L, z}\right)_{L \in \mathscr{L}(w), z \in \mathscr{C}(L, w)}, c_{0}\right),\left(\prod_{L \in \mathscr{L}(w)}\left(\prod_{z \in \mathscr{C}(L, w)} h_{z}^{e_{z}}\right) \cdot\left[\sigma_{L}\right] \Psi_{L}\right) \cdot \pi_{0}\right) \mapsto(s, t),
\end{aligned}
$$

where $t$ is parameterised according to the image of the map in Lemma 46 and $s: \Gamma \rightarrow K$ is defined by

$$
[\gamma] s= \begin{cases}x_{z}^{-1} \cdot c_{L, z} \cdot x_{[z] \sigma_{L}}, & \text { if } \gamma=\gamma_{z}^{h^{j-e_{z}}} \in \operatorname{terr}(w), e_{z}<j<\left|h_{z}\right| \text { or } j=0 \\ x_{z}^{-1} \cdot c_{L, z} \cdot x_{[z] \sigma_{L}} \cdot\left[\gamma_{[z] \sigma_{L}}\right] f, & \text { if } \gamma=\gamma_{z}^{h^{j-e_{z}}} \in \operatorname{terr}(w), 1 \leq j \leq e_{z} \\ {[\gamma] c_{0},} & \gamma \notin \operatorname{terr}(w)\end{cases}
$$

Proof. We omit the proof that $\Phi$ is well-defined in order to concentrate on the more important property that $\Phi$ is surjective. A proof for well-definedness can be deduced from the arguments below by reversing them. We show surjectivity of $\Phi$ by proving that every element of $C_{W}(w)$ can be decomposed into the components of the domain of $\Phi$.
Let $a=(s, t) \in C_{W}(w)$. Then $t \in C_{H}(h)$, since $w=w^{a}=\left(\left(s^{-1}\right)^{t} \cdot f^{t} \cdot s^{h^{-1} t}, t^{-1} \cdot h \cdot t\right)$. Moreover, $t \in \operatorname{Stab}_{C_{H}(h)}(\mathscr{P}(w))$, as $\mathscr{P}(w)=\mathscr{P}\left(w^{a}\right)=\mathscr{P}(w)^{t}$. By Lemma 46, we parameterise $t \in \operatorname{Stab}_{C_{H}(h)}(\mathscr{P}(w))$ as

$$
t=\left(\prod_{L \in \mathscr{L}(w)}\left(\prod_{z \in \mathscr{C}(L, w)} h_{z}^{e_{z}}\right) \cdot\left[\sigma_{L}\right] \Psi_{L}\right) \cdot \pi_{0}
$$

where $e_{z} \in \mathbb{Z}_{\geq 0}, \pi_{L}:=\left[\sigma_{L}\right] \Psi_{L} \in \operatorname{im}\left(\Psi_{L}\right)$, the map $\Psi_{L}$ is as in Lemma 46, and $\pi_{0} \in \operatorname{Sym}(\Gamma \backslash$ terr $(w)$ ). We now decompose $a$ into disjoint wreath product elements. For any $L \in \mathscr{L}(w)$ define $t_{L}:=\left(\prod_{z \in \mathscr{C}(L, w)} h_{z}^{e_{z}}\right) \cdot \pi_{L}$ and $t_{0}:=\pi_{0}$. For any $L \in \mathscr{L}(w)$ define $\Omega_{L}:=\biguplus_{z \in \mathscr{C}(L, w)} \operatorname{terr}(z)$ and $\Omega_{0}:=\Gamma \backslash \operatorname{terr}(w)$. Further set $s_{L}:=\left.s\right|_{\Omega_{L}} ^{\Gamma}, a_{L}:=\left(s_{L}, t_{L}\right)$ and $a_{0}:=\left(\left.s\right|_{\Omega_{0}} ^{\Gamma}, t_{0}\right)$. Then $a=a_{0} \cdot \prod_{L \in \mathscr{L}(w)} a_{L}$ and the $a_{0}, a_{L}$ are pairwise disjoint. Note that for all $L \in \mathscr{L}(w)$ we have terr $\left(a_{L}\right) \subseteq \Omega_{L}$ and terr $\left(a_{0}\right) \subseteq \Omega_{0}$ which shows $w^{a}=\prod_{L \in \mathscr{L}(w)}\left(\prod_{z \in \mathscr{C}(L, w)} z\right)^{a_{L}}$ since disjoint cycles commute. In particular, since $w=w^{a}$ and as the load of a wreath cycle is invariant under conjugation, we have $\left(\prod_{z \in \mathscr{C}(L, w)} z\right)^{a_{L}}=\prod_{z \in \mathscr{C}(L, w)} z^{a_{L}}=\prod_{z \in \mathscr{C}(L, w)} z$. Fix $z \in \mathscr{C}(L, w)$ for some $L \in \mathscr{L}(w)$. Then, for $y:=[z] \sigma_{L}$ we have $\left(f_{y}, h_{y}\right)=w_{y}=z^{a_{L}}=$ $\left(\left(s_{L}^{-1}\right)^{t_{L}} \cdot f_{z}^{t_{L}} \cdot s_{L}^{h_{z}^{-1} t_{L}}, t_{L}^{-1} \cdot h_{z} \cdot t_{L}\right)$. Note that for all $\gamma \in \Gamma$ we must have

$$
\begin{equation*}
[\gamma]\left(\left(s_{L}^{-1}\right)^{t_{L}} \cdot f_{z}^{t_{L}} \cdot s_{L}^{h_{z}^{-1} t_{L}}\right)=[\gamma] f_{y} . \tag{4}
\end{equation*}
$$

Thus

$$
\begin{aligned}
{\left[\gamma_{y}\right] f } & =\left[\gamma_{y}, y\right] \text { Yade }=\prod_{j=0}^{\left|h_{y}\right|-1}\left[\gamma_{y}\right] f_{y}^{h_{y}^{-j}}=\prod_{j=0}^{\left|h_{y}\right|-1}\left[\gamma_{y}^{h_{y}^{j}}\right] f_{y} \\
& \underline{4}^{\mid}{ }^{\left|h_{y}\right|-1} \prod_{j=0}\left[\gamma_{y}^{h_{y}^{j}}\right]\left(\left(s_{L}^{-1}\right)^{t_{L}} \cdot f_{z}^{t_{L}} \cdot s_{L}^{h_{z}^{-1} t_{L}}\right) \\
& =\prod_{j=0}^{\left|h_{z}\right|-1}\left[\gamma_{z}^{h_{z}^{j-e_{z}}}\right] s_{L}^{-1} \cdot\left[\gamma_{z}^{h_{z}^{j-e_{z}}}\right] f_{z} \cdot\left[\gamma_{z}^{h_{z}^{j-e_{z}+1}}\right] s_{L} \\
& =\left[\gamma_{z}^{h_{z}^{-e_{z}}}\right] s_{L}^{-1} \cdot\left(\prod_{j=0}^{\left|h_{z}\right|-1} \cdot\left[\gamma_{z}^{h_{z}^{j-e_{z}}}\right] f_{z}\right) \cdot\left[\gamma_{z}^{\left.h_{z}^{\left|h_{z}\right|-1-e_{z}+1}\right] s_{L}}\right. \\
& =\left[\gamma_{z}^{h_{z}^{-e_{z}}}\right] s_{L}^{-1} \cdot\left[\gamma_{z}^{h_{z}^{-e_{z}}}, z\right] \text { Yade } \cdot\left[\gamma_{z}^{h_{z}^{-e_{z}}}\right] s_{L}=\left[\gamma_{z}^{h_{z}^{-e_{z}}}\right] s_{L}^{-1} \cdot\left[\gamma_{z}\right] f \cdot\left[\gamma_{z}^{h_{z}^{-e_{z}}}\right] s_{L}
\end{aligned}
$$

which implies $\left[\gamma_{z}^{h_{z}^{-e z}}\right] s_{L} \in C_{K}\left(\left[\gamma_{z}\right] f\right) \cdot x_{z}^{-1} \cdot x_{y}=C_{K}\left(\left[\gamma_{L}\right] f\right)^{x_{z}} \cdot x_{z}^{-1} \cdot x_{y}=x_{z}^{-1} \cdot C_{K}\left(\left[\gamma_{L}\right] f\right) \cdot x_{y}$. Ву Equation 44, the component $\left[\gamma_{z}^{\left.h_{z}^{-e_{z}}\right] s_{L} \text { uniquely determines every other component of } s_{L}, ~}\right.$ on $\operatorname{terr}(z)=\left\{\gamma_{z}^{h_{z}^{0}}, \ldots, \gamma_{z}^{h_{z}^{\left|h_{z}\right|-1}}\right\}=\left\{\gamma_{z}^{h_{z}^{0-e_{z}}}, \ldots, \gamma_{z}^{h_{z}^{\left|h_{z}\right|-1-e_{z}}}\right\}$, since for all $j \in \mathbb{Z}$ we can inductively conclude:

$$
\begin{aligned}
{\left[\gamma_{z}^{h_{z}^{j+1-e_{z}}}\right] s_{L} } & =\left[\gamma_{y}^{h_{y}^{j}}\right] s_{L}^{h_{z}^{-1} t_{L}}\left[\gamma_{y}^{h_{y}^{j}}\right]\left(\left(f_{z}^{-1}\right)^{t_{L}} \cdot s_{L}^{t_{L}} \cdot f_{y}\right) \\
& =\left[\gamma_{z}^{h_{z}^{j-e_{z}}}\right] f_{z}^{-1} \cdot\left[\gamma_{z}^{h_{z}^{j-e_{z}}}\right] s_{i} \cdot\left[\gamma_{y}^{h_{y}^{j}}\right] f_{y} \\
& =\left[\gamma_{z}^{h_{z}^{j-e_{z}}}\right] f^{-1} \cdot\left[\gamma_{z}^{h_{z}^{j-e_{z}}}\right] s_{i} \cdot\left[\gamma_{y}^{h_{y}^{j}}\right] f .
\end{aligned}
$$

Let $c_{L, z} \in C_{K}\left(\left[\gamma_{L}\right] f\right)$ such that $\left[\gamma_{z}^{\left.h_{z}^{-e_{z}}\right]} s_{L}=x_{z}^{-1} \cdot c_{L, z} \cdot x_{y}\right.$. Note that for $j=0$ we have $\left[\gamma_{z}^{h^{j-e_{z}}}\right] s_{L}=x_{z}^{-1} \cdot c_{L, z} \cdot x_{[z] \sigma_{L}}$. By the above induction

$$
\left[\gamma_{z}^{h^{j-e_{z}}}\right] s_{L}=\left(\prod_{n=1}^{j}\left[\gamma_{z}^{h^{j-n-e_{z}}}\right] f^{-1}\right) \cdot x_{z}^{-1} \cdot c_{L, z} \cdot x_{[z] \sigma_{L}} \cdot\left(\prod_{n=1}^{j}\left[\gamma_{[z] \sigma_{L}}^{h^{j-n}}\right] f\right)
$$

where $0 \leq j \leq\left|h_{z}\right|-1$. Recall $[\delta] f^{-1}=1_{K}$ for all $\delta \in \operatorname{supp}\left(h_{y}\right) \backslash\left\{\gamma_{y}\right\}$. Hence, for $1 \leq j \leq e_{z}$, we have

$$
\left[\gamma_{z}^{h^{j-e_{z}}}\right] s_{L}=x_{z}^{-1} \cdot c_{L, z} \cdot x_{[z] \sigma_{L}} \cdot\left[\gamma_{[z] \sigma_{L}}\right] f
$$

and for $j>e_{z}$ we have

$$
\begin{aligned}
{\left[\gamma_{z}^{h^{j-e_{z}}}\right] s_{L} } & =\left[\gamma_{z}\right] f \cdot x_{z}^{-1} \cdot c_{z} \cdot x_{[z] \sigma_{L}} \cdot\left[\gamma_{[z] \sigma_{L}}\right] f \\
& =x_{z}^{-1} \cdot\left[\gamma_{L}\right] f \cdot x_{z} \cdot x_{z}^{-1} \cdot c_{L, z} \cdot x_{[z] \sigma_{L}} \cdot x_{[z] \sigma_{L}^{-1}} \cdot\left[\gamma_{L}\right] f \cdot x_{[z] \sigma_{L}}=x_{z}^{-1} \cdot c_{L, z} \cdot x_{[z] \sigma_{L}}
\end{aligned}
$$

which shows surjectivity of $\Phi$. The well-definedness of $\Phi$ follows by reversing the order of the arguments used.
We now show injectivity $\Phi$ : Suppose $[(c, t)] \Phi=[(b, r)] \Phi$. Then $t=r$ and by comparing the images of $\gamma_{z}$ under the base components of $[(c, t)] \Phi$ and $[(b, t)] \Phi$ one obtains $b=c$.

Note that $\Phi$ in Theorem 47 is not a group homomorphisms. However, using $\Phi$ we can obtain a generating set for $C_{W}(w)$.
Corollary 48. Assume the notation of Theorem 47, Then

$$
\{1\} \rightarrow B_{w} \stackrel{\iota}{\hookrightarrow} C_{W}(w) \stackrel{\rho}{\longrightarrow} \operatorname{Stab}_{C_{H}(h)}(\mathscr{P}(w)) \rightarrow\{1\}
$$

is a short exact sequence of groups, where

$$
\begin{gathered}
B_{w}:=\underset{L \in \mathscr{L}(w)}{\times} C_{K}\left(\left[\gamma_{L}\right] f\right)^{\{\operatorname{terr}(z): z \in \mathscr{C}(L, w)\}} \times K^{\Gamma \backslash \operatorname{terr}(w)}, \\
\iota: B_{w} \hookrightarrow C_{W}(w), c \mapsto\left[\left(c, 1_{H}\right)\right] \Phi, \text { and } \rho: C_{W}(w) \rightarrow \operatorname{Stab}_{C_{H}(h)}(\mathscr{P}(w)),(s, t) \mapsto t .
\end{gathered}
$$

In particular, if $B_{w}=\langle X\rangle$ and $\operatorname{Stab}_{C_{H}(h)}(\mathscr{P}(w))=\langle Y\rangle$, then

$$
C_{W}(w)=\langle[X] \iota \cup\{[(1, t)] \Phi: t \in Y\}\rangle
$$

Proof. It is clear that $\iota$ is a monomorphism and $\rho$ is an epimorphism. We show $\operatorname{im}(\iota)=$ $\operatorname{ker}(\rho)$, where the inclusion $\subseteq$ is obvious. Now suppose $a \in \operatorname{ker}(\rho)$. Then $a=\left(s, 1_{H}\right) \in C_{W}(w)$ for some $s \in K^{\Gamma}$ and therefore, for $\left(c, 1_{H}\right):=\left[\left(s, 1_{H}\right)\right] \Phi^{-1}$, we obtain $[c] \iota=a$. As $[[(1, t)] \Phi] \rho=$ $t$ for all $t \in Y$, the claim for the generating set follows as we have an exact sequence of groups.

Example 49. We use the notation from Example 36 and first compute $\left|C_{W_{i}}(w)\right|$ for $1 \leq i \leq$ 3 using Theorem47, For this we need to conjugate the element $w$ to an element $v$ in sparse wreath cycle decomposition, say $v:=w^{b}$. Note that in this case we have $C_{W_{i}}(w)^{b}=C_{W_{i}}(v)$. For example, using $b:=[1,(1,3,2,4)] \mathscr{E} \cdot[2,(1,3)(2,4)] \mathscr{E}$ we have

$$
v:=w^{b}=\left(\begin{array}{cccccccc}
1 & 2^{2} & 3 & 4 & 5 & 6^{6} & 7 & 8
\end{array}\right)
$$

with

Using this colouring to encode the sparse wreath cycles, we write $u$ in a disjoint sparse wreath cycle decomposition:

$$
v=v_{2,1,1} \cdot v_{2,2,1} \cdot v_{2,2,2} \cdot v_{4,2,1}
$$

Further let $v_{i, j, \ell}=\left(e_{i, j, \ell}, h_{i, j, \ell}\right)$. We choose points in the territory of each sparse wreath cycle as in Theorem 47.

$$
\gamma_{2,1,1}:=7, \quad \gamma_{2,2,1}:=1, \quad \gamma_{2,2,2}:=4, \quad \gamma_{4,2,1}:=5 .
$$

First note that the left iterated cartesian product occurring in the source of the bijection $\Phi$ defined in Theorem47 does not depend on the chosen top group, namely

$$
\begin{aligned}
& \left.\mid C_{K}([7] f)^{\{\{7\}}\right\} \times C_{K}([1] f)^{\{\{1,2\},\{3,4\}\}} \times C_{K}([5] f)^{\{\{5,6\}\}} \times K^{\Gamma \backslash \operatorname{terr}(v)} \mid \\
& \quad=\left|C_{K}((1,2)) \times C_{K}((3,4))^{2} \times C_{K}((1,2,3)) \times K\right|=4,608 .
\end{aligned}
$$

We have

$$
\begin{aligned}
& \left|C_{W_{1}}(v)\right|=4,608 \cdot\left|\operatorname{Stab}_{C_{H_{1}}(h)}(\mathscr{P}(v))\right|=4,608 \cdot 8=36,864, \\
& \left|C_{W_{2}}(v)\right|=4,608 \cdot\left|\operatorname{Stab}_{C_{H_{2}}(h)}(\mathscr{P}(v))\right|=4,608 \cdot 2=9,216, \\
& \left|C_{W_{3}}(v)\right|=\left|C_{W_{2}}(v)\right| .
\end{aligned}
$$

Let $\Phi$ be the bijection from Theorem 47 for the wreath product $W_{1}$, where we choose elements $x_{2,1,1}:=(), x_{2,2,1}:=(), x_{2,2,2}:=(1,3)(2,4)$ and $x_{4,2,1}:=()$. We choose an element from the domain of $\Phi$ using the same notation as in Theorem 47
and compute the image of $(c, t)$ under $\Phi$. For this we first need to write the element $t \in \operatorname{Stab}_{C_{H_{1}}(h)}(\mathscr{P}(w))$ in a suitable decomposition:

$$
\begin{aligned}
t & =()^{e_{2,1,1}} \cdot\left[\sigma_{2,1}\right] \Psi_{2,1} \cdot(1,2)^{e_{2,2,1}} \cdot(3,4)^{e_{2,2,2}} \cdot\left[\sigma_{2,2}\right] \Psi_{2,2} \cdot(5,6)^{e_{4,2,1}} \cdot\left[\sigma_{4,2}\right] \Psi_{4,2} \cdot \pi_{0} \\
& =(1,2) \cdot(3,4) \cdot[(1,2)] \Psi_{2,2}=(1,2) \cdot(3,4) \cdot(1,4)(2,3)
\end{aligned}
$$

Now we can compute the base component of $a:=(s, t)=[(c, t)] \Phi$. For example, the images of $s$ under $\operatorname{terr}\left(w_{2,2,1}\right)=\{1,2\}$ are:

$$
\begin{aligned}
{\left[\gamma_{2,2,1} h^{0-e_{2,2,1}}\right] s } & =[2] s=x_{2,2,1}^{-1} \cdot c_{2,2,1} \cdot x_{2,2,[1] \sigma_{2,2}} \\
& =() \cdot(3,4) \cdot(1,3)(2,4)=(1,3,2,4), \\
{\left[\gamma_{2,2,1} h^{1-e_{2,2,1}}\right] s } & =[1] s=x_{2,2,1}^{-1} \cdot c_{2,2,1} \cdot x_{2,2,[1] \sigma_{2,2}} \cdot\left[\gamma_{\left.2,2,[1] \sigma_{2,2}\right] e}\right] \\
& =() \cdot(3,4) \cdot(1,3)(2,4) \cdot(1,2)=(1,3)(2,4) .
\end{aligned}
$$

Repeating this computation for the territory of each wreath cycle and $\Gamma \backslash \operatorname{terr}(v)$ yields

$$
a=\left(\begin{array}{ccc}
1 & 2 & \stackrel{3}{4} \\
(1,3)(2,4)
\end{array},(1,3,2,4),(1,4)(2,3),(1,3,2,4),(1,3,2),(1,3,2),(1,2),(1,2,3,4) ;(1,3)(2,4)\right)
$$

## 6 Performance of an implementation

The third author implemented the disjoint wreath cycle decomposition in the GAP package WPE [12]. Building on this, he implemented algorithms using the theory in this paper for working in finite wreath products $W=K{l_{\Gamma}} H$, where $\Gamma$ is finite and $H \leq \operatorname{Sym}(\Gamma)$. The GAP-package $W P E$ provides methods to test whether two elements of $K{ }_{{ }_{\Gamma}} \operatorname{Sym}(\Gamma)$ are conjugate in $W$ and, in this case, computes a conjugating element. Moreover it provides algorithms to compute representatives of the $W$-conjugacy classes of elements and methods for efficient centraliser computations in $W$.
To highlight the efficiency of the new methods, we present sample computations. These were performed on a $1,8 \mathrm{GHz}$ IntelCore $\mathrm{i} 5-5350 \mathrm{U}$ and are presented in the following tables. The first column lists the groups we considered, the second column labelled GAP4 lists the time taken by native GAP 4.11 .1 code and the final column lists the time taken by the package WPE [12] loaded in GAP 4.11.1. For Table (a) we precomputed a set of 100 random pairs of conjugate elements (by conjugating 100 random elements by a further 100 random elements). We list the average time of computing a conjugating element. In
some cases, the computation using native GAP4-code for a single computation took too long and the computation was terminated after the time recorded in column GAP4. In this case the symbol > indicates that the computation was terminated. In Table (b) we list the average times to compute the centralisers of 100 precomputed random elements. Finally, Table (c) lists the time taken to compute a set of representatives of the conjugacy classes of elements of the groups $W$ listed in the first column and the last column contains the number of conjugacy classes of elements of that group.

| Group | GAP4 | WPE[12] |
| :---: | :---: | :---: |
| $S_{4} \backslash S_{8}$ | $<1 \mathrm{~s}$ | $<1 \mathrm{~s}$ |
| $S_{10}\left\langle M_{24}\right.$ | 22 s | $<1 \mathrm{~s}$ |
| $S_{25} \backslash S_{100}$ | $>40 \mathrm{~m}$ | $<1 \mathrm{~s}$ |
| $\operatorname{SL}(2,2)\langle\operatorname{PSp}(4,3)$ | $<1 \mathrm{~s}$ | $<1 \mathrm{~s}$ |
| $\operatorname{SL}(2,2)\langle\operatorname{PSU}(4,4)$ | $>40 \mathrm{~m}$ | $<1 \mathrm{~s}$ |
| $\operatorname{PSL}(5,3)\langle\operatorname{PSU}(6,2)$ | $>40 \mathrm{~m}$ | 20 s |

(a) Conjugacy problem

| Group | GAP4 | WPE[12] |
| :---: | :---: | :---: |
| $S_{4} \backslash S_{8}$ | $<1 \mathrm{~s}$ | $<1 \mathrm{~s}$ |
| $S_{10} \backslash M_{24}$ | 22 s | $<1 \mathrm{~s}$ |
| $S_{25} \backslash S_{100}$ | $>40 \mathrm{~m}$ | $<1 \mathrm{~s}$ |
| $\operatorname{SL}(2,2)\langle\operatorname{PSp}(4,3)$ | $<1 \mathrm{~s}$ | $<1 \mathrm{~s}$ |
| $\operatorname{SL}(2,2)\langle\operatorname{PSU}(4,4)$ | $>40 \mathrm{~m}$ | $<1 \mathrm{~s}$ |
| $\operatorname{PSL}(5,3)\langle\operatorname{PSU}(6,2)$ | $>40 \mathrm{~m}$ | 14 s |

(b) Centraliser of elements

| Group | GAP4 | WPE[12] | \#Conjugacy classes |
| :---: | :---: | :---: | :---: |
| $\mathrm{SL}(2,2)$ PSL(2,7) | $<1 \mathrm{~s}$ | $<1 \mathrm{~s}$ | 216 |
| $S_{4} \backslash S_{8}$ | 60 s | $<1 \mathrm{~s}$ | 6,765 |
| $A_{5} \backslash M_{11}$ | $>40 \mathrm{~m}$ | 125 s | 15,695 |
| $\mathrm{SU}(3,2)$ A $A_{7}$ | 35 m | 22 s | 398,592 |
| $M_{24} \backslash S_{7}$ | $>40 \mathrm{~m}$ | 145 s | $9,293,050$ |
| $S_{7}\langle\mathrm{PSL}(2,7)$ | $>40 \mathrm{~m}$ | 300 s | $15,342,750$ |

(c) Conjugacy class representatives

Figure 1: Time comparisons between native GAP4 code and the WPE package

Our experiments show that the computations in the wreath products are roughly as hard as the corresponding computations in the groups $K$ and $H$.

## Acknowledgements

We thank Colva Roney-Dougal for discovering Ore's paper [11] which initiated this project. Sebastian Krammer presented the wreath cycle decomposition in modern language for an algorithm in his M.Sc. thesis. We also thank Max Horn and Alexander Hulpke for helpful suggestions and discussions regarding the GAP-implementation. We thank anonymous referees for their helpful in-depth comments.
This is a contribution to Project-ID 286237555 - TRR 195 - by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation). The first author acknowledges financial support from the RWTH Scholarships for Doctoral Students.

## References

[1] Robert W. Baddeley, Cheryl E. Praeger, and Csaba Schneider, Transitive simple subgroups of wreath products in product action, J. Aust. Math. Soc. 77 (2004), no. 1, 55-72, DOI 10.1017/S1446788700010156.
[2] Wieb Bosma, John Cannon, and Catherine Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput. 24 (1997), no. 3-4, 235-265, DOI 10.1006/jsco.1996.0125. Computational algebra and number theory (London, 1993).
[3] John J. Cannon and Derek F. Holt, Computing conjugacy class representatives in permutation groups, J. Algebra 300 (2006), no. 1, 213-222, DOI 10.1016/j.jalgebra.2006.02.014.
[4] John D. Dixon and Brian Mortimer, Permutation groups, Graduate Texts in Mathematics, vol. 163, Springer-Verlag, New York, 1996.
[5] GAP - Groups, Algorithms, and Programming, Version 4.11.1, The GAP Group, 2021.
[6] Alexander Hulpke, Conjugacy classes in finite permutation groups via homomorphic images, Math. Comp. 69 (2000), no. 232, 1633-1651, DOI 10.1090/S0025-5718-99-01157-6.
[7] Gordon James and Adalbert Kerber, The representation theory of the symmetric group, Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley Publishing Co., Reading, Mass., 1981. With a foreword by P. M. Cohn; With an introduction by Gilbert de B. Robinson.
[8] L. G. Kovács, Primitive subgroups of wreath products in product action, Proc. London Math. Soc. (3) 58 (1989), no. 2, 306-322, DOI 10.1112/plms/s3-58.2.306.
[9] Jeffrey S. Leon, Permutation group algorithms based on partitions. I. Theory and algorithms, J. Symbolic Comput. 12 (1991), no. 4-5, 533-583, DOI 10.1016/S0747-7171(08)80103-4.Computational group theory, Part 2.
[10] Martin W. Liebeck, Cheryl E. Praeger, and Jan Saxl, On the O'Nan-Scott theorem for finite primitive permutation groups, J. Austral. Math. Soc. Ser. A 44 (1988), no. 3, 389-396.
[11] Oystein Ore, Theory of monomial groups, Trans. Amer. Math. Soc. 51 (1942), 15-64, DOI 10.2307/1989979.
[12] Friedrich Rober, WPE, Provides efficient methods for working with generic wreath products., Version 0.1, 2021. https://FriedrichRober.github.io/ WPE/
[13] Ákos Seress, Permutation group algorithms, Cambridge Tracts in Mathematics, vol. 152, Cambridge University Press, Cambridge, 2003.
[14] Wilhelm Specht, Eine Verallgemeinerung der symmetrischen Gruppe, Humboldt-Universität zu Berlin, 1932.

