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# Existence and approximation for vibro-impact problems with a time-dependent set of constraints

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We consider a discrete mechanical system subjected to perfect time-dependent unilateral constraints, which dynamics is described by a second order measure differential inclusion. The transmission of the velocity at impacts is given by a minimization property of the kinetic energy with respect to the set of kinematically admissible post-impact velocities. We construct a sequence of feasible approximate positions by using a time-stepping algorithm inspired by a kind of Euler discretization of the differential inclusion. We prove the convergence of the approximate trajectories to a solution of the Cauchy problem and we obtain as a by-product a global existence result.

*Keywords:* Discrete mechanical system; Time-dependent constraints; Differential inclusion; Inelastic shocks; Time-stepping scheme

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## 1. Introduction

We consider a mechanical system with a finite number of degrees of freedom subjected to frictionless unilateral constraints. If we denote by  $u(t)$  its representative point in generalized coordinates and by  $K(t)$  the set of admissible positions at any instant  $t$ , the dynamics of the system is described by a second order differential equation combined with the condition  $u(t) \in K(t)$ , leading to a measure differential inclusion.

Such problems occur frequently in automotive industry or aeronautics where looseness of joints may create unwanted vibrations and impacts, but they are also of crucial importance in environment for the study of the damages due to earthquakes for instance. One can find an important literature on this topic in the case of time-independent constraints (i.e. when the set of admissible positions does not depend on  $t$ ), which is more related to industrial issues, and several existence results have been obtained either by constructing a sequence of approximate trajectories and by proving their convergence to a solution of the Cauchy problem [24,6,4,5,23,11,20,9,14,22,15,10,7,8,17,19], or by using theoretical arguments based on existence results for ordinary differential equations and variational inequalities [1]. On the contrary, very few studies are available in the case of time-dependent constraints, which is more related to environmental issues like earthquakes. In [25] an existence result is established, by considering a generalization of the Yosida-type approximation already proposed in [20]. Unfortunately this technique transforms the differential

inclusion into a sequence of very stiff ordinary differential equations which are not well suited for implementation (see [21] for a more detailed discussion). More recently another existence result, based on a time-stepping approximation of the problem, has been obtained [2] when the sets  $K(t)$  are defined as a finite intersection of complements of convex sets. But this property is a very restrictive assumption and is usually not satisfied.

The aim of this work is thus to propose another time-stepping approximation of the problem and to state its convergence, which will give as a by-product a global existence result, in a more general geometrical setting than in [2]. More precisely we will first derive from the basic mechanical and geometrical description of the problem its mathematical formulation. Then we introduce a time-stepping scheme inspired by some implicit Euler's type discretization and the main steps of the convergence proof are outlined.

## 2. Formulation of the problem

We consider a discrete mechanical system which unconstrained motion is described by the following second order differential equation in  $\mathbb{R}^d$ :

$$\ddot{u} = g(t, u). \quad (1)$$

We assume that the system is subjected to time-dependent unilateral constraints i.e.

$$u(t) \in K(t) \quad \forall t \quad (2)$$

where  $K(t)$  is the set of admissible positions at instant  $t$ , given by the geometrical inequalities

$$u \in K(t) \iff f_\alpha(t, u) \geq 0, \quad \forall \alpha \in \{1, \dots, \nu\}, \quad \nu \geq 1$$

with smooth functions  $f_\alpha$ . When a contact occurs, i.e. when  $u(t) \in \partial K(t)$ , a reaction force appears. If we assume that the contact is frictionless then [12]

$$\ddot{u} - g(t, u) = R \in -N(K(t), u) \quad (3)$$

where  $N(K(t), u)$  is the normal cone to  $K(t)$  at  $u$  given by

$$N(K(t), u) = \begin{cases} \{0\} & \text{if } u \in \text{Int}(K(t)) \\ \left\{ \sum_{\alpha \in J(t, u)} \lambda_\alpha \nabla_u f_\alpha(t, u), \lambda_\alpha \leq 0 \right\} & \text{if } u \in \partial K(t) \\ \emptyset & \text{otherwise} \end{cases}$$

and  $J(t, u) = \{\alpha \in \{1, \dots, \nu\}; f_\alpha(t, u) \leq 0\}$  is the set of active constraints at the point  $(t, u)$ .

As usual for vibro-impact problems, the velocity  $\dot{u}$  may be discontinuous and the adequate framework for the solutions is the set of absolutely continuous functions  $u$  which derivative  $\dot{u}$  belongs to the space of functions of bounded variation. Indeed, if we assume that the right and left velocities exist at some instant  $t$ , we infer from (2) that

$$\dot{u}^+(t) \in T(t, u(t)), \quad \dot{u}^-(t) \in -T(t, u(t))$$

with

$$T(t, u) = \{v \in \mathbb{R}^d; \partial_t f_\alpha(t, u) + \langle \nabla_u f_\alpha(t, u), v \rangle \geq 0 \text{ for all } \alpha \in J(t, u)\}.$$

It follows that  $\dot{u}$  may be discontinuous at  $t$  if  $J(t, u(t)) \neq \emptyset$  and the reaction force  $R$  is a measure. Of course the acceleration  $\ddot{u}$  should now be understood as the Stieltjes measure  $d\dot{u}$  and the measure  $d\dot{u} - g(t, u)dt$  should vanish on  $\{t; J(t, u(t)) = \emptyset\}$ , in order to recover (1) when the constraints are not saturated. More precisely, there exist  $\nu$  scalar measures  $\lambda_\alpha$  such that

$$\begin{cases} d\dot{u} - g(\cdot, u)dt = \sum_{\alpha=1}^{\nu} \lambda_\alpha \nabla_u f_\alpha(\cdot, u) \\ \lambda_\alpha \geq 0, \quad \text{Supp}(\lambda_\alpha) \subset \{t; f_\alpha(t, u(t)) = 0\} \quad \forall \alpha \in \{1, \dots, \nu\}. \end{cases}$$

Hence, at any contact instant  $t$ , we obtain

$$\begin{aligned} \dot{u}^+(t) &\in T(t, u(t)), & \dot{u}^-(t) &\in -T(t, u(t)), \\ \dot{u}^+(t) - \dot{u}^-(t) &\in -N(K(t), u(t)). \end{aligned}$$

Unfortunately these relations do not define uniquely  $\dot{u}^+(t)$  and we have to complete the description of the problem by an impact law. We model the transmission of the velocity by assuming that

$$\dot{u}^+(t) = \text{Proj}(T(t, u(t)), \dot{u}^-(t)) \quad \text{if } J(t, u(t)) \neq \emptyset. \quad (4)$$

Observing that  $T(t, u(t))$  is the set of kinematically admissible right velocities at  $t$ , this relation corresponds to a minimizing property of the kinetic energy which generalizes the definition of standard inelastic shocks introduced by J.J. Moreau [13]. So, for any admissible initial data  $(u_0, v_0) \in K(0) \times T(0, u_0)$ , we will consider the following Cauchy problem:

**Problem (P).** Find  $u : [0, \tau] \rightarrow \mathbb{R}^d$ , with  $\tau > 0$ , such that

(P1)  $u$  is absolutely continuous on  $[0, \tau]$ ,  $\dot{u} \in BV(0, \tau; \mathbb{R}^d)$ ,

(P2)  $u(t) \in K(t)$  for all  $t \in [0, \tau]$ ,

(P3) there exist  $v$  scalar measures  $\lambda_\alpha$  such that

$$\begin{cases} d\dot{u} - g(\cdot, u)dt = \sum_{\alpha=1}^v \lambda_\alpha \nabla_u f_\alpha(\cdot, u) \\ \lambda_\alpha \geq 0, \quad \text{Supp}(\lambda_\alpha) \subset \{t \in [0, \tau]; f_\alpha(t, u(t)) = 0\} \quad \forall \alpha \in \{1, \dots, v\}, \end{cases}$$

(P4) for all  $t \in (0, \tau)$  such that  $J(t, u(t)) \neq \emptyset$  we have

$$\dot{u}^+(t) = \text{Proj}(T(t, u(t)), \dot{u}^-(t)),$$

(P5)  $u(0) = u_0, \dot{u}^+(0) = v_0$ .

Let us emphasize that the sets  $\{t \in [0, \tau]; f_\alpha(t, u(t)) = 0\}$ ,  $\alpha \in \{1, \dots, v\}$ , corresponding to the instants  $t$  for which the constraint numbered  $\alpha$  is saturated, play the role of free boundaries.

### 3. Approximation by a time-stepping scheme

Let  $h > 0$  be a given time-step and let  $t_n = nh$  for all  $n \geq 0$ . We define

- $U^{-1} = u_0 - hv_0, U^0 = u_0$ ,
- for all  $n \geq 0$

$$G^n = \frac{1}{h} \int_{t_n}^{t_{n+1}} g(s, U^n) ds \quad (5)$$

and

$$W^n = 2U^n - U^{n-1} + h^2 G^n, \quad U^{n+1} \in \text{Argmin}_{z \in K(t_{n+1})} \|W^n - z\|. \quad (6)$$

We can observe that, whenever  $K(t_{n+1})$  is a convex subset of  $\mathbb{R}^d$ , (6) is equivalent to

$$U^{n+1} = \text{Proj}(K(t_{n+1}), 2U^n - U^{n-1} + h^2 G^n).$$

Hence this scheme is a natural generalization of the one proposed in [22,15,17]. Furthermore, (5)–(6) may also be interpreted as a generalization of the algorithm proposed in [2] when  $K(t_{n+1})$  is not convex. Indeed, if we replace in (6)  $K(t_{n+1})$  by its convex approximation  $\tilde{K}(t_{n+1}, U^n)$  defined by

$$\tilde{K}(t_{n+1}, U^n) = \{q \in \mathbb{R}^d; f_\alpha(t_{n+1}, U^n) + \langle \nabla_u f_\alpha(t_{n+1}, U^n), q - U^n \rangle \geq 0 \quad \forall \alpha \in \{1, \dots, v\}\}$$

we obtain

$$U^{n+1} = \text{Proj}(\tilde{K}(t_{n+1}, U^n), 2U^n - U^{n-1} + h^2 G^n)$$

which is equivalent to

$$U^{n+1} = U^n + hV^n, \quad V^n = \text{Proj}(K_h(t_{n+1}, U^n), V^{n-1} + hG^n) \quad (7)$$

with

$$K_h(t_{n+1}, U^n) = \{v \in \mathbb{R}^d; f_\alpha(t_{n+1}, U^n) + h\langle \nabla_u f_\alpha(t_{n+1}, U^n), v \rangle \geq 0 \forall \alpha \in \{1, \dots, v\}\}.$$

We recognize here the scheme introduced in [2].

Let us observe that (6) ensures the feasibility of the approximate positions on the contrary of (7), which does not ensure  $U^n \in K(t_n)$  for all  $n \geq 0$  if the functions  $f_\alpha$ ,  $\alpha \in \{1, \dots, v\}$ , are not convex with respect to their second argument, i.e. if the sets  $K(t_n)$  are not defined as a finite intersection of complements of convex sets.

#### 4. Convergence result

For our convergence (and existence) result we adopt the same regularity assumptions for the data as in [2] but we will not assume any convexity property for the functions  $f_\alpha$ . More precisely, let  $T > 0$ . We assume:

(H1) The mappings  $f_\alpha$ ,  $\alpha \in \{1, \dots, v\}$ , belong to  $C^2([0, T] \times \mathbb{R}^d; \mathbb{R})$  and for all  $t \in [0, T]$ , the set  $K(t) = \{u \in \mathbb{R}^d; f_\alpha(t, u) \geq 0, \forall \alpha \in \{1, \dots, v\}\}$  is not empty.

We define

$$K = \{(t, u) \in [0, T] \times \mathbb{R}^d; u \in K(t)\}$$

and for any  $r > 0$ ,  $K_r$  is the neighbourhood of  $K$  given by

$$K_r = \{(s, y) \in [0, T] \times \mathbb{R}^d; \exists (t, u) \in K / |s - t| \leq r, \|y - u\| \leq r\}.$$

(H2) There exist  $r > 0$ ,  $m > 0$  and  $M > 0$  such that, for all  $(s, y) \in K_r$ :

$$\begin{aligned} m &\leq \|\nabla_u f_\alpha(s, y)\| \leq M, & |\partial_t f_\alpha(s, y)| &\leq M, \\ |\partial_t^2 f_\alpha(s, y)| &\leq M, & \|\partial_t \nabla_u f_\alpha(s, y)\| &\leq M, & \|D_u^2 f_\alpha(s, y)\| &\leq M. \end{aligned}$$

(H3) There exist  $\gamma > 0$  and  $\rho > 0$  such that, for all  $t \in [0, T]$  and for all  $u \in K(t)$ :

$$\sum_{\alpha \in J_\rho(t, u)} \lambda_\alpha \|\nabla_u f_\alpha(t, u)\| \leq \gamma \left\| \sum_{\alpha \in J_\rho(t, u)} \lambda_\alpha \nabla_u f_\alpha(t, u) \right\| \quad \forall \lambda_\alpha \geq 0, \alpha \in J_\rho(t, u)$$

where

$$J_\rho(t, u) = \{\alpha \in \{1, \dots, v\}; f_\alpha(t, u) \leq \rho\}.$$

Whenever several constraints are saturated at  $(t, u)$ , this last assumption may be interpreted as a kind of *positive* linear independence property for the vectors  $(\nabla_u f_\alpha(t, u))_{\alpha \in J(t, u)}$  which allows us to consider also cases where the active constraints are not linearly independent. It follows that this geometrical framework is more general than in [1,15,17,19].

(H4) The function  $g$  is a Caratheodory function from  $[0, T] \times \mathbb{R}^d$  with values in  $\mathbb{R}^d$  and there exist  $k_g > 0$  and  $F \in L^1(0, T; \mathbb{R})$  such that, for almost every  $t \in [0, T]$  we have

$$\begin{aligned} \|g(t, u) - g(t, \tilde{u})\| &\leq k_g \|u - \tilde{u}\|, & \|g(t, u)\| &\leq F(t) \\ \forall (u, \tilde{u}) \in (\mathbb{R}^d)^2 \text{ such that } (t, u) &\in K_r, (t, \tilde{u}) &\in K_r. \end{aligned}$$

As a consequence of these assumptions we obtain

**Proposition 1** ([18]). *For all  $n \in \{0, \dots, \lfloor T/h \rfloor - 1\}$ , there exists a family of non negative real numbers  $(\lambda_\alpha^n)_{\alpha \in \{1, \dots, v\}}$  such that  $\lambda_\alpha^n = 0$  for all  $\alpha \notin J(t_{n+1}, U^{n+1})$  and*

$$V^n - V^{n-1} - hG^n = \sum_{\alpha=1}^v \lambda_\alpha^n \nabla_u f_\alpha(t_{n+1}, U^{n+1}).$$

This relation is the discrete analogous of property (P3). Furthermore it can be reformulated as

$$\frac{V^n - V^{n-1}}{h} - G^n \in -N(K(t_{n+1}), U^{n+1}) \quad \forall n \in \{0, \dots, \lfloor T/h \rfloor - 1\}$$

which is a kind of Euler discretization of the measure differential inclusion (3).

This is not enough to ensure the good behaviour of the algorithm at impacts. Indeed, when several constraints may be saturated at the same instant  $t$ , continuity on data does not hold in general. The study of the model problem of a material point moving in an angular domain of  $\mathbb{R}^2$  shows that a small change in the initial position can lead to a completely different trajectory after an impact if the edge angle is obtuse. From a computational point of view, this bad property may create a kind of numerical unpredictability, due to round-up errors, and should be avoided.

In the case of time-independent constraints, it has been proved in [1,16] that a necessary and sufficient condition to ensure continuity on data is given by

$$\langle \nabla_u f_\alpha(t, u(t)), \nabla_u f_\beta(t, u(t)) \rangle \leq 0 \quad \forall (\alpha, \beta) \in J(t, u(t))^2, \alpha \neq \beta$$

i.e. the active constraints along the trajectory create right or acute angles. Hence we will also consider a similar geometrical assumption in our convergence result.

We define the sequence of approximate velocities  $(v_h)_{h>0}$  by

$$\begin{aligned} v_h(t) &= V^n = \frac{U^{n+1} - U^n}{h} \quad \forall t \in [nh, (n+1)h), \quad \forall n \in \{0, \dots, N(h) - 1\}, \\ v_h(t) &= V^{N(h)-1} \quad \forall t \in [N(h)h, T], \end{aligned}$$

with  $N(h) = \lfloor \frac{T}{h} \rfloor$ . Let us define also the sequence of approximate trajectories  $(u_h)_{h>0}$  by a linear interpolation of the  $U^n$ 's i.e.

$$\begin{aligned} u_h(t) &= U^n + (t - nh)V^n \quad \forall t \in [nh, (n+1)h), \quad \forall n \in \{0, \dots, N(h) - 1\}, \\ u_h(t) &= U^{N(h)} + (t - N(h)h)V^{N(h)-1} \quad \forall t \in [N(h)h, T]. \end{aligned}$$

We obtain

**Theorem 2 ([18]).** *Let us assume that (H1)–(H4) hold. Let  $(u_0, v_0) \in K(0) \times T(0, u_0)$ . Then, we can extract subsequences, still denoted  $(u_h)_{h>0}$  and  $(v_h)_{h>0}$ , and there exist  $u \in C^0([0, T]; \mathbb{R}^d)$  and  $v \in BV(0, T; \mathbb{R}^d)$  such that*

$$\begin{aligned} u_h &\rightarrow u \quad \text{strongly in } C^0([0, T]; \mathbb{R}^d), \\ v_h &\rightarrow v \quad \text{pointwise in } [0, T], \\ dv_h &\rightharpoonup dv \quad \text{weakly* in } \mathcal{M}^1(0, T; \mathbb{R}^d). \end{aligned}$$

Moreover

$$u(t) = u_0 + \int_0^t v(s) ds \quad \forall t \in [0, T]$$

and  $u$  satisfies properties (P1)–(P3). Furthermore, if for all  $t \in [0, T]$ , we have

$$\langle \nabla_u f_\alpha(t, u(t)), \nabla_u f_\beta(t, u(t)) \rangle \leq 0 \quad \forall (\alpha, \beta) \in J(t, u(t))^2, \alpha \neq \beta,$$

then  $u$  satisfies also properties (P4) and (P5) and is a solution of Problem (P) on  $[0, T]$ .

Let us recall that uniqueness is not true in general [3,24,1]. Hence, we cannot expect the convergence of the whole sequence of approximate solutions.

As usual for this kind of problems, the proof is divided in several steps. First we establish some a priori estimates for the sequence  $(v_h)_{h>0}$  in  $L^\infty(0, T; \mathbb{R}^d)$  and in  $BV(0, T; \mathbb{R}^d)$ . By applying Ascoli's and Helly's theorems we may extract a subsequence, still denoted  $(u_h)_{h>0}$  and  $(v_h)_{h>0}$ , such that

$$\begin{aligned} u_h &\rightarrow u \quad \text{strongly in } C^0([0, T]; \mathbb{R}^d), \\ v_h &\rightarrow v \quad \text{pointwise in } [0, T], \\ dv_h &\rightharpoonup dv \quad \text{weakly* in } \mathcal{M}^1(0, T; \mathbb{R}^d). \end{aligned}$$

Moreover

$$u(t) = u_0 + \int_0^t v(s) ds \quad \forall t \in [0, T]$$

and we prove that the limit trajectory is feasible, i.e.  $u(t) \in K(t)$  for all  $t \in [0, T]$ . Then using Proposition 1, we have

$$\begin{aligned} dv_h &= \sum_{n=1}^{N(h)-1} (V^n - V^{n-1}) \delta(t - nh) \\ &= \sum_{n=1}^{N(h)-1} h G^n \delta(t - nh) + \sum_{n=1}^{N(h)-1} \sum_{\alpha=1}^v \lambda_\alpha^n \nabla_u f_\alpha(t_{n+1}, U^{n+1}) \delta(t - nh) \end{aligned}$$

with

$$\lambda_\alpha^n \geq 0 \quad \forall \alpha \in \{1, \dots, v\}, \quad \lambda_\alpha^n = 0 \quad \forall \alpha \notin J(t_{n+1}, U^{n+1})$$

for all  $n \in \{0, \dots, N(h) - 1\}$ . By observing that

$$\left\| \sum_{\alpha=1}^v \lambda_\alpha^n \nabla_u f_\alpha(t_{n+1}, U^{n+1}) \right\| \leq \|V^n - V^{n-1}\| + h \|G^n\| \quad \forall n \in \{1, \dots, N(h) - 1\}$$

we obtain

$$\sum_{n=1}^{N(h)-1} |\lambda_\alpha^n| \leq \frac{\gamma}{m} (TV(v_h) + \|F\|_{L^1(0, T; \mathbb{R}^d)}) \quad \forall \alpha \in \{1, \dots, v\}.$$

Next we define

$$\lambda_{\alpha, h}(t) = \sum_{n=1}^{N(h)-1} \lambda_\alpha^n \delta(t - nh), \quad \lambda_\alpha^n \geq 0, \quad \forall \alpha \in \{1, \dots, v\}.$$

Possibly extracting another subsequence, there exist non negative scalar measures  $\lambda_\alpha$  such that, for all  $\alpha \in \{1, \dots, v\}$ ,

$$\lambda_{\alpha, h} \rightharpoonup \lambda_\alpha \quad \text{weakly* in } \mathcal{M}^1(0, T; \mathbb{R})$$

and we get

$$d\dot{u} = dv = \sum_{\alpha=1}^v \lambda_\alpha \nabla_u f_\alpha(\cdot, u) + g(\cdot, u) dt.$$

Furthermore for all  $\alpha \in \{1, \dots, v\}$  we have

$$\text{Supp}(\lambda_\alpha) \subset \{t \in [0, T]; f_\alpha(t, u(t)) = 0\}.$$

Finally there remains to study the transmission of the velocities at impacts.

Let  $t \in [0, T)$ . We already know that

$$v^+(t) \in T(t, u(t)), \quad v^-(t) \in -T(t, u(t))$$

and

$$v^+(t) - v^-(t) \in -N(K(t), u(t)) = \sum_{\alpha \in J(t, u(t))} \mathbb{R}^+ \nabla_u f_\alpha(t, u(t)).$$

We have to prove that

$$v^+(t) = \text{Proj}(T(t, u(t)), v^-(t)) \quad \text{if } J(t, u(t)) \neq \emptyset.$$

Let us assume that  $J(t, u(t)) \neq \emptyset$ . We split  $J(t, u(t))$  as

$$J(t, u(t)) = J_1(t, u(t)) \cup J_2(t, u(t))$$

with

$$J_1(t, u(t)) = \left\{ \alpha \in J(t, u(t)) ; \exists r_\alpha > 0, \exists h_\alpha \in (0, h_1] / \forall h \in (0, h_\alpha], \right. \\ \left. \forall nh \in \left[ t - \frac{r_\alpha}{4(C+1)}, t + \frac{r_\alpha}{4(C+1)} \right] \cap [0, T], f_\alpha(t_{n+1}, U^{n+1}) > 0 \right\}$$

and

$$J_2(t, u(t)) = \left\{ \alpha \in J(t, u(t)) ; \forall r_\alpha > 0, \forall h_\alpha \in (0, h_1], \exists h \in (0, h_\alpha], \right. \\ \left. \exists nh \in \left[ t - \frac{r_\alpha}{4(C+1)}, t + \frac{r_\alpha}{4(C+1)} \right] \cap [0, T] / f_\alpha(t_{n+1}, U^{n+1}) \leq 0 \right\}.$$

We infer that there exists a neighbourhood  $\mathcal{V}(t)$  of the instant  $t$  such that, for all  $h$  small enough

$$J(t_{n+1}, U^{n+1}) \subset J_2(t, u(t)) \quad \forall nh \in \mathcal{V}(t)$$

i.e.

$$V^n - V^{n-1} = hG^n + \sum_{\alpha \in J_2(t, u(t))} \lambda_\alpha^n \nabla_u f_\alpha(t_{n+1}, U^{n+1}) \quad \forall nh \in \mathcal{V}(t).$$

It follows that

$$v^+(t) - v^-(t) = \sum_{\alpha \in J_2(t, u(t))} \bar{\lambda}_\alpha \nabla_u f_\alpha(t, u(t)), \quad \bar{\lambda}_\alpha \geq 0$$

and we have to prove that, for all  $w \in T(t, u(t))$

$$\langle v^-(t) - v^+(t), w - v^+(t) \rangle = - \sum_{\alpha \in J_2(t, u(t))} \bar{\lambda}_\alpha \langle \nabla_u f_\alpha(t, u(t)), w - v^+(t) \rangle \leq 0.$$

But, for all  $\alpha \in J_2(t, u(t))$ , we have

$$\langle \nabla_u f_\alpha(t, u(t)), w - v^+(t) \rangle = \overbrace{\left( \partial_t f_\alpha(t, u(t)) + \langle \nabla_u f_\alpha(t, u(t)), w \rangle \right)}^{\geq 0} \\ - \left( \partial_t f_\alpha(t, u(t)) + \langle \nabla_u f_\alpha(t, u(t)), v^+(t) \rangle \right).$$

It follows that

$$\langle v^-(t) - v^+(t), w - v^+(t) \rangle \leq \sum_{\alpha \in J_2(t, u(t))} \bar{\lambda}_\alpha \left( \partial_t f_\alpha(t, u(t)) + \langle \nabla_u f_\alpha(t, u(t)), v^+(t) \rangle \right)$$

and we prove that

$$\partial_t f_\alpha(t, u(t)) + \langle \nabla_u f_\alpha(t, u(t)), v^+(t) \rangle = 0 \quad \alpha \in J_2(t, u(t))$$

which allows us to conclude.

## 5. Conclusion

Starting from the mechanical description of the dynamics of discrete mechanical systems subjected to frictionless time-dependent unilateral constraints, we have derived a mathematical formulation of the problem in terms of measure differential inclusions. Then feasible approximate solutions are built by using a time-stepping scheme inspired by implicit Euler's methods. Finally a convergence result is stated, leading also to a global existence result, and the main steps of the proof are outlined.



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