**Assignment of attribute weights with belief distributions for MADM under uncertainties**

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**Abstract: Multiple attribute decision making (MADM) problems often consist of various types of quantitative and qualitative attributes. Quantitative attributes can be assessed by accurate numerical values, interval values or fuzzy numbers, while qualitative attributes can be evaluated by belief distributions, linguistic variables or intuitionistic fuzzy sets. However, the determination of attribute weights is still an open issue in MADM problems until now. In the traditional objective weight assignment method, attributes are usually assessed by accurate values. In this paper, an entropy weight assignment method is proposed to dealing with the situation where the assessment of attributes can contain uncertainties, e.g., interval values, or contain both uncertainties and incompleteness, e.g., belief distributions. The advantage of the proposed method lies in that uncertainties and incompleteness contained in the interval numerical values or belief distributions can be preserved in the generated weights. Specifically, several pairs of programming models to generate the weights of attributes are constructed in three different circumstances: (1) quantitative attribute expressed by interval values; (2) incomplete belief distribution with accurate belief degrees; and (3) belief distribution constituted by interval belief degrees. The evidential reasoning approach is then utilized to aggregate the distributions of attributes based on the generated attribute weights. The normalized interval weight vector is defined, and the characteristics of the weight assignment method are discussed. The proposed method has been experimented with real data to illustrate its advantages and the potential in supporting MADM with uncertain and incomplete information.**

**Keywords: evidential reasoning; belief distribution; entropy weight assignment method; interval value; interval belief degree; incompleteness**

**1. Introduction**

Multiple attribute decision making (MADM) with various types of attributes is common in practice[1]. Quantitative and qualitative attributes which can be assessed by numerical values and subjective judgments respectively are always included in aMADM problem. The purpose of aMADM problem is to make a ranking order of several selected alternatives, or to choose the best one of them. When qualitative attributes are expressed by belief distributions (BDs)[2-6,82] or probabilistic linguistic term set (PLTS)[85], the evidential reasoning (ER) approach[1,7-13] can provide a probabilistic aggregation process where uncertainty, ignorance and ambiguity are well coped with. In recent years, the ER approach has been applied in many domains such as fault diagnosis[4,14-16], life cycle assessment[17,18], belief rule based inference[19-23], urban bus transit network assessment[24], data classification[25], medical quality assessment[26], optimal power system dispatch[27], consumer preference prediction[28], sensor data fusion[29,30] and so on. But how to create a set of relatively rational and appropriate weights of attributes is still an open issue.

As we all know, there are three categories of weight assignment methods (WAMs) considering the information used: subjective, objective and hybrid[31-33]. Subjective WAMs such as direct rating method[34-36], weighted least square method[37], AHP[38,39] and Delphi method[40,41] are based on the preferences provided by decision makers (DMs) through questionnaire survey, discussion or brainstorming. When there is no sufficient information available or the DM is lack of knowledge and expertise, subjective methods may not be applied effectively. Moreover, in group decision making (GDM)[42-47,83-85], DMs with diverse background and preference may provide inconsistent attribute weights which need to be aggregated or reach a consensus. Objective WAMs generate attribute weights from the intrinsic information contained in the assessment values. Traditional representative methods include entropy method[12,48-53,81,82], standard deviation (SD)[8,54,55], criteria importance through intercriteria correlation (CRITIC)[54-56] and maximizing deviation method[57,58]. In recent years, some new methods such as correlation coefficient and standard deviation integrated (CCSD) method[31], deviation and decision incompatibility based method[8] and combined discriminating power method[2] were proposed to coping with more complex situations. They are all based on one or both of the following two dimensions of information included in the decision matrix. One is the discrepancies of the values that different alternatives be assessed on a specific attribute, typical terms such as contrast intensity[54], incompatibility[8], discriminating power[2] or discrepancy[12] are all characterizations of this type of information. The other one reflects the conflict among the value vectors of different attributes, or in an opposite concept, denotes the correlation or interdependency between each pair of attributes. Hybrid WAMs[33,49,59] utilize both the preferences derived from DMs and the acquired assessment values when these two types of information are entirely or partially available.

Since different sets of attribute weights may generate different solutions to a MADM problem[2], how to calculate weight effectively is particularly important. In the life cycle assessment (LCA) of a complex product, the quantitative data obtained in the environmental and economic dimensions may contain uncertainties and ignorance due to various reasons; while the qualitative judgments acquired from investigation to assessing the social dimension also include ambiguities and incompleteness because of the subjectivity of DMs. Traditional objective or hybrid methods assume that the values been assessed to attributes are in the form of numerical values, e.g. accurate value[31,48,49,54,55,57] and fuzzy number[60-63]. But when the numerical values contain both uncertainties and incompleteness, how to elicit objective attribute weights is still an unanswered question. For example, incomplete interval value proposed in [24] allows ignorance and unreliability be contained in interval value assessments. The method to measure the discrepancy among interval values assessed to all alternatives respect to a specific attribute or the interdependency between each pair of attributes is pivotal for obtaining the weights. Meanwhile, the incompleteness should be reflected in the generated weights, e.g. a final reliability of attribute weights can be computed derived from the incompleteness. Two ways can be adopted to generate attribute weights in this situation. One way is to directly use one or both of the above mentioned discrepancy and interdependency measure for interval-valued assessments to calculate weights objectively. The other way is to employ optimization models to generate the uncertain range of attribute weights. In this paper, the second way is adopted for the purpose that the uncertainties and incompleteness contained in interval-valued assessments can be preserved in the generated attribute weights.

In the development of the ER approach, researches mainly focus on the issue that how to construct a reasonable aggregation rule compared with other evidence combination rules (ECRs) such as a set of D-S rules[64-67]. Although the attribute weights have been assumed to be various types, e.g., accurate value[7,13,17,24,68-70], interval value[5,10,71] and triangular fuzzy number[1,11], the method of generating attribute weights either from subjective or objective method was not fully discussed in the early two decades since ER has been proposed. In recent years, some literatures[2,8,12] have paid attention on the WAMs for the ER based MADM problem when the information obtained is in the form of BDs. But some issues still need to be further discussed. Firstly, the ignorance/incompleteness contained in the BDs should be reflected from the generated weights, which will directly determine the rationality of the final aggregation results. Secondly, interval belief distribution (IBD)[9,71-74] may be a feasible representation in GDM problems because the judgments by DMs may be inconsistent due to their different backgrounds, expertise and preferences. Moreover, the belief degrees given by individual DM himself/herself may also be imprecise on account of some uncertainties, such as the lack of knowledge. In this paper, for the generation of objective weights, optimization models are constructed based on Shannon’s information entropy[75] to tackling the situations that assessments are given in the form of accurate or interval belief degrees. It is a generalization of the entropy method for the assignment of attribute weights in that the above two situations have not been fully discussed in the previous studies.

The main contributions of the paper can be summarized as follows:

(1) The entropy weight assignment method (EWAM) is extended to coping with the situation where assessment values are presented in the form of interval values. The definition of normalized interval weight vector is then given.

(2) The entropy weight assignment method with belief distributions (EWAM with BDs) is studied. Optimization models are constructed to generate the intervals of attribute weights which are caused by the uncertainties and incompleteness contained in BDs.

(3) For the case where the decision matrix is represented by IBDs, EWAM based optimizationmodels are constructed to compute the uncertain ranges of attribute weights. The concept of complete IBD matrix is also provided.

The remainder of this paper is organized as follows. Section 2 is a brief introduction about the ER based MADM framework. In Section 3, EWAMwith uncertain numerical values is studied, the concept of normalized interval weight vector is then defined. In Section 4, EWAM with incomplete BDs and IBDs are proposed respectively. Some properties are shown and proved. Section 5 presents a case study to illustrate the methods proposed in Sections 3 and 4, and the comparisons are conducted with some existing methods. This paper is concluded in Section 6.

**2. Preliminaries**

The ER approach which was developed from D-S evidence theory[64,65,67] is one of MADM methods that can be applied in situations where uncertainties, ambiguities and incompleteness are included. The unique characteristic of the ER approach lies in that incompleteness or ignorance involved in the subjective judgments can be dealt with in a systematic and consistent way. Suppose the frame of discernment which contains linguistic evaluation grades is given as follows:

(1)

In general, is supposed to be preferred to represented by . The utility of is denoted by such that . Let and be the set of basic attributes and selected alternatives respectively. The relative weight of is denoted by such that and . The state of for alternative to grade can be described as the following expectation:

(2)

Eq.(2) is called a basic BD, where signifies the intensity to which the state of the *i*th attribute at is confirmed to , or is termed belief degree. denotes the degree of uncertainty that is assessed on , also called the degree of global ignorance/incompleteness. BD is assumed to satisfy the rationality assumption[7] in which and are commonly satisfied, and . The ER approach uses orthogonal sum operation either recursively[13] or analytically[69] to aggregate the distributions on a specific alternative and generate global BD on the frame of discernment as follows:

(3)

In Eq.(3), denotes the global belief degree that be assessed on , and represents the degree of belief unassigned to any individual evaluation grade after all the attributes in have been considered. Different from the D-S rules where the weight of each evidence is assumed to be identical such that , the ECR in the ER approach considers the difference of evidence weight which is necessary in MADM problems. Apart from the global BD to give a panoramic view about the assessment on , the ER approach also employs utility function decision making (UFDM) approach to rank alternatives straightforwardly. The maximum and minimum utility of can be generated as follows when is allocated to and ) respectively.

(4)

) (5)

Then can be judged by the interval where and . Recently, the ER approach has been developed from many aspects, e.g. ER rule with weight and reliability[12,17,76,77], ER with discrete BD[6], ER with contrary support[78] and so on.

**3. EWAM with uncertain numerical values**

Suppose *S* alternatives and *L* attributes are included in a MADM problem which can be represented by a decision matrix shown in Eq.(6). Each column signifies the assessment vector of a specific attribute respect to all the *S* alternatives, represented by ; While each row indicates the performance vector that an alternative be assessed to all *L* attributes, represented by . By aggregating the values in each performance vector , the final performance of alternative represented by can be generated, and then the comparison of the alternatives can be conducted. If is a numerical vector, simple additive weighting (SAW) method[48] can be used to generate which is also a numerical value.

(6)

**3.1 EWAM with crisp values**

Let be the value of provided that is a quantitative attribute. The EWAM with crisp values[48,50,81] includes the following three steps:

(1) Standardization of original numerical values of performance:

Linear Proportional Transformation or Standard 0-1 Transformation can be applied in the standardization. For a benefit attribute, we have

(7)

or

(8)

While for a cost attribute, we have

(9)

or

(10)

(2) Normalization of the standardized value from :

(11)

where .

(3) Generating the weights of attributes

The entropy of derived from is calculated by

(12)

where the denominator is to limit within . As is known to all, the higher entropy be assigned to an attribute, the more difficult to discriminate different alternatives on the attribute, so the weight of is calculated by

(13)

where .

The original EWAM assumes that the assessments to all attributes are given by precise numerical values. When the situation is uncertain and some quantitative attributes are provided in the form of interval values, or the assessments to qualitative attributes are given by BDs denoted by Eq.(2), how could the entropy method be employed to generate reasonable weights is to be analyzed.

**3.2 EWAM with interval numerical assessment**

Interval theory is one of the most common forms of uncertainty modeling[71]. A lot of studies have been done on MADM based on interval theory, which can be roughly split into three aspects, interval numerical assessment[24,79,80], IBD[9,71-74] and interval weight or reliability[5,10,17,71]. Two processes can be applied to define the similarity measure such as the information entropy when quantitative attributes are represented by interval numerical assessments. The first one is to directly define the entropy measure of interval values assessed to different alternatives on each attribute, followed by the generation of attribute weights. The second one is to make interval values as constraints to conduct optimization models for the generation of entropy measure and attribute weights. Suppose is given by interval value as follows:

(14)

The minimum and maximum possible value assessed to respect to all alternatives is denoted by and respectively. The entropy of the interval assessment vector on derived from can be calculated by the following pair of optimization models:

s.t. Eqs.(7) or (8), (9) or (10), (11)

where and in Eqs.(7)-(10) should be replaced by and respectively. *S* variables are included in <Model 1> for generating the range of . Let and be the optimal values of <Model 1> respectively. Then, we have .

**Property 1** The uncertainty of the generated interval entropy in <Model 1> changes continuously with the change of .

To generate the range of weights from decision matrix in Eq.(6) which is comprised of interval numerical values, the following pair of optimization models is designed.

s.t. Eqs.(7) or (8), (9) or (10), (11), (12)

Let be the weight vector of attributes, if where for and , is said to be an interval weight vector. From <Model 2>, the minimum and maximum weight of under interval assessment values can be generated as and respectively. In Eq.(14), if and , , Eq.(6) becomes a matrix only composed of precise numerical values. Then the weights generated by <Model 2> will satisfy .

**Remark 1** If and , then the interval weights are said to be valid. Invalid interval weights cannot be applied in a MADM problem.

**Definition 1** Suppose is the weight vector of attributes with for . is said to be a normalized interval weight vector if the following equations are satisfied:

(15)

(16)

Eqs.(15) and (16) can also be represented by Eqs.(17) and (18) as follows:

(17)

(18)

**Remark 2** If the uncertainty of in Eq.(14) measured by for some in the decision matrix of Eq.(6) are large to some extent, the interval weights generated by <Model 2> may not be normalized although they are valid. Thus the reliability and effectiveness of the final assessment result will be influenced.

Nevertheless, if for some attributes are large enough although the generated weight vector is normalized, the interval weights are still not appropriate to be used for the final decision-making.

**4. EWAM with uncertain or interval BDs of assessment**

The main difference between BD and probability assignment lies in that the probability can only be employed by individual elements in the frame of discernment, while BD is defined on the power set of the frame of discernment which is all the subsets of in Eq.(1). There is a special case that the belief degree be given to the set of all grades in Eq.(1), which refers to the degree of complete/global ignorance. In the ER approach, the BD is defined on all the individual elements and the set of grades. Just as Yang proved in [13], the overall BD by the ER approach is incomplete provided that the BD of at least one attribute contains ignorance no matter how the attribute weights are assigned. The attribute weights derived from BDs should also reflect the feature of ignorance contained in the BDs on basic attributes. Moreover, when qualitative attributes are given by IBDs, optimization models should be designed to generate the weights for the purpose of preserving the features of uncertainties included in IBDs.

**4.1 The 0-1 Ave-Entropy method with BDs**

For a qualitative attribute , can be given in the form of BD, fuzzy number or other types of representation. When is composed of BDs, the ER algorithm which is a nonlinear aggregation method can be applied. The weight of attributes is an important factor in MADM because different weights may produce different evaluation results. The Ave-Entropy method[12] is applicable when in Eq.(6) are all represented by BDs or have been transformed to BDs from numerical values. Three steps are included in the EWAM with BDs as follows:

(1) Transformation of to utility as follows:

(19)

) (20)

(21)

(2) Normalizing by such that . The Standard 0-1 Transformation can be used before the normalization to narrow the discrepancy among the performances of alternatives such that , . Then

(22)

(3) The entropy of is computed by , which is then used to generate the weight of by Eq.(13).

If the Standard 0-1 Transformation is used before the normalization process, we call it 0-1 Ave-Entropy method; otherwise, it is called Ave-Entropy method.

**Property 2** If Linear Proportional Transformation is used to replace the Standard 0-1 Transformation in the above EWAM with BDs, the generated weights are identical with Ave-Entropy method.

The proof of Property 2 is shown in the Appendix. When entropy[12,48-53], CRITIC method[54-56] or CCSD[2,31] is used to determine attribute weights, more than two alternatives are required if the Standard 0-1 Transformation is applied for the normalization of quantitative attributes. If only two alternatives are involved in the assessment, the smaller and bigger assessment value of the two alternatives on each attribute will be 0 and 1 after the Standard 0-1 Transformation. When an assessment is only comprised of qualitative attributes, these three methods can be used to generate weights from average utility if the Standard 0-1 Transformation is not applied even though there are only two alternatives involved. However, when the Standard 0-1 Transformation is not applied, some unreasonable results may be generated in some circumstances.

**Example 1.** Given a frame of discernment . Table 1 shows the BDs of two attributes and on two alternatives and . It is clear that all the four distributions are absolutely certain and complete assessments.

**Table 1** BDs of attributes on alternatives

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *e*1 | *H*1 | *H*2 | *H*3 | *H*4 | *H*5 |
| *a*1 | 1 | 0 | 0 | 0 | 0 |
| *a*2 | 0 | 1 | 0 | 0 | 0 |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *e*2 | *H*1 | *H*2 | *H*3 | *H*4 | *H*5 |
| *a*1 | 1 | 0 | 0 | 0 | 0 |
| *a*2 | 0 | 0 | 0 | 0 | 1 |

When the objective method is used, should probably be assigned with a higher weight than since the dissimilarity between and on is larger than . Suppose that the utilities of the five evaluation grades are set to be risk averse such that , , , , , the weights of the two attributes generated by the Ave-Entropy method are , respectively even though the Standard 0-1 Transformation is not applied. When the utilities of the five evaluation grades are set to be risk taking, the weights of and are also 0.5 and 0.5.

**Example 2.** Similar withExample 1, two attributes are assessed on two alternatives in the form of BDs which are shown in Table 2. Different from Table 1, the distributions of on and here are incomplete.

**Table 2** BDs of attributes on alternatives

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *e*1 | *H*1 | *H*2 | *H*3 | *H*4 | *H*5 |
| *a*1 | 1 | 0 | 0 | 0 | 0 |
| *a*2 | 0 | 1 | 0 | 0 | 0 |

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| *e*2 | *H*1 | *H*2 | *H*3 | *H*4 | *H*5 | H |
| *a*1 | 0.8 | 0 | 0 | 0 | 0 | 0.2 |
| *a*2 | 0 | 0 | 0 | 0 | 0.8 | 0.2 |

When the Ave-Entropy method is used, the weights of the two attributes are , respectively although the dissimilarity between and on is larger than .

**Property 3** The weights generated by the 0-1 Ave-Entropy method are continuous respect to the belief degrees of assessments.

The proof of Property 3 is shown in the Appendix. It indicates that the weights generated by the 0-1 Ave-Entropy method will not change much provided that the belief degrees vary slightly.

**4.2 The 0-1 Ave-Entropy based optimization model for assigning attribute weights**

For the purpose of considering the ignorance contained in the BDs, the utility of on represented by is assumed to be in the range of . So the following pair of optimizationmodels is constructed to generate the minimum and maximum value of .

Eqs.(19), (20)

variables are contained in <Model 3> which can be solved by Matlab or Excel, and times of calculation need to be conducted to generate the minimum and maximum values of the weights on all attributes. <Model 3> is based on the 0-1 Ave-Entropy method. If Ave-Entropy method is used in <Model 3>, is to be replaced by in the optimization model. A special case is that for all the BDs which will lead to the weights generated by the model be crisp values. This occurs when all the subjective judgments provided by DM are complete although uncertainties are contained in the BDs. Next, we take a numerical example in [12] to illustrate the above proposed optimizationmodel.

**Example 3.** Given a frame of discernment . The BDs that 5 alternatives been assessed on 6 attributes are shown in Table 3. From Table 3, we can see that for a specific attribute , and , and the ignorance contained in the assessment increases from to .

**Table 3** Belief degrees that 5 alternatives be assessed on 6 attributes

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |
|  | {(*H*1, 1)} | {(*H*2, 1)} | {(*H*3, 1)} | {(*H*4, 1)} | {(*H*5, 1)} |
|  | {(*H*1, 0.8),(*H*, 0.2)} | {(*H*2, 0.8),(*H*, 0.2)} | {(*H*3, 0.8),(*H*, 0.2)} | {(*H*4, 0.8),(*H*, 0.2)} | {(*H*5, 0.8),(*H*, 0.2)} |
|  | {(*H*1, 0.6),(*H*, 0.4)} | {(*H*2, 0.6),(*H*, 0.4)} | {(*H*3, 0.6),(*H*, 0.4)} | {(*H*4, 0.6),(*H*, 0.4)} | {(*H*5, 0.6),(*H*, 0.4)} |
|  | {(*H*1, 0.4),(*H*, 0.6)} | {(*H*2, 0.4),(*H*, 0.6)} | {(*H*3, 0.4),(*H*, 0.6)} | {(*H*4, 0.4),(*H*, 0.6)} | {(*H*5, 0.4),(*H*, 0.6)} |
|  | {(*H*1, 0.2),(*H*, 0.8)} | {(*H*2, 0.2),(*H*, 0.8)} | {(*H*3, 0.2),(*H*, 0.8)} | {(*H*4, 0.2),(*H*, 0.8)} | {(*H*5, 0.2),(*H*, 0.8)} |
|  | {(*H*1, 0.1),(*H*, 0.9)} | {(*H*2, 0.1),(*H*, 0.9)} | {(*H*3, 0.1),(*H*, 0.9)} | {(*H*4, 0.1),(*H*, 0.9)} | {(*H*5, 0.1),(*H*, 0.9)} |

According to Eqs.(19)-(21), the average utilities of the 5 alternatives on 6 attributes can be calculated. Fig.1 shows the generated weights by 0-1 Ave-Entropy, Ave-Entropy, SD method and Gini’s mean difference (GMD) method[2,41]. The horizontal and vertical axes represent the serial number and weight of attribute respectively. Taking and for example, the average utility vectors of these two attributes on the 5 alternatives are and respectively provided that the utilities of the five evaluation grades are set to be equal with example 1. So the Spearman correlation coefficient between the two vectors is 1. The correlation coefficient between any other two average utility vectors is also 1. Thus the CRITIC method cannot be used in this situation. CCSD method is also not applicable since the correlation coefficient between the utility vector on and the overall utility vector without the consideration of is 1 for . The weights generated by 0-1 Ave-Entropy are equal for all the six attributes that we call it the best well-distributed weights. The results need to be discussed because the dissimilarities of BDs on the five alternatives for each of the six attributes are really different. It is equal to the result that the ignorance in each BD is added to the nonzero belief degree. In this situation, all the 6 attributes on each alternative are given the same BD. Comparatively, the weights generated by Ave-Entropy and SD method are easy for us to differentiate the set of attributes which have a dominating role. It is an interesting thing that the weights generated by SD and GMD are the same in this example. Nevertheless, the weights generated by the four methods are crisp values that the incompleteness contained in the BDs cannot be reflected. Taking for example, the ignorance contained in each of the five alternatives is 0.9 which is the largest incompleteness compared with to . So it is more rational that the generated weight of is formulated as an uncertain value because ignorance are contained in the BDs respect to these five attributes.

**Fig. 1.** Attribute weights generated by 0-1 Ave-Entropy, Ave-Entropy, SD and GMD method

To reflect the ignorance contained in the BDs from the generated weights, <Model 3> is applied. The utilities of the five evaluation grades are set to be risk averse such that ,, , and . The weights generated by <Model 3> based on 0-1 Ave-Entropy are shown in Fig.2. They are limited to the interval of and satisfies . It can be seen that with the ignorance contained in the BDs on alternatives increased from to , the difference between and becomes larger. If the ignorance contained in one or some of the BDs are too large to some extent, the weights generated by <Model 3> may be non-normalized. Thus it is necessary to figure out the reason of large ignorance contained in the BD, and decrease the ignorance if possible for a more reliable aggregation result.

**Fig. 2.** Interval weights generated by <Model 3> subject to the 0-1 Ave-Entropy

We also generate the interval weights by <Model 3> based on Ave-Entropy method that are shown in Fig.3. Although the weights in Fig 3 are normalized, the difference between the minimum and maximum weight on any one of the six attributes is too large that would make the DM too confusing. From this point of view, the Standard 0-1 Transformation narrows the differences of weights among all the attributes. Meanwhile, it also reduces the uncertainty of interval weights provided that incompleteness are contained in BDs.

**Fig. 3.** Interval weights generated by <Model 3> subject to the Ave-Entropy

**4.3 Consideration of interval BDs**

<Model 3> proposed in Section 4.2 assumes that the BDs provided by the DM are comprised of accurate belief degrees. Just as Wang discussed in [9], acquiring precise belief degrees is not easy in some circumstances where IBD is more appropriate to represent DM’s subjective judgments. Here, the EWAM with IBDs is proposed.

Suppose the belief degree that be assessed to on evaluation grade is included in an interval such that , and [9]. Then Eq.(2) becomes an IBD, which leads Eq.(6) to be an IBD matrix. If , , the IBD becomes an accurate belief distribution.

**Remark 3**[5] Suppose an IBD is denoted as follows:

(23)

where , . is said to be logical if . Otherwise, it is illogical.

Illogical IBD cannot be used in the ER algorithm or to generate attribute weights by objective methods. In this case, the subjective judgment from DM should be corrected.

**Remark 4**[9]Suppose an IBD denoted by Eq.(23) is logical. If we have where , then the IBD is said to be complete. In this situation, . Otherwise, it is said to be an incomplete IBD or contain ignorance such that and .

There may be three situations considering the incompleteness of IBD. a) If , will always be satisfied, so we have , then is said to be absolutely incomplete. b) When and , we have and , then the IBD is either complete or incomplete. c) When and , we have and , then it is an absolutely complete IBD. In essence, situation c is corresponding to an accurate BD because . From situation a) to c), the incompleteness of IBD decreases. It should be mentioned that the situation that both and be satisfied will not happen because it is illogical.

**Definition 2** Given an IBD matrix denoted by Eqs.(6) and (23), if for alternative , , where , then the assessment on represented by is said to be an incomplete IBD vector; Otherwise, the IBD vector on is said to be complete such that .

**Definition 3** If and , in Eq.(6) is complete no matter is a quantitative or qualitative attribute, then Eq.(6) is said to be a complete decision matrix; Otherwise, it is an incomplete decision matrix. A special case is that all the attributes are qualitative attributes represented by IBDs, then Eq.(6) is said to be a complete IBD matrix if all the IBDs are complete.

The definition of incomplete numerical assessment either in the form of accurate or interval value can be referred to [24]. When all the IBD vectors () are complete, the IBD matrix is complete. According to [13], if the BD of an attribute on is incomplete, the aggregated BD on from all attributes will also be incomplete. It is also applicable to the IBDs. When the subjective judgments are presented in the form of IBDs, the entropy of can be computed by <Model 4> as follows:

Eqs.(19)-(21)

Here, another example is presented to illustrate the entropy measure of IBDs by <Model 4>.

**Example 4.** Given a frame of discernment , and the utilities of the three evaluation grades are set to be ,, respectively. The IBDs of on alternative are shown in Table 4.

**Table 4** IBDs assessed on two alternatives

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | | *H*1 | *H*2 | *H*3 |
|  |  | 0.9 | 0 | 0 |
|  | 1 | 0.1 | 0 |
|  |  | 0 | 0 | 0.9 |
|  | 0 | 0.1 | 1 |

By applying <Model 4>, the entropy measure of the IBDs on respect to is between 0.4690 to 0.6061. reaches the minimum when and which corresponding to the situation that the assessments on the two alternatives are highly inconsistent. When and , reaches the maximum. Besides, the utilities of evaluation grades also influence the entropy of to a certain degree.

The following pair of optimization models can then be constructed to generate the interval weight of .

Eqs.(19)-(21)

In <Model 5>, variables are included, and there are times of calculation needed to generate the minimum and maximum values of all attribute weights.

Additionally, in order to capture the subjective judgments of DMs, preference relations on attribute weights can be incorporated into models 2, 3 and 5. For instance, , , are representative forms of subjective preference on attribute weights. If preference relations are not included in the optimization models, and the generated weights are inconsistent with DM's subjective preferences, the weights should be adjusted.

**4.4 Sensitivity analysis**

In order to illustrate property 3, sensitivity analysis is conducted on Example 3 to measure the impact of the change of belief degrees on weights. Here, is set to be changed from 1 to 0.95, while other BDs are fixed. Fig.4 shows the result when the 0-1 Ave-Entropy method is applied. It is clear that the weight of each attribute changes very little when the step of variation for is set to be 0.01.

**Fig. 4.** Changes of attribute weights generated by 0-1 Ave-Entropy with respect to

Sensitivity analysis on Example 4 is also conducted to measure the changes of IBDs on the generated interval entropy by <Model 4>. The IBD of on is set to be where , while the IBD of on is fixed. Fig.5 shows the values of interval entropy generated by <Model 4> when increases from 0 to 0.9. It can be seen that the assessments to between and become more consistent when increases, which leads to the rise of and simultaneously.

**Fig. 5.** Changes of interval entropy generated by <Model 4> with respect to

**5. Case study**

In this section, a case study is conducted to illustrate the validity and rationality of the proposed approach mentioned in the above section. Several different methods are compared with the proposed method to give a panoramic view of the features of the given models.

**5.1 Generating weights from the proposed models**

A car selection problem adapted from [7] and [8] is chosen for the illustration. Four qualitative attributes and three quantitative attributes are included in the assessment where six cars are to be compared. The weights in [7] are supposed to be equal such that . The frame of discernment in this case consists of six evaluation grades such as Worst(W), Poor(P), Average(A), Good(G), Excellent(E) and Top(T). The original numerical values assessed to quantitative attributes and grades on qualitative attributes have all been transformed to BDs which are shown in Table 5. For the generation of the overall performance on each car, the ER algorithm is to be used to combine the BDs in Table 5.

From Table 5, we can see that some of the BDs are incomplete. For example, the assessment of Car 5 on is represented by . It means that Car 5 on ‘Acceleration’ is assessed to be good and excellent with a belief degree of 0.4, and the degree of ignorance contained in the assessment is 0.2. It will lead to the weights generated by <Model 3> be uncertain, and the overall performance on Car 5 will also be incomplete. Intuitively, the uncertainties on the weight of will be the largest when <Model 3> is applied because both Car 3 and Car 6 are assessed to be completely ignorant on .

**Table 5** BDs that 6 cars be assessed on 7 attributes

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Performance | Car 1 | Car 2 | Car 3 | Car 4 | Car 5 | Car 6 |
|  | Acceleration | {(P,0.2), (A,0.8)} | {(G,0.5), (E,0.5)} | {(E,0.75), (T,0.25)} | {(A,0.4), (G,0.6)} | {(G,0.4), (E,0.4), (Ω,0.2)} | {(G,0.25), (E,0.75)} |
|  | Braking | {(G,1.0)} | {(E,0.333), (T,0.667)} | {(G,0.5), (E,0.5)} | {(P,0.75), (A,0.25)} | {(P,1.0)} | {(E,1.0)} |
|  | Handling | {(A,0.4), (G,0.6)} | {(E,0.6), (T,0.4)} | {(A,0.4), (G,0.4), (Ω,0.2)} | {(A,1.0)} | {(G,1.0)} | {(E,0.5), (T,0.4), (Ω,0.1)} |
|  | Horsepower | {(E,0.333), (T,0.667)} | {(P,0.533), (A,0.467)} | {(G,0.462), (E,0.538)} | {(G,0.385), (E,0.615)} | {(W,0.467), (P,0.533)} | {(A,0.267), (G,0.733)} |
|  | Ride quality | {(G,0.6), (E,0.4)} | {(A,1.0)} | {(Ω,1.0)} | {(G,1.0)} | {(G,1.0)} | {(Ω,1.0)} |
|  | Powertrain | {(A,0.4), (G,0.6)} | {(G,1.0)} | {(E,0.5), (T,0.4), (Ω,0.1)} | {(A,0.4), (G,0.6)} | {(G,0.6), (E,0.4)} | {(E,0.5), (T,0.3), (Ω,0.2)} |
|  | Fuel economy | {(G,1.0)} | {(G,1.0)} | {(E,1.0)} | {(G,1.0)} | {(A,1.0)} | {(G,1.0)} |

The utilities of the six evaluation grades are set to be risk neutral such that , , , , , . It should be mentioned that in a multi-attribute group decision making (MAGDM) problem, different risk preferences of DMs may lead to the discrepancy of utility estimations on evaluation grades due to their diverse background and expertise. Moreover, the risk preference of a DM may change at different decision points due to the changes of external environment[1]. From this point of view, different utility estimation may influence the generated weights and aggregated BD on each alternative. But this is not the focus of our discussion in this paper, and it can be related to [1,11,12]. Fig.6 shows the weights generated by different methods, i.e. 0-1 Ave-Entropy, Ave-Entropy, SD, CRITIC, CCSD and GMD. The standard deviations of the generated weights by the six methods are shown in Table 6. Obviously, the weights of the seven attributes obtained by Ave-Entropy method have the greatest difference, while the weights generated by the 0-1 Ave-Entropy method have the smallest difference.

**Fig. 6.** Weights of the 7 attributes generated by different methods

**Table 6** The standard deviations of the attribute weights by different methods

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Methods | 0-1 Ave-Entropy | Ave-Entropy | SD | CRITIC | CCSD | GMD |
| Standard deviation | 0.0428 | 0.1293 | 0.0561 | 0.0548 | 0.0673 | 0.0589 |

Since the BDs on some of the attributes are incomplete, the accurate weights shown in Fig.6 cannot reflect the intrinsic features of the original information contained in BDs. So <Model 3> is applied to generate interval weights which are shown in Figs.7 and 8. The dotted lines denote the weights that are not obtained using optimization model. The difference between these two figures lies in that whether the Standard 0-1 Transformation is used in the normalization of average utilities.

**Fig. 7.** Interval weights generated by <Model 3> subject to the 0-1 Ave-Entropy method

**Fig. 8.** Interval weights generated by <Model 3> subject to the Ave-Entropy method

In both Fig.7 and Fig.8, the distance between and is the largest because both Car 3 and Car 6 are assessed to be completely ignorant on ‘Ride quality’. But the weights in Fig.8 are non-normalized interval weights because . Comparatively, the weights generated in Fig.7 are normalized interval weights which are more rational.

**5.2 Generating weights from some other methods**

Here, several objective WAM based optimization models are applied to generate attribute weights for the comparison of these methods against the proposed method.

1) SD method based on BDs

Suppose the BD matrix in Eq.(6) has been transformed to utility matrix denoted by , where signifies the utility of . Let be the normalized utility of be evaluated on . can be set to be equal to or the transformed utility by Standard 0-1 Transformation. Then the standard deviation that the utility of on all the alternatives can be computed by

(24)

In Eq.(24), if is used as to determine , we call it the Ave-*σ* method. Then we have

(25)

In order to cope with the ignorance contained in the BDs, Eq.(25) can be extended to a pair of optimization models as follows:

s.t. Eqs.(19),(20),(24)

Fig.9 shows the interval weights generated by <Model 6> based on the BDs in Table 5. Like Fig.7, the curve represented by ‘SD’ is the results generated by Ave-*σ* method where the optimization process is not applied.

**Fig. 9.** Interval weights generated by <Model 6>

2) CCSD based on BDs

Let be the overall utility of without the consideration of , then we have

(26)

where can be equal to or the transformed utility by Standard 0-1 Transformation. The correlation coefficient between the utility vector of and the overall utility vector without the consideration of is computed by

(27)

Then the following optimization model can be constructed to generate the attribute weights.

s.t. Eqs.(19),(20),(24),(27)

variables are included in <Model 7>. It should be mentioned that <Model 7> only need to be solved once to create the weights of attributes, and the weights generated by <Model 7> are crisp values which are shown in Table 7. So the incompleteness contained in the BDs from Table 5 is not reflected although the incompleteness is considered in the constraints of <Model 7>.

**Table 7** Weights generated by CCSD based optimization model

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |
| *wi* | 0.12 | 0.1555 | 0.1261 | 0.222 | 0.206 | 0.1029 | 0.0678 |

3) CRITIC based on BDs

The CRITIC method, known as Criteria Importance Through Intercriteria Correlation, considers both the contrast intensity and conflicting character of the evaluation criteria[54]. It utilizes the standard deviation to measure the contrast intensity among different alternatives respect to a specific attribute, while Spearman correlation coefficient is employed to quantify the conflict between each pair of attributes. The correlation coefficient between and is calculated by

(28)

Then the weights of attributes are generated by

(29)

If average utility is used as in Eqs.(28) and (29), it is called Ave-CRITIC method. In consideration of the ignorance included in BDs, Ave-CRITIC method is extended to the following pair of optimization models:

s.t. Eqs.(19),(20),(24),(28)

By applying <Model 8>, the interval weights can be generated and shown in Fig.10, and the curve of ‘CRITIC’ is the result by applying Ave-CRITIC method. The weights generated by <Model 8> are also normalized interval weights according to Definition 1.

**Fig. 10.** Interval weights generated by <Model 8>

4) GMD method based on BDs

The GMD of considering the utilities of BDs on all alternatives is computed by

(30)

Then the weight of can be generated as

(31)

To capture the incompleteness of assessment in the original BDs, the following GMD based optimization models are constructed.

s.t. Eqs.(19),(20),(30)

The interval weights generated by <Model 9> are shown in Fig.11. The curve of ‘GMD’ refers to the generated weights by applying Eqs.(30) and (31) where is determined by . From Figs.9 and 11, it can be seen that the interval weights generated by the GMD based model are similar with that of the SD based model.

**Fig. 11.** Interval weights generated by <Model 9>

**5.3 Comparative analysis**

**(1) Comparison with other objective WAMs**

Let the distance between the minimum and maximum values of be denoted by . From Figs.7 to 11, the features of each model can be summarized as follows. All the models generate normalized interval weights except the Ave-Entropy based optimization model. The reason lies in that in Fig.8 is too small, while 0.3506 is very large comparatively. From Eq.(17), we can see that when the Ave-Entropy based optimization model is applied, the first part is not small enough and the second part is too large which results in the non-normalized interval weights. The maximum, average and standard deviation values of by the five models are shown in the 2nd to 4th row of Table 8. From Table 8, it is clear that the Ave-Entropy based optimization model creates the maximum value of . Although the ignorance contained in the BDs of respect to the 6 cars is the largest, the value of is still too large compared with other four methods which will lead to a relatively uncertain aggregation result when the ER algorithm is used. This just reflects that the 0-1 Ave-Entropy based optimization model decreases the uncertainties of interval weights compared with the Ave-Entropy model. The last row of Table 8 shows that the standard deviation of on the 7 attributes by the Ave-Entropy based model is the largest. Thus the generated weights cannot reflect the ignorance in the original BDs effectively because the incompleteness is enlarged on and narrowed on , and .

**Table 8** The maximum, average and standard deviation values of

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | 0-1 Ave-Entropy | Ave-Entropy | SD | CRITIC | GMD |
|  | 0.1941 | 0.3506 | 0.1517 | 0.2379 | 0.158 |
|  | 0.0767 | 0.0854 | 0.061 | 0.0961 | 0.0621 |
|  | 0.06 | 0.11 | 0.0391 | 0.0649 | 0.0413 |

In Table 5, it can be seen that the extent of incompleteness contained in the BDs of the 7 attributes is . Specifically, is assessed to be completely ignorant on both Car 3 and Car 6, so the additive ignorance on is . and are assessed to be partly ignorant on Car 3 and Car 6, and the additive ignorance on these two attributes are equal such that . For , the additive ignorance is because only Car 5 is assessed to be partly ignorant. The additive ignorance on , and are 0 since the BDs of each of the three attributes on all the 7 cars do not contain ignorance. Intuitively, should be the largest among the 7 attributes, while , and are the smallest. and ought to be smaller than and larger than which is bigger than the smallest three attributes. The weights generated by the SD and GMD based models in Figs.9 and 11 show that the values of , , and are close with each other that is inconsistent with our intuition. Fig.10 which shows the weights generated by the CRITIC based model presents similar values on and . It is also irrational according to the above discussion. The weights generated by the 0-1 Ave-Entropy based model shown in Fig.7 are relatively reasonable. In Fig.7, is the largest, the value of is similar with that of , which is smaller than . The values of , and are close with each other and smaller than and . It just accords with our intuition. The comparisons of these different WAMs are shown in Table 9 as follows:

**Table 9** Comparison of several WAMs

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Method Property | Proposed | Ave-entropy | SD | CCSD | CRITIC | GMD | Incompatibility-based method |
| Normalized weights | √ | sometimes not | √ | √ | √ | √ | √ |
| Sensitivity to the ignorance of BD | moderate | relatively high |  |  |  |  |  |
| Considering the discrepancy on the risk preference of DMs |  | √ |  |  |  |  | √ |
| Optimization models included | √ |  |  | √ |  |  | √ |
| The extent of subjectivity | to a certain degree | to a certain degree | No | No | No | No | to a certain degree |
| Interval numerical value permitted | √ |  |  |  |  |  |  |
| Assessment in the form of crisp numerical value | √ |  | √ | √ | √ | √ |  |
| Assessment represented by BD | √ | √ |  |  |  |  | √ |
| Taking into account the ignorance in the original data | √ |  |  |  |  |  |  |
| Consideration of IBD | √ |  |  |  |  |  |  |

The idea of the incompatibility-based method proposed in [8] is similar with the CCSD method. Since the original SD, CCSD, CRITIC and GMD methods assume that all attributes are represented by numerical values, these methods are a purely objective weighting process. Comparatively, the proposed WAM in this paper contains qualitative or quantitative attributes which are denoted by BDs. So the DM can directly express his/her subjective judgments instead of providing numerical data even on qualitative attributes passively. Although the weights are not provided directly by the DM, the generated weights have a certain degree of subjectivity because the BD is the subjective judgment provided by the DM. Besides, preference relations on attribute weights can be incorporated into both the proposed models and the incompatibility-based method to capture the subjective judgments of DMs. From this point of view, the proposed method can be seen as a hybrid WAM.

The proposed method allows that the BDs provided by DMs contain local or global ignorance. As a result, more practical problems can be dealt with because DMs often express their judgments in incomplete manners, thus <Model 3> which is an optimization model is constructed just based on this condition. In comparison, Ave-entropy, SD, CCSD, CRITIC and GMD in the previous studies all require the assessment values to be crisp and complete. So these methods have limitations because the assessment either in the form of numerical value or subjective judgment like BD may be unreliable to a certain degree. For example, data acquisition equipment may be unstable in some complex environment in the LCA, so the unreliability of the equipment is interpreted as the incompleteness involved in the acquired data. Moreover, the data from stable equipment may also change under different working conditions, so incomplete interval value[24] is preferable for the representation of attribute assessment. For this reason, the WAMs involving interval numerical values and IBDs are proposed in <Model 2> and <Model 5> respectively, whereas the other six methods do not consider these two aspects in the previous studies. Although the method in [8] considered the incompatibility of BDs, IBDs is not discussed, and the incompleteness of the BDs on basic attributes is also not addressed.

In the proposed method, the uncertainty of the generated attribute weights is not too sensitive to the ignorance of BDs, which leads to the normalized interval weights as depicted in Fig.7. The 2nd column of Table 8 also reflects this feature of the proposed WAM. Comparatively, the Ave-entropy method which does not consists of a standard 0-1 transformation is more sensitive to the incompleteness of BDs. So it sometimes generates a non-normalized interval weight vector as shown in Fig.8 and the 3rd column of Table 8. From the above discussion, we can draw a conclusion that different forms of assessment such as crisp numerical value, complete and incomplete interval value, BD with accurate and interval belief degrees can all be dealt with to create attribute weights by the proposed models. This allows us to deal with a broader range of decision-making problems.

**(2) Comparison with some methods for GDM**

In addition to the Ave-entropy method[12] and incompatibility-based method[8], none of the above methods takes into account the discrepancy on the risk preference of DMs because GDM[83,84] is not the focus of the discussion. In recent years, the influence-guided GDM in social network has attracted the attention of many researchers on two aspects. One is opinion evolution and feedback mechanism for consensus researching process (CRP)[42,44,45,47,83,84], the other one is incomplete preference estimation[43,45,46,86] and opinion aggregation[85]. The comparisons between the proposed method and these GDM methods are specified as follows:

① Incomplete information estimation

In [45], [46], [86] and our proposed method, a common issue lies in that the information or preferences derived from experts include incomplete or missing preferences, although they are in different forms. Both [45] and [46] proposed the estimation of incomplete preferences based on trust relationships. Comparatively, we do not propose an estimation process because incomplete information is regarded as the ignorance, e.g. the local and global ignorance presented in Table 5. The BDs in Table 5 can also be seen as the aggregated opinion from a group of DMs. The estimation of incomplete preferences in [46] is derived from the trust relationships based on social network. Here, the social influence commonly existed in some GDM scenarios is not considered because individual decision-making is the focus of this paper. How to estimate incomplete preferences in a rational way for individual decision-making problems where trust relationships cannot be obtained is an interesting issue. Furthermore, in some GDM probems, experts do not interact with each other. A typical case is the selection of a president where there are a large number of voters who do not know each other.

② Opinion evolution and aggregation

Just as the above mentioned, the BD can be seen as the subjective judgment by an individual or the aggregated preference from a group of DMs. So the proposed EWAM focuses primarily on the attribute aggregation process in the context of MADM. In this situation, the opinion evolution and feedback mechanism in a social network group decision making (SNGDM) process is not stressed here. As such, it is a relatively static process compared with the dynamic GDM process[42-47,84]. The difference of opinion aggregation process between [44-47,84] and the proposed method lies in the operator and object of aggregation. Simple additive weighting (SAW) is used to construct a collective decision matrix, and the aggregation of experts’ opinions is stressed in [44-47], while the ER algorithm which represents a nonlinear operator is used on the aggregation of attribute values in this paper. The method in [17] presents the ER rule to aggregate the assessments of multiple attributes and multiple experts where the weights and reliabilities of both experts and attributes are considered. But the opinion evolution and how to generate the weights and reliabilities are still open issues in the ER based MADM approaches.

③ Dissimilarity measure

In the EWAM with BDs, one point is similar to the ‘consensus measure’ in a CRP. In a CRP, the distances between each pair of individual preference relations are employed to measure the similarity degree among DMs. Specifically, the in-degree centrality index is utilized to determine the importance degree of a DM in the social network. The in-degree centrality index is generated from the weighted adjacent matrix where each element denotes the trust strength from a DM to another one. In fact, the trust strength reflects the correlation between each pair of DMs to some extent. In the proposed method, the entropy measure is used to quantify the dissimilarity of BDs among all the alternatives on each attribute. The entropy of an attribute which is the measure of dissonance can also be seen as the ‘average distance’ between each pair of alternatives respect to a designated attribute.

④ Preference representation

In the above mentioned literatures, the DM’s preferences are represented in different ways using distinct strucrues, such as interval-valued intruitionistic fuzzy preference relations (IVIFPRs)[44], fuzzy preference relations (FPR)[45,46], PLTS[85] and distributed linguistic trust[47]. Our future research may expand to the situation of MAGDM, where a group of DMs present their judgments in the form of BDs on some attributes, together with some different types of preference representations on other attributes. The CRP could then be discussed under this circumstance. Moreover, the preference estimaiton is also to be included in the EWAM.

Fig.12 shows the EWAM proposed in this paper and the whole decision-making procedure. Three steps are specified as follows:

① The first step is the preparation of the MADM problem. Three major tasks are to be done: (1) Confirmation of the MADM problem, which refers to the identification of DMs and a set of alternatives. (2) Then the attribute set should be constructed according to specific standard or elicited from an individual or a group of DMs. (3) With the generated set of attributes, the values on both quantitative and qualitative attributes are to be acquired. The values of quantitative attributes can be obtained from equipment, statistical data, investigation, etc., while qualitative attributes are usually provided by DMs subjectively.

② The second step is to generate attribute weights from the values acquired. Four different situations may arise as follows: (1) If only quantitative attributes represented by accurate numerical values are included in the MADM problem, traditional EWAM shown in Eqs.(7)-(13) can be used to calculate attribute weights. (2) When some of the quantitative attributes are measured by interval values, <Model 2> is applied to generate the minimum and maximum weights of attributes. (3) If qualitative attributes represented by BDs are included, 0-1 Ave-Entropy method is used provided that the BDs are all complete assessments. (4) Otherwise, <Model 3> is to be utilized to generate interval weights when some of the BDs are incomplete. (5) The last is the most uncertain situation that IBDs are included, which leads to <Model 5> be applied.

③ The third step is the decision-making process. If only an individual is involved in the MADM problem, the ER approach[1,7-13] can be directly employed to aggregate the BDs on qualitative attributes together with the numerical values on quantitative attributes. When a group of DMs are involved, the weights and reliabilities of DMs should be firstly generated in an appropriate way[17]. Then CRP[42-47,84] and opinion aggregation can be implemented. Since weights and reliabilities of DMs and attributes should all be tackled in a rational way, ER rule[17,76,78] is suitable to be applied in the aggregation. Finally, a combined BD with acceptable adjustment cost and consensus level is generated, following by a comparison or selection of alternatives.

**Preparation**

**Decision-making process**

Opinion aggregation

Feedback mechanism

Ref.[17,76]

Ref.[44-47]

Individual decision making

**Generating attribute weights**

Incomplete

Complete

Acquiring attribute values

Quantitative attribute

Qualitative attribute

Crisp value

Interval value

BD

IBD

Traditional EWAM

Model 2

0-1 Ave-Entropy

Model 3

Model 5

Eq.(6)

Eqs.(7)-(13)

ER algorithm

GDM

Acquiring the weights and reliabilities of DMs

CRP

ER rule

Combined BD

Selection of alternatives

Confirmation of the MADM problem

Selection of attribute set

Fig.12 EWAM and the decision-making process

**6. Conclusions**

Uncertainty is ubiquitous in practical MADM problems due to the complexity of decision-making circumstances such as unreliability of data sources, subjectivity of judgment by individual person or different backgrounds and expertise of DMs. How to elicit attribute weights in a rational and appropriate way from various kinds of uncertain available information is significant. In this paper, EWAM is proposed to tackling with the situations where attributes are assessed by interval numerical values, BDs with accurate belief degrees and IBDs. Several pairs of entropy based optimization models are constructed to generate attribute weights in an objective way under these circumstances. The properties of the proposed models are discussed. The advantage of EWAM lies in that the uncertainties and incompleteness contained in the original assessment information are preserved in each of the three situations from the generated weights. Some comparisons with other WAMs are conducted to illustrate the effectiveness of the proposed models. The ER approach can thus be used to cope with MADM problems which consists of precise numerical values, interval values, BDs and IBDs on quantitative and qualitative attributes provided that attribute weights are difficult to be generated subjectively. Just as the above mentioned, future researches would be extended to GDM in social network, and a combination of several objective and subjective WAMs is also to be implemented in an appropriate way for a more flexible process to deal with complex situations.

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**References**

[1]M. Zhou, X.B. Liu, J.B. Yang, C. Fang. Group Evidential Reasoning Approach for MADA under Fuzziness and Uncertainties, International Journal of Computational Intelligence Systems, 2013, 6(3): 423-441.

[2]Chao Fu, Dong-Ling Xu, Min Xue. Determining attribute weights for multiple attribute decision analysis with discriminating power in belief distributions. Knowledge-Based Systems, 2018, 143: 127-141.

[3]Yin L, Deng X, Deng Y. The negation of a basic probability assignment. IEEE Transactions on Fuzzy Systems, 2019, 27(1): 135-143.

[4]Xiaobin Xu, Ping Liu, Yanbo Sun, et al. Fault diagnosis based on the updating strategy of interval-valued belief structures. Chinese Journal of Electronic, 2014, 23(4): 753-760.

[5]Yager, R.R. Dempster-Shafer belief structures with interval valued focal weights. International Journal of Intelligent Systems, 2001, 16: 497-512.

[6]Shengqun Chen, Yingming Wang, Hailiu Shi , Meijing Zhang, Yang Lin. Evidential reasoning with discrete belief structures. Information Fusion, 2018, 41: 91-104.

[7]J.B. Yang. Rule and utility based evidential reasoning approach for multiattribute decision analysis under uncertainties. European Journal of Operational Research, 2001, 131: 31-61.

[8]K.S. Chin, C. Fu, Y.M. Wang. A method of determining attribute weights in evidential reasoning approach based on incompatibility among attributes. Computers & Industrial Engineering, 2015, 87:150-162.

[9]Y.M. Wang, J.B. Yang, D.L. Xu, K.S. Chin. The evidential reasoning approach for multiple attribute decision analysis using interval belief degrees. European Journal of Operational Research, 2006, 175: 35-66.

[10]M. Guo, J.B. Yang, K.S. Chin, H.W. Wang. Evidential reasoning based preference programming for multiple attribute decision analysis under uncertainty. European Journal of Operational Research, 2007, 182(3): 1294-1312.

[11]M. Zhou, X.B. Liu, J.B. Yang. Evidential Reasoning Based Nonlinear Programming Model for MCDA under Fuzzy Weights and Utilities. International Journal of Intelligent Systems, 2010, 25(1): 31-58.

[12]Mi Zhou, Xin-Bao Liu, Jian-Bo Yang, Yu-Wang Chen, Jian Wu. Evidential reasoning approach with multiple kinds of attributes and entropy-based weight assignment. Knowledge-Based Systems, 2019, 163: 358-375.

[13]J.B. Yang, D.L. Xu. On the evidential reasoning algorithm for multiple attribute decision analysis under uncertainty. IEEE Trans. Syst., Man, Cybern, -Part A: Systems and Humans, 2002, 32(3): 289-304.

[14]Xiaobin Xu, Shibao Li, Xiaojing Shen, et al. The optimal design of industrial alarm systems based on evidence theory. Control Engineering Practice, 2016, 46: 142-156.

[15]Xiaobin Xu, Zhen Zhang, Dongling Xu, et al. Interval-valued evidence updating with reliability and sensitivity analysis for fault diagnosis. International Journal of Computational Intelligence Systems, 2016, 9(3): 396-415.

[16]Xiaobin Xu, Haishan Feng, Chenglin Wen. An information fusion method of fault diagnosis based on interval basic probability assignment. Chinese Journal of Electronics, 2012, 20(2): 255-260.

[17]M. Zhou, X.B. Liu, Y.W. Chen, J.B. Yang. Evidential reasoning rule for MADM with both weights and reliabilities in group decision making. Knowledge-Based Systems, 2018, 143: 142-161.

[18]Xin-Bao Liu, Jun Pei, Lin Liu, Hao Cheng, Mi Zhou, Panos M.Pardalos. Optimization and Management in Manufacturing Engineering-Resource Collaborative Optimization and Management through the Internet of Things. Springer, 2017.

[19]Chang L.L, Zhou Y, Jiang J, et al. Structure learning for belief rule base expert system: A comparative study. Knowledge-Based Systems, 2013, 39: 159-172.

[20]Chang L.L, Zhou Z.J, Chen Y.W, et al. Belief rule base structure and parameter joint optimization under disjunctive assumption for nonlinear complex system modeling. IEEE Transactions on Systems, Man, and Cybernetics: Systems, 2017, 99: 1-13.

[21]Jian-Bin Sun, Jimmy Xiangji Huang, Lei-Lei Chang, Jiang Jiang, Yue-Jin Tan. BRBcast: A new approach to belief rule-based system parameter learning via extended causal strength logic. Information Sciences, 2018, 444:51–71.

[22]Z.G. Zhou, F. Liu, L.L. Li, L.C. Jiao, Z.J. Zhou, J.B. Yang, Z.L. Wang. A cooperative belief rule based decision support system for lymph node metastasis diagnosis in gastric cancer. Knowledge Based Systems, 2015, 85: 62-70.

[23]Chang Leilei, Zhou Zhijie, Chen Yuwang, Xu Xiaobin, Sun Jianbin, Liao Tianjun, Tan Xu. Akaike Information Criterion-based conjunctive belief rule base learning for complex system modeling. Knowledge-Based Systems, 2018, 161: 47-64.

[24]M. Zhou, X.B. Liu, J.B. Yang. Evidential reasoning approach for MADM based on incomplete interval value. Journal of Intelligent & Fuzzy Systems, 2017, 33: 3707-3721.

[25]X.B. Xu, J. Zheng, J.B. Yang, D.L. Xu, Y.W. Chen. Data classification using evidence reasoning rule. Knowledge Based Systems, 2017, 116: 144-151.

[26]G.L. Kong, D.L. Xu, X.M. Ma. Combined medical quality assessment using the evidential reasoning approach. Expert Systems with Application, 2015, 42: 5522-5530.

[27]Y.Z. Li, Q.H. Wu, L. Jiang, J.B. Yang, D.L. Xu. Optimal power system dispatch with wind power integrated using nonlinear interval optimization and evidential reasoning approach. IEEE Trans. Power Syst. 2016, 31(3): 2246-2254.

[28]Ying Yang, Chao Fu, Yu-Wang Chen, Dong-Ling Xu, Shan-Lin Yang. A belief rule based expert system for predicting consumer preference in new product development. Knowledge-Based Systems, 2016, 94: 105-113.

[29]Song Y, Deng Y. A new method to measure the divergence in evidential sensor data fusion. International Journal of Distributed Sensor Networks, 2019, 15(4): 1-8.

[30]Xiao F. Multi-sensor data fusion based on the belief divergence measure of evidences and the belief entropy. Information Fusion, 2019, 46: 23-32.

[31]Y.M. Wang, Y. Luo. Integration of correlations with standard deviations for determining attribute weights in multiple attribute decision making. Math. Comput. Model. 2010, 51(1-2): 1-12.

[32]C. Fu, Y.M. Wang. An interval difference based evidential reasoning approach with unknown attribute weights and utilities of assessment grades. Computers & Industrial Engineering, 2015, 81: 109-117.

[33]G.L. Yang, J.B. Yang, D.L. Xu, M. Khoveyni. A three-stage hybrid approach for weight assignment in MADM. Omega, 2017, 71: 93-105.

[34]P.A. Bottomley, J.R. Doyle. A comparison of three weight elicitation methods: Good, better, and best. Omega, 2001, 29: 553-560.

[35]Paul A. Bottomley, John R. Doyle. Comparing the validity of numerical judgements elicited by direct rating and point allocation: Insights from objectively verifiable perceptual tasks. European Journal of Operational Research, 2013, 228: 148-157.

[36]John R. Doyle, Rodney H. Green, Paul A. Bottomley. Judging relative importance: direct rating and point allocation are not equivalent. Organizational behavior and human decision processes, 1997, 70(1): 65-72.

[37]A.T.W. Chu, R.E. Kalaba, K. Spingarn. A comparison of two methods for determining the weights of belonging to fuzzy sets. Journal of Optimisation Theory and Application, 1979, 27: 531-538.

[38]T.L. Saaty. The Analytic Hierarchy Process. New York: McGraw-Hill, 1980.

[39]Pornwasin Sirisawat, Tossapol Kiatcharoenpol. Fuzzy AHP-TOPSIS approaches to prioritizing solutions for reverse logistics barriers. Computers & Industrial Engineering, 2018, 117: 303-318.

[40]C.L. Hwang, M.J. Lin. Group Decision Making under Multiple Criteria: Methods and Applications. Springer, Berlin, 1987.

[41]S. Kotz , N.L. Johnson. Encyclopedia of Statistical Sciences. Wiley, New York, 1982.

[42]Jian Wu, Xue Li, Francisco Chiclana, Ronald R. Yager. An attitudinal trust recommendation mechanism to balance consensus and harmony in group decision making. IEEE Transactions on Fuzzy Systems DOI 10.1109/TFUZZ.2019.2895564.

[43]Jian Wu, Jiali Chang, Qingwei Cao, Changyong Liang. A trust propagation and collaborative filtering based method for incomplete information in social network group decision making with type-2 linguistic trust. Computers & Industrial Engineering, 2019, 127: 853-864.

[44]Jian Wu, Qi Sun, Hamido Fujita, Francisco Chiclana. An attitudinal consensus degree to control feedback mechanism in group decision making with different adjustment cost. Knowledge-Based Systems, 2019, 164 (15): 265-273.

[45]Yucheng Dong, Quanbo Zha, Hengjie Zhang, Gang Kou, Hamido Fujita, Francisco Chiclana, Enrique Herrera-Viedma. Consensus Reaching in Social Network Group Decision Making: Research Paradigms and Challenges. Knowledge-Based Systems, 2018, 162(15): 3-13.

[46]Capuano, Nicola   Chiclana, Francisco   Fujita, Hamido   Herrera-Viedma, Enrique   Loia, Vincenzo. Fuzzy Group Decision Making with Incomplete Information Guided by Social Influence. IEEE Transaction on Fuzzy Systems, 2018, 26(3): 1704-1718.

[47]Jian Wu, Lifang Dai, Francisco Chiclana, Hamido Fujita, Enrique Herrera-Viedma. A minimum adjustment cost feedback mechanism based consensus model for group decision making under social network with distributed linguistic trust. J. Information Fusion, 2018, 41: 232-242.

[48]C.L. Hwang, K. Yoon. Multiple attribute decision making: methods and applications. Berlin: Springer-Verlag, 1981.

[49]J. Ma, Z.P. Fan and L.H. Huang. A subjective and objective integrated approach to determine attribute weights. European Journal of Operational Research, 1999, 112(2): 397-404.

[50]Qiang Xiao, Ruichun He, Changxi Ma, Wei Zhang. Evaluation of urban taxi-carpooling matching schemes based on entropy weight fuzzy matter-element. Applied Soft Computing, 2019, 81, 105493.

[51]Hongshi Xu, Chao Ma, Jijian Lian, Kui Xu, Evance Chaima. Urban flooding risk assessment based on an integrated k-means cluster algorithm and improved entropy weight method in the region of Haikou, China. Journal of Hydrology, 2018, 563: 975-986.

[52]Malin Song, Qingyuan Zhu, Jun Peng, Ernesto D.R. Santibanez Gonzalez. Improving the evaluation of cross efficiencies: A method based on Shannon entropy weight. Computers & Industrial Engineering, 2017, 112: 99-106.

[53]Chuan Yue. Entropy-based weights on decision makers in group decision-making setting with hybrid preference representations. Applied Soft Computing, 2017, 60: 737-749.

[54]Diakoulaki, D., Mavrotas, G., & Papayannakis, L. Determining objective weights in multiple criteria problems: The critic method. Computers & Operations Research, 1995, 22: 763-770.

[55]Hepu Deng, Chung-Hsing Yeh, Robert J. Willis. Inter-company comparison using modified TOPSIS with objective weights. Computers & Operations Research, 2000, 27: 963-973.

[56]Reza Rostamzadeh, Mehdi Keshavarz Ghorabaee, Kannan Govindan, Ahmad Esmaeili, Hossein Bodaghi Khajeh Nobar. Evaluation of sustainable supply chain risk management using an integrated fuzzy TOPSIS- CRITIC approach. Journal of Cleaner Production, 2018, 175: 651-669.

[57]Y.M. Wang. Using the method of maximizing deviations to make decision for multi-indices. System Engineering and Electronics, 1998, 7: 24-26. 31.

[58]Kun Qian, Yihui Luan. Weighted measures based on maximizing deviation for alignment-free sequence comparison. Physica A, 2017, 481: 235-242.

[59]Sen Liu, FelixT.S. Chan,Wenxue Ran. Decision making for the selection of cloud vendor: An improved approach under group decision-making with integrated weights and objective/subjective attributes. Expert systems with Applications, 2016, 55: 37-47.

[60]Huimin Zhang, Liying Yu. MADM method based on cross-entropy and extended TOPSIS with interval-valued intuitionistic fuzzy sets. Knowledge-Based Systems, 2012, 30: 115-120.

[61]Jian-Zhang Wu, Qiang Zhang. Multicriteria decision making method based on intuitionistic fuzzy weighted entropy. Expert Systems with Applications, 2011, 38: 916-922.

[62]Ting-Yu Chen, Chia-Hang Li. Objective weights with intuitionistic fuzzy entropy measures and computational experiment analysis. Applied Soft Computing, 2011, 11: 5411-5423.

[63]Ting-Yu Chen, Chia-Hang Li. Determining objective weights with intuitionistic fuzzy entropy measures: A comparative analysis. Information Sciences, 2010, 180: 4207-4222.

[64]Yager R.R. On the Dempster-Shafer framework and new combination rules. Information Sciences, 1987, 41: 93-137.

[65]Yujuan Wang, Kezhen Zhang, Yong Deng. Base belief function: an efficient method of conflict management. Journal of Ambient Intelligence and Humanized Computing, 2019, 10(9): 3427-3437.

[66]Smets P, Kennes R. The transferable belief model. Artificial Intelligence, 1994, 66(2): 191-234.

[67]Yangxue Li, Yong Deng. Intuitionistic Evidence Sets. IEEE Access, 2019, 7(1): 106417- 106426.

[68]Yang J.B, Wang Y.M, Xu D.L, Chin K.S. The evidential reasoning approach for MADA under both probabilistic and fuzzy uncertainties. European Journal of Operational Research, 2006, 171: 309-343.

[69]Y.M. Wang, J.B. Yang, D.L. Xu. Environmental impact assessment using the evidential reasoning approach. European Journal of Operational Research, 2006, 174: 1885-1913.

[70]X.B. Liu, M. Zhou, J.B. Yang, S.L. Yang. Assessment of strategic R&D projects for car manufacturers based on the evidential reasoning approach. International Journal of Computational Intelligence Systems, 2008, 1: 24-49.

[71]M.J. Zhang , Y.M. Wang , L.H. Li , S.Q. Chen. A general evidential reasoning algorithm for multi-attribute decision analysis under interval uncertainty. European Journal of Operational Research, 2017, 257: 1005-1015.

[72]Fu, C, & Yang, S. L. Analyzing the applicability of Dempster’s rule to the combination of interval-valued belief structures. Expert Systems with Applications, 2011, *38*(4): 4291-4301.

[73]Song, Y. F, Wang, X. D, Lei, L, Xue, A. J. Combination of interval-valued belief structures based on intuitionistic fuzzy set. Knowledge-Based Systems, 2014, 67: 61-70.

[74]Ying-Ming Wang, Jian-Bo Yang, Dong-Ling Xu, Kwai-Sang Chin. On the combination and normalization of interval-valued belief structures. Information Sciences, 2007, 177(5): 1230-1247.

[75]Shannon CE. A mathematical theory of communication. ACM SIGMOBILE Mobile Computing and Communications Review, 2001, 5(1): 3-55.

[76]Yang J.B, Xu D.L. Evidential reasoning rule for evidence combination. Artificial Intelligence, 2013, 205: 1-29.

[77]Yuan-Wei Du, Ying-Ming Wang, Man Qin. New evidential reasoning rule with both weight and reliability for evidence combination. Computers & Industrial Engineering, 2018, 124: 493-508.

[78]Yuan-Wei Du, Ying-Ming Wang. Evidence combination rule with contrary support in the evidential reasoning approach. Expert Systems with Applications, 2017, 88: 193-204.

[79]T.Y. Chen. Multiple criteria decision analysis using a likelihood-based outranking method based on interval-valued intuitionistic fuzzy sets. Information Sciences, 2014, 286: 188-208.

[80]X.B. Liu, F. Pei, J.B. Yang, S.L. Yang. An MAGDM Approach Combining Numerical Values with Uncertain Linguistic Information and Its Application in Evaluation of R&D Projects. International Journal of Computational Intelligence Systems, 2010, 3(5): 575-589.

[81]Shuobo Xu, Dishi Xu, Lele Liu. Construction of regional informatization ecological environment based on the entropy weight modified AHP hierarchy model. Sustainable Computing: Informatics and Systems, 2019, 22: 26-31.

[82]Fuyuan Xiao. EFMCDM: Evidential fuzzy multicriteria decision making based on belief entropy. IEEE Transactions on Fuzzy Systems, 2019. DOI: 10.1109/TFUZZ.2019.2936368.

[83]Orestes Appel, Francisco Chiclana, Jenny Carter, Hamido Fujita. A consensus approach to the sentiment analysis problem driven by support-based IOWA majority. International Journal of Intelligent Systems, 2017, 32: 947-965.

[84]Jian Wu, Francisco Chiclana, Hamido Fujita, Enrique Herrera-Viedma. A visual interaction consensus model for social network group decision making with trust propagation. Knowledge-Based Systems, 2017, 122: 39-50.

[85]Huchang Liao, Lisheng Jiang, Benjamin Lev, Hamido Fujita. Novel operations of PLTSs based on the disparity degrees of linguistic terms and their use in designing the probabilistic linguistic ELECTRE III method. Applied Soft Computing, 2019, 80: 450-464.

[86]Fanyong Meng, Jie Tang, Hamido Fujita. Linguistic intuitionistic fuzzy preference relations and their application to multi-criteria decision making. Information Fusion, 2019, 46: 77-90.

**Appendix**

A-1 Proof of Property 2

If Linear Proportional Transformation is used before the normalization, we have

Obviously, is identical with that Linear Proportional Transformation or Standard 0-1 Transformation is not used. So the generated weights are the same with the Ave-Entropy method.

A-2 The calculation process of weights in Example 1

According to Eqs.(19) and (20), the maximum and minimum utilities of on are calculated as follows:

Then the average utility of on can be computed by Eq.(21) as follows:

Similarly, the maximum, minimum and average utilities of on are computed by

According to the Ave-Entropy method, the normalization process of the utility is conducted as follows:

Then the entropy of is computed by

The entropy of can be calculated according to the above procedure such that . So the weight of and are generated by Eq.(13) as follows:

,

A-3 The calculation process of weights in Example 2

In Example 2, the BDs of and on is identical with Example 1. So we have . According to Eqs.(19)-(21), the maximum, minimum and average utilities of on are computed as follows:

The maximum, minimum and average utilities of on are computed as follows:

Then the entropy of is computed by

So the weight of and are generated by Eq.(13) as follows:

,

A-4 Proof of Property 3

Suppose changes to where is very small. Since and , we have . Then . From Eqs.(19)-(21), we have

Here, for attribute , it is obvious that for , and . From Eq.(22), we have

(A.1)

(A.2)

There may be three situations which are shown as follows:

1) If , and , then we have

In this situation, Eqs.(A.1) and (A.2) can be represented by

Considering Eq.(22), , , so we have and for . Then

(A.3)

From Eq.(13), we can see that the change of is little provided that is small enough.

2) If , , then we have , and . In this situation, Eqs.(A.1) and (A.2) can be represented by

(A.4)

Since , then .

(A.5)

Similarly, we have for . Given Eqs.(A.3) and (13), we conclude that the change of is little provided that is small enough.

3) If , , then we have , and , . If is small enough, there at least exists another alternative such that . So although is not the minimum utility anymore. For attribute , if , where is small enough, we could not clearly discriminate the alternatives respect to , and should better be excluded from the specific assessment problem. Otherwise, . So it is similar with the first situation.

It should be mentioned that when is small enough, the situation that and will not happen.