

# On the $\mathbb{F}_2$ -linear relations of Mersenne Twister pseudorandom number generators

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## Abstract

Sequence generators obtained by linear recursions over the two-element field  $\mathbb{F}_2$ , i.e.,  **$\mathbb{F}_2$ -linear generators**, are widely used as pseudorandom number generators. For example, the Mersenne Twister MT19937 is one of the most successful applications. An advantage of such generators is that we can assess them **quickly** by using theoretical criteria, such as the dimension of equidistribution with  $v$ -bit accuracy. To compute these dimensions, several **polynomial-time lattice reduction algorithms** have been proposed **in the case of  $\mathbb{F}_2$ -linear generators**.

In this paper, **in order to assess non-random bit patterns in dimensions that are higher than the dimension of equidistribution with  $v$ -bit accuracy**, we focus on the relationship between points in the Couture–L’Ecuyer dual lattices and  $\mathbb{F}_2$ -linear relations on the most significant  $v$  bits of output sequences, and consider a new figure of merit  $N_v$  based on the minimum weight of  $\mathbb{F}_2$ -linear relations whose degrees are minimal for  $v$ . **Next, we numerically** show that MT19937 has low-weight  $\mathbb{F}_2$ -linear relations in dimensions higher than 623, and show that some output vectors with specific lags are rejected or have small  $p$ -values in birthday spacings tests. We also report that some variants of Mersenne Twister, such as WELL generators, are significantly improved from the perspective of  $N_v$ .

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## 1. Introduction

The *Mersenne Twister* MT19937 is a pseudorandom number generator developed by Matsumoto and Nishimura [25]. This generator has the following advantages: (i) Its generation speed is very fast; (ii) it has a large period of  $2^{19937} - 1$ ; (iii) it has high-dimensional equidistribution property (i.e., 623-dimensionally equidistributed). The algorithm of the Mersenne Twister is based on a linear recurrence relation over the two-element field  $\mathbb{F}_2$ . For such a class of generators (so-called  $\mathbb{F}_2$ -linear generators), the following two quality criteria are well-known: (i) the dimension of equidistribution with  $v$ -bit accuracy  $k(v)$  for each  $v$  ( $1 \leq v \leq w$  and  $w$  indicates the word size of machines) is large enough and (ii) the number  $N_1$  of nonzero terms in a characteristic polynomial is large enough (see Section 3 for details). From this perspective, MT19937 was much superior to all other classical pseudorandom number generators when it appeared, and it has since been the most widely used generator in Monte Carlo simulations.

However, it may not be sufficient to use only the above two criteria for assessing pseudorandom number generators. A motivation of this paper is to detect non-random bit patterns in dimensions that are higher than  $k(v)$ . In such dimensions, we always have certain bits of output whose sum becomes 0 over  $\mathbb{F}_2$ , in the case of  $\mathbb{F}_2$ -linear generators. Such relations are said to be  $\mathbb{F}_2$ -linear relations, but they are usually hidden. In fact, when there exist  $\mathbb{F}_2$ -linear relations whose numbers of terms are small (e.g, the number of terms  $\leq 6$ ) in a low-dimensional projection, it is likely to observe some deficiencies for small sample sizes. As a previous work, Matsumoto and Nishimura [26] gave theoretical justification for this fact in terms of coding theory. Thus, we should avoid such  $\mathbb{F}_2$ -linear relations.

In this paper, in order to assess non-random bit patterns, we focus on the  $\mathbb{F}_2$ -linear relations whose degrees are minimal for the most significant  $v$  bits given, which correspond to the shortest vectors in the Couture–L’Ecuyer dual lattice [2] for computing  $k(v)$ , and we develop a new figure of merit  $N_v$  based on the minimum number of terms of the  $\mathbb{F}_2$ -linear relations. The value  $N_v$  can be considered as a quality criterion in dimensions higher than  $k(v)$  and as a multi-dimensional generalization of  $N_1$ . Next, we assess the Mersenne Twister MT19937 and its variants in terms of  $N_v$ ’s, and show that  $N_v$ ’s of MT19937 are small, relative to the WELL generators [33]. We also report that MT19937 has some deviations in birthday spacings test [21, 11, 13, 14] for non-successive output values, in accordance with the existence of  $\mathbb{F}_2$ -linear

relations. As far as the author knows, there has been no report of deviations of MT19937 except for linear complexity tests and poor initialization until now (see [14, 27, 33]).

The rest of this paper is organized as follows. In Section 2, we recall the framework of  $\mathbb{F}_2$ -linear generators. In Section 3, we explain the terminologies of  $k(v)$  and  $N_1$ . In Section 4, we briefly survey the Couture–L’Ecuyer dual lattice method [2] for computing  $k(v)$ , which will be used in later sections. In Section 5, we show the relationship between  $\mathbb{F}_2$ -linear relations and points in the Couture–L’Ecuyer dual lattices, define a new figure of merit  $N_v$ , and give an algorithm for computing  $N_v$  using Gray codes. In Section 6, we analyze the Mersenne Twister MT19937 in terms of both  $N_v$ ’s and  $\mathbb{F}_2$ -linear relations. In Section 7, we report some deviations of MT19937 in the birthday spacings tests with selected lags. Section 8 is devoted to the analysis of other  $\mathbb{F}_2$ -linear generators, such as the WELL generators. We also introduce a new tempering parameter of MT19937 in order to optimize  $k(v)$  as an improvement of the author’s previous work [8]. Our conclusions are presented in Section 9.

## 2. $\mathbb{F}_2$ -linear generators

Mersenne Twister generators belong to a general class of pseudorandom number generators based on the following matrix recurrences over the two-element field  $\mathbb{F}_2 := \{0, 1\}$ :

$$\mathbf{x}_i := \mathbf{A}\mathbf{x}_{i-1}, \quad (1)$$

$$\mathbf{y}_i := \mathbf{B}\mathbf{x}_i, \quad (2)$$

$$u_i := \sum_{l=1}^w y_{i,l-1}2^{-l} = 0.y_{i,0}y_{i,1}\cdots y_{i,w-1}, \quad (3)$$

where  $\mathbf{x}_i = {}^t(x_{i,0}, \dots, x_{i,p-1}) \in \mathbb{F}_2^p$  is the  $p$ -bit *state vector* at step  $i$  ( *$t$  denotes the transpose of a vector*);  $\mathbf{y}_i = {}^t(y_{i,0}, \dots, y_{i,w-1}) \in \mathbb{F}_2^w$  is the  $w$ -bit *output vector* at step  $i$ ;  $p$  and  $w$  are positive integers ( *$w$  indicates the word size of machines*),  $\mathbf{A}$  is a  $p \times p$  *transition matrix* with elements in  $\mathbb{F}_2$ , and  $\mathbf{B}$  is a  $w \times p$  *output transformation matrix* with elements in  $\mathbb{F}_2$ . We identify a  $w$ -dimensional vector  $\mathbf{y}_i$  and a binary expansion  $u_i$  with an unsigned  $w$ -bit binary integer and a real number in the interval  $[0, 1)$ , respectively. The output sequence  $\{u_i\}$  is supposed to imitate independent random variables



we often use the terminology of the *dimension of equidistribution with  $v$ -bit accuracy*. Let  $\Psi_k$  be the multiset of  $k$ -dimensional vectors, from all possible initial states  $\mathbf{x}_0$ :

$$\Psi_k := \{(u_0, \dots, u_{k-1}) \mid \mathbf{x}_0 \in \mathbb{F}_2^p\} \subset [0, 1)^k.$$

Let us **divide** axis  $[0, 1)$  into  $2^v$  pieces. Then,  $[0, 1)^k$  is divided into  $2^{kv}$  cubic cells of equal size. The generator is said to be  *$k$ -dimensionally equidistributed with  $v$ -bit accuracy* if each cell contains exactly the same number of points of  $\Psi_k$ , i.e.,  $2^{p-kv}$  points. The largest value of  $k$  with this property is called the *dimension of equidistribution with  $v$ -bit accuracy*, denoted by  $k(v)$ . As a criterion of uniformity, larger  $k(v)$  for each  $1 \leq v \leq w$  is desirable (see [37]). We have a trivial upper bound  $k(v) \leq \lfloor p/v \rfloor$ . The gap  $d(v) := \lfloor p/v \rfloor - k(v)$  is called the *dimension defect* at  $v$ , and their sum  $\Delta := \sum_{v=1}^w (\lfloor p/v \rfloor - k(v))$  is called the *total dimension defect*. If  $\Delta = 0$ , the generator is said to be *maximally equidistributed*. Note that MT19937 has  $\Delta = 6750$ .

As a secondary requirement, we may consider whether the number  $N_1$  of nonzero coefficients for  $P(z)$  is large enough or not (see [1, 39]). For example, **generators for which  $P(z)$  is a trinomial or a pentanomial fail statistical tests [19, 24, 26], so that they should be avoided.** MT19937 has  $N_1 = 135$ , **so that it has long-lasting impact from poor initialization, e.g.,  $\mathbf{x}_0$  that contains only a few bits set to 1 (see [33]).** WELL generators [33] have  $N_1 \approx p/2$ , and overcome the above drawback. **We note that such phenomena of initialization will not happen practically if we take care of initialization routines.**

#### 4. Lattice structures

We briefly recall a lattice method for computing  $k(v)$  in terms of the Couture–L’Ecuyer dual lattices **by following [2, 12].** Recently, the faster algorithms [9, 10] using the original lattices [3, 36] (not the dual lattices) were proposed. The aim of this paper is to extract other information from the dual lattices, so that we revisit the Couture–L’Ecuyer dual lattices.

Let  $K$  denote the formal power series field  $K := \mathbb{F}_2((z^{-1})) = \{\sum_{i=i_0}^{\infty} a_i z^{-i} \mid a_i \in \mathbb{F}_2, i_0 \in \mathbb{Z}\}$ . For  $a(z) = \sum_{i=i_0}^{\infty} a_i z^{-i} \in K$ , **we put a standard norm by  $|a(z)| := \max\{-i \in \mathbb{Z} \mid a_i \neq 0\}$  if  $a(z) \neq 0$  and  $|a(z)| := -\infty$  if  $a(z) = 0$ .** For a vector  $\mathbf{a}(z) = {}^t(a_0(z), a_1(z), \dots, a_{v-1}(z)) \in K^v$ , we define its *norm* (or its *length*) by  $\|\mathbf{a}(z)\| := \max_{1 \leq i \leq v} |a_{i-1}(z)|$ .

A subset  $L \subset K^v$  is said to be an  $\mathbb{F}_2[z]$ -*lattice* if there exists a  $K$ -linear basis  $\{\mathbf{v}_1(z), \mathbf{v}_2(z), \dots, \mathbf{v}_v(z)\}$  of  $K^v$  such that  $L$  is their span over  $\mathbb{F}_2[t]$ , i.e.,

$L = \langle \mathbf{v}_1(z), \mathbf{v}_2(z), \dots, \mathbf{v}_v(z) \rangle_{\mathbb{F}_2[t]}$ . Such a set of vectors is called a *basis* of  $L$ . A reduced basis  $\{\mathbf{v}_1(z), \dots, \mathbf{v}_v(z)\}$  is defined as follows: (i)  $\mathbf{v}_1(z)$  is a nonzero shortest vector in  $L$ , and (ii) for  $l = 2, \dots, v$ ,  $\mathbf{v}_l(z)$  is a shortest vector among the set of vectors  $\mathbf{v}(z)$  in  $L$  such that  $\mathbf{v}_1(z), \dots, \mathbf{v}_{l-1}(z), \mathbf{v}(z)$  are linearly independent over  $K$ . It is not unique, but the numbers  $\nu_l := \|\mathbf{v}_l(z)\|$  ( $l = 1, \dots, v$ ) are uniquely determined by a given lattice  $L$ , and  $\nu_1, \dots, \nu_v$  are called the *successive minima* of  $L$  (see [20]).

We consider an  $\mathbb{F}_2$ -linear generator. For a given nonzero initial state  $\mathbf{x}_0 \in \mathbb{F}_2^p$ , we define the following formal power series  $G_{l-1}(z)$  of  $l$ th bits of the integer output (i.e.,  $y_{0,l-1}, y_{1,l-1}, y_{2,l-1}, \dots$ ):

$$G_{l-1}(z) := \sum_{i=0}^{\infty} y_{i,l-1} z^{-i-1} = y_{0,l-1} z^{-1} + y_{1,l-1} z^{-2} + y_{2,l-1} z^{-3} + \dots \in \mathbb{F}_2((z^{-1})).$$

Note that  $G_{l-1}(z)$  has a rational form  $G_{l-1}(z) = h_{l-1}(z)/P(z)$ , where  $h_{l-1}(z) \in \mathbb{F}_2[z]$  and  $\deg h_{l-1}(z) < \deg P(z)$ . If  $P(z)$  is irreducible, let  $h_0^{-1}(z)$  be a polynomial that is a multiplicative inverse to  $h_0(z)$  modulo  $P(z)$ . We set  $\bar{h}_{l-1}(z) := h_0^{-1}(z)h_{l-1}(z) \bmod P(z)$  ( $2 \leq l \leq v$ ). We consider the following vectors:  $\mathbf{w}_1(z) := {}^t(P(z), 0, 0, \dots, 0)$ ,  $\mathbf{w}_2(z) := {}^t(-\bar{h}_1(z), 1, 0, \dots, 0)$ ,  $\mathbf{w}_3(z) := {}^t(-\bar{h}_2(z), 0, 1, \dots, 0)$ ,  $\dots$ ,  $\mathbf{w}_v(z) := {}^t(-\bar{h}_{v-1}(z), 0, 0, \dots, 1)$ , and construct an  $\mathbb{F}_2[z]$ -lattice  $\mathcal{L}_v^* := \langle \mathbf{w}_1(z), \dots, \mathbf{w}_v(z) \rangle_{\mathbb{F}_2[z]} \subset \mathbb{F}_2^v[z]$ , which is said to be the Couture–L’Ecuyer *dual lattice* [2].

**Theorem 1 ([2]).** *Consider an  $\mathbb{F}_2$ -linear generator started from a nonzero initial state vector. Assume that the characteristic polynomial  $P(z)$  of  $\mathbf{A}$  is primitive. Then,  $k(v) = \nu_1^*$ , where  $\nu_1^*$  is the first successive minimum of  $\mathcal{L}_v^*$ .*

We can obtain a reduced basis by using some polynomial-time lattice basis reduction algorithms (e.g., [7, 18, 28, 35]).

## 5. A new figure of merit for $\mathbb{F}_2$ -linear generators

Usually, when we assess an  $\mathbb{F}_2$ -linear generator, we only see the length of a shortest vector (i.e.,  $k(v)$ ) as the first filter, and abandon the other information. In this section, we focus on the polynomial elements of vectors in  $\mathcal{L}_v^*$ , and develop a new figure of merit  $N_v$  as a quality criterion in dimensions that are higher than  $k(v)$  and as a multi-dimensional generalization of  $N_1$ .

First, we arrange the relationship between  $\mathbb{F}_2$ -linear relations appeared on the most significant  $v$  bits and points in  $\mathcal{L}_v^*$ .

**Proposition 2.** *There exists an  $\mathbb{F}_2$ -linear relation*

$$\sum_{l=1}^v \sum_{j=0}^{k-1} w_{j,l-1} y_{i+j,l-1} = 0 \text{ for all } i \geq 0, \quad (7)$$

if and only if  ${}^t(w_0(z)), \dots, w_{v-1}(z) \in \mathcal{L}_v^*$ , where  $w_{l-1}(z) := \sum_{j=0}^{k-1} w_{j,l-1} z^j \in \mathbb{F}_2[z]$ .

PROOF. We consider a linear combination

$$G_0(z)w_0(z) + \dots + G_{l-1}(z)w_{l-1}(z). \quad (8)$$

For  $i \geq 0$ , the coefficient of  $z^{-i-1}$  in (8) is  $\sum_{l=1}^v \sum_{j=0}^{k-1} w_{j,l-1} y_{i+j,l-1}$ , so that the coefficients of the negative power are all zero if and only if (7) holds. Then, (8) is a polynomial. On the other hand, (8) is also described as  $(h_0(z)w_0(z) + \dots + h_{v-1}(z)w_{v-1}(z))/P(z)$ . Thus, (7) is equivalent to

$$h_0(z)w_0(z) + \dots + h_{v-1}(z)w_{v-1}(z) \equiv 0 \pmod{P(z)}. \quad (9)$$

Here, we assume (7). By multiplying (9) by  $h_0^{-1}(z)$ , we have  $w_0(z) \equiv -\bar{h}_1(z)w_1(z) - \dots - \bar{h}_{v-1}(z)w_{v-1}(z) \pmod{P(z)}$ . In (9), each of polynomial solutions  ${}^t(w_0(z), \dots, w_{v-1}(z))$  is written as  $-a(z)\mathbf{w}_1(z) + h_1(z)\mathbf{w}_2(z) + \dots + h_{v-1}(z)\mathbf{w}_v(z) = {}^t(-a(z)P(z) - \bar{h}_1(z)w_1(z) - \dots - \bar{h}_{v-1}(z)w_{v-1}(z), w_1(z), \dots, w_{v-1}(z))$  for a suitable  $a(z) \in \mathbb{F}_2[z]$ . Hence,  ${}^t(w_0(z), \dots, w_{v-1}(z)) \in \mathcal{L}_v^*$ .

Conversely, it is easy to see that all of  $\mathbb{F}_2[z]$ -linear combinations of  $\mathbf{w}_1(z), \dots, \mathbf{w}_v(z)$  satisfy (9), because  $\mathbf{w}_1(z), \dots, \mathbf{w}_{v-1}(z)$  are solutions in (9), respectively. Thus, the proposition follows.

Using the above proposition, from vectors of  $\mathcal{L}_v^*$ , we obtain information on  $\mathbb{F}_2$ -linear relations in dimensions that are higher than  $k(v)$ . In particular, a nonzero shortest vector of  $\mathcal{L}_v^*$  corresponds to a non-trivial  $\mathbb{F}_2$ -linear relation

$$\sum_{l=1}^v \sum_{j=0}^{k(v)} w_{j,l-1} y_{i+j,l-1} = 0 \text{ for all } i \geq 0, \quad (10)$$

whose degree is minimal for the most significant  $v$  bits given. We call (10) a *minimal  $\mathbb{F}_2$ -linear relation with  $v$ -bit accuracy*. In general, such a minimal  $\mathbb{F}_2$ -linear relation is not unique, because a shortest vector is not unique. Furthermore, we have no non-trivial  $\mathbb{F}_2$ -linear relation  $\sum_{l=1}^v \sum_{j=0}^k w_{j,l-1} y_{i+j,l-1} = 0$

for  $k < k(v)$ . All of the minimal  $\mathbb{F}_2$ -linear relations with  $v$ -bit accuracy are included in all the vectors whose dimensions are higher than  $k(v)$ , so that the  $(k(v) + 1)$ -dimensional case appears to be the most important.

Here, to assess the  $\mathbb{F}_2$ -linear generators, let us consider whether or not the minimal  $\mathbb{F}_2$ -linear relations have simple regularity. The simplest way of checking this is to enumerate the number of nonzero coefficients  $w_{j,l-1}$  in (10). We call this the *weight*. When there exist low-weight  $\mathbb{F}_2$ -linear relations, the generator may have risks in some situations (see Remark 1). Therefore, we define the *minimum weight*  $N_v$  by the lowest weight for all the minimal  $\mathbb{F}_2$ -linear relations with  $v$ -bit accuracy in (10), and propose  $N_v$  as a new figure of merit for  $\mathbb{F}_2$ -linear generators. When  $v = 1$ , the minimal  $\mathbb{F}_2$ -linear relation (10) coincides with a characteristic polynomial  $P(z)$ , so that  $N_v$  equals the number  $N_1$  of nonzero coefficients of  $P(z)$ . Hence, we can interpret  $N_v$  as a multi-dimensional generalization of  $N_1$ .

For practical use, we give an algorithm for computing  $N_v$  as follows. Let  $\{\tilde{\mathbf{w}}_1(z), \dots, \tilde{\mathbf{w}}_v(z)\}$  be a reduced basis of  $\mathcal{L}_v^*$ . From the uniqueness of the successive minima, we have an integer  $v' \in \{1, \dots, v\}$  such that  $\|\tilde{\mathbf{w}}_1(z)\| = \dots = \|\tilde{\mathbf{w}}_{v'}(z)\| < \|\tilde{\mathbf{w}}_{v'+1}(z)\| \leq \dots \leq \|\tilde{\mathbf{w}}_v(z)\|$ . Then, all the shortest vectors are described by

$$\{c_1 \tilde{\mathbf{w}}_1(z) + \dots + c_{v'} \tilde{\mathbf{w}}_{v'}(z) \mid {}^t(c_1, \dots, c_{v'}) \in \mathbb{F}_2^{v'} \setminus {}^t(0, \dots, 0)\}, \quad (11)$$

and they correspond to all the minimal  $\mathbb{F}_2$ -linear relations with  $v$ -bit accuracy. The number of the shortest vectors is  $2^{v'} - 1$ . Here, if we give coefficients  $c_1, \dots, c_{v'}$  by  $v'$ -bit Gray code order, it is possible to obtain another shortest vector by executing the addition only once. In addition, if  $v'$  is small and  $p$  is not too large (e.g.,  $v' \leq 32$  and  $p \leq 19937$ ), we can compute the minimum weight  $N_v$  within a practical time period.

**Remark 1.** *As theoretical justification*, we mention a strong relationship between our figure of merit  $N_v$  and the *weight discrepancy test* proposed by Matsumoto and Nishimura [26]. For simplicity, consider  $k$  successive output values with  $v$ -bit accuracy, where  $k > k(v)$ , and let  $\Phi$  be the map from the state vectors to  $m := v \times k$  bits in the outputs:

$$\Phi : \mathbb{F}_2^p \rightarrow \mathbb{F}_2^m, \quad \mathbf{x}_0 \mapsto (\text{trunc}_v(\mathbf{y}_0), \text{trunc}_v(\mathbf{y}_1), \dots, \text{trunc}_v(\mathbf{y}_{k-1})). \quad (12)$$

The map  $\Phi$  is  $\mathbb{F}_2$ -linear, so that the image  $C \subset \mathbb{F}_2^m$  is a linear subspace. In coding theory,  $C$  is said to be a *linear code*. The *dual code*  $C^\perp$  of  $C$  is defined

by

$$C^\perp := \{\mathbf{c}' \in \mathbb{F}_2^m \mid \langle \mathbf{c}', \mathbf{c} \rangle = 0 \text{ for all } \mathbf{c} \in C\},$$

where  $\langle \mathbf{c}', \mathbf{c} \rangle = \sum_{i=1}^m c'_i c_i$  is an inner product for  $\mathbf{c}' = {}^t(c'_1, \dots, c'_m) \in \mathbb{F}_2^m$  and  $\mathbf{c} = {}^t(c_1, \dots, c_m) \in \mathbb{F}_2^m$ . Note that  $C^\perp$  contains the set of  $\mathbb{F}_2$ -linear relations on  $m = k \times v$  bits (see [26]).

The weight discrepancy test is a theoretical test for estimating the deviation of the number of 1's on  $m = v \times k$  bits in (12) from the binomial distribution. Matsumoto and Nishimura [26] gave a formula for computing a risky sample size from a weight enumerator polynomial of  $C$ , which is computed via a weight enumerator polynomial of  $C^\perp$  with  $k$  being slightly greater than  $k(v)$ , and via inversion by the MacWilliams identity. Their paper implies that if the minimum weight of vectors of  $C^\perp$  is more than 15 or 20, a given generator is safe, but if the weights are too low, e.g.,  $\leq 6$ , (and especially if such  $\mathbb{F}_2$ -linear relations concentrate on a low-dimensional projection), there is a possibility of detecting deviations. In fact, we can identify  $C^\perp$  with a set of all the vectors whose lengths are shorter than  $k$ , so that  $N_v$  coincides with the minimum weight of vectors in  $C^\perp$  in the case where  $k = k(v) + 1$ . Thus, when  $N_v$  is small, a given generator may have some risks.

A drawback of  $N_v$  is that we have to execute exhaustive searches in (11) because the weight enumeration (or finding the minimum weight  $N_v$ ) is NP-hard [38]. However, from the viewpoint of speed and memory efficiency, the use of the Couture–L'Ecuyer dual lattice method appears to be much superior to the use of the Gaussian elimination on a  $p \times m$  matrix in [26] when we construct a basis of  $C^\perp$  (as an  $\mathbb{F}_2$ -linear vector space) for a large  $p$ .

## 6. $\mathbb{F}_2$ -linear relations of Mersenne Twister MT19937

In this section, we numerically analyze 32-bit Mersenne Twister MT19937 (i.e.,  $w = 32$ ) in terms of the method in Section 5. Tables 1 and 2 list the successive minima  $\nu_1^*, \nu_2^*, \dots, \nu_v^*$  of  $\mathcal{L}_v^*$ , the dimension defect  $d(v)$  at  $v$ , and our new figure of merit  $N_v$  for each  $1 \leq v \leq 32$ . From Theorem 1, note that  $\nu_1^* = k(v)$ . As a result,  $N_v$ 's for lower bits are small.

To conduct statistical tests in the next section, we introduce the minimal  $\mathbb{F}_2$ -linear relations with 21-bit and 12-bit accuracy, for example. First, we analyze the minimal  $\mathbb{F}_2$ -linear relations with 21-bit accuracy. By checking all the nonzero shortest vectors in  $\mathcal{L}_{21}^*$ , we obtain the following low-weight

Table 1: The successive minima,  $d(v)$ , and  $N_v$  of MT19937.

	$\mathcal{L}_1^*$	$\mathcal{L}_2^*$	$\mathcal{L}_3^*$	$\mathcal{L}_4^*$	$\mathcal{L}_5^*$	$\mathcal{L}_6^*$	$\mathcal{L}_7^*$	$\mathcal{L}_8^*$	$\mathcal{L}_9^*$	$\mathcal{L}_{10}^*$	$\mathcal{L}_{11}^*$	$\mathcal{L}_{12}^*$	$\mathcal{L}_{13}^*$	$\mathcal{L}_{14}^*$	$\mathcal{L}_{15}^*$	$\mathcal{L}_{16}^*$
$\nu_1^*$	19937	9968	6240	4984	3738	3115	2493	2492	1869	1869	1248	1246	1246	1246	1246	1246
$\nu_2^*$		9969	6848	4984	3738	3115	2493	2492	1869	1869	1868	1246	1246	1246	1246	1246
$\nu_3^*$			6849	4984	3738	3115	2493	2492	1870	1869	1869	1247	1246	1246	1246	1246
$\nu_4^*$				4985	3739	3116	3114	2492	1870	1869	1869	1247	1246	1246	1246	1246
$\nu_5^*$					4984	3738	3114	2492	2491	1869	1869	1868	1246	1246	1246	1246
$\nu_6^*$						3738	3115	2492	2492	1869	1869	1869	1247	1246	1246	1246
$\nu_7^*$							3115	2492	2492	1870	1869	1869	1247	1246	1246	1246
$\nu_8^*$								2493	2492	1870	1869	1869	1868	1246	1246	1246
$\nu_9^*$									2492	2491	1869	1869	1869	1247	1246	1246
$\nu_{10}^*$										2492	1869	1869	1869	1247	1246	1246
$\nu_{11}^*$											1869	1869	1869	1868	1246	1246
$\nu_{12}^*$												1869	1869	1869	1247	1246
$\nu_{13}^*$													1869	1869	1247	1246
$\nu_{14}^*$														1869	1868	1246
$\nu_{15}^*$															1869	1246
$\nu_{16}^*$																1247
$d(v)$	0	0	405	0	249	207	355	0	346	124	564	415	287	178	83	0
$N_v$	135	10020	393	128	44	57	38	15	10	10	40	5	5	5	5	5

$\mathbb{F}_2$ -linear relations: the six-term linear relation

$$y_{i,1} + y_{i,16} + y_{i+396,2} + y_{i+396,17} + y_{i+623,2} + y_{i+623,17} = 0,$$

the seven-term linear relations

$$y_{i,7} + y_{i,14} + y_{i,15} + y_{i+396,8} + y_{i+396,16} + y_{i+623,8} + y_{i+623,16} = 0,$$

$$y_{i,3} + y_{i+396,1} + y_{i+396,4} + y_{i+396,19} + y_{i+623,1} + y_{i+623,4} + y_{i+623,19} = 0,$$

$$y_{i,2} + y_{i,9} + y_{i,10} + y_{i,17} + y_{i,20} + y_{i+396,11} + y_{i+623,11} = 0,$$

and so on. In particular, all of the minimal  $\mathbb{F}_2$ -linear relations concentrate on the three non-successive output values  $\{\mathbf{y}_i, \mathbf{y}_{i+396}, \mathbf{y}_{i+623}\}$ .

In addition to the above, we analyze the minimal  $\mathbb{F}_2$ -linear relations with 12-bit accuracy. In this case, we have three minimal  $\mathbb{F}_2$ -linear relations, i.e., the five-term linear relation

$$y_{i,2} + y_{i+792,4} + y_{i+792,11} + y_{i+1246,4} + y_{i+1246,11} = 0, \quad (13)$$

and the 18-term and the 19-term linear relations. They appear only on the five non-successive output values  $\{\mathbf{y}_i, \mathbf{y}_{i+396}, \mathbf{y}_{i+623}, \mathbf{y}_{i+792}, \mathbf{y}_{i+1246}\}$ .

Consequently, we aim to determine whether there are any observable deviations for such non-successive output values.

**Remark 2.** Niederreiter [30, 31] proposed the *multiple-recursive matrix method* as a general class of pseudorandom number generators. In this framework, we can describe Mersenne Twisters in (4)–(6) by the following matrix linear recurrence:

$$\mathbf{y}_i = \mathbf{y}_{i+n_2-n_1} + \mathbf{T}\tilde{\mathbf{A}} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_r \end{pmatrix} \mathbf{T}^{-1} \mathbf{y}_{i+1-n_1} + \mathbf{T}\tilde{\mathbf{A}} \begin{pmatrix} \mathbf{I}_{w-r} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{T}^{-1} \mathbf{y}_{i-n_1}.$$

Table 2: The successive minima,  $d(v)$ , and  $N_v$  of MT19937 (continued).

	$\mathcal{L}_{17}^*$	$\mathcal{L}_{18}^*$	$\mathcal{L}_{19}^*$	$\mathcal{L}_{20}^*$	$\mathcal{L}_{21}^*$	$\mathcal{L}_{22}^*$	$\mathcal{L}_{23}^*$	$\mathcal{L}_{24}^*$	$\mathcal{L}_{25}^*$	$\mathcal{L}_{26}^*$	$\mathcal{L}_{27}^*$	$\mathcal{L}_{28}^*$	$\mathcal{L}_{29}^*$	$\mathcal{L}_{30}^*$	$\mathcal{L}_{31}^*$	$\mathcal{L}_{32}^*$
$\nu_1^*$	623	623	623	623	623	623	623	623	623	623	623	623	623	623	623	623
$\nu_2^*$	623	623	623	623	623	623	623	623	623	623	623	623	623	623	623	623
$\nu_3^*$	1246	623	623	623	623	623	623	623	623	623	623	623	623	623	623	623
$\nu_4^*$	1246	623	623	623	623	623	623	623	623	623	623	623	623	623	623	623
$\nu_5^*$	1246	1246	623	623	623	623	623	623	623	623	623	623	623	623	623	623
$\nu_6^*$	1246	1246	624	623	623	623	623	623	623	623	623	623	623	623	623	623
$\nu_7^*$	1246	1246	1246	623	623	623	623	623	623	623	623	623	623	623	623	623
$\nu_8^*$	1246	1246	1246	624	623	623	623	623	623	623	623	623	623	623	623	623
$\nu_9^*$	1246	1246	1246	1246	623	623	623	623	623	623	623	623	623	623	623	623
$\nu_{10}^*$	1246	1246	1246	1246	624	623	623	623	623	623	623	623	623	623	623	623
$\nu_{11}^*$	1246	1246	1246	1246	1246	623	623	623	623	623	623	623	623	623	623	623
$\nu_{12}^*$	1246	1246	1246	1246	1246	624	623	623	623	623	623	623	623	623	623	623
$\nu_{13}^*$	1246	1246	1246	1246	1246	1246	623	623	623	623	623	623	623	623	623	623
$\nu_{14}^*$	1246	1246	1246	1246	1246	1246	624	623	623	623	623	623	623	623	623	623
$\nu_{15}^*$	1246	1246	1246	1246	1246	1246	1246	623	623	623	623	623	623	623	623	623
$\nu_{16}^*$	1246	1246	1246	1246	1246	1246	1246	624	623	623	623	623	623	623	623	623
$\nu_{17}^*$	1247	1246	1246	1246	1246	1246	1246	1246	623	623	623	623	623	623	623	623
$\nu_{18}^*$		1247	1246	1246	1246	1246	1246	1246	624	623	623	623	623	623	623	623
$\nu_{19}^*$			1246	1246	1246	1246	1246	1246	1246	623	623	623	623	623	623	623
$\nu_{20}^*$				1246	1246	1246	1246	1246	1246	624	623	623	623	623	623	623
$\nu_{21}^*$					1246	1246	1246	1246	1246	1246	623	623	623	623	623	623
$\nu_{22}^*$						1246	1246	1246	1246	1246	624	623	623	623	623	623
$\nu_{23}^*$							1246	1246	1246	1246	1246	623	623	623	623	623
$\nu_{24}^*$								1246	1246	1246	1246	624	623	623	623	623
$\nu_{25}^*$									1246	1246	1246	1246	623	623	623	623
$\nu_{26}^*$										1246	1246	1246	624	623	623	623
$\nu_{27}^*$											1246	1246	1246	623	623	623
$\nu_{28}^*$												1246	1246	624	623	623
$\nu_{29}^*$													1246	1246	623	623
$\nu_{30}^*$														1246	624	623
$\nu_{31}^*$															1246	623
$\nu_{32}^*$																624
$d(v)$	549	484	426	373	326	283	243	207	174	143	115	89	64	41	20	0
$N_v$	7	6	6	6	6	6	6	6	6	6	6	6	6	6	5	5

From a comparison of the lower  $r$  coordinates, it is easy to see that there exist  $\mathbb{F}_2$ -linear relations among  $\{\mathbf{y}_i, \mathbf{y}_{i+n_2-n_1}, \mathbf{y}_{i+1-n_1}\}$ . The Couture–L’Ecuyer dual lattice method gives explicit  $\mathbb{F}_2$ -linear relations without direct matrix computations, and it is applicable not only for Mersenne Twisters defined by (4)–(6) but also for general  $\mathbb{F}_2$ -linear generators (1)–(3).

## 7. Birthday spacings tests for non-successive output values

In this section, we report statistical tests for non-successive output values of MT19937. We conduct the birthday spacings test, which was proposed by Marsaglia [21], further studied in [11, 13], and implemented in the TestU01 package [14]. We consider the techniques that are similar to [15, 16] for some multiple recursive generators (e.g., [5]).

Following the notations of [13, 17], we introduce the testing procedure. We fix two positive integers,  $n$  and  $t$ , and generate  $n$  “independent” points  $\mathbf{u}_0, \dots, \mathbf{u}_{n-1}$  in the  $t$ -dimensional hypercube  $[0, 1)^t$ . For the hypercube, we partition it into  $d^t$  cubic boxes of equal size by dividing  $[0, 1)$  into  $d$  equal segments. These boxes are numbered from 0 to  $d^t - 1$  in lexicographic order.

Let  $I_1 \leq I_2 \leq \dots \leq I_n$  be the numbers of the boxes where these points have fallen, sorted by increasing order. Define the spacings  $S_j := I_{j+1} - I_j$ , for  $j = 1, \dots, n-1$ . Let  $Y$  be the total number of collisions of these spacings, i.e., the number of values of  $j \in \{1, \dots, n-2\}$  such that  $S_{(j+i)} = S_{(i)}$ , where  $S_{(1)}, \dots, S_{(n-1)}$  are the spacings sorted by increasing order. We test the null hypothesis  $\mathcal{H}_0$ : the PRNG produces i.i.d.  $U(0, 1)$  random variables. If  $d^t$  is large and  $\lambda = n^3/(4d^t)$  is not too large,  $Y$  is approximately a Poisson distribution with mean  $\lambda$  under  $\mathcal{H}_0$ . We generate independent  $N$  replications of  $Y$ , add them, and compute the  $p$ -value by using the sum, which is approximately a Poisson distribution with mean  $N\lambda$ , under  $\mathcal{H}_0$ . If  $d = 2^v$ , note that the  $t$ -dimensional output with  $v$ -bit accuracy is tested.

To extract non-successive output values, let us consider the  $t$ -dimensional output vectors constructed as  $\mathbf{u}_i = (u_{(j_t+1)i+j_1}, \dots, u_{(j_t+1)i+j_t})$  for  $i = 0, \dots, n-1$  with lags  $I = \{j_1, \dots, j_t\}$ .

First, we conduct experiments with the parameter set  $(N, n, d, t) = (5, 20000000, 2^{21}, 3)$ , which is just No. 12 of Crush in TestU01. Then, the three-dimensional output with 21-bit accuracy is tested. We choose  $I = \{0, 396, 623\}$ . The second row in Table 3 gives right  $p$ -values for five initial states, and all the  $p$ -values are  $< 10^{-15}$ . Thus, MT19937 with  $I = \{0, 396, 623\}$  decisively fails the birthday spacings tests, **in accordance with** low-weight minimal  $\mathbb{F}_2$ -linear relations with 21-bit accuracy in Section 6. It takes approximately eight minutes on an Intel Core i7-3770 3.90 GHz computer (with the gcc compiler with the `-O3` optimization flag on a Linux operating system) for each test.

Next, we **also** conduct the birthday spacings tests five times with the parameter set  $(N, n, d, t) = (5, 15000000, 2^{12}, 5)$ . Thus, the five-dimensional output with 12-bit accuracy is tested. The third row in Table 3 shows small deviations for the points with  $I = \{0, 396, 623, 792, 1246\}$ . It takes approximately 11 minutes for each test in the above environment. Furthermore, we focus on the five-term  $\mathbb{F}_2$ -linear relation (13). **The last row of Table 3 shows similar deviations of the birthday spacings tests for  $(N, n, d, t) = (5, 20000000, 2^{21}, 3)$  and  $I = \{0, 792, 1246\}$ .** In general, note that the discovery of such bad lag sets  $I$  is not trivial when we use traditional criteria, such as  $k(v)$  and  $N_1$ .

## 8. Some variants of Mersenne Twister generators

We analyze other  $\mathbb{F}_2$ -linear generators whose periods are  $2^{19937} - 1$  (i.e.,  $p = 19937$  and  $w = 32$ ). First, we investigate the WELL generators [33],

Table 3: The  $p$ -values on the birthday spacings tests with selected lags  $I$  for MT19937.

	1st	2nd	3rd	4th	5th
$I = \{0, 396, 623\}$	$1.7 \times 10^{-16}$	$1.8 \times 10^{-18}$	$3.1 \times 10^{-21}$	$8.5 \times 10^{-17}$	$1.4 \times 10^{-21}$
$I = \{0, 396, 623, 792, 1246\}$	$4.8 \times 10^{-5}$	0.01	$1.5 \times 10^{-4}$	$1.1 \times 10^{-4}$	$8.5 \times 10^{-4}$
$I = \{0, 792, 1246\}$	$2.0 \times 10^{-4}$	$3.0 \times 10^{-7}$	$3.9 \times 10^{-6}$	$9.2 \times 10^{-5}$	$4.5 \times 10^{-6}$

which are variants of Mersenne Twister and have almost optimal  $k(v)$  and  $N_1$ . A key idea of the improvement is to construct a more complicated transition matrix  $\mathbf{A}$  in (4) by using linear recurrences with a double loop, instead of that with a single loop in Remark 2 (see [34] for details). Panneton et al. [33] list the parameters of WELL generators WELL19937a, which has  $\Delta = 4$ , and WELL19937c, which has  $\Delta = 0$  (i.e., maximally equidistributed) by adding the Matsumoto–Kurita tempering [23]. The author [8] also introduced more simplified temperings required to attain the maximal equidistribution. (We discovered a typo in Table 4 of [8]. The bitmask 4202000 should be corrected to 4202010.) All of the WELL generators have  $N_1 = 8585$  and  $N_v > 9500$  ( $2 \leq v \leq 32$ ), so that we have no low-weight minimal  $\mathbb{F}_2$ -linear relations in (10). This means that all the minimal  $\mathbb{F}_2$ -linear relations lie on a large number of coordinates (i.e.,  $\geq \lceil N_v/v \rceil$ ). In other words, among the  $((k(v) + 1) \times v)$  bits that correspond to the image of the map of (12) with  $k = k(v) + 1$ , there do not exist  $\mathbb{F}_2$ -linear relations concentrating on low-dimensional projections. Thus, as far as we focus on the  $((k(v) + 1) \times v)$  bits (in the same setting as MT19937), there are no corresponding bad lag sets  $I = \{j_1, \dots, j_t\}$  for which the birthday spacings tests are rejected, because  $t$  is too large. In this respect, the WELL generators are much superior to MT19937.

As another improvement, the author [8] constructed a maximally equidistributed Mersenne Twister MEMT19937 by replacing the tempering in (5) and (6) with a more complicated output transformation  $\mathbf{B}$ , which consists of a linear combination of some part of the state vector. However, the author recently noted the following drawbacks: (i) MEMT19937 is sometimes slower than WEL19937a on some recent platforms; (ii) MEMT19937 has a small value  $N_{32} = 26$ . Again, we search for a better parameter set by using [9]. By trial-and-error, as well as in [8], we obtain a simple linear transformation:

$$\mathbf{z} \leftarrow \mathbf{m}_i,$$

$$\begin{aligned}
\mathbf{z} &\leftarrow \mathbf{z} \oplus (\mathbf{m}_{i-473} \& \text{b219beab}), \\
\mathbf{z} &\leftarrow \mathbf{z} \oplus (\mathbf{z} \ll 8), \\
\mathbf{z} &\leftarrow \mathbf{z} \oplus (\mathbf{z} \ll 14), \\
\mathbf{y}_i &\leftarrow \mathbf{z} \oplus (\mathbf{m}_{i-588} \& \text{56bde52a}),
\end{aligned}$$

where the state vector  $\mathbf{x}_i$  is decomposed into  $\mathbf{x}_i =: (\mathbf{m}_i, \mathbf{m}_{i-1}, \dots, \mathbf{m}_{i-n_1+2}, \text{trunc}_{w-r}(\mathbf{m}_{i-n_1+1}))$ ,  $\oplus$  denotes bitwise exclusive-or,  $\&$  bitwise AND,  $(\mathbf{z} \ll s_1)$  the  $s_1$  bit left-shift,  $(\mathbf{z} \gg s_2)$  the  $s_2$  bit right-shift, and **b219beab** and **56bde52a** are hexadecimal notations. We replace the tempering (5) and (6) with the above, and we then obtain a maximally equidistributed generator that has fewer operations than MEMT19937 and has almost the same number of operations as MT19937. We name this MEMT19937-II. MEMT19937-II has  $N_1 = 135$  and  $N_v > 9000$  ( $2 \leq v \leq 32$ ), namely,  $N_v$ 's significantly increase. This generator passes the Big Crush suits in the TestU01 statistical test library, with the exception of two linear complexity tests (the test number 80 and 81), and these rejections are common among  $\mathbb{F}_2$ -linear generators, such as the Mersenne Twister and WELL generators (see [14]). Regarding linear complexity tests, Deng, Lu, and Chen [6] recently proposed the combined Mersenne Twister CMT19937, which passes the battery of tests including linear complexity tests in TestU01, by adding the outputs as real numbers (modulo 1), as described in [4]. Now, MEMT19937-II is available at the author's homepage <http://www3.ocn.ne.jp/~harase/megenerators2.html>.

Here, we measure the speed to generate  $10^9$  32-bit unsigned integers on two different 64-bit CPUs: Intel Core i7-3770 3.90 GHz and AMD Phenom II X6 1045T 2.70 GHz. We use the gcc compiler with the `-O3` optimization flag on Linux operating systems. In comparison, we also conduct experiments with MT19937ar, Shawn Cokus' other implementation MT19937ar-cok (both are obtained from [22]), WELL19937a, and MEMT19937. Table 4 gives a summary of the CPU time (in seconds) and  $\Delta$ . MT19937ar-cok is the fastest, but the maximally equidistributed generator MEMT19937-II is comparable to or faster than MT19937ar on the two platforms.

Finally, we conduct the birthday spacings tests for non-successive output values of MEMT19937-II. Table 5 gives a summary of the birthday spacings tests with the same parameter sets for three- and five-dimensional non-successive outputs in Section 7. A simple improvement of  $\mathbf{B}$  also increases  $N_v$ 's, and the bad lag sets  $I$  disappear, so that the deviations of tests are avoided.

Table 4: CPU time (sec) taken to generate  $10^9$  pseudorandom numbers and total dimension defects  $\Delta$ .

	Intel Core i7	AMD Phenom II	$\Delta$
MT19937ar-cok	3.126	4.199	6750
MEMT19937-II	4.093	5.841	0
MT19937ar	4.771	6.106	6750
WELL19937a	4.953	6.678	4
MEMT19937	5.111	7.812	0

Table 5: The  $p$ -values of the birthday spacings tests five times for MEMT19937-II.

	1st	2nd	3rd	4th	5th
$I = \{0, 396, 623\}$	0.11	0.32	0.87	0.63	0.03
$I = \{0, 396, 623, 792, 1246\}$	0.87	0.80	0.28	0.33	0.10
$I = \{0, 792, 1246\}$	0.76	0.63	0.85	0.78	0.67

## 9. Conclusions

To assess  $\mathbb{F}_2$ -linear pseudorandom number generators, we have discussed the relationship between  $\mathbb{F}_2$ -linear relations and the Couture–L’Ecuyer dual lattices, and have proposed the new figure of merit  $N_v$  based on the minimum weight of  $\mathbb{F}$ -linear relations for most significant  $v$  bits in  $(k(v)+1)$ -dimensional output vectors. We presented an algorithm for computing  $N_v$ , and applied it to the Mersenne Twister MT19937. The experimental results showed that MT19937 has low-weight  $\mathbb{F}_2$ -linear relations, and is rejected for birthday spacings tests with specific lags, and the reason appears to be the existence of such  $\mathbb{F}_2$ -linear relations. To avoid such phenomena, some improvements of Mersenne Twister generators were also discussed.

The above result of MT19937 will not affect most Monte Carlo simulations because real simulations are not synchronized with bad lag sets  $I$ . However, in general, when strange phenomena occur in simulations, and when these are due to the regularity of pseudorandom number generators, it is significantly difficult for experimenters to determine the reason for occurrence of the strange phenomena. Thus, when designing pseudorandom number generators, it is important to perform assessments in as many situations as possible beforehand. In this respect, the Couture–L’Ecuyer lattices are powerful tools

not only for computing  $k(v)$  but also for detecting hidden structural defects of  $\mathbb{F}_2$ -linear generators.

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