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High-order filtered schemes for first order time dependent linear and non-linear partial differential equations.

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Abstract

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1. Introduction

In this paper, we aim to solve first order time dependent partial differential equations (PDEs) in particular hyperbolic conservation law and Hamilton-Jacobi (HJ) equation by high-order filtered scheme. It is well known that, in 1D, there is a strong link between time-dependent HJ equations and hyperbolic

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[1]Since 2015.

conservation laws. To be more precise, the viscosity solution of the evolutive HJ equation is the primitive of entropy solution of the corresponding conservation law. Due to this link several schemes have been developed to solve hyperbolic conservation law (see references [8],[12],[13],[14]) and many of them extended for HJ equations. For instance, well-known high-order essentially non-oscillatory (ENO) scheme have been introduced by A. Harten et al. in [15] for conservation laws, and then extended to HJ equation by Osher and Shu [17]. ENO schemes have been shown to have high-order accuracy numerically however no general convergence results are available. The interest for these schemes is due to the fact that they should be high-order accurate if they converge. In [2], Barles and Souganidis have given a general frame work for the convergence of approximated solution towards the viscosity solution under generic monotonicity, stability and consistency assumptions. Recently filter scheme has been introduce in [11] to solve Monge-Ampere equation, and adapted for the stationary and time-dependent first order HJ equations in [3, 16, 4, 18]. Proposed scheme in [4] is written in explicit time marching form ("fully explicit" schemes) which is well adapted to time-dependent equations, while the setting of [11] or [16] can be better adapted to solve stationary equations. In our work, we follow the filtered scheme from [4]. This framework enables the development of simple schemes that have high-order consistency in both space and time. Filter can stabilize an unstable scheme and achieves higher-order accuracy. It is well known by the Godunov theorem that monotone scheme can atmost first order hence one has to look for the non-monotonicity. Then it is difficult to combine non-monotonicity and converges to the viscosity solution. In [4], convergence results and the error estimate have been proved for stationary and time-dependent HJ equations.

In this paper, we present several examples with filtration of different schemes up to 3D. For the monotone scheme we will use semi-Lagrangian (SL) schemes (by Courant, Isaacson and Rees [6]) and finite difference scheme (by Crandall and Lions in [8] with the convergence result) for HJ equations. For high-order scheme we will use second and third order schemes. We will compare the proposed filtered scheme with the high-order scheme used in filtration and ENO scheme via several numerical tests up to 3D.

Organization of paper. In Section 2, we will present the model problem and recall filtered scheme from [4] with the limiter. In Section 3, we will present some numerical examples of second and third order filtered scheme upto three-dimensions. In section 4, we will conclude and finally Appendix 5 contains some theoretical outline.

2. Filtered scheme

We recall the filtered scheme from [4] for the following model problem:

$$\partial_t v + H(x, \nabla v) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d \tag{1}$$

$$v(0, x) = v_0(x), \quad x \in \mathbb{R}^d, \tag{2}$$

the typical assumptions on Hamiltonian H and the initial data $v_0(x)$ are:

A1. $H(\cdot, \cdot, \cdot)$ is uniformly continuous in all the variables.

A2. $H(x, v, \cdot)$ is convex and coercive.

A3. $H(x, \cdot, \nabla v)$ is monotone.

A4. $v_0(x)$ is Lipschitz continuous

The above assumptions guarantee existence and uniqueness in the framework of weak solutions in viscosity sense [1, 7]. For simplicity, we present scheme in 1D and can be easily adapt to the higher dimension (filtered scheme for 2D has been presented in [18]). The basic idea of filter scheme is the combination the of low order and high-order scheme. This allows us to construct finite difference schemes which are easy to implement and behave like a monotone scheme in the singular region and as a high-order scheme where the solution is smooth. We use the discontinuous filter function which has been used in [16, 4, 18] for which the filtered scheme is still an "ϵ-monotone" scheme (see (17)). In our case, we justify the use of this discontinuous filter to obtain a high order numerical behaviour of the scheme in the L^∞ norm. We observe that using instead the continuous filter initially introduced in [11] leads to only first order behaviour although for steady equations both filter gives similar results.

Discretization: Let $\Delta t > 0$ be a time step (in the form of $\Delta t = \frac{T}{N}$ for some $N \geq 1$), and $\Delta x > 0$ be a space step. A uniform mesh in time is defined by $t_n := n\Delta t$, $n \in [0, \ldots, N]$, and in space by the nodes $x_j := j\Delta x$, $j \in \mathbb{Z}$. Hence the filtered scheme (for more details see [4]) is then defined as

$$u_j^{n+1} \equiv S^F(u^n)_j := S^M(u^n)_j + \epsilon \Delta t F\bigg(\frac{S^A(u^n)_j - S^M(u^n)_j}{\epsilon \Delta t}\bigg), \tag{3}$$

where $\epsilon = \epsilon_{\Delta t, \Delta x} > 0$ is a parameter satisfying

$$\lim_{(\Delta t, \Delta x) \to 0} \epsilon = 0. \tag{4}$$

Where S^M is a monotone scheme here we will consider two cases for the monotone schemes.

- **Case 1:** S^M is based on a first order finite difference scheme [8]. Hence the monotone finite difference scheme written as

$$S^M(u^n)_j := S^M(u^n)(x_j) := u_j^n - \Delta t \; h^M(x_j, D^- u_j^n, D^+ u_j^n), \; D^{\pm} u_j^n := \pm \frac{u_{j\pm 1}^n - u_j^n}{\Delta x}, \tag{5}$$

where h^M is numerical monotone Hamiltonian which satisfies following properties:

A5. h^M is a Lipschitz continuous function.

A6. (*consistency*) $\forall \; x, \; p, \; h^M(x, p, p) = H(x, p)$.

A7. (*monotonicity*) for any functions u, v, such that $u \leq v \implies S^M(u) \leq S^M(v)$. Consistency property (A5) with (A6) implies that for any $v \in C^2([0, T] \times \mathbb{R})$, there exists a constant $C_M \geq 0$ independent of Δx such that

$$\bigg| h^M(x, D^- v(x), D^+ v(x)) - H(x, v_x) \bigg| \leq C_M \Delta x \|\partial_{xx} v\|_\infty. \tag{6}$$

Hence the consistency error estimate:

$$\begin{aligned} \mathcal{E}_{S^M}(v)(t,x) &:= \bigg| \frac{v(t + \Delta t, x) - S^M(v(t,.))(x)}{\Delta t} - \big(v_t(t,x) + H(x, v_x(t,x))\big) \bigg| \\ &\leq \; C_M \bigg(\Delta t \|\partial_{tt} v\|_\infty + \Delta x \|\partial_{xx} v\|_\infty \bigg). \end{aligned} \tag{7}$$

Remark 2.1. *Assuming (A5), it is easily shown that the monotonicity property (A7) is equivalent to that $h^M = h^M(x, p^-, p^+)$ satisfies, a.e. $(x, p^-, p^+) \in \mathbb{R}^3$:*

$$\frac{\partial h^M}{\partial p^-} \geq 0, \quad \frac{\partial h^M}{\partial p^+} \leq 0, \tag{8}$$

(also denoted $h^M = h^M(\cdot,\uparrow,\downarrow)$), and the CFL condition

$$\frac{\Delta t}{\Delta x}\left(\frac{\partial h^M}{\partial p^-}(x,p^-,p^+) - \frac{\partial h^M}{\partial p^+}(x,p^-,p^+)\right) \le 1. \tag{9}$$

When using finite difference schemes, it is assumed that the CFL condition (9) is satisfied, and that can be written equivalently in the form

$$c_0\frac{\Delta t}{\Delta x} \le 1, \tag{10}$$

where c_0 is a constant independent of Δt and Δx.

Case 2: S^M based on a semi-Lagrangian (SL) scheme. Let $I_1[u]$ denote the P_1-interpolation of a function u in dimension one on the mesh $G = \{x_j\}$, i.e.

$$I_1[u](x) = \frac{x_{j+1}-x}{\Delta x}u_j + \frac{x-x_j}{\Delta x}u_{j+1} \quad \text{for} \quad x\in[x_j,x_{j+1}] \tag{11}$$

Then the SL scheme for (1) is

$$u_j^{n+1} = \min_{a\in R}\{I[u^n](x_j - a\Delta t)\Delta t H^*(a)\}, \tag{12}$$

where $H^*(p) = \sup_{p\in R}\{p\cdot q - H(p)\}$ is the Legendre-Fenchel conjugate ([5, 9]). SL approximation mimics the method of characteristics looking for the foot of the characteristic curve passing through every node, and following this curve for a single time step. Above SL scheme with P_1-interpolation is monotone stable and works for the large Courant number and for more details we refer reader to see [10].

• S^A is a high-order scheme. We consider an iterative scheme of “high–order” in the form written as

$$S^A(u^n)(x) = u^n(x) - \Delta t h^A(x, D^{k,-}u^n(x),\dots,D^-u^n(x),D^+u^n(x),\dots,D^{k,+}u^n(x)), \tag{13}$$

where h^A corresponds to a “high-order” numerical Hamiltonian, we assume that

A8. h^A is Lipschitz continuous.

$$D^{\ell,\pm}u(x) := \pm\frac{u^n(x\pm\ell\Delta x) - u^n(x)}{\Delta x} \quad \text{for} \quad \ell = 1,\dots,k.$$

To simplify the notation we may write (13) in the more compact form

$$S^A(u^n)(x) = u^n(x) - \Delta t h^A\big(x, D^\pm u^n(x)\big) \tag{14}$$

even if there is a dependency on ℓ in $(D^{\ell,\pm}u^n(x))_{\ell=1,\dots,k}$.

The high-order consistency implies, for all $\ell\in[1,\dots,k]$, and for $v\in C^{\ell+1}(\mathbb{R})$,

$$\left|h^A(x,\dots,D^-v,D^+v,\dots) - H(x,v_x)\right| \le C_{A,\ell}\|\partial_x^{\ell+1}v\|_\infty\Delta x^\ell.$$

(Centered scheme) A typical example with $k = 2$ is obtained with the centered TVD (Total Variation Diminishing) approximation in space and the Runge-Kutta 2nd order scheme in time (or Heun scheme):

$$S_0(u^n)_j := u_j^n - \Delta t H(x_j, \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}),\ S^A(u) := \frac{1}{2}(u + S_0(S_0(u))) \tag{15}$$

• F is the filter function. We consider the following filter function which has been introduce in [16, 4] and used in [18]:

$$F(x) := x 1_{|x|\leq 1} = \begin{cases} x & \text{if } |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

The idea of filter function is to keep the high–order scheme when $|h^A - h^M| \leq \epsilon$ (because then $|S^A - S^M|/(\tau\epsilon) \leq 1$ and $S^F = S^M + \tau\epsilon F(\frac{S^A - S^M}{\tau\epsilon}) \equiv S^A$), whereas $F = 0$ and $S^F = S^M$ if that bound is not satisfied, i.e., the scheme is simply given by the monotone scheme itself.

Filtered scheme is **"ϵ-monotone"** in the sense that

$$u_j \leq v_j,\ \forall j, \quad \Rightarrow \quad S^F(u)_j \leq S^F(v)_j + \epsilon\tau \|F\|_{L^\infty},\ \forall j. \tag{17}$$

with $\epsilon \to 0$ as $(\Delta t, \Delta x) \to 0$. This implies the convergence of the scheme (see Appendix 5) by Barles-Souganidis convergence theorem (see [2]).

2.1. Adding a limiter

Furthermore, It has been already mentioned in [4] that in case of nonlinear PDEs when we filtered high-order scheme with the monotone scheme then filtered scheme switches back to first order after a few time steps. Then a limiting process has been introduced in [4] to obtain high order accuracy and that is made precise in the case of front-propagation models. This limiting process was not needed in [11, 16] for the treatment of steady equations. Filtered scheme may let small errors occur near extrema, when two possible directions of propagation occur in the same cell. This is the case for instance near a minima for an eikonal equation. In order to improve the scheme near extrema, we used the same limiter which was proposed in [4]. It will be needed only at extrema. We recall the limiter from [4]. Let us consider the case of front propagation, i.e., equation of type (1), with the following Hamiltonian

$$H(x, v_x) = \max_{a \in A} \big(f(x,a) v_x \big) \tag{18}$$

In the one-dimensional case, the cell centered in x_j may need a correction if there is a local minima and if

$$\min_a f(x_j,a) \le 0 \quad \text{and} \quad \max_a f(x_j,a) \ge 0. \tag{19}$$

We decide to "mark" such cells. For a marked cell, the numerical solution should u_j^{n+1} not go below the local minima around the point, i.e., we want

$$u_j^{n+1} \ge u_{min,j} := \min(u_{j-1}^n, u_j^n, u_{j+1}^n), \tag{20}$$

and, in the same way, we want to impose that

$$u_j^{n+1} \le u_{max,j} := \max(u_{j-1}^n, u_j^n, u_{j+1}^n), \tag{21}$$

as it would be the case in order to have the L^∞ stability for an advection equation. If we consider the high-order scheme to be of the form $u_j^{n+1} = u_j^n - \Delta t h^A(u^n)$, then the limiting process amounts to saying that

$$h^A(u^n)_j \le h_j^{max} := \frac{u_j^n - u_{min,j}}{\Delta t} \text{ and } h^A(u^n)_j \ge h_j^{min} := \frac{u_j^n - u_{max,j}}{\Delta t}.$$

This amounts to define a limited $\bar{h}^A$ such that

$$\begin{cases} \bar{h}^A(u^n)_j := \min\Big(\max(h^A(u^n)_j,\, h_j^{min}),\, h_j^{max}\Big), & \text{if (19) holds at mesh point } x_j, \\ \bar{h}_j^A :\equiv h_j^A \text{ otherwise.} \end{cases}$$

Then the filtering process is the same, using $\bar{h}^A$ instead of h^A in the definition of S^F.

For two dimensional equations a similar limiter could be developped in order to make the scheme more efficient at singular regions. However, for the numerical tests of the next section (in two and three dimensions) we will simply limit the scheme by using an equivalent of (20)-(21). Hence, instead of the scheme value $u_{ij}^{n+1} = S^A(u^n)_{ij}$ for the high–order scheme, we will update the value by

$$u_{ij}^{n+1} = \min(\max(S^A(u^n)_{ij}, u_{ij}^{min}), u_{ij}^{max}), \tag{22}$$

where $u_{ij}^{min} = \min(u_{ij}^n, u_{i\pm1,j}^n, u_{i,j\pm1}^n)$ and $u_{ij}^{max} = \max(u_{ij}^n, u_{i\pm1,j}^n, u_{i,j\pm1}^n)$. Moreover, the filtered scheme (3) needs the use of a filtering parameter "ϵ" that must be chosen in order to switch between the high-order scheme and the monotone scheme in a convenient way. A natural upper bound for the parameter is given in [11, 16, 4], of order $O(\sqrt{\Delta x})$ and precise lower bound has been given in [4] (see the Appendix 5). In our simulations, we will use $\epsilon = c_1 \Delta x$ where c_1 is a constant dependent on the second derivative of the data in order to obtain numerically a high order behaviour, and therefore our choice is similar to [4] and slightly different from the one of [16]. Error estimates for filtered scheme has been obtained for general time-dependent HJ equations, of order $O(\sqrt{\Delta x})$ where Δx is the spatial mesh size, under a standard CFL (10) condition on the time step.

3. Numerical examples

This section is dedicated to the numerical examples in several dimensions. Here we compare high-order scheme alone with the filtered scheme and ENO scheme (of same order). We will be more precise with CFL number and the order of scheme used in every example. Every example have been chosen to give different feature of the scheme. In Example 3.1 and 3.2 we are solving advection and eikonal equation in 1D with periodic boundary condition and error calculations are global. Example 3.3, solves eikonal equation with non-smooth initial data and Example 3.4 with smooth initial data with variable velocities in 2D. Last example of the paper is eikonal equation in 3D with smooth fronts. In this example ENO scheme is very slow as compare to filtered scheme we also added the CPU time of the filtered and ENO scheme. Example 3.3 onward we are using Dirichlet boundary conditions and we have calculated local error in the L^2 norms in the sub-domain D, at a given time t_n, corresponds to

$$e_{L^2_{loc}} := \left(\Delta x \sum_{\{i,\ x_i \in D\}} |v(t_n, x_i) - u_i^n|^2 \right)^{1/2}$$

and similarly L^1 and Ł$^\infty$ errors also comparable. Mx, My, Mz and Nt are the number of nodes in the x, y, z and t respectively.

Example 3.1. 1D Advection equation

$$v_t + v_x = 0, \quad t > 0,\ x \in (-2, 2), \tag{23}$$

$$v(0, x) = v_0(x) = \max\left(0, 1 - |x|^2\right)^4, \quad x \in (-2, 2). \tag{24}$$

Final time $T = 0.3$, CFL is 0.37 and filtering parameter $\epsilon = 4\Delta x$. This smooth initial data is chosen in order to have at least a 3rd order continuous derivative at $x = \pm 1$. For the monotone scheme S^M we are using upwind Hamiltonian ($h^M(v_x) = v_x = Dv_j^-$) with Euler forward in time. For high-order scheme we are testing two cases (second

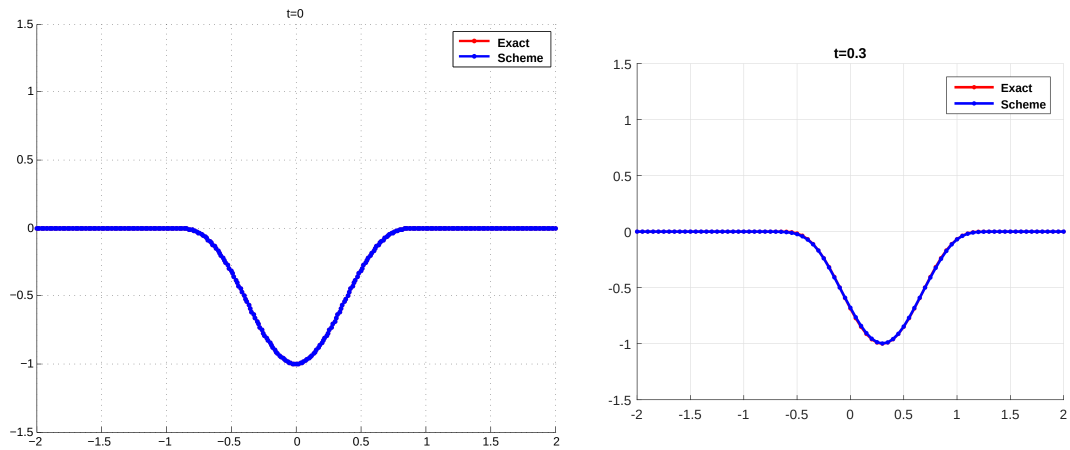

Figure 1: Example 3.1, On the left initial data (24) and on the right solution by filtered scheme.

and third order schemes).

(1) *Second order scheme: Here the high-order scheme S^A is central finite difference (Centered) scheme in space and TVD (Total Variation Diminishing) Runge-Kutte2 (as in* (15)*) in time. Results are given in Table 1 for the errors in L^2 norms, we compared the centered scheme, the second ENO (ENO2) scheme with RK2 in time (for more detail see Appendix 5 and [17].)*

(2) *A third order shceme: Here high-order scheme S^A is a third order scheme. The derivative v_x estimated using a third order backward difference in space i.e.*

$$h^A(v)_j := v_x(x_j) \equiv \frac{1}{\Delta x}\left(\frac{11}{6}v(x_j) - 3v(x_{j-1}) + \frac{3}{2}v(x_{j-2}) - \frac{1}{3}v(x_{j-3})\right), \tag{25}$$

with usual TVD-RK3 in time as in [12] (see (37) *in the Appendix 5). Results are given in Table 2 are the full errors Table. It is indeed also observed near to third-order convergence. This is only true for small enough CFL numbers though (CFL≤ 0.35), otherwise it was numerically observed a switch to second order.*

Example 3.2. (1D Advection + Eikonal equation).

$$v_t + \frac{1}{2}v_x + |v_x| = 0, \quad t > 0,\ x \in (-2, 2), \tag{26}$$

final time $T = 0.3$ with the $CFL = 0.37$ and initial data (24)*. We are using SL scheme with P_1 interpolation as monotone scheme S^M as defined in* (12)*. For high-order scheme S^A we use backward third order discretization* (25) *in space with TVD*

		Filter $\epsilon = 4\Delta x$		Centered		ENO2	
Mx	Nt	L^2 error	order	L^2 error	order	L^2 error	order
40	8	1.26E-02	1.98	1.26E-02	1.98	2.29E-02	1.79
80	16	3.07E-03	2.03	3.07E-03	2.03	5.96E-03	1.95
160	32	7.66E-04	2.00	7.66E-04	2.00	1.51E-03	1.98
320	64	1.90E-04	2.01	1.90E-04	2.01	3.77E-04	2.00
640	128	4.76E-05	2.00	4.76E-05	2.00	9.41E-05	2.00

Table 1: (Example 3.1.) Global L^2 errors for Filter, Centered scheme and ENO (2nd order) scheme with RK2 in time.

Mx	Nt	L^1 error	order	L^2 error	order	L^∞ error	order
41	8	1.67E-02	2.78	1.19E-02	2.71	1.41E-02	2.64
81	16	2.21E-03	2.92	1.60E-03	2.89	1.86E-03	2.93
161	32	2.77E-04	2.99	2.07E-04	2.95	2.87E-04	2.69
321	64	3.43E-05	3.02	2.64E-05	2.97	4.78E-05	2.58
641	128	4.51E-06	2.93	3.43E-06	2.94	7.26E-06	2.72

Table 2: (Example 3.1.) Global Errors for the third order filter scheme ($\epsilon = 4\Delta x$).

RK3 (37) *in time as defined in the previous example. This is a non-linear PDE which involve with advection and Eikonal term* ($|u_x| = \max_{a\in\{-1,1\}}(av_x)$) *and for this case filtered scheme switches to first order near extrema. In order to have high-order we added a limiter as defined in Section 2.1. As expected semi-Lagrangian scheme with* P_1 *interpolation shows first order behavior. It is clear from the error Table 3 that filtered scheme alone is only first order however when we add limiter then order improves.*

Example 3.3. 2D Eikonal equation with non-smooth initial data.

$$v_t + |\nabla v| = 0, \quad , t > 0 \ (x, y) \in (-3, 3)^2, \tag{27}$$

$$v(0, x, y) = v_0(x, y) = \|(x, y)\|_\infty - r_0, \ (x, y) \in (-3, 3)^2 \tag{28}$$

The initial condition square centered at origin with the sides $r_0 = 1$. *We choose* $\epsilon = 10\Delta x$ *with CFL is* 0.37. *In the monotone scheme we will use Lax-Friedrich flux*

		Filter+Limiter $\epsilon = 4\Delta x$		Third-order		SL-P_1	
$Mx = My$	Nt	L^2 error	order	L^2 error	order	L^2 error	order
41	8	12.6E-02	0.89	3.36E-02	0.99	3.20E-02	0.82
81	15	2.85E-03	2.26	1.72E-02	0.96	1.65E-02	0.95
161	30	5.61E-04	2.35	8.72E-03	0.98	8.57E-03	0.95
321	59	7.97E-05	2.82	4.39E-03	0.99	4.32E-03	0.99
641	118	1.16E-05	2.03	2.20E-03	1.00	2.18E-03	0.99

Table 3: (Example 3.2.) Global L^2 errors for filter scheme with limiter, third-order scheme and semi-Lagrangian scheme with P_1 interpolation.

i.e.

$$h^{M,LF}\left(\phi_1^- \phi_1^+, \phi_2^-, \phi_2^+\right) = H\left(\frac{\phi_1^- + \phi_1^+}{2}, \frac{\phi_2^- + \phi_2^+}{2}\right) - \frac{C_x}{2}\left(\phi_1^+ - \phi_1^-\right) - \frac{C_y}{2}\left(\phi_2^+ - \phi_2^-\right), \tag{29}$$

where $C_x = \max_{A \le \phi_1 \le B} |H_{\phi_1}(\phi_1, \phi_2)|$, $C_y = \max_{A \le \phi_2 \le B} |H_{\phi_2}(\phi_1, \phi_2)|$ *and* $H_i(\phi_1, \phi_2)$ *is the partial derivative of* H *with respect to* i*-th argument, or the Lipschitz constant of* H *with respect to the* i*-th argument and* $A = (\phi_1^-, \phi_1^+)$, $B = (\phi_2^-, \phi_2^+)$ *with the CFL condition* (10). *Centered scheme with TVD Runge-Kutta 2 in time.*

$$S^{A,1}(\phi_{ij}^n) := \phi_{ij}^n - \Delta t h\left(\frac{\phi(x_i + \Delta x, y) - \phi(x_i - \Delta x, y)}{2\Delta x}, \frac{\phi(x, y_i + \Delta x) - \phi(x, y_i - \Delta x)}{2\Delta x}\right),$$

$$S^A(\phi_{ij}) := \frac{1}{2}\left(\phi_{ij}^n + S^{A,1}(S^{A,1}(\phi_{ij}^n))\right). \tag{30}$$

We also added 2d limiter here (22). *The motivation of showing this example is that we start with the front with sharp corners and the evolution proceeds in the outward direction. Initially front has sharp corners but after the evolution it becomes smooth thats why local errors have been calculated. We have given the full error table of filtered scheme in Table 4.*

Example 3.4. *(***2D Eikonal equation with variable velocities.)** *We are solving 2D Eikonal equation 27 in the same domain as in Example 3.3 with the smooth initial data*

$$v(x, y) = 0.5 - 0.5 \max\left(0, \frac{1 - x^2 - y^2}{1 - r_0^2}\right)^4, \tag{31}$$

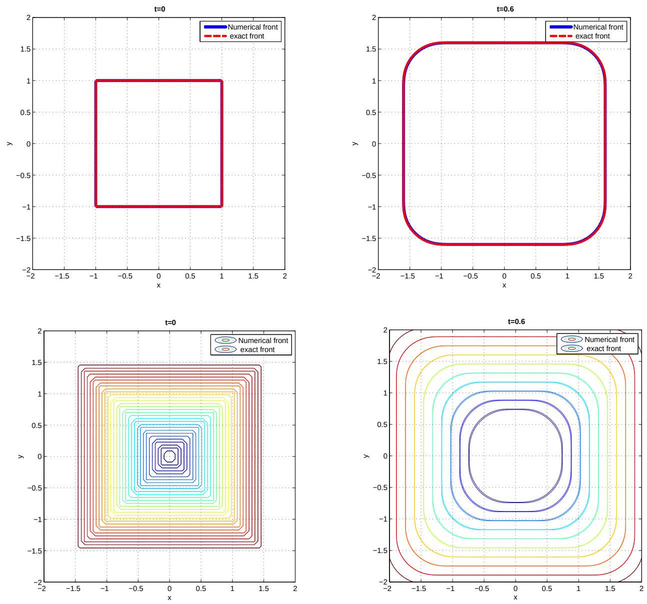

Figure 2: Example 3.3, Square.

		L_1-Error		L_2-Error		L_∞-Error	
$Mx = My$	Nt	error	order	error	order	error	order
100	50	6.89E-03	2.23	6.65E-03	2.12	9.36E-03	2.09
200	100	1.80E-03	1.93	1.84E-03	1.86	3.53E-03	1.41
400	200	3.02E-04	2.58	3.56E-04	2.37	1.10E-03	1.68
800	400	7.52E-05	2.01	8.72E-05	2.03	2.20E-04	2.32

Table 4: Example 3.3, local errors filtered scheme and RK2 in time where $\epsilon = 10\Delta x$ and with CFL=0.37.

and CFL is 0.37. Moreover, we assume the velocity $f(x, y)$ to be Lipschitz continuous. Numerical tests are performed here for the following different variable velocities. Here we will present numerical solution without the error tables. In the monotone scheme we use Lax-Friedrich flux (29) *and for high-order scheme we use centered scheme with TVD Runge-Kutte 2* (30) *in time. We are dealing non-linear PDE hence in order to improve the accuracy we added 2d limiter* (22)*.*

(i) $f(x, y) = |x|$ *in the Fig. 4 solved by the filtered scheme with* $\epsilon = 20\Delta x$ *and T=1.*

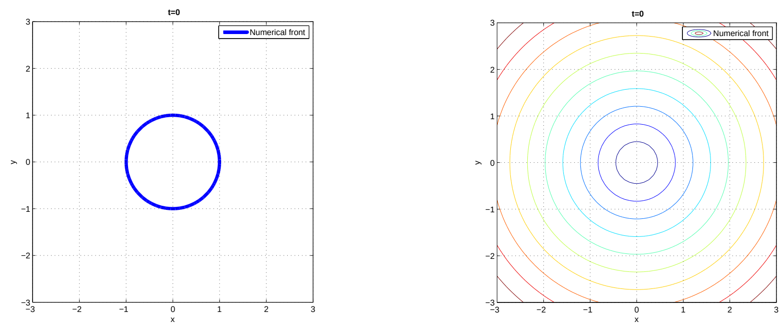

Figure 3: Example 3.4 Initial data (31)

(ii) $f(x,y) = |y|$ *in the Fig. 5 solved by the filtered scheme with* $\epsilon = 20\Delta x$ *and T=1.*

(iii) $f(x,y) = |x| + |y|$ *in the Fig. 6 solved by the filtered scheme with* $\epsilon = 20\Delta x$ *and T=0.8.*

(iv) $f(x,y) = (cos(\frac{\pi}{6}), sin(\frac{\pi}{6}))$ *in the Fig. 7 solved by the filtered scheme with* $\epsilon = 20\Delta x$ *and* $T = 0.6$.

(v) $f(x,y) = (f_1, f_2) = (|x|cos(\frac{\pi}{6}), |y|sin(\frac{\pi}{6}))$ *in the Fig. 8 solved by filtered scheme with* $\epsilon = 20\Delta x$ *and* $T = 0.6$.

(vi) $f(x,y) = (f_1, f_2) = (|x|cos(\frac{\pi}{6}), |x|sin(\frac{\pi}{6}))$ *in the Fig. 9 solved by filtered scheme with* $\epsilon = 20\Delta x$ *and* $T = 0.6$.

Note that after few time steps front expand and the solution is not smooth anymore even though initial data was smooth. So that we cannot expect filtered scheme to have high-order behavior everywhere. Hence filter scheme shows nice expansion of front and locally second order. The Fig. 4 and 5 show the direction of velocity of propagation $f(x,y)$ *in the direction of* x *and* y *axis respectively. On the other hand Fig. 6, 7, 8 and 9 are different direction of propagation.*

Example 3.5. (3D Eikonal equation) *We are solving same 3D Eikonal equation as in Example 3.3. This is the last example of the paper. Motivation to present this example, is that if we have more than two fronts then still filtered scheme is second order. In this example we have five spheres*

$$v_k(x,y,z) = r_0 - r_0 \max\left(0, \frac{1-(x-x_k)^2-(y-y_k)^2-(z-z_k)^2}{1-r_0^2}\right)^4$$

$k = 1, ..., 5$ *they all have same radius* $r_0 = 0.25$. *For* $k = 1, ..., 5$ *centers* (x_k, y_k, z_k) *are* $(1,0,0)$, $(-1,0,0)$, $(0,0,0)$, $(-1,0,0)$, $(0,-1,0)$, $(0,1,0)$. *Computations are done*

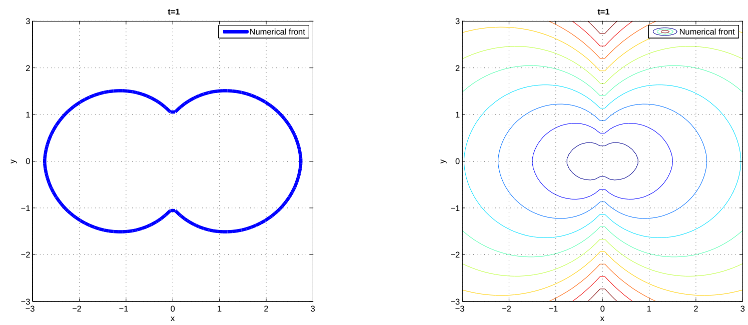

Figure 4: Example 3.4 (i) , $f(x, y) = |x|$ and T=1 solved by the filtered scheme.

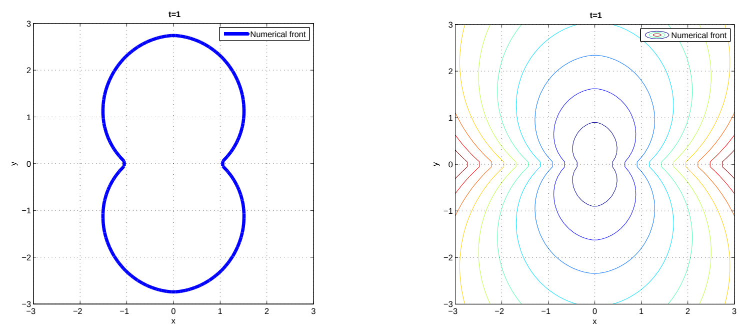

Figure 5: Example 3.4 (ii), $f(x, y) = |y|$ and T=1 solved by filtered scheme.

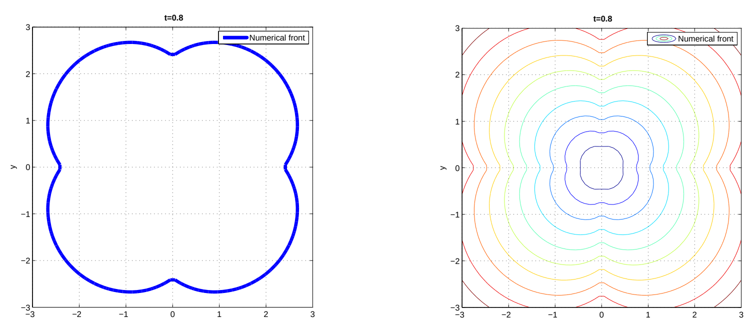

Figure 6: Example 3.4 (iii), $f(x, y) = |x| + |y|$ solved by filtered scheme and T=0.8.

on the domain $\Omega = (-2, 2)^3$*, CFL is 0.37 and* $\epsilon = 20\Delta x$*. Centered finite difference is not stable and filtered scheme is faster than the ENO2 scheme. In the Table 5 we presented* L^2 *local errors (the results are similar for the* L^1 *and the* L^∞ *errors) and we also added the CPU time and also* $Mx = My = Mz = M$*. Error calculations are local away from singularity.*

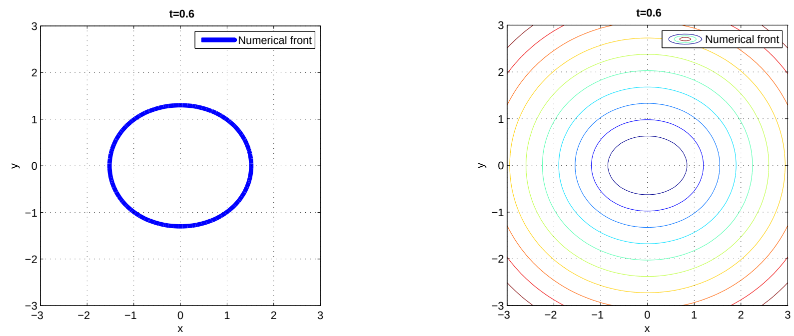

Figure 7: Example 3.4 (iv), $f(x,y) = (cos(\frac{\pi}{6}), sin(\frac{\pi}{6}))$ solved by filtered scheme and T=0.6.

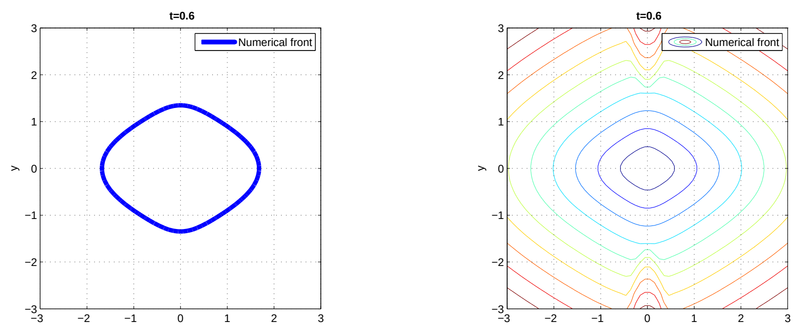

Figure 8: Example 3.4 (v), $f(x,y) = (|x|cos(\frac{\pi}{6}), |y|sin(\frac{\pi}{6}))$ solved by filtered scheme and T=0.6.

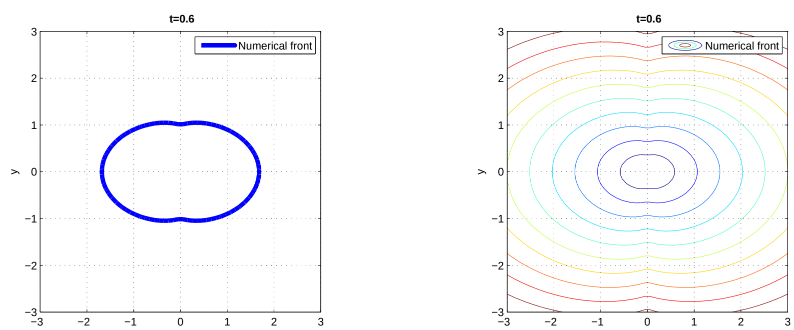

Figure 9: Example 3.4 (vi), $f(x,y) = (|x|cos(\frac{\pi}{6}), |y|sin(\frac{\pi}{6}))$ solved by filtered scheme and T=0.6.

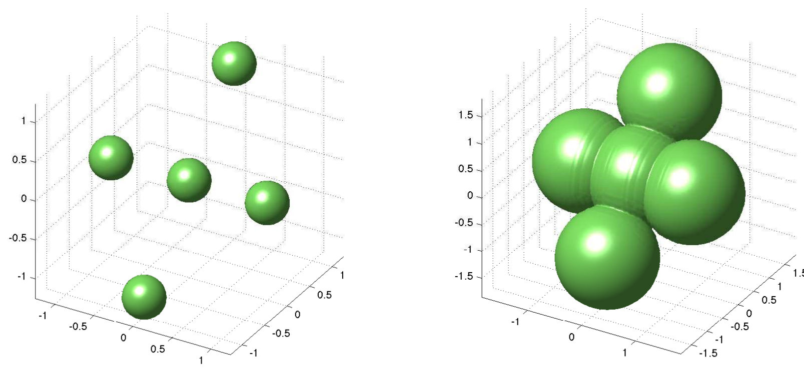

Figure 10: Example 3.5, on the left we have initial configuration of five spheres of radius $r_0 = 0.25$ and on the right expanded fronts at time $T = 0.6$.

Errors		filtered ($\epsilon = 20\Delta x$)			centered		ENO		
M	Nt	L^2 error	order	CPU $_{\text{time}}$	L^2 error	order	L^2 error	order	CPU $_{\text{time}}$
25	13	1.43E-01	-	1.30	1.69E-01	-	1.30E-01	-	1.60
50	26	6.37E-02	1.17	5.78	1.54E-01	0.14	4.18E-02	1.64	9.46
100	52	1.50E-02	2.09	130.5	1.46E-01	0.08	1.20E-02	1.79	204.6
200	104	3.95E-03	1.92	1.3E+03	2.25E+01	-7.46	3.75E-03	1.68	5.2E+03

Table 5: (Example 3.5) local errors ENO scheme CFL=0.37 $T = 0.6$.

4. Conclusion

We have solved several examples upto three-dimension for non-linear PDEs by filtered scheme. Filtered scheme constructed to take the advantage of the low and and high-order methods. When solution is smooth filtered scheme switches to high-order otherwise switches to low-order. The approach in general can be apply to filter different schemes. In Example 3.1 we have solved advection equation by second and third filtered scheme where the monotone scheme was upwind and SL scheme and high-order scheme was centered scheme and backward third order discretization in space. Resultant scheme is high order as expected. We also solved eikonal equation upto three-dimension. Notice that when we solved eikonal equation we added a limiter. It remains to improve the choice of the filtering parameter ϵ, and the limiting process is only detailed here in 1D but not in 2D. The third order behavior is not obtained in some particular cases. This is the subject of ongoing works. However we emphasize that in most cases we observe second order behavior with a relatively simple scheme, together with a provable convergence and error estimates (see Apendix 5) .

5. Appendix

Theorem 1. *Convergence Theorem. Let Hamiltonian H and initial data v_0 be Lipschitz continuous (A1)-(A4). S^M be the monotone scheme (either finite difference scheme* (5) *with monotone and consistent numerical Hamiltonian or semi-Lagrangian scheme* (12)*) satisfies (A5)-(A7). Let S^A be any "high-order" scheme* (14) *(possibly unstable). Let $v_j^n := v(t_n, x_j)$ where v is the exact solution of* (1)*. Assume switching parameter*

$$0 < \epsilon \le c_0\sqrt{\Delta x} \tag{32}$$

for some constant $c_0 > 0$.

(i) The scheme u^n satisfies the Crandall-Lions estimate

$$\|u^n - v^n\|_\infty \le C\sqrt{\Delta x}, \quad \forall\ n = 0, ..., N. \tag{33}$$

for some constant C independent of Δx.

(ii) (First order convergence for classical solutions.) If furthermore the exact solution v belongs to $C^2([0,T]\times\mathbb{R})$, and $\epsilon \le c_0\Delta x$ (instead of (32)*), then, we have*

$$\|u^n - v^n\|_\infty \le C\Delta x, \quad n = 0, ..., N, \tag{34}$$

for some constant C independent of Δx.

(iii) (Local high-order consistency.) Assume that S^A is a high-order scheme satisfying (A8) for some $k \ge 2$. Let $1 \le \ell \le k$ and v be a $C^{\ell+1}$ function in a neighborhood of a point $(t,x) \in (0,T)\times\mathbb{R}$. Assume that

$$(C_{A,1} + C_M)\left(\|v_{tt}\|_\infty\,\tau + \|v_{xx}\|_\infty\,\Delta x\right) \le \epsilon. \tag{35}$$

Then, for sufficiently small $t_n - t$, $x_j - x$, Δt, Δx, it holds

$$S^F(v^n)_j = S^A(v^n)_j$$

and, in particular, a local high-order consistency error for the filtered scheme S^F holds:

$$\mathcal{E}_{S^F}(v^n)_j \equiv \mathcal{E}_{S^A}(v^n)_j = O(\Delta x^{\ell})$$

(the consistency error $\mathcal{E}_{S^A}$ is defined in (15)*.*

For the proof of the above theorem we refer reader to see [4].

Bound for the switching parameter ϵ: • Choose $\epsilon \leq c_0\sqrt{\Delta x}$ for some constant $c_0 > 0$ in orderthat the convergence and error estimate result holds (see Theorem 1).
• Choose $\epsilon \geq c_1 \Delta x$, where c_1 is sufficiently large. This constant should be chosen roughly such that

$$\frac{1}{2}\|v_{xx}\|_\infty \left\| \frac{\partial h^M}{\partial u^+}(., v_x, v_x) - \frac{\partial h^M}{\partial u^-}(., v_x, v_x) \right\|_\infty \leq c_1.$$

where the range of values of v_x and v_{xx} can be estimated, in general, from the values of $(v_0)_x$, $(v_0)_{xx}$ and the Hamiltonian function H. Then the scheme is expected to switch to the high-order scheme where the solution is regular. For more details we refer reader to see [4].

An essentially non-oscillatory (ENO) scheme of second order We recall here a simple second order ENO method based on the work of Osher and Shu [17] for HJ equation. ENO procedure is a first strategy of reconstruction which has been developed in order to reduce Gibb's oscillations. ENO interpolation cuts essentially such oscillations and retain a high-order of accuracy where the solution is smooth. Here we will give an idea of second order ENO reconstruction and in the same manner one can generalized for any order. Here we follow the same notation and discretization from described in section 2). Let m be the minmod function defined by

$$m(a,b) = \begin{cases} a & \text{if } |a| \leq |b|,\ ab > 0 \\ b & \text{if } |b| < |b|,\ ab > 0 \\ 0 & \text{if } ab \leq 0 \end{cases} \tag{36}$$

(other functions can be considered such as $m(a,b) = a$ if $|a| \leq |b|$ and $m(a,b) = b$ otherwise). Let $D^{\pm}u_j = \pm(u_{j\pm1} - u_j)/\Delta x$ and

$$D^2 u_j := \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}.$$

Then the right and left ENO approximation of the derivative can be defined by

$$\bar{D}^{\pm}u_j = D^{\pm}u_j \mp \frac{1}{2}\Delta x\ m(D^2u_j, D^2u_{j\pm1})$$

and the ENO (Euler forward) scheme by

$$S_0(u)_j := u_j - \tau h^M(x_j,\, \bar{D}^- u_j,\, \bar{D}^+ u_j).$$

The corresponding RK2 scheme can then be defined by $S(u) = \frac{1}{2}(u + S_0(S_0(u)))$. **TVD RK3 scheme:** Here we are recalling third order TVD Runge Kutta scheme from [12]

$$u_j^{n,1} := u_j^n - \Delta t h(x_j, D^{\mp} u_j^n). \tag{37}$$

$$u_j^{n,2} := \frac{3}{4}(u_j^n + \frac{1}{4}u^{n,1} - \Delta t h(x_j, D^{\mp} u_j^{n,1}). \tag{38}$$

$$u_j^{n+1} = \frac{1}{3}u_j^n + \frac{2}{3}u^{n,2} - \frac{2}{3}\Delta t h(x_j, D^{\mp} u_j^{n,2}), \tag{39}$$

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MATCOM-S-16-00174 muscript
"High-order filtered schemes for first order time dependent linear and non-linear partial differential equations."
by Smita Sahu

Answers to Referee's Comments (Report #1)

Replace "Examples" with "examples" in the title of section 4.

Corrected.

*Response to Reviewers

MATCOM-D-16-00136R1 Manuscript
"High-order filtered schemes for first order time dependent linear and non-linear partial differential equations."
by Smita Sahu

The paper has been carefully revised according to the comments and suggestions of the reviewer(s) and Editor, We thank the Editor and the reviewer(s) for their careful reading of the paper and for their suggestions. Below is the detailed description of those changes.

- We mentioned all changes in colour blue.
- We checked and corrected all the misprints Referee 1 pointed to our attention.
- Reference [18], modified.
- We added two references [5,9] for Legendre-Fenchel conjugate.
- We have removed the Fig 1 (in order to save some space for the appendix).

Answers to Referee's Comments (Report #3)

The paper is potentially an interesting one, but it requires a major revision. The novelty of the proposed study against existing work in the recent literature could be clarified better. The comparison against other solution approaches could be extended. Finally, the choice of the test examples could be motivated more in detail to state the relevance of the numerical results.

The text needs to be edited in several points for typsetting mistakes (e.g., p. 2, last paragraph, "In the [4]" should be "In [4]"; p. 4, 2nd paragraph, "based a first" should be "based on a first"; "we consider" should be "We consider"; p. 6, Section 4, 1st paragraph, "We will" should be "we will"; p. 14, Example 4.4, "eq:eikonal2d" should be **??**; p. 18, Section 5, "thee" should be "three").

Punctuation is incorrect in several points as well, e.g., after Eqns. (1)-(2), after Eqn. (10), after the equation below Eqn. (11), after Eqn. (22), etc.

Some sentences need to be reworded, e.g. in Section 2 "In section 4, dedicated to numerical examples. Where we present ..."; in Example 4.1 "Same example with third order finite difference discretization in space and TVD-RK3 scheme in time [10]."; in Section 5, "Last two examples on the paper in the dimension three." and "Filtered scheme have nice evolution and very fast."

Answer: We have carefully revised the paper. All the changes has been mentioned in blue colour.

A reference might be added for the Legendre-Fenchel conjugate at p. 5.

Answer: We added two references ([1, 2]) in page 5 just before the line 90.

References

[1] L. Corrias. Fast legendre-fenchel transform and applications to hamilton. *SIAM J. Numer. Anal.*, 33:1534–1558, 1996.

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*Response to Reviewers

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"High-order filtered schemes for first order time dependent linear and non-linear partial differential equations."
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The paper has been carefully revised according to the comments and suggestions of the reviewer(s) and Editor, We thank the Editor and the reviewer(s) for their careful reading of the paper and for their suggestions. Below is the detailed description of those changes.

- We mentioned all changes in blue colour.
- We checked and corrected all the misprints Referee 1 pointed to our attention.
- Reference [18], modified.
- We added two references [5,9] for Legendre-Fenchel conjugate.
- We inserted some more details to point out the various steps and all the changes are mentioned in blue colour.
- We inserted, an Appendix. In the appendix we stated the convergence result, bounds for filtering parameter, ENO and TVD Runge Kutta schemes.
- We have removed the Fig 1 (in order to save some space for the appendix).

Answers to Referee's Comments (Report #2)

The paper deals with "filtered schemes" which present a kind of blending procedures which combine monotone low-order and non-monotone high-order schemes. Though there are various types of blending described in the CFD literature, the particular form of filtering presented in the paper for solving Eiconal and Hamilton -Jacobi equations has the potential for being useful for constructing non-linear monotone locally high-order schemes. However, the paper suffers from serious drawbacks. In my opinion, they are as follows.

1. The main output of the paper are numerical examples while theoretical background is rather poor. For a reader convenience, more information concerning the filtered schemes is needed. For example, the filtering defined by Eq.(3) differs from that presented in Ref. [14]. The main theoretical outlines should be included either in the body of the manuscript or in the Appendix.

Answer: Yes the filtered scheme is differs from [2] beacause we are using filtered scheme from [1]. We added a paragraph in the introduction on page 2 just after the line 20. We also added an Appendix.

2. The information concerning third-order schemes is not presented properly. The sentences at the page 9 "Because of this non-linearity filtered scheme switches to first order near the extrema. We need to add a limiter to get high-order scheme" should be clarified. In

fact, switching to the first-order in unsmooth regions is due to the filtering definition given by Eq.(3); it is quite natural. It is unclear how limiters which are monotonization tools can increase approximation orders. In this context, more information and some discussion concerning Tables 4 and 5 are needed. For better understanding, it is worth comparing the graphics of the solutions of the original third-order scheme, the filtered third-order scheme without the limiter and the filtered third-order scheme with the limiter.

Answer: We added a subsection (Adding a limiter) on page 6. In Table 3 we added the word Filter+ limiter in the second column. It important to note that in this example here third order scheme alone is not third order. When we filtered third order with semi-Lagrangian scheme with P_1-interpolation we don't see improvement in the order of the scheme. Because in this example PDEs involve with eikonal (front propagation) term and in subsection 2.1 we have explained the problem and addition of a limiter. Third order scheme with limiter improves the order. However we stated in conclusion about this observation.

3. The presentation of the manuscript needs considerable improvement (constants M and N in Tables 1-7 are not defined; the scheme at the bottom of page 5 is labeled as iterative without any indication of the iterative procedure; there are spelling and grammar errors in the text; the italic format and the usual one are mixed in the text in the unclear manner etc.). Thus the revision of the full text is needed. In my opinion, the manuscript can be published in the Journal only if its in-depth revision will be carried out.

Answer: Paper has been carefully revised and all the modifications in blue colour. We have added a paragraph in Section 3 and Mx, My, Mz and Nt are the number of nodes in the x, y, z and t respectively. We have combined the Table 3,4,5 (which were the full table of semi-Lagrangian, third order and third-order filtered scheme) in to the Table 3 (which is a combined table of all three schemes).

References

[1] O. Bokanowski, M. Falcone, and S. Sahu. An efficient filtered scheme for some first order time-dependent Hamilton-Jacobi equations. *SIAM Journal on Scientific Computing*, 38(1):A171–A195, 2016.

[2] A. M. Oberman and T. Salvador. Filtered schemes for Hamilton-Jacobi equations: a simple construction of convergent accurate difference schemes. *J. Comput. Phys.*, 284:367–388, 2015.