Segre-Driven Radicality Testing

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Abstract

We present a probabilistic algorithm to test if a homogeneous polynomial ideal I defining a scheme X in \mathbb{P}^n is radical using Segre classes and other geometric notions from intersection theory. Its worst case complexity depends on the geometry of X. If the scheme X has reduced isolated primary components and no embedded components supported the singular locus of $X_{\text{red}} = \mathbb{V}(\sqrt{I})$, then the worst case complexity is doubly exponential in n; in all the other cases the complexity is singly exponential. The realm of the ideals for which our radical testing procedure requires only single exponential time includes examples which are often considered pathological, such as the ones drawn from the famous Mayr-Meyer set of ideals which exhibit doubly exponential complexity for the ideal membership problem.

1 Introduction

We consider the problem of testing if an ideal is radical, or in other words if the associated scheme is reduced. More precisely, for a homogeneous ideal $I = \langle f_1, \ldots, f_r \rangle$ in $\mathbb{C}[x_0, x_1, \ldots, x_n]$ with d being the maximum degree of the polynomials f_i , we present a (probabilistic) algorithm to test if the scheme $X \subset \mathbb{P}^n$ defined by I is reduced, i.e., to test if I is radical (up to saturation by the irrelevant ideal). When X has reduced isolated primary components that in addition have no embedded components outside of the singular locus of $X_{\text{red}} = \mathbb{V}(\sqrt{I})$, then the radical is computed via a single ideal saturation and the worst case complexity becomes doubly exponential in n. In all the other cases the radical is not explicitly computed and the worst case complexity is singly exponential in n. To understand what types of embedded components we can deal with, while maintaining the singly exponential complexity bound, consider the following example.

Example 1 (Embedded components outside the singular locus of the radical). We work in \mathbb{P}^2 with coordinates x, y, z and we consider the scheme X defined by the ideal

$$I = \langle -x^2y^2 + y^3z, -x^4 + x^2yz \rangle = \langle x^2 - yz \rangle \cap \langle y^2, x^4 - x^2yz \rangle.$$

In this case the only isolated primary component of X is the reduced component $X_{red} = \mathbb{V}(x^2 - yz)$; also X_{red} is a smooth curve in \mathbb{P}^2 and so its singular locus is empty. However, X has an embedded component $\mathbb{V}(y^2, x^4 - x^2yz)$, supported on the point [0:0:1]. This embedded component would be detected by our algorithm using methods which have singly exponential worst case complexity in the number of variables.

A more interesting example of an ideal I defining a scheme X with embedded components outside the singular locus of X_{red} is furnished by the homogeneous version of the Mayr-Meyer ideals [28] introduced by Bayer and Stillman [6]. The Mayr-Meyer family of ideals are generated by polynomials of degree $\mathcal{O}(d)$ in $\mathcal{O}(n)$ variables. Ideals in this family have the property that for some polynomial $f \in I$ the polynomials

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 r_i which solve the ideal membership problem via the expression $f = \sum_i r_i f_i$ are such that $\deg(r_i)$ is doubly exponential in n, i.e., $\mathcal{O}(d^{2^n})$. In [6, §2] a family of Mayr-Meyer ideals J_n is described in a ring with 10n variables and these generators are homogenized to give a homogeneous ideal J'_n in 10n + 1 variables. Consider the n = 2 case; in this case the homogeneous ideals J'_2 are ideals in a ring with the 21 variables $S_0, S_1, F_0, F_1, a_0, a_1, b_0, b_1, c_0, c_1, d_0, d_1, e_0, e_1, f_0, f_1, g_0, g_1, h_0, h_1, z$. One of these ideals is as follows (additional examples may be generated using the Macaulay2 [16] function mayr in [17]):

$$I = \langle S_0h_0 - F_1z, S_0g_0 - F_0h_0, S_0f_0 - F_0h_0, F_0e_0 - F_0h_0, S_0e_0 - S_1z, F_0d_0h_0 - F_1z^2, F_0a_0h_0 - S_1z^2, F_0c_0g_0 - F_0h_0z, F_0b_0f_0 - F_0h_0z, F_0a_0f_0 - F_0d_0g_0, F_0c_0d_1f_0h_1 - F_0d_0g_0h_1z, F_0c_0c_1f_0g_1 - F_0d_0g_0g_1z, F_0b_1c_0f_0f_1 - F_0d_0f_1g_0z, F_0a_1c_0e_1f_0 - F_0d_0e_1g_0z \rangle.$$
(1)

The ideal I defines a scheme $X \subset \mathbb{P}^{20}$ of codimension three. The irreducible component $\mathbb{V}(F_0, S_0, z)$ contains the embedded components $\mathbb{V}(z, g_0, f_0, F_0, S_0, F_1e_0 - S_1h_0)$ and $\mathbb{V}(z, f_0 - g_0, e_0 - h_0, d_0, c_0, b_0, a_0, F_0, S_1 - F_1, S_0)$; neither of these components is contained in the singular locus of $X_{\text{red}} = \mathbb{V}(\sqrt{I})$. While it seems to us interesting that many pathological examples, such as the Mayr-Meyer examples, with potentially numerous and complicated embedded components, can be considered by our algorithm in time at most singly exponential in the number of variables, it is worth noticing that many desirable (and perhaps more mundane seeming) examples, such as all ideals which are radical, require us to perform an operation (namely a single ideal saturation) which has a doubly exponential worst case bound. The latter case of course includes an ideal generated by generic polynomials, which is expected to be radical, so this is in that sense the "most common" case. There does, however, seem to be some hope that this saturation operation could be avoided in some way, as it in fact computes much more information than we strictly need for our test. Nevertheless, if we have the information that the ideal is "generic", then we can combine our approach with other algorithms that have single exponential complexity bounds in this case, e.g., [24, 4].

We now give a brief conceptual overview of our approach. For a homogeneous ideal I in $\mathbb{C}[x_0,\ldots,x_n]$ we consider the scheme $X = \mathbb{V}(I) \subset \mathbb{P}^n$ associated to it. We first sample (at least one) generic point p_i in each isolated primary component W_i of X and compute the multiplicity of p_i in W_i via a calculation which requires only the computation of the degree of an ideal (we do this in singly exponential time). If a generic point has multiplicity greater than one, then the associated component is not reduced and I is not radical. If all isolated primary components are reduced, then we compute the singularity subscheme of X, $Sing(X) = \mathbb{V}(J)$ which is defined by an ideal J whose generators consist of the $\operatorname{codim}(X) \times \operatorname{codim}(X)$ minors of the Jacobian matrix of a generating set of I. This scheme Sing(X) has primary components whose support is either an embedded component of X not contained in the singular locus of X_{red} or else are supported on the singular locus of X_{red} . We use a multiplicity based test to identify the presence of embedded components of the first type. If no non-reduced structures in X have been identified at this stage of the algorithm we see that $\sqrt{I} = I : J^{\infty}$; this computation is the **only** computation involved which has a doubly exponential worst case bound. At this point one could test if \sqrt{I} equals I using standard methods. Instead, using the computed generators of \sqrt{I} we compute the Segre class $s(X_{red}, \mathbb{P}^n)$ and using the generators of I we compute the Segre class $s(X, \mathbb{P}^n)$, both these operations require singly exponential time. Then, using a result we show in §3.1, we can conclude that X is a reduced scheme in \mathbb{P}^n if and only if these two Segre classes agree (in terms of ideals this translates to $I: \langle x_0, \ldots, x_n \rangle^{\infty}$ being radical). We should also mention that we present a method to compute the multiplicity of an irreducible variety contained in some lower dimensional isolated primary component of a scheme (which is new); see Sections 3.4 and 3.5 and also (12). This method requires only the computation of dimension and the degree of a polynomial ideal and makes no assumptions on their structure. We have trialed our algorithms on several test cases using Macaulay 2 [16] where we also exploit the various routines for Segre class computations based on [18].

Finally we remark that, while we have not been able to construct a method to compute the Segre class $s(X_{red}, \mathbb{P}^n) = s(\mathbb{V}(I : J^{\infty}), \mathbb{P}^n)$ without the generators of the ideal $\sqrt{I} = I : J^{\infty}$ in general; this question seems worth further consideration. For this reason, we employ the Segre class based test in the final step of our algorithm rather than some more standard method to check equality of ideals. In particular, given a

singly exponential algorithm to compute $s(\mathbb{V}(I : J^{\infty}), \mathbb{P}^n)$ using only the generators of I and J, our test for X being reduced becomes singly exponential in all cases. We note that, for example, if $I : J^{\infty}$ is a complete intersection, given the degrees of a set of generators, the computation of $s(\mathbb{V}(I : J^{\infty}), \mathbb{P}^n)$ in fact requires only constant time. Such facts, along with the fact that Segre class computations tend to require only the computation of degrees of ideals, give us hope that a singly exponential bound for the computation of $s(\mathbb{V}(I : J^{\infty}), \mathbb{P}^n)$ may be possible using only the generators of I and J.

Historically, many different algorithms to test if an ideal I is radical have been presented in the literature; in most of the cases they test for radicality in the process of computing the radical or the primary decomposition. Roughly speaking, these algorithms tend to consist of a step which computes the radical \sqrt{I} , a step which computes a reduced Gröbner basis for both the ideal and its radical (in the same term order), and a step which checks if I and \sqrt{I} have the same reduced Gröbner basis. For the step which computes the radical of I, the best known algorithms have (worst case) bounds *doubly exponential* in the number of variables or in the dimension of the ideal I, e.g., the algorithm of [26] has complexity $(rd)^{2^{O(n^2)}}$, and that of [23] has complexity doubly exponential in the dimension of I; other algorithms have similar or worse bounds in the general case. In certain special cases, such as when I is unmixed [23, Proposition 4.1], I is a complexity bounds (in the number of variables n). The complexity of computing the Gröbner basis for the second step of testing if I is radical is analogous to radical computation, though in practice the computation of \sqrt{I} is often much more difficult than computing a reduced Gröbner basis for I. Another approach of testing if an ideal is radical is to use the algorithm [11] that computes the primary decomposition, see also [15, 33, 9]. This approach also has doubly exponential worst case complexity in the number of variables.

Terminology and Notation

Since our algorithm arises from geometric ideas in intersection theory we will frequently find it useful to employ more geometric terminology (as we have in the introduction); we now make this terminology precise. We will, for the most part, work over an algebraically closed field of chacterisitic zero which we will denote k; usually this will be \mathbb{C} or the algebraic closure of the rationals, $\overline{\mathbb{Q}}$. In particular, given a polynomial ideal Iin $k[x_0, \ldots, x_n]$ we will think of it as defining a subscheme X of either \mathbb{P}^n if I is homogeneous, or k^{n+1} if not; in both cases we will write $\mathbb{V}(I)$ for this scheme X defined by I. A variety will be a reduced scheme (we do not assume that a variety is irreducible). When the ideal I has primary decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$, then we will refer to the schemes $\mathbb{V}(\mathfrak{q}_i)$ as *primary components* of X. Similarly we will refer to the scheme $\mathbb{V}(\mathfrak{q}_i)$ as an *isolated primary component*, if $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ is a minimal prime. The irreducible varieties $\mathbb{V}(\mathfrak{p}_i)$ for minimal primes \mathfrak{p}_i will be called *irreducible components*, while the irreducible varieties $\mathbb{V}(\mathfrak{p}_j)$ for embedded primes $\mathfrak{p}_j = \sqrt{\mathfrak{q}_j}$ will be called *embedded components*. We write X_{red} for the reduced subscheme associated to the scheme X, i.e., X_{red} denotes the variety defined by \sqrt{I} .

Outline of the paper. In the next section, Section 2, we present the necessary background for Segre class computations and summarize symbolic methods for sampling at least one point in each irreducible component of a subscheme of afffine space. Section 3 contains a collection of various results needed by our algorithm. In particular we present the connection of Segre classes with the integral closure of ideals, we show how to compute the degree of the isolated primary components of a scheme, and how to compute the multiplicities. Finally, in Section 4 we present our algorithm along with its proof correctness and its complexity analysis.

2 Background

We briefly review several important concepts and results which will be used extensively in later sections. In particular we review the notion of Segre classes and their association to the so called *algebraic* or *Hilbert-Samuel multiplicity*. We will also review how we can sample and represent symbolically points on a variety

using the Rational Univariant Representation (RUR).

2.1 Segre Classes and Algebraic Multiplicity

The algorithm that we present in Section 4 makes extensive use of ideas from (computational) intersection theory [13], and in particular of Segre classes. Below we give a brief overview of the relevant objects.

In general the Segre class s(X, Y) is defined for pairs of schemes $X \subset Y$ and is an element of the *Chow* group of X (see [13, §4] for details). We will restrict our discussion to the case where X and Y are subschemes of a projective space \mathbb{P}^n and will instead (via pushforward) consider the Segre class s(X, Y) as an element of the *Chow ring* of the ambient projective space \mathbb{P}^n , $A^*(\mathbb{P}^n)$. More explicitly, if H is the rational equivalence class of a hyperplane in \mathbb{P}^n , then we will represent elements in the Chow ring $A^*(\mathbb{P}^n)$ as polynomials in H with integer coefficients via the isomorphism $A^*(\mathbb{P}^n) \cong \mathbb{Z}[H]/(H^{n+1})$; in this representation the rational equivalence class of an irreducible variety V of codimension c is $[V] = \deg(V) \cdot H^c \in A^*(\mathbb{P}^n)$, where $\deg(V)$ denotes the *degree* of the variety V. Hence, in our setting the Segre class s(X, Y) will be represented as a polynomial of degree at most n in H with integer coefficients.

Consider subschemes X and Y of \mathbb{P}^n and let $\operatorname{Bl}_X Y$ denote the blowup of Y along X. This comes equipped with a map $\pi : \operatorname{Bl}_X Y \to Y$ and an exceptional divisor $E = \pi^{-1}(X)$. In the case where Y is a irreducible variety and $I_X = (f_0, \ldots, f_r)$ is the ideal defining the scheme X we have that $\operatorname{Bl}_X Y$ is isomorphic to the graph Γ of f_0, \ldots, f_r . More specifically, the graph is

$$\Gamma = \overline{\{(y,z) \mid y \in Y, \ z = (f_0(y): \dots : f_r(y))\}} \subset \mathbb{P}^n \times \mathbb{P}^r,$$

and $\Gamma \cong Bl_X Y$. In this setting π is the projection map $\Gamma \to Y$ from the graph onto Y and the exceptional divisor E is the inverse image of X under this projection map. The blowup (and hence the graph) captures information about how X sits inside Y and in particular quantifies how singular the embedding of X in Y is.

Informally speaking, the Segre class attempts to extract key parts of this information by considering how $E = \pi^{-1}(X)$ intersects with a similar scheme which is perturbed to be in general position inside of Γ . How 'general' of a position we can put a version of $\pi^{-1}(X)$ into inside Γ is significantly determined by how closely X and Y are related. We now give a formal definition of the Segre class s(X, Y).

Definition 2. Let X and Y are subschemes of \mathbb{P}^n . We have a blowup diagram

$$E \longrightarrow \operatorname{Bl}_X Y$$

$$\downarrow^{\eta} \qquad \qquad \downarrow^{\pi}$$

$$X \longrightarrow Y,$$

where E is the exceptional divisor. The Segre class of X in Y is

$$s(X,Y) = \eta_*((1-E+E^2-\dots) \frown [E]) \in A_*(X).$$

We will abuse notation and also write s(X, Y) for the pushforward to $A^*(\mathbb{P}^n)$.

We now briefly review results of [18] which give a explicit and computable expression for the Segre class. Using the notation above, consider the rational map $pr_X : Y \to \mathbb{P}^r$ which is defined by $pr_X : p \mapsto (f_0(p) : \cdots f_r(p))$. We then have the following diagram:



Computationally, rather than trying to understand the self intersections of $\pi^{-1}(X)$ we will instead seek to understand $pr_X^{-1}(\mathbb{P}^{r-\dim(Y)-i}) - X$ for $i = 0, \ldots, \dim(Y)$. In particular we will study the *projective degrees*, $\mathfrak{d}_i(X, Y)$, which are the coefficients appearing in the class

$$G(X,Y) = \sum_{i=0}^{\dim(Y)} [pr_X^{-1}(\mathbb{P}^{r-\dim(Y)-i}) - X] = \sum_{i=0}^{\dim(Y)} \mathfrak{d}_i(X,Y) H^{n-i}.$$

We now give explicit expressions for these objects in terms of polynomial ideals in $k[x] = k[x_0, ..., x_n]$, the homogenous coordinate ring of \mathbb{P}^n . As above, k is an algebraically closed field of characteristic zero.

Definition 3 (Projective Degrees). Consider the subschemes $X \subset Y \subset \mathbb{P}^n$. Let X be defined by the homogeneous ideal $I_X = (f_0, \ldots, f_r)$ and let d be the maximum degree of the defining polynomials of X. We can, without loss of generality, assume that $\deg(f_i) = d$ for all $0 \le i \le r$. Define the projection of Y along X as the rational map

$$\begin{aligned} \pi_X : & Y & \dashrightarrow & \mathbb{P}^r \\ & p & \mapsto & (f_0(p) : \cdots : f_r(p)). \end{aligned}$$

$$(3)$$

The projective degrees of π_X are the sequence of integers $(\mathfrak{d}_0(X,Y),\ldots,\mathfrak{d}_{\dim(Y)}(X,Y))$, where

$$\mathfrak{d}_i(X,Y) = \deg\left(\pi_X^{-1}\left(\mathbb{P}^{r-(\dim(Y)-i)}\right) - X\right).$$
(4)

Equivalently (via [18, Proposition 3.3]), we can define the projective degrees of π_X as

$$\mathfrak{d}_i(X,Y) = \deg(Y \cap L^{(i)} \cap \mathcal{U} - X),\tag{5}$$

where $\mathcal{U} = \mathbb{V}(P_1, \ldots, P_{\dim(Y)-i})$ with $P_j = \sum_{\nu=0}^r \lambda_{\nu} f_{\nu}$ for generic $\lambda_{\nu} \in k$ and $L^{(i)} \subset \mathbb{P}^n$ is a generic linear space of codimension *i*.

Using the latter characterization of projective degrees, we can express them with respect to the ideals corresponding to the subschemes $X \subset Y \subset \mathbb{P}^n$, that is $I_X = \langle f_0, \ldots, f_r \rangle$ and I_Y . Let t be a new variable; then, for $0 \leq i \leq r$, we define the family of ideals

$$\mathcal{I}_{i} = I_{Y} + \left\langle \sum_{j=0}^{r} \lambda_{1,j} f_{j}, \dots, \sum_{j=0}^{r} \lambda_{\dim(Y)-i,j} f_{j}, \ell_{1}(\boldsymbol{x}), \dots, \ell_{i}(\boldsymbol{x}), \ell_{0}(\boldsymbol{x}) - 1, 1 - t \sum_{j=0}^{r} \lambda_{0,j} f_{j} \right\rangle \subset k[\boldsymbol{x}][t],$$
(6)

where $\ell_{\nu}(\boldsymbol{x}) = \sum_{j=0}^{n} \theta_{\nu,j} x_j$ for generic $\theta_{l,j} \in k$, and generic $\lambda_{l,j} \in k$. We also write $\mathcal{I}_i(X, Y)$ to denote the dependency on the subschemes X and Y. By [18, Theorem 3.5] the projective degree of dimension *i* can be computed as

$$\mathfrak{d}_i(X,Y) = \dim_k \left(k[\boldsymbol{x},t]/\mathcal{I}_i \right). \tag{7}$$

In [18, §3] an explicit formula for the Segre class s(X, Y) is given which depends only on the numbers $\mathfrak{d}_i(X, Y)$ obtained via the computation in (7) and the degree d of the generators of I_X (though the final Segre class does not depend on d). The projective degrees can also be used to compute the *algebraic multiplicity* of a variety inside a scheme.

Let $X \,\subset \mathbb{P}^n$ be an irreducible (and reduced) subvariety and let $Y \subset \mathbb{P}^n$ be a pure dimensional subscheme with corresponding ideals $I_X = \langle f_0, \ldots, f_r \rangle$ and I_Y in $k[x] = k[x_0, \ldots, x_n]$. The algebraic or Hilbert-Samuel multiplicity of X on Y, denoted $e_X Y$ is the integer coefficient of [X] in the Segre class s(X,Y). This multiplicity is more classically defined via the Hilbert-Samuel polynomial of the local ring $(k[x_0, \ldots, x_n]/I_Y)_{I_X}$, where the subscript denotes localization at the prime ideal I_X , see, e.g., [13, Example 4.3.1,Example 4.3.4] or [10, Chapter 12]. In practice we will compute this multiplicity as follows. Let d be the maximum degree among a set of generators of I_X . By [18, Theorem 5.2] we have that

$$e_X Y = \frac{\deg(Y) d^{\dim(Y) - \dim(X)} - \mathfrak{d}_{\dim(X)}(X, Y)}{\deg(X)}, \tag{8}$$

where $\mathfrak{d}_{\dim(X)}(X, Y)$ is the dimension X projective degree of X in Y, see (7), (5), or Definition 3.

It is a classical result of Samuel [32, II §6.2b] (see also [13, Ex. 12.4.5(b)]) that $e_X Y = 1$ if and only a generic point in X is not contained in the singularity subscheme of Y. We state this as a proposition below.

Proposition 4 (Samuel [32]). Let $X \subset \mathbb{P}^n$ be an irreducible subvariety and let $Y \subset \mathbb{P}^n$ be a pure dimensional subscheme. Then $e_X Y = 1$ if and only if a generic point in Y is reduced and X is not contained in the singular locus of Y_{red} .

This fact along with a formula derived in §3.4 will be adapted to furnish a test which tells us when an isolated primary component of a scheme is generically reduced.

In the final section we will additionally need to work with varieties X which are not irreducible, in this case it will be most convenient to write our criterion in terms of the $\dim(X)$ part of the Segre class. Let Y be a pure dimensional subscheme of \mathbb{P}^n and let X be a closed subscheme of Y. Let d be the maximum degree of the equations defining X. With the notations above the result of [18, Corollary 3.14] gives the following:

$$\{s(X,Y)\}_{\dim(X)} = d^{\dim(Y) - \dim(X)} \deg(Y) - \mathfrak{d}_{\dim(X)}(X,Y).$$

$$\tag{9}$$

2.2 Rational Univariate Representation and Computing One Point in Each Irreducible Component

An important part of the main algorithm presented in Section 4 is the ability to sample points from each irreducible component of a scheme $X \subset \mathbb{C}^n$. Approximate point samples could be furnished using methods such as homotopy continuation from numeric algebraic geometry [34], we however opt for symbolic methods, e.g., [30], in our presentation as these methods have well understood worst case complexities. We note that our algorithm certainly could be implemented using numerical methods instead, see also Remark 16. In the symbolic setting, it is important for the sampling algorithms we employ to have an efficient and exact representation of the sampled points. For this we exploit the *rational univariate representation (RUR)* [31], see also [1, 27].

Briefly, given a zero dimensional ideal in $\mathbb{Q}[x_1, \ldots, x_n]$, RUR represents the coordinates of the associated points as a univariate rational function evaluated at the roots of univariate polynomial. It is of the form $R(t), x_1 = \frac{R_1(t)}{R_0(t)}, \ldots, x_n = \frac{R_n(t)}{R_0(t)}$ where $R, R_0, R_1, \ldots, R_n \in \mathbb{Q}[t]$ and t is a new variable. The following theorem provides a brief presentation of RUR and summarizes some of its important properties.

We note in the Theorem statement below we consider the (reduced) zero set associated to a polynomial system (or polynomial ideal). These methods do not require that the input polynomials define a radical ideal but will only furnish information about the zero set of the input polynomials, i.e., about the associated reduced subscheme.

Theorem 5. Consider the solution set $W = \{x \in \mathbb{C}^n \mid f_1(x) = \cdots = f_n(x) = 0\}$, where the polynomials $f_i \in \mathbb{Z}[x_1, \ldots, x_n]$ are dense of degree d and maximum coefficient bitsize τ . Then, there is an algorithm that computes univariate polynomials of $R, R_1, \ldots, R_n \in \mathbb{Z}[t]$ such that:

- 1. The degrees of R, R_1, \ldots, R_n are all bounded by d^n and their bitsize by $\mathcal{O}(d^n + nd^{n-1}\tau)$.
- 2. For any root ξ of R(t) = 0, the tuple $r(\xi) := \left(\frac{R_1(\xi)}{R'(\xi)}, \dots, \frac{R_n(\xi)}{R'(\xi)}\right) \in (\mathbb{C}^*)^n$ is a point in W; R' is the derivative of R with respect to t. We call this representation the Rational Univariate Representation (*RUR*) [31].
- 3. Every $r(\xi)$ corresponds to an isolated point of W or to a point on an irreducible component of W(of dimension greater than 0). In addition, the set of points $\{r(\xi) \in \mathbb{C}^n | R(\xi) = 0\}$ contains all the 0-dimensional irreducible components of W in \mathbb{C}^n .
- 4. We compute the RUR by performing $d^{\mathcal{O}(n)}$ arithmetic operations.

Proof. For the bounds on the bitsize of the coefficients and a generalization to the sparse case and how we compute the RUR using resultants we refer the reader to [27, Theorem 4.3]. For item 3 we refer the reader to [30].

To compute the RUR in the case where the system is not 0-dimensional we use Canny's generalized characteristic polynomial [8], or the toric variant by Rojas [30, 29], that relya on Macaulay and sparse resultant matrices, respectively, see [27]. The complexity bound follows from [7].

We remark that the above result is presented in the case of a square system, however if the system is not square, then one may simply take a general linear combination of the generators to get a square system and verify which sampled points satisfy the original system. To get samples from all irreducible components of higher dimensions we may then add a generic linear polynomial to the defining ideal (i.e., intersect the scheme with a generic hyperplane) and repeat this procedure. Such considerations are discussed in more detail in the references.

The algorithms (and the corresponding mathematical results) that support Theorem 5 allow us to sample at least one (generic) point from each irreducible component of an arbitrary subscheme of \mathbb{C}^n . The symbolic algorithms that support these computations rely on resultant or Gröbner basis computations and careful analysis of the bitsize of the involved polynomial computations. We refer the interested reader to [12] for an approach based on Bézoutian matrices, to [27, 30] for an approach based on resultant matrices and to [22] for an implementation. This leads us to the following corollary.

Corollary 6. Let I be a polynomial ideal generated by polynomials of degree at most d in $\mathbb{Q}[x_1, \ldots, x_n]$ defining a scheme $X \subset \mathbb{C}^n$. Then, we can obtain a collection of RUR of points containing at least one generic point in each irreducible component of X in at most $d^{\mathcal{O}(n)}$ arithmetic operations.

3 A Collection of Results Which Enable Our Algorithm

In this section we gather together several results which will be employed by our main algorithm which is presented in Section 4. We first consider, in §3.1, some results regarding the relationships between the radical of an ideal, its integral closure, and a Segre class associated to the corresponding scheme. Following this, in §3.2, we show how the degree of isolated primary components of any dimension can be computed in worst case singly exponential time with respect to the number of variables. Then, in §3.3, we give a result which lets us compute the radical of an ideal in one saturation when all isolated primary components of the associated scheme are reduced. Finally, in §3.4 and §3.5, we look at multiplicity and (partial) Segre class computations relative to isolated primary components of any dimension and how these computations can be done in a generic affine patch of projective space.

3.1 Segre Classes and the Integral Closure of Ideals

We will now give a test which will let us determine if an ideal I is equal to its radical (upto a factor of the irrelevant ideal) via a Segre class computation. En-route we shall be led to consider the *integral closure* \overline{I} of the ideal I. We shall in particular be interested in the case where $\overline{I} = \sqrt{I}$.

We begin by reviewing several key results from the literature. First, we present a result of Gaffney and Gassler [14], that we slightly modify to fit in our notational framework. This result makes clear the connection between Segre classes and integral closures of ideals. One can also deduce this result from other more recent works, such as [2, 3] that use a more similar terminology to ours.

Proposition 7 (Corollary 4.9 of [14]). Let X and Y be subschemes of \mathbb{P}^n defined by ideals I_X and I_Y in $k[x_0, \ldots, x_n]$ with $X_{\text{red}} = Y_{\text{red}}$. Then, we have that $s(X, \mathbb{P}^n) = s(Y, \mathbb{P}^n)$ if and only if

$$\overline{I_X}$$
: $\langle x_0, \dots, x_n \rangle^{\infty} = \overline{I_Y}$: $\langle x_0, \dots, x_n \rangle^{\infty}$

To connect the result above to the problem of checking if an ideal is equal to its radical we will need to in turn explore connections between the radical of a polynomial ideal and its integral closure. Our main sources here are [35] and [20]. It is shown in [35, Remark 1.6] that if J, L, and W are ideals in an integral domain R with $\overline{J} = \overline{L}$, then we have that $(JW) : L \subseteq \overline{W}$. In our setting this gives the following corollary.

Corollary 8. Let $R = k[x_0, ..., x_n]$ and let I be any proper ideal such that $\sqrt{I} = \overline{I}$. Then we have that:

- $I. \ (\sqrt{I})^2: I \subset \sqrt{I},$
- 2. $(\sqrt{I})^2 \subseteq I$.

Proof. Part 1 follows immediately from [35, Remark 1.6] and the fact that $\sqrt{I} = \overline{I}$ by taking $J = W = \sqrt{I}$ and L = I (note $\overline{L} = \overline{J} = \sqrt{I}$ by assumption).

Since I is a proper ideal of R it follows that \sqrt{I} is also a proper ideal, and hence part 2 follows by properties of the colon ideal.

Below we state a characterization of the integral closure of an ideal in a Noetherian ring.

Proposition 9 (Corollary 6.8.11 of [20]). Let R be a Noetherian ring, I and ideal in R and r any element of R. Then $r \in \overline{I}$ if and only if there exists an integer ν such that for all integers $m > \nu$ we have $r^m \in I^{m-\nu}$.

We now show that an ideal whose radical and integral closure agree must be radical, this will allow us to directly employ Proposition 7 to test if an ideal is equal to its radical.

Proposition 10. Let $R = k[x_0, ..., x_n]$ and let I be any ideal such that $\overline{I} = \sqrt{I}$. Then $I = \sqrt{I}$, that is I is radical.

Proof. Suppose that I is not radical but $\overline{I} = \sqrt{I}$; we will show this yields a contradiction. If $I \neq \sqrt{I}$ then $\sqrt{I} - I$ is non-empty, hence we may choose some $r \neq 0$ in $\sqrt{I} - I$. Then we have that $r \in \sqrt{I} = \overline{I}$, but $r \notin I$. Further by Corollary 8 we have that $\sqrt{I}^2 \subseteq I$, hence $r^2 \in I$. Since R is a UFD we have $r = r_1^{a_1} \cdots r_c^{a_c}$ is the unique expression for r, further we may choose $r \in \sqrt{I} - I$ minimal in that sense that $r \in \sqrt{I}$, but $(r_1^{a_1} \cdots r_c^{a_c})/r_i \notin \sqrt{I}$ for any i; we may choose r in this way since if one of these expressions were in \sqrt{I} we could simply take this factor as r instead.

Claim: For any $\nu \ge 0$ we have $r^m \notin I^{m-\nu}$ for $m > 2\nu$. Note that, if this claim holds then by Proposition 9 we have that $r \notin \overline{I} = \sqrt{I}$, but this contradicts our assumption that $\sqrt{I} - I$ is non-empty.

We now prove the above claim: for any $\nu \ge 0$ we have $r^m \notin I^{m-\nu}$ for $m > 2\nu$. Since $r \notin I$ but $r^2 = r_1^{2a_1} \cdots r_c^{2a_c} \in I$, then $r^4 = r_1^{4a_1} \cdots r_c^{4a_c} \in I^2$ is the smallest power of r in I^2 , since R is a UFD. In particular note that if $r^3 = r_1^{3a_1} \cdots r_c^{3a_c}$ were in I^2 then this would imply $r \in I$ since we know that $(r_1^{a_1} \cdots r_c^{a_c})/r_i \notin \sqrt{I}$ for all r_i . Similarly r^6 is the smallest power of r in I^3 , and by induction, $r^{\ell} \notin I^{\mu}$ for $\ell < 2\mu$. Now take $\ell = m$, $\mu = m - \nu$, then we have that $r^m \notin I^{m-\nu}$ for any m and ν such that $m < 2(m - \nu) = 2m - 2\nu$, or equivalently, such that $2\nu < m$. In other words, for any $\nu \ge 0$ we have $r^m \notin I^{m-\nu}$ for $m > 2\nu$; which is what we wished to show. Hence the claim is proved and the result follows as discussed above.

This result, along with Proposition 7 combine to yeild a method to test if an ideal is equal to its radical *without* using a Gröbner basis. We summarize this in the corollary below.

Corollary 11. Let I be a homogeneous polynomial ideal in $k[x_0, \ldots, x_n]$, let $X \subset \mathbb{P}^n$ be the scheme defined by I and let X_{red} be the reduced scheme defined by \sqrt{I} . Then $I : \langle x_0, \ldots, x_n \rangle^{\infty}$ is radical if and only if $s(X, \mathbb{P}^n) = s(X_{red}, \mathbb{P}^n)$. We note that while the result of Corollary 11 removes the need to use a Gröbner basis calculation (or some other doubly exponential method) to test if \sqrt{I} is equal to I (up to saturation by the irrelevant ideal), it does not immediately remove the requirement to compute \sqrt{I} . In many cases $s(X_{\text{red}}, \mathbb{P}^n)$ can in fact be computed with objectively less information than a generating set of \sqrt{I} , in general however it is not clear how to compute $s(X_{\text{red}}, \mathbb{P}^n)$ without a generating set for \sqrt{I} . The result presented in §3.3 reduces this problem (in the situations where we need to solve it) to that of computing $s(\mathbb{V}(I : J^{\infty}), \mathbb{P}^n)$, for J generated by the $\operatorname{codim}(X) \times \operatorname{codim}(X)$ minors of the Jacobian of I, using only a generating set of I and J; as yet we are also unable to furnish this computation however. This remains the main barrier to the algorithm presented in Section 4 having a singly exponential worst case bound (in the number of variables) in all cases.

3.2 The Degree of Isolated Primary Components of a Scheme in Singly Exponential Time

In this subsection we describe how we can use methods to compute a geometric equidimensional decomposition such as [21], along with zero dimensional Gröbner basis computation, to obtain the sum of the degrees of all isolated primary components of a scheme of a given dimension. Given an ideal $I = \langle f_1, \ldots, f_r \rangle$ in the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ where deg $(f_i) + 1 \leq d$ for all *i* which defines a scheme $X = \mathbb{V}(I) \subseteq \mathbb{C}^n$, we will show in Proposition 13, that this degree computation has worst case complexity bounded by $d^{\mathcal{O}(n^3)}$. We note that this computation takes as input only the generators of the ideal I.

The algorithm of [21] takes as input the ideal I and outputs (a straight-line program) defining (distinct) polynomial ideals K_{i_1}, \ldots, K_{i_m} where these ideals give an equidimensional decomposition of the variety $X_{\text{red}} = \mathbb{V}(\sqrt{I})$, i.e.,

$$X_{\mathrm{red}} = (\mathbb{V}(K_{i_1}))_{\mathrm{red}} \cup \cdots \cup (\mathbb{V}(K_{i_m}))_{\mathrm{red}},$$

where $(\mathbb{V}(K_{i_j}))_{\text{red}}$ is the union of all irreducible components of X_{red} of dimension i_j .

Without loss of generality we may assume that $i_1 > \cdots > i_m$. Let L_j be a general linear form in $\mathbb{C}[x_1, \ldots, x_n]$. Suppose that $K_{i_j} = \langle g_1, \ldots, g_r \rangle$ and define the polynomial $P(K_{i_j}, T)$ in $\mathbb{C}[x_1, \ldots, x_n, T]$ as

$$P(K_{i_j}, T) := 1 - T \cdot \sum_{\nu=1}^r \lambda_{\nu} g_{\nu}.$$
 (10)

Let $i_{\nu} \ge \mu \ge 0$ where ν is minimal and $i_1 \ge \mu \ge 0$ holds. Define the (isolated) degree in dimension μ of I as

$$\deg_{\mu}(I) := \dim_{\mathbb{C}} \mathbb{C}[x_1, \dots, x_n, T_1, \dots, T_{\nu}]/\mathfrak{J}(\mu), \tag{11}$$

where

$$\mathfrak{J}(\mu) := I + \langle L_1, \dots, L_\mu \rangle + \langle P(K_{i_1}, T_1), \dots, P(K_{i_\nu}, T_\nu) \rangle.$$

If X is the subscheme of \mathbb{C}^n defined by I we will define

$$\deg_{\mu}(X) := \deg_{\mu}(I).$$

Proposition 12. Let I be a polynomial ideal in $\mathbb{Q}[x_1, \ldots, x_n]$. Suppose that the ideal \mathfrak{I}_{μ} is the intersection of all μ -dimensional isolated primary components of I. Then, with $\deg_{\mu}(I)$ as in (11), we have

$$\deg_{\mu}(I) = \deg(\mathfrak{I}_{\mu}).$$

Proof. Note that this result is essentially an application of the standard Rabinowitsch trick. Let X be the scheme defined by I in $\mathbb{C}^{n+\nu}$. By construction all primary components of X of dimension greater than μ have an empty intersection with the scheme defined by $\langle P(K_{i_1}, T_1), \ldots, P(K_{i_\nu}, T_\nu) \rangle$, note this also applies to any embedded components of X of any dimension which are contained in isolated primary components of X of dimension greater than μ . Note for points (x_1, \ldots, x_n) not in $(\mathbb{V}(K_{i_l}))_{\text{red}}$ the polynomial $P(K_{i_l}, T)$ has a single solution of multiplicity one, hence intersection with these hypersurfaces can be thought of a intersection with hyperplanes for isolated primary components of X of dimension less than or equal to μ . Since the linear forms L_j are general and there are μ of them the conclusion follows since the only points which remain in the zero dimensional scheme defined by $\mathfrak{J}(\mu)$ came from those in the scheme defined by \mathfrak{I}_{μ} .

Finally we note that the expression in (11) can be computed in singly exponential time in the number of variables. More specifically we have the following result.

Proposition 13. Consider a polynomial ideal $I = \langle f_1, \ldots, f_r \rangle$ in the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ where $\deg(f_i) + 1 \leq d$. For any $\mu \leq \dim(I)$, the worst case complexity of computing the expression $\deg_{\mu}(I)$ is bounded in $d^{\mathcal{O}(n^3)}$.

Proof. From results in [21] we know the degree of the polynomials g_{ν} appearing in (10) is bounded by d^{n^2} . The output of [21] consists of polynomials in straight-line programs representation. However, using evaluation/interpolation we can convert them to a dense representation, using the bound on their degree.

Since the ideal $\mathfrak{J}(\mu)$ has dimension zero by construction, using standard bounds on the computation of Gröbner basis in dimension zero [19, 25] (or for other methods to compute the degree of a zero dimensional ideal) gives $(d^{n^2})^{2n-1}$, giving $d^{\mathcal{O}(n^3)}$.

3.3 Finding the Radical of an Ideal with Prime Isolated Primary Components

In this subsection we consider a scheme X such that all isolated primary components are reduced and prove a compact formula in terms of a certain saturation for the ideal of the reduced scheme X_{red} . The following proposition is a summary of the result we will use in the algorithm in Section 4, this result is essentially just by definition, and no doubt well known, but we include a short proof for completeness.

Proposition 14. Let I be a homogeneous ideal in $k[x_0, ..., x_n]$ defining a scheme $X = \mathbb{V}(I) \subset \mathbb{P}^n$ with $c = \operatorname{codim}(X)$. Let J = I + K where K is the ideal generated by all $(c \times c)$ -minors of the Jacobian matrix of I. Suppose that all isolated primary components of X are reduced, then $X_{red} = \mathbb{V}(I : J^\infty)$; i.e., $\sqrt{I} = I : J^\infty$ up to saturation by the irrelevant ideal. In other words $X_{red} = \overline{X} - \mathbb{V}(J)$.

Proof. Relabeling a primary decomposition of X we may write

$$X = (W_1^{\text{iso}} \cup W_1^{\text{emb}}) \cup \dots \cup (W_r^{\text{iso}} \cup W_r^{\text{emb}}),$$

where W_i^{iso} is an isolated primary component and W_i^{emb} is the union of all primary components which correspond to embedded components contained in the irreducible component corresponding to W_i^{iso} . Further since, by assumption, all isolated primary components are reduced we must have that

$$X_{\mathrm{red}} = W_1^{\mathrm{iso}} \cup \dots \cup W_r^{\mathrm{iso}}$$

Now consider $\overline{X - \mathbb{V}(J)}$, we have

$$\overline{X - \mathbb{V}(J)} = \overline{\left((W_1^{\text{iso}} \cup W_1^{\text{emb}}) \cup \dots \cup (W_r^{\text{iso}} \cup W_r^{\text{emb}}) \right) - \mathbb{V}(J)}$$
$$= \overline{(W_1^{\text{iso}} \cup W_1^{\text{emb}}) - \mathbb{V}(J)} \cup \dots \cup \overline{(W_r^{\text{iso}} \cup W_r^{\text{emb}}) - \mathbb{V}(J)}$$
$$= W_1^{\text{iso}} \cup \dots \cup W_r^{\text{iso}}.$$

The last equality follows since W_i^{iso} is irreducible and reduced (being reduced means $W_i^{\text{iso}} \cap \mathbb{V}(J) \subsetneq W_i^{\text{iso}}$) and since we must have that $W_i^{\text{emb}} \subset \mathbb{V}(J)$ for all *i*. Hence, $X_{\text{red}} = \overline{X - \mathbb{V}(J)}$.

3.4 Multiplicity of a Subvariety in Isolated Primary Components

Using the dimension sensitive degree function from §3.2 and the multiplicity formula (8) we obtain a method to compute the multiplicity of an irreducible variety contained in the union of all ν -dimensional isolated primary components of a scheme $Y \subset \mathbb{P}^n$ with multiple components of different dimensions.

Let $Y \subset \mathbb{P}^n$ be an arbitrary subscheme. Let W be the union of all isolated primary components of Y of dimension ν and let $X = \mathbb{V}(f_0, \ldots, f_r) \subset \mathbb{P}^n$ be an irreducible subvariety of W with the generators chosen such that $\deg(f_i) = d$. Define the ideal

$$\mathcal{I}_{\nu}(X) = I_Y + \left\langle \sum_{j=0}^r \lambda_{1,j} f_j, \dots, \sum_{j=0}^r \lambda_{\nu-\dim(X),j} f_j, \ell_0(\boldsymbol{x}) - 1, 1 - t \sum_{j=0}^r \lambda_{0,j} f_j \right\rangle \subset k[\boldsymbol{x}][t].$$

Since Y has isolated primary components of dimension ν the scheme $\mathbb{V}(\mathcal{I}_{0,\nu})$ has isolated primary components of dimension equal to $\dim(X)$. Then, since $X \subset W$ we have

$$e_X W = \frac{\deg_{\nu}(Y) d^{\nu - \dim(X)} - \deg_{\dim(X)}(\mathcal{I}_{\nu}(X))}{\deg(X)}.$$
(12)

We note that this formula follows immediately from (8) since the functions \deg_{ν} and $\deg_{\dim(X)}$, respectively, will capture the degrees of the ν and $\dim(X)$ dimensional isolated primary components only, respectively. Hence, since X is contained in the pure ν -dimensional scheme W, the primary components of other dimensions play no role and we immediately obtain $e_X W$. For identical reasons we obtain an analogous version of (9) which we state below:

$$\{s(X,W)\}_{\dim(X)} = d^{\nu - \dim(X)} \deg_{\nu}(Y) - \deg_{\dim(X)}(\mathcal{I}_{\nu}(X)).$$
(13)

We note that these computations take as input only the equations defining the schemes X and Y and that their complexity is determined by that of the degree function of §3.2, see Proposition 13. We can also extend this to the case where X is an arbitrary variety not contained in W, but contained in Y, using [13, Example 4.3.4], or the similar result for Segre classes, [13, Lemma 4.2], however this extension will not be needed for our purposes here. It is worth noting, however, that in particular the result of [13, Lemma 4.2] implies that if $X \cap W$ is empty then s(X, W) = 0 and that if $\dim(X \cap W) < \dim(X)$ then $\{s(X, W)\}_{\dim(X)} = 0$, and further it follows from results in [18] that all these properties hold for the formula given above, e.g. (9) and (13).

3.5 Affine Version of Multiplicity

From the definition of the Hilbert-Samuel multiplicity in terms of the Hilbert-Samuel polynomial one can easily deduce that the multiplicity of a projective variety in a subscheme of projective space agrees with the multiplicity of the resulting pair of affine varieties on a generic affine patch. Below we will see that our method for computing these multiplicities works similarly. This will be useful when we need to employ sampling techniques which are meant for affine varieties in our main algorithm.

Let $X \,\subset W \,\subset \mathbb{P}^n$ with X a variety and W the union of all dimension ν isolated primary components of Y. Suppose that $I_X = \langle f_0, \ldots, f_r \rangle$ and without loss of generality assume $d := \deg(f_i)$ for all i. Take $\ell_0(x) \in \mathbb{C}[x_0, \ldots, x_n]$ to be a general linear form and set $\hat{I}_X = I_X + \langle \ell_0 - 1 \rangle$ and $\hat{I}_Y = I_Y + \langle \ell_0 - 1 \rangle$. Finally, let $\hat{Y} \subset \mathbb{C}^{n+1}$ be the scheme defined by the ideal $I_Y + \langle \ell_0 - 1 \rangle$, let \hat{W} be the pure ν -dimensional subscheme of \hat{Y} arising from W and let $\hat{X} = V(\hat{I}_X) \subset \mathbb{C}^{n+1}$. Note that \hat{X} is a subvariety of \hat{W} and that the degrees and dimensions are unchanged, i.e., $\deg(X) = \deg(\hat{X})$, $\deg_{\mu}(Y) = \deg_{\mu}(\hat{Y})$ for all μ such that Y has an isolated primary component of dimension μ , $\dim(X) = \dim(\hat{X})$, and $\dim(Y) = \dim(\hat{Y})$. We can think of this procedure of moving to the ideal \hat{I} from I as a combination of dehomogenizing with respect to a general point at infinity and linearly embedding \mathbb{C}^n into \mathbb{C}^{n+1} . Now let $\phi_i = f_i$ for $i = 1, \ldots, r$ and $\phi_{r+1} = \ell_0 - 1$, set

$$\hat{\mathcal{I}}_{\nu}(X) = \hat{I}_{Y} + \left\langle \sum_{j=0}^{r} \lambda_{1,j} \phi_{j}, \dots, \sum_{j=0}^{r} \lambda_{\nu-\dim(X),j} \phi_{j}, \ell_{0}(\boldsymbol{x}) - 1, 1 - t \sum_{j=0}^{r} \lambda_{0,j} \phi_{j} \right\rangle \subset k[\boldsymbol{x}][t],$$

and as in the previous subsection

$$\mathcal{I}_{\nu}(X) = I_Y + \left\langle \sum_{j=0}^r \lambda_{1,j} f_j, \dots, \sum_{j=0}^r \lambda_{\nu-\dim(X),j} f_j, \ell_0(\boldsymbol{x}) - 1, 1 - t \sum_{j=0}^r \lambda_{0,j} f_j \right\rangle \subset k[\boldsymbol{x}][t].$$

Observe that $\dim(\mathcal{I}_i) = \dim(\hat{\mathcal{I}}_i)$ and that $\deg_{\nu}(\mathcal{I}_i) = \deg_{\nu}(\hat{\mathcal{I}}_i)$. Hence if we define

$$e_{\hat{X}}\hat{W} := \frac{\deg_{\nu}(\hat{Y})d^{\nu-\dim(\hat{X})} - \deg_{\dim(X)}(\hat{I}_{\nu}(X))}{\deg(\hat{X})},$$
(14)

then using (12) we immediately obtain that $e_{\hat{X}}\hat{W} = e_X W$. This equivalence will be used extensively in the sequel to allow us to freely switch between projective and affine formulations. This also immediately gives the following formula for the dimension $\dim(X)$ part of the Segre class of X in W:

$$\{s(X,W)\}_{\dim(X)} = \deg_{\nu}(\hat{Y})d^{\nu-\dim(\hat{X})} - \deg_{\dim(X)}(\hat{I}_{\nu}(X)),$$
(15)

where we no longer need to assume that X is irreducible or reduced, we only require X is a subscheme of W. Note, in Algorithm 1 we will abuse notation and write $\{s(\hat{X}, \hat{W})\}_{\dim(X)}$ to mean the expression $\{s(X, W)\}_{\dim(X)}$ computed via the affine formula (15) above.

4 Testing if an Ideal is Radical

In this section we present our main algorithm, Algorithm 1, for testing if a scheme Y in \mathbb{P}^n is reduced or, equivalently, if the defining ideal is radical up to saturation by the irrelevant ideal. First, in §4.1, we give the idealized version (i.e., we assume that we can work over \mathbb{C}) and that for a point sampled from an irreducible component we obtain the associated maximal ideal. Next, in §4.2, we explain how we would adapt Algorithm 1 to perform computations with (groups of) sampled points represented in the Rational Univariate Representation (RUR), as it is required for a realistic symbolic implementation.

4.1 Algorithm

Here we present our main algorithm, Algorithm 1, to test if a homogeneous ideal is radical, up to saturation via the irrelevant ideal. We begin with a proof of correctness.

Theorem 15. Given a homogeneous ideal I in $k[x_0, ..., x_n]$ Algorithm 1 correctly tests if $I : \langle x_0, ..., x_n \rangle^{\infty}$ is radical and termiates in finite time.

Proof. The fact that Line 7 detects all isolated primary components which are non-reduced follows from Proposition 4, combined with (14) and (15), and the fact that we work with a single reduced point in the first factor of the Segre class. Now we consider the case where Line 16 detects embedded components which are not contained in the singular locus of $Y_{\rm red}$. First, note that if q is a reduced point in the singular subscheme of Y, then either q is contained in an embedded component which is embedded outside of the singular locus of $Y_{\rm red}$ or q is inside the singular locus of $Y_{\rm red}$. Note the loop in Line 14 starts in top dimension. Suppose that ν is the largest value such that q is contained in W_{ν} (in the notation of the algorithm). If q is inside the singular locus of $Y_{\rm red}$ then its multiplicity will always be greater than one in W_{ν} and $s(q, W_{\rho}) = 0$ for $\rho > \nu$ since q cannot be contained in these. If q is not inside the singular locus of Y_{red} then it must be inside some embedded component of Y. In this case q will also be contained in some higher dimensional isolated primary component, but then since we know all the isolated primary components are reduced, and since multiplicity is computed relative to the top dimensional component, we will obtain that the multiplicity of q is one inside this highest dimensional component in which it is contained by Proposition 4. Hence we will have correctly identified the existence of any embedded components outside of the singular locus of Y_{red} . The correctness of the remainder of the algorithm follows from the combination of Proposition 14, for Line 21, and Corollary 11, for Lines 22–25. The algorithm terminates since there are finitely many components.

Algorithm 1: IS_RADICAL **Input:** A homogeneous ideal $I = \langle f_0, \ldots, f_r \rangle \subset \mathbb{C}[x_0, \ldots, x_n]$ defining a subscheme Y of \mathbb{P}^n . **Output:** TRUE if Y is reduced (i.e., if $I : \langle x_0, \ldots, x_n \rangle^{\infty}$ is radical) and FALSE otherwise. 1 Set $\ell_0(x)$ to be a generic (random) homogeneous linear form in $\mathbb{C}[x_0, \ldots, x_n]$; 2 Set $I = I + \langle \ell_0 - 1 \rangle$; Set N = I; Set \hat{Y} to be the subscheme of \mathbb{C}^{n+1} associated to N: **3** for $\nu = \dim(\hat{Y}), \ldots, 0$ do Compute (at least) one generic point in each dim. ν irreducible component of \hat{Y} via the method 4 summarized in Corollary 6. Call the resulting collection of points $\{p_1, \ldots, p_r\} \subset \mathbb{C}^{n+1}$; for $p \in \{p_1, ..., p_r\}$ do 5 Let W(p) denote the isolated primary component of \hat{Y} containing p; 6 7 Compute $\{s(p, W(p))\}_0$ using (15); if $\{s(p, W(p))\}_0 \neq 1$ then 8 RETURN FALSE ; 9 10 Set $c = \operatorname{codim}(Y, \mathbb{P}^n);$ 11 Set $J = I + \mathfrak{Jac}_c(I)$ where $\mathfrak{Jac}_c(I)$ is the ideal generated by the $c \times c$ minors of the Jacobian matrix of $f_0, ..., f_r$; 12 Using the method of Corollary 6 compute (at least) one generic point in each irreducible component of the singular subscheme $\mathbb{V}(J)$ of \hat{Y} , call this collection of points $\{q_1, \ldots, q_s\} \subset \mathbb{C}^{n+1}$; 13 for $q \in \{q_1, ..., q_r\}$ do for W_{ν} the union of isolated primary comp. of \hat{Y} of dim. ν , starting from top dim. $\nu = \dim(Y)$ do 14 Compute $\{s(q, W_{\nu})\}_0$ using (15); 15 if $\{s(q, W_{\nu})\}_0 = 1$ then 16 q is contained in an embedded component of Y but not in the singular locus of Y_{red} ; 17 RETURN FALSE ; 18 if $\{s(q, W_{\nu})\}_0 > 0$ then 19 BREAK to line 13 and select the subsequent point q in the list; 20 21 Compute the saturation $K = I : J^{\infty}$ and let $Y_{red} = \mathbb{V}(K)$, note K is equal to \sqrt{I} by Proposition 14; 22 if $s(Y_{red}, \mathbb{P}^n) \neq s(Y, \mathbb{P}^n)$ then RETURN FALSE; 23 24 else RETURN TRUE; 25

Remark 16 (Numerical Version of Algorithm 1). One could implement a version of Algorithm 1 using methods from numerical algebraic geometry [34] for point sampling and Segre class computations (via (15)). For both tasks, implementations such as the stand alone packages Bertini [5], PHCPack [36], or the NumericalAlgebraicGeometry Macaulay2 [16] package could be employed. The only step which could not be implemented, at least straightforwardly, numerically is Line 21. However, we do not focus on this case here. We instead focus on the case where points are represented symbolically since our primary aim is to give complexity bounds and these are well understood for the symbolic methods such as [30] which we employ (via the adaptations discussed in §4.2 below, see Remark 21 specifically). To furnish a reliable numerical implementation would require one to establish bounds on the precision needed in the points sampled in Lines 4 and 12 to ensure that the numeric versions of the degree computations in (15) would produce the correct results using the approximate points (since (15) uses the defining equations of the point, which would in the numeric case be approximate).

We now prove a worst case complexity bound for the operations preformed in Algorithm 1 above. As noted in the Introduction this bound depends on the geometry of the input scheme $Y \subset \mathbb{P}^n$.

Theorem 17. Let $I = \langle f_0, \ldots, f_r \rangle$ be a homogeneous ideal in $k[x_0, \ldots, x_n]$ with $\deg(f_i) \leq d$ defining a scheme Y in \mathbb{P}^n . If Y has reduced isolated primary components and has no embedded components outside of the singular locus of $Y_{\text{red}} = \mathbb{V}(\sqrt{I})$, then Algorithm 1 computes the radical via a single ideal saturation and its worst case complexity is doubly exponential in n. In all other cases, Algorithm 1 has worst case complexity bounded in $d^{\mathcal{O}(n^4)}$, that is singly exponential in n.

Proof. That the saturation is only computed if Y has only reduced isolated primary components and has no embedded components supported outside of the singular locus of $Y_{\text{red}} = \mathbb{V}(\sqrt{I})$ follows from the proof of Theorem 15. The complexity of the saturation in this case is that of a Gröbner basis in an elimination order, i.e., doubly exponential in n, e.g., [38]. In all other cases the complexity is that of the deg_{ν} function applied to an ideal generated by polynomials of degree no more than d^n , as this is the upper bound on the degree of the generators of the ideal J defined in line 11 of Algorithm 1. The conclusion then follows by Proposition 13.

Remark 18 (The minors of the Jacobian). We should note that Algorithm 1 also computes, Line 11, all the $c \times c$ minors of the Jacobian. Each minor requires a determinant computation; this costs $\mathcal{O}(c^{\omega})$, where ω is the exponent of the complexity of matrix multiplication [37]. There, are $\binom{r+1}{c} \cdot \binom{n+1}{c} \leq (16rn)^c$ minors; since $c \leq n$, this number is bounded by $(16rn)^n$. As r, the number of polynomials, is part of the input this bound is still singly exponential in n. Even more, it is reasonable to assume that for all practical cases $r = \mathcal{O}(d^n)$.

4.2 Using Points Represented in the Rational Univariate Representation

In the previous subsection we assumed that all points could be represented exactly as maximal ideals in our ambient coordinate ring. In practice on a computer we will in fact work in the ring $\mathbb{Q}[x_1, \ldots, x_m]$ and will need a way to represent each point which may appear in our algorithm in this ring. We will use the *rational univariate representation (RUR)*, see Theorem 5.

Consider zero dimensional ideals defining sets of points in \mathbb{C}^m ; usually for our purposes below m = n+2. We now consider the representation of points in RUR and how this interacts with multiplicity computation (14) and Sergre class computation (15). To setup the context and the notation for a 0-dimensional ideal $\mathcal{I} \subset \mathbb{Q}[x_1, \ldots, x_m]$ the RUR of $(\mathbb{V}(\mathcal{I}))_{\text{red}}$ is the ideal

$$J = \left\langle R(\theta), x_1 - \frac{A_1(\theta)}{R'(\theta)}, \dots, x_m - \frac{A_m(\theta)}{R'(\theta)} \right\rangle \subset \mathbb{Q}(\theta)[x_1, \dots, x_m],$$

where $R, A_1, \ldots, A_m \in \mathbb{Q}[\theta]$ are square-free polynomials. Equivalently, we will consider the RUR of $(\mathbb{V}(\mathcal{I}))_{\text{red}}$ as the ideal

$$\mathcal{J} = \left\langle R(\theta), 1 - T \cdot R'(\theta), x_1 R'(\theta) - A_1(\theta), \dots, x_m R'(\theta) - A_m(\theta) \right\rangle \subset \mathbb{Q}[x_1, \dots, x_m, \theta, T].$$

Further, there is a Q-algebra isomorphism between $\mathbb{Q}[x_1, \ldots, x_m]/\sqrt{\mathcal{I}}$ and $\mathbb{Q}(\theta)[x_1, \ldots, x_m]/J$, and between $\mathbb{Q}[x_1, \ldots, x_m]/\sqrt{\mathcal{I}}$ and $\mathbb{Q}[x_1, \ldots, x_m, \theta, T]/\mathcal{J}$, and the variety $(\mathbb{V}(\mathcal{I}))_{\text{red}}$ consists of deg $(R(\theta))$ reduced points [31].

The following result let us compute the zero dimensional part of the Segre class of a set of points inside a scheme when the set of points is represented using a RUR.

Theorem 19. Let $I = \langle f_1, \ldots, f_r \rangle$ be a homogeneous ideal in $\mathbb{Q}[x_0, \ldots, x_n]$ defining a scheme Y in \mathbb{P}^n and let X be a zero dimensional variety (i.e., a union of reduced points) contained in Y. Suppose $I_0 = \langle g_0, \ldots, g_\mu \rangle$ is the radical ideal defining X. Fix a (general) dehomogenization of I via the linear form $\ell_0(x)$ corresponding to a generic affine patch of \mathbb{P}^n ; this gives the scheme \hat{Y} in \mathbb{C}^{n+1} defined by the ideal $\hat{I} = \langle f_1, \ldots, f_r, \ell_0 - 1 \rangle$ and a set of points $\hat{X} \subset \mathbb{C}^{n+1}$ defined by the ideal $\hat{I}_0 = I_0 + \langle \ell_0 - 1 \rangle$. Let

$$\mathcal{J} = \left\langle R(\theta), 1 - T \cdot R'(\theta), x_1 R'(\theta) - A_1(\theta), \dots, x_n R'(\theta) - A_n(\theta) \right\rangle = \left\langle g_0, g_1, \dots, g_{n+1} \right\rangle$$

be a polynomial ideal in $\mathbb{Q}[x_0, \ldots, x_n, \theta, T]$ giving a RUR of \hat{X} . Finally consider the polynomial ring $\mathbb{Q}[x_0, \ldots, x_n, \theta, t, T]$ and define the ideal

$$\mathcal{I} = \tilde{I} + \langle P_1, \dots, P_{\dim(Y)}, 1 - tP_0 \rangle \subset \mathbb{Q}[x_0, \dots, x_n, \theta, t, T],$$

where $P_j = \sum_{i=0}^{\mu} \lambda_i^{(j)} g_i$ for general $\lambda_i^{(j)} \in \mathbb{Q}$. Then

 $\{s(X,Y)\}_0 = \deg(Y) \cdot \deg(R(\theta))^{\dim(Y)} - \dim_{\mathbb{Q}} \left(\mathbb{Q}[x_0,\ldots,x_n,\theta,t,T]/\mathcal{I}\right).$

Proof. Pick a set of homogeneous generators for $I_0 = \langle w_1, \dots, w_s \rangle$ with $\deg(w_i) = \deg(R(\theta))$ and set

$$\mathfrak{V} = \hat{I} + \left\langle \sum_{i=1}^{s} \rho_i^{(1)} w_i, \dots, \sum_{i=1}^{s} \rho_i^{(\dim(Y))} w_i, 1 - t \left(\sum_{i=1}^{s} \rho_i^{(0)} w_i \right) \right\rangle \subset \mathbb{Q}[x_0, \dots, x_n, t],$$

for general $\rho_i^{(j)} \in \mathbb{Q}$. Since $\deg(w_i) = \deg(R(\theta))$ then by (6), (7), and (9) we have that that

$$\{s(X,Y)\}_0 = \deg(Y) \cdot \deg(R(\theta))^{\dim(Y)} - \dim_{\mathbb{Q}} (\mathbb{Q}[x_0,\ldots,x_n,t]/\mathfrak{V}).$$

Hence to prove the result it is sufficient to establish an equality between $\dim_{\mathbb{Q}}(\mathbb{Q}[x_0,\ldots,x_n,t]/\mathfrak{V})$ and $\dim_{\mathbb{Q}}(\mathbb{Q}[x_0,\ldots,x_n,\theta,t,T]/\mathcal{I}).$

By the definition of the RUR we have a Q-algebra isomorphism, which we will denote Φ , between $\mathbb{Q}[x_0, \ldots, x_n, t]/\langle w_1, \ldots, w_s \rangle$ and $\mathbb{Q}[x_0, \ldots, x_n, t, \theta, T]/\mathcal{J}$, c.f. [31, 1]. This isomorphism induces a map, which we also call Φ , between the quotient rings $\mathbb{Q}[x_0, \ldots, x_n, t]/\hat{I}$ and $\mathbb{Q}[x_0, \ldots, x_n, t, \theta, T]/\hat{I}$. Necessarily this induced map

$$\Phi: \mathbb{Q}[x_0, \dots, x_n, t]/\hat{I} \to \mathbb{Q}[x_0, \dots, x_n, t, \theta, T]/\hat{I}$$

must map a general polynomial of a fixed degree in $\langle w_1, \ldots, w_s \rangle$ to a general polynomial of the same degree in \mathcal{J} ; in particular we have that

$$\Phi\left(\sum_{i=1}^{s} \rho_i w_i\right) = \sum_{i=0}^{n+1} \lambda_i g_i,$$

for some general constants λ_i which will be determined by the general ρ_i . Since dimensions of the Q-vector spaces being considered do not depend on the choice of constants, provided they are general, it follows that

$$\dim_{\mathbb{Q}} \left(\mathbb{Q}[x_0, \dots, x_n, t] / \mathfrak{V} \right) = \dim_{\mathbb{Q}} \left(\Phi(\mathbb{Q}[x_0, \dots, x_n, t] / \mathfrak{V}) \right) = \dim_{\mathbb{Q}} \left(\mathbb{Q}[x_0, \dots, x_n, \theta, t, T] / \mathcal{I} \right).$$

In other words, the image of Φ applied to the quotient ring $\mathbb{Q}[x_0, \ldots, x_n, t]/\mathfrak{V}$ may differ from the quotient ring $\mathbb{Q}[x_0, \ldots, x_n, \theta, t, T]/\mathcal{I}$, but only up to a choice of general constants, which does not effect the resulting vector space dimension.

This result in conjunction with Proposition 20 below, which can be used used in Lines 8 and 16 of Algorithm 1 via the RUR of a zero dimensional variety \hat{X} . As noted in the proof this criterion is in effect testing all the multiplicities of dimension zero components at once.

Proposition 20. Let Y be a scheme in \mathbb{P}^n , let W be the union of all ν -dimensional primary components of Y and let X a dimension zero variety fully contained in W and such that no point in X is in the singular locus of W_{red} . Then $\{s(X,W)\}_0 > \deg(X)$ if and only if W is not reduced.

Proof. By [13, Example 4.3.4] we know that $\{s(X, W)\}_0$, the coefficient of the dimension 0 part of the Segre class, is the sum of the multiplicities of W along each point in X. For W to be reduced we require that it is reduced at a generic point (i.e., one outside the singular locus of W_{red}), in light of Proposition 4 we see that in particular we need $e_pW = 1$ for each $p \in X$ (since dim(X) = 0), hence taken together, for W to be reduced and all points in X are smooth in W_{red} (again via Proposition 4) $e_pW = 1$, for all $p \in X$; it follows that in this case we necessarily have $\{s(X, W)\}_0 = \deg(X)$.

Remark 21 (Using the RUR in Algorithm 1). *To use the RUR of sampled points in Algorithm 1 we make the following alterations:*

- *p* in Line 6 and *q* in Line 13 are now both zero-dimensional varieties equal to the union of some set of reduced points represented as a RUR;
- *in Line 7 we use Theorem 19 in conjunction with* (15) *to obtain the Segre class and the criterion in Line 8 becomes* $\{s(p, W(p))\}_0 > \deg(p)$ by Proposition 20;
- in Line 15 we use Theorem 19 in conjunction with (15) to obtain the Segre class and the criterion in Line 16 becomes $\{s(q, W_{\nu})\}_0 = \deg(q)$ by Proposition 20.

The rest of the algorithm remains as presented in Algorithm 1.

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