A Smooth Estimator for MC/QMC Methods in Finance

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Abstract

We investigate the effect of martingale control as a smoother for MC/QMC methods. Numerical results of estimating low-biased solutions for American put option prices under the Black-Scholes model demonstrate that using QMC methods can be problematic. But it can be fixed by adding a (local) martingale control variate into the least-squares estimator to gain accuracy and efficiency. In examples of estimating European option prices under muitl-factor stochastic volatility models, randomized QMC methods improve the variance by merely a single digit. After adding a martingale control, the variance reduction ratio raise up to 700 times for randomized QMC and about 50 times for MC simulations. When the delta estimation problem is considered, the efficiency of the martingale control variate method decreases. We propose an importance sampling method which performs better particularly in the presence of rare events.

Keywords: Option pricing; Multi-factor stochastic volatility models; control variate method; Monte Carlo and quasi-Monte Carlo methods.

1 Introduction

The evaluation of financial derivatives are central problems in modern finance. In the seminal work of Black and Scholes [4], the fair price of a European-style derivative, denoted by P, can be presented as a conditional expectation under the risk-neutral probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, I\!\!P^\star)$

$$P(t,S_t) = I\!\!E^\star \left\{ e^{-r(T-t)} H(S_T) | \mathcal{F}_t \right\}, \tag{1}$$

where the underlying risky asset S_t is governed by the geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dW_t^\star. \tag{2}$$

Other notations are defined as follows: t the current time, $T < +\infty$ the maturity, r the risk-free interest rate, σ the volatility, $W_t^\star$ the standard Brownian motion, $H(x)$ the payoff function satisfying the usual integrability condition. For example, if $H(x) = \max\{x-K, 0\} \equiv (x-K)^+$ for the strike price $K > 0$, it is a call payoff; if $H(x) = \max\{K-x, 0\} \equiv (K-x)^+$, it is a put payoff. A financial contract with the call or put payoff exercised at the maturity date is called a European call option or a European put option respectively.

From the simulation point of view, it is straightforward to construct the basic Monte Carlo (MC for short) estimator

$$\frac{1}{N} \sum_{i=1}^{N} e^{-rT} H(S_T^{(i)}) \tag{3}$$

where N is the total number of sample paths and $S_T^{(i)}$ denotes the i-th independent replication of the random variable S_T.
Our main interest in this paper is to improve the accuracy of the estimate obtained from (3) by variance reduction techniques or by Quasi Monte Carlo (QMC for short) method. Motivated from stochastic financial theory, every option contract defined in (1) and (2) can be perfectly replicated a the hedging portfolio such that

$$P(0, S_0) = e^{-rT} H(S_T) - \mathcal{M}(P; T) \tag{4}$$

where $\mathcal{M}(P;T)$ is a zero-centered (hedging) martingale

$$\mathcal{M}(P; t) = \int_0^t e^{-rs} \frac{\partial P}{\partial x}\left(s, S_s\right) \sigma S_s dW_s^\star. \tag{5}$$

Equation (4) can be understood as a martingale representation by applying Ito's lemma to the discounted option price process $e^{-rt}P(t, S_t)$ provided that the derivative price function $P(t,x)$ is once differentiable in time t and twice differentiable in the asset price x. Ideally, if one was able to sample perfectly for S_T and $\mathcal{M}(P)$, then one can run a single Monte Carlo simulation on the right hand side of (4) to obtain the derivative price $P(0, S_0)$. In reality, if one was able to calculate $\mathcal{M}(P;T)$ perfectly, the partial derivative $\frac{\partial P}{\partial x}(t,x)$ would be known so that the option price $P(t,x)$ could be known in advance. Therefore Equation (4) is not feasible for a direct computation for the option price. Nevertheless by employing a zero-centered martingale as a control we can formulate the unbiased control variate estimator

$$\frac{1}{N} \sum_{i=1}^N \left[e^{-rT} H(S_T^{(i)}) - \mathcal{M}^{(i)}(\tilde{P}; T) \right] \tag{6}$$

for the option price $P_0 = I\!\!E^\star \left\{ e^{-rT} H(S_T) - \mathcal{M}(\tilde{P};T) | \mathcal{F}_0 \right\}$ where the new martingale control $\mathcal{M}(\tilde{P};T)$ consists of the price approximation $\tilde{P}$ to the actual option price P. In financial interpretation $\mathcal{M}(\tilde{P};T)$ represents the delta hedging portfolio accumulated up to time T, so the term $\mathcal{M}(\tilde{P};T)$ is called the hedging martingale by the price $\tilde{P}$ and the estimator defined in (6) is called the martingale control variate estimator.
Empirical studies such as [16] on hedging options document the robustness and effectiveness of using the Black-Scholes delta hedging strategy. Therefore, a financial intuition about the effectiveness of the martingale control variate $e^{-rT}H(S_T) - \mathcal{M}(\tilde{P};T)$ is that if the delta $\frac{\partial \tilde{P}}{\partial x}(t,x)$ is close to the actual hedging strategy $\frac{\partial P}{\partial x}(t,x)$, fluctuations of the replicating error will be small so that the variance of the estimator (6) should be small. In this current study, we adopt the Black-Scholes option price, denoted by P_{BS}, to approximate the actual option price. Other variance reduction techniques with less or no financial implication include *conditional* Monte Carlo [27], *importance sampling* [13, 7], direct sampling [3], etc. We refer to [14] and references therein.
All Monte Carlo methods mentioned so far are fundamentally related to pseudo random sequences. As an alternative integral methods using quasi-random sequences (or called low-discrepancy sequences) have drawn lots of attentions in recent years because its theoretical rate of convergence is $\mathcal{O}(1/n^{1-\varepsilon})$ for all $\varepsilon > 0$ subjected to the dimensionality and the regularity of the integrand [23]. Despite the regularity of the integrand function corresponding to the payoff $H(S_T)$ is generally poor [14], there are still many applications of using QMC or randomized QMC as a computational tool in finance. Many developed QMC techniques are motivated from financial applications [2, 19, 20, 26]. In next section we give a counterexample of using QMC method to estimate lower bound solutions of American option prices. After combining the martingale control variate with QMC, we find that the accuracy of the

American option price estimate is significantly improved. However when we consider the delta estimation problem in Section 4, the efficiency of the martingale control variate method starts to decrease. This is due to the regularity of the delta payoff is worse than its option payoff. Alternatively, we propose an importance sampling method and show efficiency gain from the rare event simulation.
In this paper, we investigate primarily the effect of martingale control variate under MC/QMC methods. The evaluation of option prices under multifactor stochastic volatility models are also explored. Several numerical experiments are conducted to compare the variance reduction performance for the martingale control variate method mentioned above with or without randomized QMC methods.
The organization of this paper is the following. In Section 2, we consider the high and low biased estimation problem for American options under the Black-Scholes model, and give an example that pricing low-biased American options by QMC can be infeasible. This can be fixed when martingale control variates are added into the least-squares estimator. In Section 3 we introduce the class of multifactor stochastic volatility models and the construction of martingale control by means of perturbation techniques. Numerical experiments by Monte Carlo method and quasi-Monte Carlo methods are presented for one-factor Heston model and generic two-factor stochastic volatility models. We test several combinations of martingale control variate methods with or without QMC methods, including the Sobol' sequence and L'Ecuyer type good lattice points together with the Brownian bridge sampling technique. In Section 4, we consider the delta estimation problem, treated by martingale control variate method, importance sampling method, and the central difference method.

2 High and Low Biased Estimates of American Option Price by MC/QMC

The right to early exercise a contingent claim is an important feature for derivative trading. An American option offers its holder, not the seller, the right but not the obligation to exercise the contract any time prior to maturity during its contract life time. Based on the no arbitrage argument, the American option price at time t, denoted by P_t, with maturity $T < \infty$ is considered as an optimal stopping time problem [8] defined by

$$P_t = \sup_{t \leq \tau \leq T} I\!E^\star \left\{ e^{-r(\tau - t)} H\left(S_\tau\right) | \mathcal{F}_t \right\}, \tag{7}$$

where τ denotes a bounded stopping time.
To estimate low-biased American option prices, Longstaff and Schwartz [22] took a primal (dynamic programming) approach and proposed a least-square regression to estimate the *continuation value* at each in-the-money asset price state. By comparing the continuation value and the instant exercise payoff, their method exploits a decision rule, denoted by $\underline{\tau}$, for early exercise along each sample path generated. As a fact that $\underline{\tau}$ being a suboptimal stopping rule, Longstaff-Schwartz' method induces a low-biased estimate for American option price at time 0

$$I\!E^\star \left\{ e^{-r\underline{\tau}} H\left(S_{\underline{\tau}}\right) | \mathcal{F}_0 \right\}. \leq P_0 \tag{8}$$

By the optional sampling theorem [21] we can use a locally hedging martingale $\mathcal{M}\left(\tilde{P}; \underline{\tau}\right)$ to preserve the low-biased estimate (8) by

$$I\!E^\star \left\{ e^{-r\underline{\tau}} H\left(S_{\underline{\tau}}\right) - \mathcal{M}\left(\tilde{P}; \underline{\tau}\right) | \mathcal{F}_0 \right\}, \tag{9}$$

where $\tilde{P}$ is an approximation of the American option price. By the spirit of hedging martingale discussed in Section 1, we consider

$$\mathcal{M}\left(P_E;\underline{\tau}\right)=\int_0^{\underline{\tau}} e^{-rs}\frac{\partial P_E}{\partial x}(s,S_s)\sigma S_s dW_s^{\star}, \tag{10}$$

where P_E denotes the counterpart European option price. In the case of the American put option, P_0 is unknown but its approximation P_E admits a closed-form solution, known as the Black-Scholes formula. Its delta is given by

$$\frac{\partial P_E}{\partial x}(t,x;T,K,r,\sigma)=\mathcal{N}\left(\frac{\ln(x/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)-1,$$

where $\mathcal{N}(x)$ denotes the cumulative normal integral function.
To estimate high-biased American option prices, Rogers [25] obtained a dual approach to solve the *inf-sup* problem over martingales:

$$P_0=\inf_{M\in H_0^1} I\!E^{\star}\left\{\sup_{0\le t\le T}\left(e^{-rt}\,H(S_t)-M_t\right)|\mathcal{F}_0\right\}, \tag{11}$$

where the martingale $M.$ belongs to

$$H_0^1=\left\{(M_t)_{0\le t\le T}:\text{ martingales with }\sup_{0\le t\le T}|M_t|\in L^1\text{ and }M_0=0\right\}.$$

Therefore, for any martingale $M\in H_0^1$, for example $M_t=\mathcal{M}\left(\tilde{P};t\right)$, a high-biased estimate of the American option price is deduced:

$$P_0\le I\!E^{\star}\left\{\sup_{0\le t\le T}\left(e^{-rt}\,H(S_t)-\mathcal{M}\left(\tilde{P};t\right)\right)|\mathcal{F}_0\right\}. \tag{12}$$

Note that when the approximate price $\tilde{P}$ equals to the true American option price P, then the equality holds. This ca be shown from the Doob-Meyer decomposition [25, 9]. From (9) and (12), we see the high and low biased estimates are

$$I\!E^{\star}\left\{e^{-r\underline{\tau}}H\left(S_{\underline{\tau}}\right)-\mathcal{M}\left(\tilde{P};\underline{\tau}\right)|\mathcal{F}_0\right\}\le P_0\le I\!E^{\star}\left\{\sup_{0\le t\le T}\left(e^{-rt}\,H(S_t)-\mathcal{M}\left(\tilde{P};t\right)\right)|\mathcal{F}_0\right\}. \tag{13}$$

We remark that these martingales shown on the left-had term and the right-hand term above can be chosen differently based on user's interest. In the following numerical example, we use the same martingale, defined in (10), by choosing the option price approximation $\tilde{P}$as the counterpart European option price P_E.

As an example we consider a pricing problem at time 0 for the American put option with parameters $K=100, r=0.06, T=0.5$, and $\sigma=0.4$. Numerical results of the low-biased estimates by MC/QMC with or without hedging martingales and high-biased estimates with hedging martingales are demonstrated in Table 1. The first column illustrates a set of different initial asset price S_0. The true American option prices corresponding to S_0 are given in Column 6, depicted from from Table 1 of [25]. Monte Carlo simulations are implemented by sample size $N=5000$ and time step size (Euler discretization) $\Delta t=0.01$. By using the least squares method, Column 2 and Column 3 illustrate low-biased estimates and their standard errors (in parenthesis) obtained from MC estimator related to Equation (8) and MC+CV estimator related to Equation (9) respectively. We observe that (1)

Table 1: Comparisons of low-biased estimates (Column 2-5), the actual American option prices (Column 6), and high-biased estimates (Column 7-8). MC denotes the basic Monte Carlo estimates. MC+CV denotes the control variate estimates with the hedging martingale $\mathcal{M}(P_E; \underline{\tau})$ being the additive control. Standard errors are shown in the parenthesis. QMC and QMC+CV denote calculations of Equation (8) and (9,12) using quasi sequences respectively.

S_0	LSM MC	LSM MC+CV	LSM QMC	LSM QMC+CV	$P_0(true)$	Dual MC+CV	Dual QMC+CV
80	20.7368 (0.2353)	20.6876 (0.0124)	20.9435	20.6626	21.6059	21.947 (0.0107)	21.9764
85	17.3596 (0.2244)	17.3586 (0.0134)	17.8031	17.3321	18.0374	18.325 (0.0128)	18.3590
90	14.3871 (0.2125)	14.3930 (0.0139)	15.0429	14.4030	14.9187	15.132 (0.0143)	15.1988
95	11.8719 (0.1995)	11.8434 (0.0148)	12.6472	11.8795	12.2314	12.371 (0.0148)	12.4660
100	9.8529 (0.1881)	9.6898 (0.0157)	10.5380	9.6942	9.946	10.147 (0.0153)	10.1433
105	7.9586 (0.1684)	7.8029 (0.0154)	8.7117	7.8351	8.0281	8.181 (0.0151)	8.1941
110	6.2166 (0.1518)	6.2606 (0.0150)	7.1663	6.2949	6.4352	6.612 (0.0149)	6.5708
115	5.0815 (0.1367)	5.0081 (0.0144)	5.8568	5.0221	5.1265	5.269 (0.0141)	5.2282
120	4.0885 (0.1245)	3.9389 (0.0146)	4.7480	3.9699	4.0611	4.198 (0.0134)	4.1358

those estimates are indeed below the true prices (2) the standard errors are significantly reduced after adding the martingale control $\mathcal{M}(P_E; \underline{\tau})$. For QMC methods we use 5000 Niederreiter sequences of dimension 100. In column 4 we see clearly that in most situations, except $S_0 = 80$ and 85, low-biaes QMC estimates are unreasonably greater than the true American prices. The striking part is that after adding hedging martingales, low-biased QMC+CV estimates shown in Column 5 are indeed below the true price. These results strongly indicate that the hedging martingale plays the role of a smoother for MC/QMC methods.

By using the dual method, Column 7 illustrates high-biased estimates and their standard errors (in parenthesis) obtained from MC+CV estimator related to Equation (12). Note that the dual formulation (11) of the American option price has naturally a martingale control embedded, so there is no need to discuss the case without hedging martingales. We observe that (1) those estimates are indeed above the true prices (2) the standard errors are as the same order of Column 3 with martingale control. For QMC+CV method we see all high-biased QMC+CV estimates are comparable with results obtained by MC+CV. This shows the reliability of QMC methods in evaluating high-biased estimates.

Because the complexity of American option pricing problems is high, we stress the smooth effect of hedging martingales by considering European option pricing problems under multifactor stochastic volatility models. For readers interested in the variance and error analysis for pricing American options by the primal (least squares method) and the dual method, please refer to [9].

3 Monte Carlo Pricing under Multi-factor Stochastic Volatility Models

Stochastic volatility models have been an important class of diffusions as an extend of the Black-Scholes model. Stochastic volatility models are convenient to capture some stylized facts appearing either from historical data or implied (or derivatives) data. See [1, 11] and references therein. Under stochastic volatility models, closed-form solutions for typical option pricing problems barely exist even for European options. Numerical methods become essential. In this section we describe muiti-factor stochastic volatility models and the construction of approximate hedging martingales by means of perturbation techniques.

Under a risk-neutral probability measure $I\!P^\star$ parametrized by the combined market price volatility premium (Λ_1, Λ_2), the multi-factor stochastic volatility model is defined by

$$
\begin{aligned}
dS_t &= rS_t dt + \sigma_t S_t dW_t^{(0)*}, \\
\sigma_t &= f(Y_t, Z_t), \\
dY_t &= \left[\frac{1}{\varepsilon}c_1(Y_t) + \frac{g_1(Y_t)}{\sqrt{\varepsilon}}\Lambda_1(Y_t, Z_t)\right] dt + \frac{g_1(Y_t)}{\sqrt{\varepsilon}}\left(\rho_1 dW_t^{(0)*} + \sqrt{1-\rho_1^2}\, dW_t^{(1)*}\right), \\
dZ_t &= \left[\delta c_2(Z_t) + \sqrt{\delta} g_2(Z_t)\Lambda_2(Y_t, Z_t)\right] dt \\
&\quad + \sqrt{\delta} g_2(Z_t)\left(\rho_2 dW_t^{(0)*} + \rho_{12} dW_t^{(1)*} + \sqrt{1-\rho_2^2-\rho_{12}^2}\, dW_t^{(2)*}\right),
\end{aligned}
\tag{14}
$$

where S_t is the underlying asset price process with a constant risk-free interest rate r. Its random volatility σ_t is driven by two stochastic processes Y_t and Z_t varying on the *time scales* ε and $1/\delta$, respectively. The vector $\left(W_t^{(0)*}, W_t^{(1)*}, W_t^{(2)*}\right)$ consists of three independent standard Brownian motions. Instant correlation coefficients ρ_1, ρ_2, and ρ_{12} satisfy $|\rho_1| \leq 1$ and $|\rho_2^2 + \rho_{12}^2| \leq 1$. The volatility function f is assumed to be smooth bounded. Coefficient functions of processes Y_t and Z_t, namely (c_1, g_1, Λ_1) and (c_2, g_2, Λ_2) are assumed to be smooth such that they satisfy the existence and uniqueness conditions for stochastic differential equations. Mean-reverting processes such as Ornstein-Uhlenbeck (OU) processes or square-root processes are typical examples to model driving volatility processes [12, 17]. Under this setup, the joint process (S_t, Y_t, Z_t) is Markovian.

Given the multi-factor stochastic volatility model (14), the price of a plain European option with the payoff function H and expiry T is defined by

$$
P^{\varepsilon,\delta}(t, x, y, z) = I\!E^\star{}_{t,x,y,z}\left\{e^{-r(T-t)}H(S_T)\right\}, \tag{15}
$$

where $I\!E^\star{}_{t,x,y,z}$ is a short notation for the expectation with respect to $I\!P^\star$ conditioning on the current states $S_t = x, Y_t = y, Z_t = z$. A basic Monte Carlo simulation estimates the option price $P^{\varepsilon,\delta}(0, S_0, Y_0, Z_0)$ at time 0 by the sample mean

$$
\frac{1}{N}\sum_{i=1}^{N} e^{-rT}H(S_T^{(i)}) \tag{16}
$$

where N is the total number of sample paths and $S_T^{(i)}$ denotes the i-th simulated stock price at time T. Variance reduction techniques are particularly important to accelerate the computing efficiency of the basic Monte Carlo pricing estimator (16). Next we briefly review the construction of a generic algorithm, i.e. martingale control variate method, recently proposed by Fouque and Han [8].

3.1 Construction of Martingale Control Variates

Assuming that the European option price $P^{\varepsilon,\delta}(t,S_t,Y_t,Z_t)$ is twice differentiable in state space and once differentiable in time, we apply Ito's lemma to its discounted price $e^{-rt}P^{\varepsilon,\delta}$, then integrate from time 0 to the maturity T. After canceling out the pricing partial differential equation in some non-martingale terms, the following martingale representation can be obtained [8]

$$P^{\varepsilon,\delta}(0,S_0,Y_0,Z_0) = e^{-rT}H(S_T) - \mathcal{M}_0(P^{\varepsilon,\delta}) - \frac{1}{\sqrt{\varepsilon}}\mathcal{M}_1(P^{\varepsilon,\delta}) - \sqrt{\delta}\mathcal{M}_2(P^{\varepsilon,\delta}), \tag{17}$$

where zero-centered martingales are given by

$$\mathcal{M}_0(P^{\varepsilon,\delta}) = \int_0^T e^{-rs}\frac{\partial P^{\varepsilon,\delta}}{\partial x}(s,S_s,Y_s,Z_s)f(Y_s,Z_s)S_s dW_s^{(0)*}, \tag{18}$$

$$\mathcal{M}_1(P^{\varepsilon,\delta}) = \int_0^T e^{-rs}\frac{\partial P^{\varepsilon,\delta}}{\partial y}(s,S_s,Y_s,Z_s)g_1(Y_s) d\tilde{W}_s^{(1)*}, \tag{19}$$

$$\mathcal{M}_2(P^{\varepsilon,\delta}) = \int_0^T e^{-rs}\frac{\partial P^{\varepsilon,\delta}}{\partial z}(s,S_s,Y_s,Z_s)g_2(Z_s) d\tilde{W}_s^{(2)*}, \tag{20}$$

where the Brownian motions are

$$\tilde{W}_s^{(1)*} = \rho_1 W_s^{(0)*} + \sqrt{1-\rho_1^2}W_s^{(1)*},$$
$$\tilde{W}_s^{(2)*} = \rho_2 W_s^{(0)*} + \rho_{12}W_s^{(1)*} + \sqrt{1-\rho_2^2-\rho_{12}^2}W_s^{(2)*}.$$

These martingales play the role of "perfect" controls for Monte Carlo simulations. Namely, if the martingales (18), (19), and (20) can be exactly computed, then one can just generate one sample path to evaluate the option price through Equation (17). Unfortunately the gradient components $\left(\frac{\partial P^{\varepsilon,\delta}}{\partial x}, \frac{\partial P^{\varepsilon,\delta}}{\partial y}, \frac{\partial P^{\varepsilon,\delta}}{\partial z}\right)$ of the option price appearing in the martingales is not possibly known in advance when the option price $P^{\varepsilon,\delta}$ itself is exactly what we want to estimate. However, one can choose an approximate option price to substitute $P^{\varepsilon,\delta}$ used in the martingales (18, 19, 20) and still retain martingale properties.

By an application of singular and regular perturbation techniques, the first order approximation derived in [12] is

$$P^{\varepsilon,\delta}(t,x,y,z) \approx P_{BS}(t,x;\bar{\sigma}(z)) \tag{21}$$

where $P_{BS}(t,x;\bar{\sigma}(z))$ denotes the solution of the Black-Scholes partial differential equation with the constant volatility $\bar{\sigma}(z)$ and the terminal condition $P_{BS}(T,x) = H(x)$. The z-dependent *effective volatility* $\bar{\sigma}(z)$ is defined as the square root of an averaging of the variance function f^2 with respect to the limiting distribution of Y_t :

$$\bar{\sigma}^2(z) = \int f^2(y,z)d\Phi(y) = \left\langle f^2(y,z)\right\rangle, \tag{22}$$

where $\Phi(y)$ denotes the invariant distribution of the fast varying process Y_t while setting the volatility premium Λ_1 as zero. We use the bracket to represent such average. In the OU case, we choose that $c_1(y) = m_1 - y$ and $g_1(y) = \nu_1\sqrt{2}$ with $\Lambda_1 = 0$ such that $1/\varepsilon$ is the rate of mean reversion, m_1 is the long run mean, and ν_1 is the long run standard deviation. Its invariant distribution Φ is normal with mean m_1 and variance ν_1^2. We refer to [12] for detailed discussions of such models.

Note that the approximate option price $P_{BS}(t,x;\bar{\sigma}(z))$ is independent of the variable y such that the term $\mathcal{M}_1(P_{BS})$ diminishes. Since the approximate martingale $\mathcal{M}_2(P_{BS})$ for (20) is small of order $\sqrt{\delta}$. Intuitively we can neglect this term as well. We then select the stochastic integral $\mathcal{M}_0(P_{BS})$ as the main control for the estimator (16) and formulate the following martingale control variate estimator:

$$\frac{1}{N}\sum_{i=1}^{N}\left[e^{-rT}H(S_T^{(i)}) - \mathcal{M}_0^{(i)}(P_{BS})\right]. \tag{23}$$

This is the approach taken by Fouque and Han [8], in which the proposed martingale control variate method is numerically superior to an importance sampling method in [7] for pricing European options. Under OU-type processes to model (Y_t, Z_t) in (14) with $0 < \varepsilon, \delta \ll 1$, the variance of the martingale control variate for European options is small of order ε and δ. This asymptotic result is shown in [8].

3.2 Variance and Error Analysis for MC/QMC

Based on the fact that random volatility is fluctuating around the long run mean of driving volatilities, we consider a simplified model which is helpful to explain the effect of martingale control as a smoother for MC/QMC methods. We assume that under the risk-neutral probability measure S_t^ϵ is a perturbed risky asset price defined by

$$dS_t^\varepsilon = rS_t^\varepsilon dt + \sigma_t^\varepsilon S_t^\varepsilon dW_t^*, \tag{24}$$

where the perturbed volatility around the constant σ is $\sigma_t^\varepsilon = \sigma + \varepsilon\, g_t$. We assume that g_t is a deterministic and bounded function such that σ_t^ε is positive and bounded away from 0. A European option, say call with the strike K, is defined by

$$P^\varepsilon\left(t, S_t^\varepsilon; \sigma^\varepsilon\right) = I\!E^\star{}_{t,S_t^\varepsilon}\left\{e^{-r(T-t)}(S_T^\varepsilon - K)^+\right\}, \tag{25}$$

where the averaged volatility is defined by $\bar{\sigma}^\varepsilon = \sqrt{\frac{1}{T-t}\int_t^T (\sigma_s^\varepsilon)^2 ds}$.

Lemma 1 *Let $\mathcal{N}(x)$ be the cumulative integral function and*

$$d_1(t,x;\sigma) = \frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Then there exists a constant C such that

$$\mathcal{N}(d_1(t,x;\bar{\sigma}^\varepsilon)) - \mathcal{N}(d_1(t,x;\sigma)) < C\varepsilon$$

for any $\varepsilon > 0, 0 \le t \le T$, and $x > 0$.

Proof: We observe that for $0 \le t < T$ and $x > 0$, $\mathcal{N}(d_1(t,x;\sigma))$ is a smooth function in σ and its partial derivative $\frac{\partial \mathcal{N}(d_1(t,x;\sigma))}{\partial x}$ is uniformly bounded. By the mean value theorem, there exists a σ' between σ and $\bar{\sigma}^\varepsilon$ such that

$$\begin{aligned}
&\mathcal{N}(d_1(t,x;\bar{\sigma}^\varepsilon)) - \mathcal{N}(d_1(t,x;\sigma)) \\
&= \frac{\partial \mathcal{N}(d_1(t,x;\sigma'))}{\partial x}\left(\bar{\sigma}^\varepsilon - \sigma\right) \\
&< C\frac{(\bar{\sigma}^\varepsilon)^2 - \sigma^2}{\bar{\sigma}^\varepsilon + \sigma} \\
&< C\,\varepsilon.
\end{aligned}$$

Here we abuse the notation C to indicate the existence of some constant.

Theorem 2 *(Variance Analysis) Suppose that the payoff function H is a call and the volatility process σ_t^ε is defined above. For any fixed initial state $(0, S_0^\varepsilon = x)$, there exist $\varepsilon > 0$ small enough and a positive C such that*

$$Var\left(e^{-rT}\left(S_T^\varepsilon - K\right)^+ - \mathcal{M}(P_{BS}; T)\right) \leq C\varepsilon.$$

Proof: Using the martingale representation

$$e^{-rT}(S_T^\varepsilon - K)^+ - P^\varepsilon(0, S_0^\varepsilon) = \int_0^T e^{-rt}\frac{\partial P^\varepsilon}{\partial x}(t, S_t^\varepsilon; \bar{\sigma}^\varepsilon)\sigma_t^\varepsilon S_t^\varepsilon dW_t^* \tag{26}$$

we obtain

$$\begin{aligned} & Var\left\{e^{-rT}(S_T^\varepsilon - K)^+ - \mathcal{M}(P_{BS}; T)\right\} \\ = \; & Var\left\{\int_0^T e^{-rt}\left(\frac{\partial P^\varepsilon}{\partial x}(t, S_t^\varepsilon; \bar{\sigma}^\varepsilon) - \frac{\partial P_{BS}}{\partial x}(t, S_t^\varepsilon; \sigma)\right)\sigma_t^\varepsilon S_t^\varepsilon dW_t^*\right\} \\ = \; & I\!\!E^\star\left\{\int_0^T \left(\mathcal{N}(d_1(t, S_t^\varepsilon; \bar{\sigma}^\varepsilon)) - \mathcal{N}(d_1(t, S_t^\varepsilon; \sigma))\right)^2 \left(\sigma_t^\varepsilon\right)^2 \left(e^{-rt}S_t^\varepsilon\right)^2 dt\right\}. \end{aligned}$$

The last equation is obtained from Ito's isometry theorem. By Lemma 1 and the fact that $I\!\!E^\star\left\{(e^{-rt}S_t^\varepsilon)^2\right\}$ is bounded [8]. Then we get $Var\left\{e^{-rT}(S_T^\varepsilon - K)^+ - \mathcal{M}(P_{BS}; T)\right\} \leq C\varepsilon$.

Remark: The financial interpretation of the martingale control term

$$\mathcal{M}(P_{BS}; T) = \int_0^T e^{-rt}\frac{\partial P_{BS}}{\partial x}(t, S_t^\varepsilon; \sigma)\sigma_t^\varepsilon S_t^\varepsilon dW_t^*$$

corresponds to the cumulative cost of a *delta* hedging strategy. From the risk management [18] viewpoint, daily hedge for derivatives are important for financial institutions in practice. Empirical studies [16] has suggested that the delta hedge under Black-Scholes model performs better than those strategies derived from complex pricing models. This empirical evidence is consistent to the effectiveness and robustness of our martingale control variate method, both theoretically and numerically discussed in this paper. In addition, this martingale control variate method can be easily extended to hitting time problems like barrier options and optimal stopping time problems like American options as discussed in [8].

From Equation (26), we see

$$e^{-rT}\left(S_T^\varepsilon - K\right)^+ - \mathcal{M}(P_{BS}; T) - P^\varepsilon(0, S_0^\varepsilon) = \int_0^T e^{-rt}\left(\frac{\partial P^\varepsilon}{\partial x}(t, S_t^\varepsilon; \bar{\sigma}^\varepsilon) - \frac{\partial P_{BS}}{\partial x}(t, S_t^\varepsilon; \sigma)\right)\sigma_t^\varepsilon S_t^\varepsilon dW_t^*. \tag{27}$$

This equation is very helpful to consider an error analysis for the QMC estimate of

$$I\!\!E^\star\left\{e^{-rT}\left(S_T^\varepsilon - K\right)^+ - \mathcal{M}(P_{BS}; T)\right\}.$$

We use Proinov bound [23] to analyze the error of martingale control variate estimator because this bound primarily requires the countinuity of the integrand. In the implementation of QMC methods, we first discretize the stochastic integral (27) by Euler scheme as a Riemann sum of normal variables, then define an integral problem over the hypercube space. The integrand is a countinuous function. Moreover since the integrand always contains the delta error $\frac{\partial P^\varepsilon}{\partial x}(t, S_t^\varepsilon; \bar{\sigma}^\varepsilon) - \frac{\partial P_{BS}}{\partial x}(t, S_t^\varepsilon; \sigma)$ which is of $\mathcal{O}(\varepsilon)$ as shown in Lemma 1. We therefore assure that the error of QMC estimates is small of $\mathcal{O}(\varepsilon)$.

From these analysis, one can see that the variance of a delta-hedged contract, $Var\left(e^{-rT}\left(S_T^\varepsilon - K\right)^+ - \mathcal{M}(P_{BS};T)\right)$ is small of order ε, so is the error of QMC estimate, while the variance of unhedged (or naked) contact is of order 1. These theoretical results are confirmed by those *jumps* of huge variance reduction illustrated in the following numerical results, generated from the one-factor Heston model and multi-factor stochastic volatility model.

3.3 One-factor SV Model: Heston Model

A widely applied mean-reverting process is Heston model [17] which incorporates a square-root diffusion to model the random volatility process. Heston model is an one-factor stochastic volatility model so that it can be viewed as a special case of our multi-factor SV model (14) after the square-root process is rescaled in the following way:

$$dY_t = \frac{1}{\varepsilon}\left(\theta - Y_t\right)dt + \frac{2\kappa}{\sqrt{\varepsilon}}\sqrt{Y_t}\left(\rho_1 dW_t^{(0)*} + \sqrt{1-\rho_1^2}dW_t^{(1)*}\right) \tag{28}$$

and $Z_t = 0$ for all $t \geq 0$. The volatility function is defined by $f(y,z) = \sqrt{y}$. The driving volatility (28) admits a limiting distribution while ε approaches zero. It is known as the central Chi-square random variable

$$\kappa^2 \chi_\nu(0) \tag{29}$$

with the degree of freedom $\nu = \theta/\kappa^2$. Its density function is denoted by

$$\phi(x) = \frac{1}{\kappa^2}\frac{1}{2^{\nu/2}\Gamma(\frac{\nu}{2})}\left(\frac{x}{\kappa^2}\right)^{\nu/2-1}\exp\left(\frac{-x}{2\kappa^2}\right) \tag{30}$$

for $x > 0$, where $\Gamma(\cdot)$ is the Gamma function.
Under the risk-neutral pricing probability measure, the European option price is

$$P^\varepsilon(t,x,y) = I\!E^\star{}_{t,x,y}\left\{e^{-r(T-t)}H(S_T)\right\}.$$

Following the singular perturbation expansion [12], one can derive an asymptotic result

$$P^\varepsilon(t,x,y) \approx P_{BS}(t,x;\bar{\sigma}) \tag{31}$$

when $\varepsilon \ll 1$. As defined in (22) but without the z-component, the effective volatility $\bar{\sigma}$ is defined from $\bar{\sigma}^2 = \int f(y)^2\phi(y)dy$. By change of variable $z = \frac{y}{\kappa^2}$, the effective variance

$$\begin{aligned}\bar{\sigma}^2 &= \frac{\kappa^2}{2^{\nu/2}\Gamma(\frac{\nu}{2})}\int_0^\infty z^{\nu/2}e^{-z/2}dz \\ &= \kappa^2\nu \\ &= \theta\end{aligned} \tag{32}$$

is deduced. We readily observe that the effective variance $\bar{\sigma}^2$ is exactly the long-run mean θ of the square root process (28).
Under the one-factor Heston model specified in Table 2 and 3, the effective variance is $\bar{\sigma}^2 = \theta = 0.3$. Several variance ratios are illustrated in Table 4 and show significant reduction gained from the martingale control variate estimator (23) versus the basic Monte Carlo estimator (16). Notice that the variance reduction ratios performs better when ε are either small (close to 1/50) or large (close to 1/0.1). In the small ε regime, one can use singular perturbation analysis to argue that the approximate

Table 2: Parameters used in the rescaled Heston's model (28).

r	θ	κ	ρ	$f(y)$
5%	0.09	0.1	-0.3	$\sqrt{y}$

Table 3: Initial conditions and European call option parameters.

$\$S_0$	Y_0	$\$K$	T years
50	0.09	50	1

option price $P_{BS}(t, x; \bar{\sigma})$ is close to the true option price. Hence the variance reduction is effective. In the large ε regime, the process Y_t is slowly moving around $Y_0 = 0.3$. This initial value is particularly chosen so that the driving volatility process $\sqrt{Y_t}$ is close to the effective volatility $\bar{\sigma} = \sqrt{\theta}$ when $0 \leq t \ll \varepsilon$. Therefore, one can apply regular perturbation argument to explain the accuracy of the option price approximation and the variance reduction effect. Both singular and regular perturbation analysis can be found in [12].

3.4 Generic Two-Factor SV Model

In this section, we will compare efficiencies for pricing European call option using control variates technique developed in the previous section, combined with Monte Carlo and quasi-Monte Carlo methods.

We assume that the underlying asset S is given by (1). In our computations, we use C++ on Unix as our programming language. The pseudo random number generator we used is $ran2()$ in [26]. In our comparisons, the sample sizes for MC method are 10240, 20480, 40960, 81920, 163840, and 327680, respectively; and those for Sobol' sequence related methods are 1024, 2048, 4096, 8192, 16384, and 32768, respectively, each with 10 random shifts; and the sample sizes for L'Ecuyer's type lattice rule points (LTLRP for short) related methods are 1021, 2039, 4093, 8191, 16381, and 32749, respectively, and again, each with 10 random shifts.

In the following examples, we divide the time interval $[0, T]$ into $m = 128$ subintervals. In Table 2, the first column labeled as N indicates the number of Monte Carlo simulations or the Quasi-Monte Carlo points. The second column labeled as MC indicates the option price estimates (standard errors in the parenthesis) based on the basic MC estimator (16). All rest columns record variance reduction ratios between many specific MC/QMC methods and the basic MC estimates. For example, the third column labeled as MC+CV indicates the variance reduction ratios as the squares of the standard errors in the second column versus the standard errors obtained from the martingale control variate estimation (23). The fourth column labeled as Sobol' indicates the variance reduction rations as the squares of the standard errors in the second column versus the sandard errors obtained from the estimation (16) by randomized Sobol' sequence.
As an example we consider a European call option with payoff $H(S_T) = \max(S_T - K, 0) = (S_T - K)^+$, where K is the strike price. We take input variables and parameters as follows: $S_0 = \$55$, $K = \$50, r = 0.1 = 10\%$, $T = 1$ year, $m_1 = m_2 = -0.8$, $\nu_1 = 0.5$, $\nu_2 = 0.8$, $\rho_1 = \rho_2 = -0.2$, $\rho_{12} = 0.0$, $y_0 = z_0 = -1.0$, where we specify driving volatility processes as $c_1(y) = m_1 - y, c_2(z) = m_2 - z, \Lambda_1(y, z) = \Lambda_2(y, z) = 0, g_1(y) = \nu_1, g_2(z) = \nu_2$. For ε and δ, we take $\varepsilon = 1/50$, $\delta = 0.5$. The results are listed in Tables 2 and 3, where MC+CV stands for Monte Carlo method using control variate technique,

Table 4: Comparison of simulated European call option prices and their standard errors for different time scale ε in (28). Let "MC price" denote the sample mean obtained from the basic Monte Carlo simulation; " MC+ CV price" denote the sample mean obtained from the Monte Carlo simulation using martingale control variates. Their standard errors are shown in parenthesis next to the option price estimates. Let V^{MC} denote the sample variance obtained from the basic Monte Carlo method, and $V^{MC+CV}(P_{BS}(\bar{\sigma}))$ denote the sample variance computed from the martingale control variates method with the option price approximation $P_{BS}(\bar{\sigma})$ in (31).

ε	MC price (std err)	MC+CV price (std err)	$V^{MC}/V^{MC+CV}(P_{BS}(\bar{\sigma}))$
1/50	7.0771 (0.1080)	7.0883 (0.0064)	287
1/25	6.9862 (0.1072)	7.0646 (0.0069)	244
1/10	7.1530 (0.1085)	7.0466 (0.0079)	187
1	7.1588 (0.1065)	7.0244 (0.0096)	122
5	7.0064 (0.1059)	7.0236 (0.0092)	132
10	7.0785 (0.1090)	7.0577 (0.0080)	185
25	7.1145 (0.1101)	7.0729 (0.0068)	266
50	6.9737 (0.1095)	7.0737 (0.0063)	298

Sobol+BB means the quasi-Monte Carlo method using Sobol' sequence with Brownian bridge sampling technique, LTLRP for QMC method using L'Ecuyer type lattice rule points, etc.

Table 5: Comparison of simulated European call option values and variance reduction ratios for $\varepsilon = 1/50$, $\delta = 0.5$

N	MC	MC+CV	Sobol'	Sobol+CV	Sobol+BB	Sobol+CV+BB
1024	11.839(0.126)	45.8	5.0	339.3	2.6	129.3
2048	11.837(0.090)	48.0	2.3	304.8	4.0	138.4
4096	11.862(0.064)	48.4	1.8	124.8	2.5	158.4
8192	11.804(0.045)	47.4	2.3	124.0	2.9	148.4
16384	11.816(0.032)	47.1	1.4	176.1	7.7	115.5
32768	11.857(0.022)	48.1	1.7	235.9	4.5	479.9

Table 5: Comparison of simulated European call option values and variance reduction ratios for $\varepsilon = 1/50$, $\delta = 0.5$ (continued).

N	LTLRP	LTLRP+CV	LTLRP+BB	LTLRP+CV+BB
1021	2.0	75.5	7.3	687.9
2039	3.1	135.1	7.0	298.5
4093	3.1	143.9	2.2	140.1
8191	4.2	347.8	4.9	286.0
16381	3.1	227.9	7.8	94.8
32749	6.4	728.7	15.1	741.6

Table 6: Comparison of time (in seconds) used in the simulation of the above European option

N	MC	MC+CV	Sobol'	Sobol+CV	Sobol+BB	Sobol+CV+BB
1024	7	10	7	9	7	12
2048	13	19	13	17	13	18
4096	26	40	26	35	28	39
8192	54	82	56	70	56	78
16384	109	167	107	139	107	157
32768	225	316	222	301	218	318

Table 6: Comparison of time (in seconds) used in simulation of the above European option (continued).

N	LGLP	LGLP+CV	LGLP+BB	LGLP+CV+BB
1021	5	9	7	11
2039	11	17	13	21
4093	24	34	27	40
8191	45	72	56	78
16381	90	167	106	160
32749	184	293	219	311

From Table 5, we observed the following facts. Using the control variate technique, the variance reduction ratios are around 48 for pseudo-random sequences. Without control variate, both Sobol' sequence and L'Ecuyer type lattice rule points, even combined with Brownian bridge sampling technique, the variance reduction ratios are only a few times better than the MC sampling at most. However, when combined with control variate, the variance reduction ratios for the Sobol' sequence vary from about 124 to 339 for Sobol'+CV and from 115 to 480 for Sobol'+CV+BB; and the variance reduction ratios for the L'Ecuyer type lattice rule points range from about 75 to 729 for LTLRP+CV and from 94 to 742 for LTLRP+CV+BB. This implicitly indicates that the new controlled payoff $e^{-rT}\left(S_T-K\right)^+ - \mathcal{M}_0(P_{BS})$ is smoother than the original call payoff $e^{-rT}\left(S_T-K\right)^+$. It can be easily seen that under the Black-Scholes model with the constant volatility σ, the controlled payoff is exactly equal to the Black-Scholes option price $P_{BS}(0,S_0;\sigma)$, which is a constant so as a smooth function; while the original call payoff function is only continuous and even not differentible.
Another interesting observation is that the variance reduction ratios do not always increase when the two low-discrepancy sequences are combined with control variate and Brownian bridge sampling, compared with when they are combined with control variate without Brownian bridge sampling.

Regarding time used in simulations, from Table 6 we observed that the time differences among methods without control variates are not significant, but the time differences between methods with and without control variates are not ignorable. Similar conclusions are true regarding time used in simulations for other cases.

4 Delta Estimation

Estimating the sensitivity of option prices over state variables and model parameters are important for risk management. In this section we consider only the first-order partial derivative of option price with respect to the underlying risk asset price, namely *delta*. We adopt (1) pathwise differentiation with control variate method or importance sampling, and (2) central difference approximation to formula our problems. As in previous sections, we can construct martingale control variates for Monte Carlo simulations and a combination of martingale control variates with Sobol sequence in randomized QMC method.

By pathwise differentiation (see [14] for instance), the chain rule can be applied to the payoff $(S_T - K)^+$ in (15) so that

$$\frac{\partial P^{\varepsilon,\delta}}{\partial S_0}(0, S_0, Y_0, Z_0) = I\!E^\star \left\{ e^{-rT} \mathbf{I}_{\{S_T > K\}} \frac{\partial S_T}{\partial S_0} \mid S_0, Y_0, Z_0 \right\}$$

is obtained. Since

$$e^{-rT} \frac{\partial S_T}{\partial S_0} = e^{\int_0^T \sigma_t dW_t^{(0)*} - \frac{1}{2}\int_0^T \sigma_t^2 dt} \tag{33}$$

is an exponential martingale, one can construct a $I\!P^\star$-equivalent probability measure $\tilde{P}$ by Girsanov Theorem. As a result, under the new measure $\tilde{P}$ the delta $\frac{\partial P^{\varepsilon,\delta}}{\partial S_0}(0, S_0, Y_0, Z_0)$ has a probabilistic representation of the digital-type option

$$P_D^{\varepsilon,\delta}(0, S_0, Y_0, Z_0) := \frac{\partial P^{\varepsilon,\delta}}{\partial S_0}(0, S_0, Y_0, Z_0) = \tilde{E} \left\{ \mathbf{I}_{\{S_T > K\}} \mid S_0, Y_0, Z_0 \right\}, \tag{34}$$

where the dynamics of S_t must follow

$$dS_t = \left(r + \sigma_t^2\right) S_t dt + \sigma_t S_t d\tilde{W}_t^{(0)}, \tag{35}$$

with $\tilde{W}^{(0)}$ being a standard Brownian motion under $\tilde{P}$. The dynamics of Y_t and Z_t will change according to the drift change of $W_t^{(0)}$.

Remark: When volatility is constant, denoted by σ, then it is easy to derive a closed-form solution $P_D := E\left\{\mathbf{I}(S_T > K) | S_0 = x\right\} = \mathcal{N}(d)$, where $d = \frac{\ln(S_0/K) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$.

4.1 Martingale Control Variate Method

Following the same argument of option price approximation, or see Appendix in [8], the digital call option $P_D^{\varepsilon,\delta}(0, S_0, Y_0, Z_0)$ admits the *homogenized* approximation $\bar{P}_D(\bar{S}_0, Z_0) := \bar{E}\left\{ \mathbf{I}_{\{\bar{S}_T > K\}} \mid \bar{S}_0 = S_0, Z_0 \right\}$, where the "homogenized" stock price $\bar{S}_t$ satisfies

$$d\bar{S}_t = \left(r + \bar{\sigma}^2(Z_t)\right) \bar{S}_t dt + \bar{\sigma}(Z_t) \bar{S}_t d\bar{W}_t^{(0)}$$

with $\bar{W}_t^{(0)}$ being a standard Brownian motion [11]. In fact, the homogenized approximation $\bar{E}\left\{ \mathbf{I}_{\{\bar{S}_T > K\}} \mid \bar{S}_t, Z_t \right\}$ is a probabilistic representation of the homogenized "delta", $\frac{\partial P_{BS}}{\partial x}$, where P_{BS} defined in Section **??**. The martingale control for the digital call option price (34) can be constructed as in Section **??** so that similar martingale control variate estimator is obtained as

$$\frac{1}{N} \sum_{k=1}^{N} \left[\mathbf{I}_{\left\{ S_T^{(k)} > K \right\}} - \mathcal{M}^{(k)}(\bar{P}_D, T) \right]. \tag{36}$$

Numerical results of variance reduction by MC/QMC to estimate *delta* can be found in Table 7. All model parameters, initial conditions and mean-reverting rates are chosen the same in previous section except $r = 0.05; \sigma_t = \exp((Y_t + Z_t)/2))$. The number of time discretization is 100 and the total number of replications is 10,000.

Table 7: Comparison of variance reduction ratios to estimate the Δ of an European call option by martingale control variate method and Importance Sampling method

K	MC	MC+CV	Sobol	Sobol+CV	MC+IS	Sobol + IS
20	0.9909 (0.00094964)	4.0442	1.0456	3.5274	8.2146	8.7948
50	0.7178 (0.0045)	7.3590	1.0072	7.4577	2.2454	2.2603
100	0.1698 (0.0038)	6.0083	1.0066	5.7454	2.5965	2.4808
200	0.0076 (0.000868)	3.1060	0.9174	3.8311	5.222	4.6259
300	0.0008 (0.000283)	3.1461	0.8183	4.6609	6.4951	4.4322

4.2 Importance Sampling Method

As seen from the derivation in (34), the delta of a European option is a probability such as a default or survival probability. There have been some extensive studies on estimating these probabilities by importance sampling techniques particularly in the presence of rare events. See references [5, 14] for possible techniques and theories to handle these problems.

The basic idea of an "efficient" importance sampling is the following:
(1) apply the change of measure to construct a unbiased estimator under the new probability measure, so that a rare event under the original probability measure is no longer rare under the new measure,
(2) the second moment of the new estimator has certain optimal property, i.e., in some scaling regime, the square of the probability is at the same order of the second moment. Next we introduce the efficient importance sampling method, developed by Han and Vestal [15], which is applicable for problems related to credit risk.

Consider a change of measure for the prominent Brownian motion

$$P_D^{\varepsilon,\delta}(0, S_0, Y_0, Z_0) = \hat{E}\left\{\mathbf{I}\left(S_T > K\right)\, Q_T | S_0, Y_0, Z_0\right\}, \tag{37}$$

where $Q_T = \exp\left(h\,\hat{W}_T - h^2/2\,T\right)$ is a particularly chosen Radon-Nykodymn derivative. In general the exponent should be in integral form, but we choose the integrand $h = \frac{\log(S_0/K)+(r+\bar{\sigma}^2)\,T}{\bar{\sigma}\,T}$ as an effective drift-change parameter so that $\hat{W}_t^{(0)} = W_t^{(0)} + h\,t$ is a standard Brownian motion under the new probability measure. For detailed properties such as asymptotic analysis can be found in [15]. According to (37), the importance sampling estimator is defined by

$$\frac{1}{N}\sum_{k=1}^{N} \mathbf{I}\left(S_T^{(k)} > K\right)\, Q_T^{(k)}, \tag{38}$$

where the underlying risky-asset process is governed by

$$dS_t \;=\; \left(r + \sigma_t^2 - \sigma_t\,h\right) S_t dt + \sigma_t S_t d\hat{W}_t^{(0)},$$

and the driving volatility processes should be changed correspondingly. Note that if the probability of the event $S_T > K$ is not rare, we can consider its complement, estimate $E\left\{\mathbf{I}(S_T < K)\right\}$, then subtract it by 1 in order to obtain better performance in variance reduction.

4.3 Central Difference Method

Another way to approximate the *delta* is by central difference. A small increment $\Delta_S > 0$ is chosen to discretize the partial derivative by

$$P_D^{\varepsilon,\delta} = \frac{\partial P^{\varepsilon,\delta}}{\partial S_0} \approx \frac{P^{\varepsilon,\delta}(0, S_0 + \Delta_S/2, Y_0, Z_0) - P^{\varepsilon,\delta}(0, S_0 - \Delta_S/2, Y_0, Z_0)}{\Delta_S}.$$

Each European option price corresponding to different initial stock price $S_0 + \Delta_S/2$ and $S_0 - \Delta_S/2$ respectively is computed by the martingale control variate method with MC/QMC. Numerical results of variance reduction by MC/QMC to estimate *delta* can be found in Table 9.

Table 8: Comparison of variance reduction ratios to approximate the Δ of an European call option by Central Difference Scheme

N	MC	MC+CV	Sobol	Sobol+CV
1024	0.8490(0.00507)	15.8	2.6	25.7
2048	0.8354(0.00357)	14.7	1.8	7.9
4096	0.8378(0.00253)	14.7	2.8	21.6
8192	0.8355(0.00179)	14.9	5.5	13.6
16384	0.8381(0.00126)	14.7	4.8	19.4
32768	0.8384(0.00090)	14.7	2.8	15.2

In summary, unlike the European call option cases, QMC method doesn't make a great benefit in variance reduction in both pathwise differentiation and central difference approximation. This is because the regularity of the *delta* function is worse than the call function. In contrast, the importance sampling methods can be useful for estimating delta, particularly in the case of rare events, e.g. small of large K. Because the digital payoff is not even continuous, we see that combing our importance sampling estimator with QMC doesn't provide better performance in variance reduction.

Regarding to the computing time in our numerical experiments, the importance sampling estimator (38) takes about half of the martingale control variate (36). This is because the martingale control $\mathcal{M}(P_{BS};T)$ is a pathwise control, which requires a series of (approximate) deltas to integrate along each simulated price trajectory. But the only unknown parameter h within Q_T of (38) is determined *a priori* so that the computing time of our importance sampling estimator is much less and actually comparable with the basic Monte Carlo estimator.

5 Conclusion

Using (randomized) QMC methods to estimate high dimensional problems of option pricing may not be effective as shown in many examples presented. Based on the delta hedging strategy in trading financial derivatives, the value process of a hedging portfolio is considered as a martingale control in order to reduce the risk (replication error) of traded derivatives. For MC/QMC methods the role of the martingale control is a smoother so that significant variance reduction ratios can be obtained. We give an explanation of the effect of the smoother under a perturbed volatility model. When the payoff function degenerates such as the delta estimation problem, the performance of the martingale control variate method decreases as well. An alternative importance sampling method is proposed to gain significant variance reduction when the event estimated is rare.

Acknowledgement: We are grateful to a financial support by the Institute for Mathematics and its Applications (IMA) at the University of Minnesota, USA. The work of C-H Han is supported by NSC grant 95-2115-M-007-017-MY2, Taiwan; and the work of Y. Lai was partially supported by an Natural Sciences and Engineering Research Council (NSERC) of Canada grant.

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