

Structured matrices in the application of bivariate interpolation to curve implicitization

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Abstract

A nonstandard application of bivariate polynomial interpolation is discussed: the implicitization of a rational algebraic curve given by its parametric equations. Three different approaches using the same interpolation space are considered, and their respective computational complexities are analyzed. Although the techniques employed are usually associated to numerical analysis, in this case all the computations are carried out using exact rational arithmetic. The power of the Kronecker product of matrices in this application is stressed.

Key words: Interpolation; Vandermonde matrix; Kronecker product; Computer Aided Geometric Design; Resultant

1 Introduction

Curve implicitization, which consists of finding the implicit equation of a curve C given by a rational parametrization, is an important problem in computer aided geometric design, and several theoretical results which help to its solution have been developed in the fields of classical algebraic geometry and computer algebra. Some of the methods for the effective computation of the implicit equation are based on interpolation.

We will consider three types of interpolation problems which share the same interpolation space. The different ways of choosing the interpolation nodes will lead to linear systems with very different coefficient matrices: an unstructured

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one, the transpose of a Vandermonde matrix, and the Kronecker product of two Vandermonde matrices.

The Kronecker product structure of the matrix in the third method will make much less expensive the computational cost of the process [12, 22] and will introduce a high degree of *parallelism*.

In this application of interpolation *exact arithmetic* is used, so stability is not as important as efficiency and we do not have to worry about matters which are very important in numerical analysis such as the ill-conditioning of Vandermonde matrices [9]. In fact, the algorithms to be described use techniques from numerical linear algebra but all the computations are carried out in exact arithmetic, which allows them to be applied with no difficulty even when polynomials of very high degree are involved.

Finally, the function to be interpolated is itself a bivariate polynomial, so there will be no interpolation error, and consequently in this case the use of interpolation will not be related to approximation theory.

In order to make our exposition as clear and complete as possible, we will begin with a very small example which will be useful to illustrate the different approaches to the problem. Let us consider the hyperbola with parametric equations

$$(x(t), y(t)) = \left(\frac{1+t}{2+t}, \frac{3+t}{4+t} \right).$$

Its implicit equation is

$$2 - 3y - x + 2xy = 0.$$

In the following sections it will be shown how this implicit equation can be obtained.

The first problem to be addressed is the choice of an appropriate interpolation space. Let us observe that the implicit equation of any degree n polynomial rational parametric curve, with the same denominator in both components of the parametrization, is a degree n algebraic curve (see Section 15.4 of [20]). However, this is not necessarily true when two different denominators are considered in the parametrization. In our example, both parametric equations have degree 1 while the implicit equation has total degree 2. This is the reason why, when considering the general situation of two different denominators, it is more natural to consider the *coordinate degree* (i.e. the degree in x , the degree in y) instead of the total degree. The following result (see [15] and [21]) gives us the precise form of the implicit equation.

Theorem 1. *Let $P = \left(x(t) = \frac{u_1(t)}{v_1(t)}, y(t) = \frac{u_2(t)}{v_2(t)} \right)$ be a proper rational parametrization of the irreducible curve C defined by $F(x, y)$, and let*

$\gcd(u_1, v_1) = \gcd(u_2, v_2) = 1$. Then $\max\{\deg_t(u_1), \deg_t(v_1)\} = \deg_y(F)$ and $\max\{\deg_t(u_2), \deg_t(v_2)\} = \deg_x(F)$.

Theorem 1 tells us that the polynomial $F(x, y)$ defining the implicit equation of the curve C belongs to the polynomial space $\Pi_{m,n}(x, y)$, where $m = \max\{\deg_t(u_2), \deg_t(v_2)\}$ and $n = \max\{\deg_t(u_1), \deg_t(v_1)\}$. The dimension of $\Pi_{m,n}(x, y)$ is $N = (m + 1)(n + 1)$, and a basis is given by

$$\begin{aligned} & \{x^i y^j | i = 0, \dots, m; j = 0, \dots, n\} = \\ & = \{1, y, \dots, y^n, x, xy, \dots, xy^n, \dots, x^m, x^m y, \dots, x^m y^n\}. \end{aligned}$$

Moreover $\deg_x(F(x, y)) = m$ and $\deg_y(F(x, y)) = n$, and therefore there is no interpolation space $\Pi_{r,s}(x, y)$ with $r < m$ or $s < n$ such that $F(x, y)$ belongs to $\Pi_{r,s}(x, y)$.

In addition, the selection of the interpolation space $\Pi_{m,n}(x, y)$ is also suitable because in practice the implicit representation of a rational parametric curve is a dense polynomial [13].

So in the application of interpolation we are considering the interpolation space is given and we have to choose appropriate Lagrange interpolation nodes. As we will see, for obtaining a linear system with a structured coefficient matrix we will need the introduction of *resultants*, a tool which is widely used in computer algebra and has a variety of applications [4].

In the next three sections we will discuss three different approaches to the implicitization problem by using interpolation, and in Section 5 we will analyze their respective computational complexities.

2 An interpolation problem with an unstructured coefficient matrix

The first approach, included in [11] along with other approaches to implicitization, leads to a non-structured interpolation problem whose solution is not unique.

Since the dimension of the linear space of solutions will be seen to be almost surely 1, the implicit equation can be computed in the following way. As we know the interpolation space (it is $\Pi_{m,n}(x, y)$), we may evaluate the

parametrization $P = (x(t), y(t))$ at some finite set of values of t (for example $t = 0, 1, 2, \dots$) for obtaining N distinct rational interpolation nodes

$$\{(x_i, y_i) = (x(t_i), y(t_i)) : i = 1, \dots, N\}.$$

Then we can formulate an interpolation problem using those interpolation nodes, in this case with *all the interpolation data equal to zero*. So, we have an interpolation problem in which the corresponding linear system is a non-structured homogeneous one, and we know from Section 1 that a nontrivial solution always exists.

If we denote by A the coefficient matrix of this homogeneous linear system and $r = \text{rank}(A)$ then the set of all solutions of $Ax = 0$ is the nullspace of A , and its dimension $N - r$ is great or equal than 1.

Let us recall that Bézout's theorem states that *two plane curves of degree $m+n$ without common components have at most $(m+n)^2$ common complex points*. As a consequence of this theorem, the probability of having $\text{rank}(A) < N - 1$ is negligible, since we are prescribing $N = (m+1)(n+1)$ interpolation conditions at rational points. However, if that happens we can add a new interpolation condition to find the correct implicit equation.

In our example the rank is necessarily equal to 3 because $m = n = 1$ and therefore $(m+n)^2 = 4 = (m+1)(n+1)$. If we consider $t = 0, 1, 2, 3$ the interpolation nodes in the corresponding order are

$$\left\{ \left(\frac{1}{2}, \frac{3}{4} \right), \left(\frac{2}{3}, \frac{4}{5} \right), \left(\frac{3}{4}, \frac{5}{6} \right), \left(\frac{4}{5}, \frac{6}{7} \right) \right\},$$

and the coefficient matrix of the homogeneous linear system $Ac = 0$ is

$$A = \begin{pmatrix} 1 & \frac{3}{4} & \frac{1}{2} & \frac{3}{8} \\ 1 & \frac{4}{5} & \frac{2}{3} & \frac{8}{15} \\ 1 & \frac{5}{6} & \frac{3}{4} & \frac{5}{8} \\ 1 & \frac{6}{7} & \frac{4}{5} & \frac{24}{35} \end{pmatrix}.$$

The solution vector is

$$c = (2, -3, -1, 2)^T$$

or any multiple of it, and so the implicit equation of the given curve is

$$2 - 3y - x + 2xy = 0.$$

3 The use of resultants and the Vandermonde matrix

The computation of the implicit equation of a rationally parametrized curve can be carried out by computing the *resultant* of two polynomials. The following theorem provides the way of doing it (see [15] and [21]).

Theorem 2. *Let $P = \left(x(t) = \frac{u_1(t)}{v_1(t)}, y(t) = \frac{u_2(t)}{v_2(t)} \right)$ be a proper rational parametrization of an irreducible curve C , with $\gcd(u_1, v_1) = \gcd(u_2, v_2) = 1$. Then the polynomial defining C is $\text{Res}_t(u_1(t) - xv_1(t), u_2(t) - yv_2(t))$ (the resultant with respect to t of the polynomials $u_1(t) - xv_1(t)$ and $u_2(t) - yv_2(t)$).*

Taking this theorem into account, the polynomial defining the implicit equation of the curve of the example introduced in Section 1 is

$$\text{Res}_t((1+t) - x(2+t), (3+t) - y(4+t)),$$

which is precisely

$$2 - 3y - x + 2xy.$$

The resultant of two polynomials can be computed, for example, by using the command `resultant` of the symbolic computation system *Maple*.

However, let us point out here that, in general, the computation of $\text{Res}_t(u_1(t) - xv_1(t), u_2(t) - yv_2(t))$ is not a trivial task. It is the determinant of the Sylvester (or Bézout) matrix of $u_1(t) - xv_1(t)$ and $u_2(t) - yv_2(t)$ (see [20], for example), and therefore its computation requires the expansion of the determinant of a matrix whose entries are polynomials in the variables x and y . The symbolic expansion of a determinant is a computer algebra problem which requires a lot of time and space to be solved due to the problem of *intermediate expression swell*. This fact is recognized in [13] where it is said that the *bottleneck* of the algorithm for implicitizing rational surfaces is the symbolic expansion of the determinant. As it can be read in [5], one of the most interesting approaches for the symbolic expansion of the determinant is based on interpolation, and it is the one presented in [13] and [14].

In that approach the N interpolation data are obtained by evaluating the Sylvester (or Bézout) matrix at the N interpolation nodes

$$\{(p_1^k, p_2^k) : k = 0, \dots, N-1; \quad p_1, p_2 \text{ distinct primes}\},$$

and computing the corresponding constant determinants. This clever selection of the nodes reduces the solution of the interpolation problem to the solution of a linear system of order N whose coefficient matrix is the transpose of a

nonsingular (since different monomials evaluate to different values) Vandermonde matrix (see, for example, [17] for the expression of the Vandermonde matrix).

We illustrate this approach with our example.

We start by showing the Sylvester matrix of the polynomials $p(t) = 1 + t - x(2 + t)$ and $q(t) = 3 + t - y(4 + t)$, which is

$$S = \begin{pmatrix} 1 - x & 1 - 2x \\ 1 - y & 3 - 4y \end{pmatrix}.$$

In order to compute its determinant by using this interpolation approach, we consider $p_1 = 2$ and $p_2 = 3$, and so the $N = 4$ interpolation nodes are:

$$\{(1, 1), (2, 3), (4, 9), (8, 27)\}.$$

The vector with the interpolation data in the corresponding order is

$$b = (0, 3, 43, 345)^T,$$

and the coefficient matrix of the linear system is

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 6 \\ 1 & 9 & 4 & 36 \\ 1 & 27 & 8 & 216 \end{pmatrix}.$$

The solution of the linear system $Ac = b$ is the vector

$$c = (2, -3, -1, 2)^T,$$

and therefore the implicit equation of the curve is

$$2 - 3y - x + 2xy = 0.$$

In [13], where this approach is presented for the case of surface implicitization, it is indicated that in this way the problem reduces to interpolating a univariate polynomial. However, it must be observed that the coefficient matrix of the linear system is not a Vandermonde matrix associated with a univariate Lagrange interpolation problem: it is the transpose of such a matrix.

The approach of [13] has its roots in computer algebra (see, for example, [23]), and so the solution of Vandermonde systems by Björck-Pereyra algorithms [1]

is not considered there. A recent extension of the Björck-Pereyra approach (closely related to bidiagonal factorizations of the inverse of the matrix) to Vandermonde-like matrices can be seen in Chapter 22 of [9].

4 A choice of nodes leading to the Kronecker product

Another approach for computing the implicit equation of a plane rationally parametrized curve by means of resultants and interpolation is the one introduced in [15].

In that paper, the polynomial defining the implicit equation of the curve, that is, $Res_t(u_1(t) - xv_1(t), u_2(t) - yv_2(t))$, is computed by using a bivariate interpolation technique in which the nodes are arranged forming a tensor product grid. This choice of the interpolation nodes is specially appropriate for the interpolation space we are working with, because it reduces the solution of the interpolation problem to the solution of a linear system of order N whose coefficient matrix is the Kronecker product

$$A = V_x \otimes V_y,$$

(where the Kronecker product $B \otimes D$ is defined by blocks as $(b_{kl}D)$, with $B = (b_{kl})$) with V_x being the Vandermonde matrix generated by the first component of the interpolation nodes and V_y being the Vandermonde matrix generated by the second component of the interpolation nodes (see [15] for the details). In addition, the algorithm included there reduces the solution of this linear system with Kronecker product structure to solving $m+1$ Vandermonde linear systems with the same matrix V_y and $n+1$ Vandermonde linear systems with the same matrix V_x . In this way, the solution of a bivariate interpolation problem is reduced to the solution of only univariate interpolation problems in the variables x and y .

A *Maple* implementation of the complete algorithm is also included in the paper. As every linear system to be solved is a Vandermonde linear system, the algorithm uses the Björck-Pereyra algorithm [1, 8] for solving them, since it takes advantage of the special structure of the coefficient matrices V_x and V_y .

The specific choice of the interpolation nodes proposed in the paper is

$$\{(x_i, y_j) = (i, j) : i = 0, \dots, m; j = 0, \dots, n\},$$

in the same order as the interpolation basis.

Now we apply this technique to our example.

The interpolation nodes considered in this case are:

$$\{(0, 0), (0, 1), (1, 0), (1, 1)\}.$$

The vector with the interpolation data in the corresponding order is:

$$b = (2, -1, 1, 0)^T.$$

The coefficient matrix of the linear system is:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The solution of the linear system $Ac = b$ is the vector

$$c = (2, -3, -1, 2)^T,$$

and therefore, the implicit equation of the curve is

$$2 - 3y - x + 2xy = 0.$$

5 Some remarks on the computational complexity

We start this section by presenting a bigger (but still small) example of curve implicitization in which we will show the different behaviour of the three different approaches we have described in the previous sections.

Let

$$P = (x(t), y(t)) = \left(\frac{2t^2 + 2t + 1}{t^3 + 5}, \frac{t^3 - 3t^2 + t - 1}{t^2 - 3} \right)$$

be a rational parametrization of a curve C whose implicit equation is given by the polynomial

$$F(x, y) = -53 + 42y - 74y^2 + 172x + 707xy + 121xy^2 + 37xy^3 - 652x^2 - 1156x^2y - 490x^2y^2 - 34x^2y^3 + 626x^3 + 396x^3y + 432x^3y^2 - 2x^3y^3.$$

When computing $F(x, y)$ by means of the approach described in Section 2, the implicitization problem is reduced to the solution of a homogeneous linear system of order 16 with a non-structured coefficient matrix A . As this matrix is too big for including it here, we just show one of its largest entries:

$$A_{15,16} = \frac{761421163154846949}{149346877368718693}.$$

After solving this linear system the vector with the coefficients of $F(x, y)$ is obtained.

The other two methods need the Sylvester matrix of the polynomials $p(t) = 2t^2 + 2t + 1 - x(t^3 + 5)$ and $q(t) = t^3 - 3t^2 + t - 1 - y(t^2 - 3)$ for computing the implicit equation of C . This Sylvester matrix is:

$$S = \begin{pmatrix} -x & 2 & 2 & -5x + 1 & 0 & 0 \\ 0 & -x & 2 & 2 & -5x + 1 & 0 \\ 0 & 0 & -x & 2 & 2 & -5x + 1 \\ 1 & -y - 3 & 1 & 3y - 1 & 0 & 0 \\ 0 & 1 & -y - 3 & 1 & 3y - 1 & 0 \\ 0 & 0 & 1 & -y - 3 & 1 & 3y - 1 \end{pmatrix}.$$

When using the approach described in Section 3, the implicitization problem is reduced to the solution of a non-homogeneous linear system $Ac = b$ of order 16 where A is the transpose of a Vandermonde matrix. As we have done before, we only present the greatest entry of the matrix, which corresponds to the evaluation of the monomial x^3y^3 at the interpolation node $(2^{15}, 3^{15})$:

$$A_{16,16} = 103945637534048876111514866313854976.$$

As for the vector b with the interpolation data, its largest component is

$$b_{16} = -207995995871362988895940143529893921.$$

The solution of this linear system in which very large numbers are involved is the vector with the coefficients of $F(x, y)$.

Finally, when we compute the implicit equation of C by means of the method described in Section 4, the problem is reduced to the solution of the non-homogeneous linear system $Ac = b$ of order 16 where A is the Kronecker

product of two Vandermonde matrices of order 4,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix},$$

and the vector b containing the interpolation data is

$$(-53, -85, -265, -593, 93, 72, 35, -12, 2691, 4277, 8723, 15561, 11497, 21242, 44579, 80014)^T.$$

In this case only 8 linear systems with the Vandermonde matrix of order 4 presented above have to be solved for obtaining the coefficients of $F(x, y)$.

Remark. Let us observe here that, as it can be read in [20], the implicit equation of a rational curve can also be computed by using techniques based on computing Gröbner bases with the pure lexicographical ordering. However, although these techniques are very important from a theoretical point of view, they are not so effective in practice because they are very time and space consuming, even for problems of moderate degree. An example of this situation is the example introduced in this section. When we tried to compute the implicit equation of C by using the *Maple* command for computing Gröbner bases `gbasis` no answer was obtained after a lot of minutes of computation, and in addition a lot of memory space (more than 70 megabytes without finishing the computation) was required.

In this sense, it can be read in [7] that the complexity theory for Gröbner bases *gives rise to the pessimistic view that these methods for polynomial ideals are not useful in practice, except for rather small cases*. In [20] it is recognized that the use of Gröbner bases for surface implicitization *is not very computationally efficient*, and this fact is also observed in a different application of Gröbner bases in [3]. As illustrated with our example in this section, the problems with Gröbner bases appear also in the case of curve implicitization and with small degrees.

As for the approaches based on interpolation, it must be observed that the numbers involved in the first and in the second approach are much larger than the numbers involved in the third one (see the example at the beginning of this section), which can make the computations slower.

Now we briefly analyze the computational complexity of the three methods for curve implicitization presented in this paper, beginning with the stage

corresponding to the solution of the linear systems. We will do it in terms of arithmetic operations, and for the sake of simplicity we assume $m = n$.

- The computational complexity of the first approach is the computational complexity of solving a non-structured linear system of order N : $O(N^3)$.
- The computational complexity of the second approach is the computational complexity of solving a linear system whose coefficient matrix is the transpose of a Vandermonde matrix of order N : $O(N^2)$ if the Björck-Pereyra algorithm is used [1, 8].
- The computational complexity of the third approach is the computational complexity of solving $2(n+1)$ Vandermonde linear systems of order $(n+1)$: $O((n+1)^3) = O(N^{3/2})$ if the Björck-Pereyra algorithm is used for solving each Vandermonde linear system [1, 8].

Let us point out that in the first approach, in addition to the higher computational cost of solving the linear system, first it is necessary to evaluate the parametric equations to obtain the interpolation nodes, and then the coefficient matrix of the linear system (which is not a structured matrix) must be constructed and stored.

As for the computation of the interpolation data, its complexity is the same both in the second and in the third approach, and it is the complexity of computing $(n+1)^2$ determinants of order $O(n)$: $O(n^5) = O(N^{5/2})$. Let us point out here that, although this complexity is greater than the complexity of solving the linear systems, this stage can easily be parallelized because each datum can be computed separately.

The design of *parallel algorithms* for computer algebra problems such as resultant computation and elimination of variables has been recently considered in [2] and [10]. In this sense, let us observe that the approach introduced in Section 4 has a high degree of intrinsic parallelism present not only in the computation of the interpolation data but also in the computation of the coefficients of the interpolating polynomial, since each one of the linear systems with matrix V_y can be solved independently and the same happens to each one of the linear systems with matrix V_x (see [15]).

Although it is true that this measure of computational complexity -which is the standard one in numerical computations- is not a complete measure of the computational complexity in exact arithmetic, where the computational cost also depends on the size and structure of the numbers, it nevertheless can serve to have useful estimates of the computational complexity. In this sense, we can read in [19] how the reduction of the order of the resultant matrix from $2n$ to n may lead to faster computations.

In addition, it must be pointed out that the two implicitization methods based

on interpolation and resultants we have described can also take advantage of this reduction in the order of the resultant matrix because they can be used with any other resultant matrix, for example, the resultant matrix obtained when using the method of moving curves [19].

Finally, it must be noticed that in the situation described in the approach of Section 4, that is, when the nodes are arranged forming a Cartesian product grid, explicit formulas (involving the *Lagrange basis*) exist for computing the interpolation polynomial [18].

Nevertheless, the existence of an explicit formula does not imply there is no computational cost in applying it. In this sense we can recall the classical paper [6], where it is stressed that the fact that we have Cramer's rule does not make the practical solution of linear systems a *trivial and dull* task.

In our case, it must be taken into account that the computation of the interpolation polynomial *in the monomial basis* by using an explicit formula has a computational cost (see [7] for the cost in the univariate case). It must be observed that our approach described in Section 4 has a complexity of the same order as the algorithm presented in a more general setting in [16].

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