L1 scheme on graded mesh for subdiffusion equation with nonlocal diffusion term

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Abstract

The solution of time fractional partial differential equations in general exhibit a weak singularity near the initial time. In this article we propose a method for solving time fractional diffusion equation with nonlocal diffusion term. The proposed method comprises L1 scheme on graded mesh, finite element method and Newton's method. We discuss the well-posedness of the weak formulation at discrete level and derive a priori error estimates for fully-discrete formulation in $L^2(\Omega)$ and $H^1(\Omega)$ norms. Finally, some numerical experiments are conducted to validate the theoretical findings.

Keywords: Nonlocal problem; initial singularity; L1 scheme; graded mesh; error estimate.

AMS(MOS): 65M12, 65M60, 35R11.

1 Introduction

The study of nonlocal problems has gained considerable attentions in recent years (see, [1–4,7] and references therein). Author in [4] consider the following parabolic nonlocal problem

$$u_t - a(l(u)) \Delta u + f(u) = h$$
 in $Q = \Omega \times (0, T)$,
 $u(x,t) = 0$ on $\Sigma = \Gamma \times (0, T)$, (1)
 $u(x,0) = u_0(x)$ in Ω ,

where Ω is a smooth bounded open subset of \mathbb{R}^d $(d \geq 1)$ with smooth boundary and $l:L^2(\Omega) \longrightarrow \mathbb{R}$ is a continuous linear form. This type of problems, besides its mathematical motivation, arises from physical situations related to migration of a population of bacterias in a container (say domain Ω) where u describes the density of the population of bacteria. The velocity of migration (\vec{v}) of this population is proportional to the gradient of the density with a positive factor 'a' depending on entire

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population, that is $\vec{v} = a \nabla u$, $a = a \left(\int_{\Omega} u \, dx \right)$. Authors in [5] have mentioned that fractional order differential equations are naturally related to systems with memory which exists in most biological systems. Also, in the nature there exist many processes that can not be adequately described with classical exponential law which is corresponding to integer order derivative. For example, the dynamics of population densities can follow a law that behaves like the exponential one but changes slowly or faster than the exponential function (for more details see [6]). This motivates us to consider following time fractional partial differential equation (PDE) with nonlocal diffusion term: Find u such that

$${}_{0}^{c}D_{t}^{\alpha}u(x,t) - a(l(u)) \Delta u(x,t) = f(u) \quad \text{in} \quad \Omega \times (0,T], \tag{2a}$$

$$u(x,t) = 0$$
 on $\partial \Omega \times (0,T]$, (2b)

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega, \tag{2c}$$

where Ω is an open bounded subset of \mathbb{R}^d (d=1 or 2) with smooth boundary $\partial\Omega$, $l(u) = \int_{\Omega} u(x,t) dx$ and the Caputo fractional derivative ${}_0^c D_t^{\alpha} u(x,t)$ $(0 < \alpha < 1)$ is defined as [16]

$${}_{0}^{c}D_{t}^{\alpha}u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} (t-s)^{-\alpha} \frac{\partial u(x,s)}{\partial s} ds, \quad t > 0.$$
(3)

Problem (2a)- (2c) is a time fractional version of the integer order nonlocal parabolic problem given in [1,2,4]. Authors in [13–15] have studied similar nonlocal problem using finite element method with uniform time grid. Due to wide range of applications, the development of effective numerical methods for time-fractional parabolic PDEs is of great importance. Many efficient time-stepping schemes have been proposed in the literature for linear time fractional PDEs. Mainly, these schemes can be divided into two class: L1 type schemes and convolution quadrature (CQ) (See [8,9,11,12,19,24–31] and references therein). An important consideration in the development of numerical methods for fractional diffusion problems is that solution u is weakly singular in time near time t = 0 [10,11,23,24,28,30]. Authors in [24] consider the following time fractional diffusion equation:

$${}_{0}^{c}D_{t}^{\alpha}u(x,t) - \Delta u(x,t) = f(u) \quad \text{in} \quad \Omega \times (0,T],$$

$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,T],$$

$$u(x,0) = u_{0}(x), \quad \text{in} \quad \Omega,$$

$$(4)$$

They show that if $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ then the solution to problem (4) satisfies $\|\partial_t u\|_{L^2(\Omega)} \leq Ct^{\alpha-1}$. The spaces $L^2(\Omega)$ and $H^m(\Omega)$, for m=1,2 are defined in the next section. In our work also, we assume that the solution to problem (2) satisfies

 $||u^{(l)}(t)||_{L^2(\Omega)} \lesssim 1 + t^{\alpha-l}$, for l = 0, 1, 2 and $t \in (0, T]$. In case of weak singularity near t=0, the L1 scheme on uniform mesh for problem (2) gives $\mathcal{O}(\tau^{\alpha})$ order of convergence in maximum norm in time, where τ denotes the time step size (see the numerical section) [24]. To overcome this issue, we analyse the L1 scheme on graded mesh [10,11] for the problem (2) having initial singularity at t=0. We have shown that with appropriate choice of the grading parameter r, one can recover the optimal convergence order $2-\alpha$ in temporal direction. Another concern of this work is to derive the optimal order of convergence in $L^2(\Omega)$ and $H^1(\Omega)$ norms in spatial direction. Authors in [19] have pointed that due to initial singularity of the solution and the discrete convolution form in numerical Caputo derivative, the traditional $H^1(\Omega)$ - norm analysis (corresponding to the case for a classical diffusion equation) to the time fractional diffusion problem always leads to suboptimal estimates. A similar conclusion is also drawn in [32], where authors have used direct discontinuous Galerkin method for solving the time fractional diffusion equation. For the derivation of optimal error estimate in $H^1(\Omega)$ norm, we follow the idea given in [9, 19]. To the best of our knowledge this is the first attempt when L1 scheme on graded mesh is used for solving the subdiffusion equation with nonlocal diffusion term. The main contribution of the present work are summarized below.

- To handle the weak singularity in the solution, we approximate ${}_{0}^{c}D_{t}^{\alpha}u(x,t)$ by well known L1 scheme on graded temporal mesh [11].
- For nonlocal term and nonlinearity in right-hand side, we use Newton's method.
- We derive a priori bound for the fully discrete solution in $L^2(\Omega)$, $H_0^1(\Omega)$ norms and prove the existence-uniqueness of fully discrete solution.
- In previous papers [13–15], the authors have shown convergence in $L^2(\Omega)$ norm using uniform mesh. We prove optimal rate of convergence in $L^2(\Omega)$ and $H^1(\Omega)$ norms using graded mesh.

Throughout the paper, C denote a generic constant (not necessarily same at different occurrences) while C_i , i = 1, 2, 3...15, are fixed constants; all these constants are independent of mesh parameters h and N.

The rest of the paper is organized as follows: In Section 2, first we recall some basic definitions and then define the weak formulation of given problem (2). In Section 3, we give fully-discrete scheme and derive the *a priori* bound for the fully-discrete scheme. We prove the existence-uniqueness of fully-discrete solution in section 4. Error analysis of our proposed scheme is presented in Section 5. Finally, numerical results in Section 6 confirm our theoretical estimates.

2 Preliminaries and weak formulation

Let $L^2(\Omega)$ be the space of square integrable functions on Ω with inner product $(g_1,g_2)=\int_{\Omega}g_1(x)g_2(x)\,dx$ and norm $\|g_1\|=\left(\int_{\Omega}|g_1(x)|^2\,dx\right)^{\frac{1}{2}}$. For a non-negative integer $m,H^m(\Omega)$ denote the Sobolev space on Ω with the norm $\|w\|_m=\left(\sum_{0\leq\alpha\leq m}\|\frac{\partial^\alpha w}{\partial x^\alpha}\|^2\right)^{1/2}$. Also, we write $\|w\|_{L^\infty(H^m)}=\sup_{0< t\leq T}\|w\|_m$ and

$$H_0^1(\Omega) := \left\{ v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega \right\}.$$

The weak formulation of given problem (2) is: find $u(\cdot,t) \in H_0^1(\Omega)$ such that for each $t \in (0,T]$ one has

$$\binom{c}{0}D_t^{\alpha}u, v) + a(l(u))(\nabla u, \nabla v) = (f(u), v), \quad \forall v \in H_0^1(\Omega).$$

$$u(x, 0) = u_0(x), \quad \text{in } \Omega.$$
(5)

In our further analysis, we need following hypotheses on given data.

- H1: $a : \mathbb{R} \to \mathbb{R}$ is bounded with $0 < m_1 \le a(x) \le m_2 < \infty, \ \forall x \in \mathbb{R}$.
- H2: $a: \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant L > 0, i.e.,

$$|a(x_1) - a(x_2)| \le L |x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}.$$
 (6)

• H3: $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant K > 0, i.e., $|f(u_1) - f(u_2)| \le K|u_1 - u_2|$, for $u_1, u_2 \in \mathbb{R}$.

Under the hypotheses H1-H3, it can be shown that there exist a unique weak solution of the problem (5). For the proof, one can use Faedo-Galerkin method in a standard way ([7,13,15,23]). Note that for the derivation of error estimates, we require some additional regularity of solution u and it is mentioned in Section 5.

3 Fully-discrete formulation and a priori bound

For the spatial discretization, we use Galerkin finite element method (FEM). Let Ω_h be a quasi uniform partition of Ω into disjoint subintervals in 1D and triangles in 2D with step size h. Let M > 0 be the finite integer and X_h be the M-dimensional subspace of $H_0^1(\Omega)$ such that X_h consists of continuous functions on closure $\overline{\Omega}$ of Ω which are linear in each $T_k \in \Omega_h$ and vanishes on $\partial \Omega$. *i.e.*,

$$X_h := \left\{ v \in C^0(\bar{\Omega}) : v_{|T_k} \in P_1(T_k), \ \forall \ T_k \in \Omega_h \ \text{and} \ v = 0 \text{ on } \partial \Omega \right\}.$$

For each i = 1, 2, ..., M, let $\phi_i(x)$ be the pyramid function in X_h which takes the value 1 at i - th node and vanishes at other node points. Then $\{\phi_i\}_{i=1}^M$ forms a basis for the space X_h .

For temporal discretization, we use L1 scheme on graded mesh [10,11]. For this let $N \in \mathbb{N}$ and $0 = t_0 < t_1 < t_2 < ... < t_N = T$ be a partition of the time interval [0,T] such that $t_n = T(\frac{n}{N})^r$, for n = 0, 1, ..., N, where $r \ge 1$ is the mesh grading parameter (for r = 1, the mesh is uniform). Also, the step size is given by $\tau_n = t_n - t_{n-1}$, for n = 1, 2, ..., N. For $1 \le n \le N$, let U^n denote the approximate value of u at the node point t_n .

Now, the L1-approximation to Caputo fractional derivative on the graded mesh [10–12] is given below.

$${}_{0}^{c}D_{t_{n}}^{\alpha}u = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t_{n}} (t_{n}-s)^{-\alpha} \frac{\partial u(x,s)}{\partial s} ds$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} w_{n,k} \left(u(x,t_{k+1}) - u(x,t_{k}) \right) + E_{n}, \quad \forall n = 1, 2, ..., N,$$
(7)

where

$$w_{n,k} = \frac{1}{\tau_{k+1}} \int_{t_{k-1}}^{t_k} (t_n - s)^{-\alpha} ds, \quad \forall k = 1, 2, ..., n.$$
 (8)

For any sequence $\{v_n\}_{n=1}^N$, let us define a discrete Caputo fractional differential operator D_N^{α} as

$$D_N^{\alpha} v^n := \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} w_{n,k} \left(v^{k+1} - v^k \right), \quad \forall n = 1, 2, ..., N,$$

$$= \frac{d_{n,1}}{\Gamma(2-\alpha)} v^n - \frac{d_{n,n}}{\Gamma(2-\alpha)} v^0 + \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{n-1} (d_{n,k+1} - d_{n,k}) v^{n-k},$$
(9)

where

$$d_{n,k} = \frac{(t_n - t_{n-k})^{1-\alpha} - (t_n - t_{n-k+1})^{1-\alpha}}{\tau_{n-k+1}}, \quad \text{for } 1 \le k \le n \le N.$$
 (10)

In particular, $d_{n,1} = \tau_n^{-\alpha}$. From mean value theorem, it follows that

$$d_{n,k+1} \le d_{n,k}, \quad \text{for } 0 \le k \le n-1 \le N-1.$$
 (11)

Let $\zeta^n := {}^c_0 D^{\alpha}_{t_n} u(x, t_n) - D^{\alpha}_N u(x, t_n).$

Lemma 3.1. [8, 9, 11] Assume that $|u^{(q)}(t)| \lesssim 1 + t^{\alpha-q}$, for q = 0, 1, 2 and $t \in (0, T]$. Then there exists a constant C such that for n = 1, 2, ..., N,

$$\|\zeta^n\|_1 \le C n^{-\min\{2-\alpha, r\alpha\}}. \tag{12}$$

Now, following the idea given in [9], we define coefficients $b_{n-i}^{(n)}$ and $p_{n-i}^{(n)}$ as follows: for $1 \le i \le n \le N$,

$$b_{n-i}^{(n)} := \frac{d_{n, n-i+1}}{\Gamma(2-\alpha)}, \text{ for } 1 \le i \le n \le N.$$
 (13)

and

$$p_{n-i}^{(n)} := \begin{cases} \Gamma(2-\alpha) \, \tau_i^{\alpha} \sum_{k=i+1}^n \left(b_{k-i-1}^{(k)} - b_{k-i}^{(k)} \right) \, p_{n-k}^{(n)}, & \text{if } i = 1, 2, ..., (n-1), \\ \Gamma(2-\alpha) \, \tau_n^{\alpha}, & \text{if } i = n. \end{cases}$$
(14)

Lemma 3.2. [9, 32] For n = 1, 2, ..., N, one has

$$\sum_{s=1}^{n} p_{n-s}^{(n)} s^{-\min\{2-\alpha, r\alpha\}} \le C N^{-\min\{2-\alpha, r\alpha\}}.$$
 (15)

Lemma 3.3. [18] For n = 1, 2, ..., N, one has

$$\sum_{s=1}^{n} p_{n-s}^{(n)} \frac{t_s^{-\alpha}}{\Gamma(1-\alpha)} \le 1. \tag{16}$$

Using the above notations, the fully-discrete scheme for the given problem is as follows: For each $1 \leq n \leq N$, find $U^n \in X_h$ such that

$$(D_N^{\alpha}U^n, v_h) + a(l(U^n))(\nabla U^n, \nabla v_h) = (f(U^n), v_h), \quad \forall v_h \in X_h,$$

$$U^0 = u_h^0,$$
(17)

where u_h^0 is some approximation of $u_0(x)$.

From (17), we have for $1 \le i \le M$,

$$(D_N^{\alpha}U^n, \phi_i) + a(l(U^n))(\nabla U^n, \nabla \phi_i) = (f(U^n), \phi_i).$$
(18)

From (9) and (18), we get

$$(U^{n}, \phi_{i}) + \frac{1}{d_{n,1}} \left(-d_{n,n}U^{0} + \sum_{k=1}^{n-1} (d_{n,k+1} - d_{n,k})U^{n-k}, \phi_{i} \right) + \frac{\Gamma(2-\alpha)}{d_{n,1}} a(l(U^{n}))$$

$$(\nabla U^{n}, \nabla \phi_{i}) = \frac{\Gamma(2-\alpha)}{d_{n,1}} (f(U^{n}), \phi_{i}),$$

$$(19)$$

where weights $d_{n,k+1}$ $(1 \le k \le n \le N)$ are given in (10). Since $U^n \in X_h$, $\exists \alpha_j^n \in \mathbb{R}$ such that

$$U^n = \sum_{j=1}^M \alpha_j^n \phi_j. \tag{20}$$

Set $\bar{\alpha}^n = [\alpha_1^n, \alpha_2^n, ..., \alpha_M^n]$. Putting above value of U^n in (19), we get system of non-linear algebraic equations

$$F_i(\bar{\alpha}^n) = F_i(U^n) = 0, \quad \text{for } 1 \le i \le M, \tag{21}$$

where

$$F_{i}(U^{n}) = (U^{n}, \phi_{i}) + \frac{1}{d_{n,1}} \left(-d_{n,n}U^{0} + \sum_{k=1}^{n-1} (d_{n,k+1} - d_{n,k})U^{n-k}, \phi_{i} \right) + \frac{\Gamma(2-\alpha)}{d_{n,1}} a(l(U^{n})) (\nabla U^{n}, \nabla \phi_{i}) - \frac{\Gamma(2-\alpha)}{d_{n,1}} (f(U^{n}), \phi_{i}).$$
(22)

If we use Newton's method for finding α_j^n , $1 \leq j \leq M$ then elements of the Jacobian matrix J_1 take the following form

$$\frac{\partial F_i}{\partial \alpha_j^n}(U^n) = (\phi_j, \, \phi_i) - \frac{\Gamma(2-\alpha)}{d_{n,1}} \left(f'(U^n)\phi_j, \, \phi_i \right) + \frac{\Gamma(2-\alpha)}{d_{n,1}} \, a(l(U^n)) \left(\nabla \phi_j, \, \nabla \phi_i \right) + \frac{\Gamma(2-\alpha)}{d_{n,1}} \, a'(l(U^n)) \, l'(U^n) \left(\int_{\Omega} \phi_j \, dx \right) \left(\nabla U^n, \, \nabla \phi_i \right). \tag{23}$$

We can observe that because of fourth term in the right hand side of (23), sparsity of the Jacobian matrix J_1 is lost [2]. In order to retain the sparsity of the Jacobian matrix, we reformulate our problem as follows [17]: Find $U^n \in X_h$ and $d \in \mathbb{R}$ such that

$$l(U^n) - d = 0, (24)$$

$$(U^{n}, \phi_{i}) + \frac{1}{d_{n,1}} \left(-d_{n,n}U^{0} + \sum_{k=1}^{n-1} (d_{n,k+1} - d_{n,k})U^{n-k}, \phi_{i} \right) + \frac{\Gamma(2 - \alpha)}{d_{n,1}} a(d) \left(\nabla U^{n}, \nabla \phi_{i} \right) - \frac{\Gamma(2 - \alpha)}{d_{n,1}} \left(f(U^{n}), \phi_{i} \right) = 0.$$
(25)

For applying the Newton's method in the reformulated problem, we rewrite equations (24) and (25) as follows:

$$F_i(U^n, d) = 0, \text{ for } 1 \le i \le M + 1,$$
 (26)

where

$$F_{i}(U^{n}, d) = (U^{n}, \phi_{i}) + \frac{1}{d_{n,1}} \left(-d_{n,n}U^{0} + \sum_{k=1}^{n-1} (d_{n,k+1} - d_{n,k})U^{n-k}, \phi_{i} \right) + \frac{\Gamma(2 - \alpha)}{d_{n,1}} a(d) \left(\nabla U^{n}, \nabla \phi_{i} \right) - \frac{\Gamma(2 - \alpha)}{d_{n,1}} \left(f(U^{n}), \phi_{i} \right), \text{ for } 1 \leq i \leq M,$$

$$(27)$$

$$F_{M+1}(U^n, d) = l(U^n) - d. (28)$$

Now, using Newton's method for the system of equations (26) and (??), we get the following matrix equation:

$$J \begin{bmatrix} \alpha^n \\ \beta \end{bmatrix} = \begin{bmatrix} A & b \\ c & \gamma \end{bmatrix} \begin{bmatrix} \alpha^n \\ \beta \end{bmatrix} = \begin{bmatrix} \bar{F} \\ F_{M+1} \end{bmatrix}, \tag{29}$$

where J denotes the Jacobian matrix, $\alpha^n = [\alpha_1^n, \alpha_2^n, ..., \alpha_M^n]^T$, $\bar{F} = [F_1, F_2, ..., F_M]^T$ and entries $A = A_{M \times M}$, $b = b_{M \times 1}$ and $c = c_{1 \times M}$ are given below.

$$A_{ij} = (\phi_j, \phi_i) + \frac{\Gamma(2 - \alpha)}{d_{n,1}} a(d) (\nabla \phi_j, \nabla \phi_i) - \frac{\Gamma(2 - \alpha)}{d_{n,1}} (f'(U^n)\phi_j, \phi_i), \quad 1 \le i, j \le M,$$

$$b_{i1} = \frac{\Gamma(2 - \alpha)}{d_{n,1}} a'(d) (\nabla U^n, \nabla \phi_i), \quad 1 \le i \le M,$$

$$c_{1j} = \int_{\Omega} \phi_j dx, \quad 1 \le j \le M,$$

$$\gamma = -1.$$
(30)

Note that the sparsity of matrix A is same as the sparsity of the Galerkin matrix corresponding to following semi-linear equation.

$$(w,v) + (\nabla w, \nabla v) = (f(w), v). \tag{31}$$

So, A is a sparse matrix. Hence, J is also a sparse matrix [2]. Also, (29) admits a unique solution [17].

In the following theorem we show that solution of (21) is equivalent to the solution of (24)-(25).

Theorem 3.1. If (U^n, d) is a solution of (24)-(25), then U^n is a solution of (21). Conversely, if U^n is a solution of (21), then (U^n, d) is a solution of (24)-(25).

Proof. Suppose (U^n, d) is a solution of (24)-(25), then $d = l(U^n)$ and putting this in (25), we get

$$(U^{n}, \phi_{i}) + \frac{1}{d_{n,1}} \left(-d_{n,n}U^{0} + \sum_{k=1}^{n-1} (d_{n,k+1} - d_{n,k}) U^{n-k}, \phi_{i} \right) + \frac{\Gamma(2-\alpha)}{d_{n,1}} a(l(U^{n})) (\nabla U^{n}, \nabla \phi_{i}) - \frac{\Gamma(2-\alpha)}{d_{n,1}} (f(U^{n}), \phi_{i}) = 0.$$

Hence, U^n is the solution of (21). The converse is obvious.

3.1 A priori bound

In this section we provide a priori bound for the fully discrete solution U^n . First, we write the following coercivity property of L1 scheme.

Lemma 3.4. [9] Let the functions $v^n = v(\cdot, t_n)$ be in $L^2(\Omega)$, for n = 0, 1, ..., N. Then, the discrete L1 scheme satisfies

$$(D_N^{\alpha} v^n, v^n) \ge \frac{1}{2} D_N^{\alpha} ||v^n||^2, \quad \text{for } n = 1, 2, ..., N.$$
 (32)

For deriving a priori estimates and a priori error estimates for fully-discrete solution U^n , we need following discrete fractional Grönwall inequality.

Lemma 3.5. [9, 19] Let $(\xi^n)_{n=1}^N$, $(\eta^n)_{n=1}^N$ and $(\lambda_l)_{l=1}^{N-1}$ be given non-negative sequences. Assume that there exists a constant Λ (independent of the step sizes) such that $\Lambda \geq \sum_{l=0}^{N-1} \lambda_l$, and the maximum step size satisfies

$$\max_{1 \le n \le N} \tau_n \le \frac{1}{\sqrt[\alpha]{2\Gamma(2-\alpha)\Lambda}}.$$
(33)

Then, for any non-negative sequence $(v^k)_{k=0}^N$ such that

$$D_N^{\alpha}(v^n)^2 \le \sum_{s=1}^n \lambda_{n-s} (v^s)^2 + \xi^n v^n + (\eta^n)^2, \quad \text{for } 1 \le n \le N,$$
 (34)

it holds that for $1 \le n \le N$,

$$v^{n} \leq 2E_{\alpha}(2 \Lambda t_{n}^{\alpha}) \left[v^{0} + \max_{1 \leq j \leq n} \sum_{s=1}^{j} p_{j-s}^{(j)} \xi^{s} + \sqrt{\Gamma(1-\alpha)} \max_{1 \leq s \leq n} \left\{ t_{s}^{\frac{\alpha}{2}} \eta^{s} \right\} \right].$$
 (35)

Theorem 3.2. Let U^n be the solution of (17), then U^n satisfies following estimates.

$$||U^n|| \le C(||U^0|| + 1),$$
 (36)

$$\|\nabla U^n\| \le C\left(\|\nabla U^0\| + 1\right),\tag{37}$$

where n = 1, 2, ..., N.

Proof. Putting $v_h = U^n$ in (17) to get

$$(D_N^{\alpha}U^n, U^n) + a(l(U^n))(\nabla U^n, \nabla U^n) = (f(U^n), U^n). \tag{38}$$

Using the bound of a and Cauchy-Schwarz inequality in equation (38), we have

$$(D_N^{\alpha} U^n, U^n) + m ||\nabla U^n||^2 \le ||f(U^n)|| \, ||U^n||. \tag{39}$$

Lipschitz continuity of f yields

$$\left| \| f(U^n) \| - \| f(U^0) \| \right| \le \| f(U^n) - f(U^0) \| \le K \left(\| U^n \| + \| U^0 \| \right).$$

Therefore,

$$||f(U^n)|| \le ||f(U^0)|| + K(||U^n|| + ||U^0||) \le C_1(1 + ||U^n||), \tag{40}$$

where

$$C_1 = \max \left\{ K, \| f(U^0) \| + K \| U^0 \| \right\}. \tag{41}$$

From (39) and (40), we get

$$(D_N^{\alpha} U^n, U^n) \le C_1 (1 + ||U^n||) ||U^n||. \tag{42}$$

Also from Lemma 3.4, we know that

$$(D_N^{\alpha}U^n, U^n) \ge \frac{1}{2} D_N^{\alpha} ||U^n||^2.$$

Thus, equation (42) can be written as

$$D_N^{\alpha} \|U^n\|^2 \le 2C_1 \|U^n\|^2 + 2C_1 \|U^n\|. \tag{43}$$

Using Lemma 3.5 (with $v^n = ||U^n||$, $\lambda_0 = 2C_1$, $\lambda_i = 0$ for i = 1, 2, ..., (n-1), $\xi^n = 2C_1$ and $\eta^n = 0$) in (43), we obtain

$$||U^{n}|| \leq 2 E_{\alpha}(4 C_{1} t_{n}^{\alpha}) \left(||U^{0}|| + \max_{1 \leq j \leq n} \sum_{s=1}^{j} p_{j-s}^{(j)} \left(2 C_{1} \right) \right)$$

$$\leq 2 E_{\alpha}(4 C_{1} t_{n}^{\alpha}) \left(||U^{0}|| + \Gamma(1-\alpha) \max_{1 \leq j \leq n} \left\{ \left(2 C_{1} t_{j}^{\alpha} \right) \sum_{s=1}^{j} p_{j-s}^{(j)} \frac{t_{s}^{-\alpha}}{\Gamma(1-\alpha)} \right\} \right). \tag{44}$$

Using Lemma 3.3 in equation (44), we obtain

$$||U^{n}|| \leq 2 E_{\alpha} (4 C_{1} t_{n}^{\alpha}) \left(||U^{0}|| + \Gamma (1 - \alpha) \max_{1 \leq j \leq n} \left\{ t_{j}^{\alpha} 2 C_{1} \right\} \right)$$

$$< 2 E_{\alpha} (4 C_{1} t_{n}^{\alpha}) \left(||U^{0}|| + 2 C_{1} \Gamma (1 - \alpha) T^{\alpha} \right).$$

$$(45)$$

Hence,

$$||U^n|| \le C(||U^0|| + 1), \tag{46}$$

where $C = 2 E_{\alpha} (4 C_1 t_n^{\alpha}) \max \{1, 2 \Gamma(1-\alpha) C_1 T^{\alpha} \}$.

Next, we take $v_h = D_N^{\alpha} U^n$ in (17) to get

$$(D_N^{\alpha}U^n, D_N^{\alpha}U^n) + a(l(U^n))(\nabla U^n, \nabla (D_N^{\alpha}U^n)) = (f(U^n), D_N^{\alpha}U^n).$$
 (47)

Dividing both the sides of (47) by $a(l(U^n))$, we get

$$\frac{1}{a(l(U^n))} \left(D_N^{\alpha} U^n, \, D_N^{\alpha} U^n \right) + \left(\nabla U^n, \, D_N^{\alpha} (\nabla U^n) \right) = \frac{1}{a(l(U^n))} \left(f(U^n), \, D_N^{\alpha} U^n \right). \tag{48}$$

Now, in equation (48) we use bound of a and Cauchy-Schwarz inequality to obtain

$$\frac{1}{m_2} \|D_N^{\alpha} U^n\|^2 + (\nabla U^n, D_N^{\alpha} (\nabla U^n)) \le \frac{1}{m_1} \|f(U^n)\| \|D_N^{\alpha} U^n\|. \tag{49}$$

For a, b > 0, using the inequality $ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$ (with $\epsilon = m_2$) in (49), we have

$$\frac{1}{m_2} \|D_N^{\alpha} U^n\|^2 + \left(\nabla U^n, D_N^{\alpha}(\nabla U^n)\right) \le \frac{m_2}{2 m_1^2} \|f(U^n)\|^2 + \frac{1}{2 m_2} \|D_N^{\alpha} U^n\|^2.$$

This gives us

$$(\nabla U^n, D_N^{\alpha}(\nabla U^n)) \le \frac{m_2}{2 m_1^2} \|f(U^n)\|^2.$$
 (50)

From equation (40) and Poincaré inequality, we can get

$$||f(U^n)|| \le C_1 (1 + ||U^n||) \le C_1 (1 + C_2 ||\nabla U^n||) \le C_3 (1 + ||\nabla U^n||),$$
 (51)

where C_2 is a constant which appears in Poincaré inequality and $C_3 = C_1 \max \{1, C_2\}$. Also, Lemma 3.4 gives

$$\left(\nabla U^n, D_N^{\alpha}(\nabla U^n)\right) \ge \frac{1}{2} D_N^{\alpha} \|\nabla U^n\|^2. \tag{52}$$

From (50) and (52), we obtain

$$D_N^{\alpha} \|\nabla U^n\|^2 \le \frac{C_3^2 m_2}{m_1^2} \left(1 + \|\nabla U^n\|\right)^2 \le C_4 \left(1 + \|\nabla U^n\|^2\right), \tag{53}$$

where constant C_4 is depending on C_3 , m_1 and m_2 .

Using Lemma 3.5 (with $v^n = \|\nabla U^n\|^2$, $\lambda_0 = C_4$, $\lambda_i = 0$, $\forall i = 1, 2, ..., (n-1)$, $\xi^n = 0$ and $\eta^n = C_4$) in (53) to get

$$\|\nabla U^{n}\|^{2} \leq 2 E_{\alpha}(2 C_{4} t_{n}^{\alpha}) \left(\|U^{0}\|^{2} + \sqrt{\Gamma(1-\alpha)} \max_{1 \leq s \leq n} \left\{ t_{s}^{\frac{\alpha}{2}} C_{4} \right\} \right)$$

$$\leq 2 E_{\alpha}(2 C_{4} t_{n}^{\alpha}) \left(\|U^{0}\|^{2} + C_{4} \sqrt{\Gamma(1-\alpha)} T^{\frac{\alpha}{2}} \right).$$
(54)

Therefore,

$$\|\nabla U^n\|^2 \le C(\|\nabla U^0\|^2 + 1),$$
 (55)

where $C = 2 E_{\alpha} (2 C_4 t_n^{\alpha}) \max \left\{ 1, C_4 \sqrt{\Gamma(1-\alpha)} T^{\frac{\alpha}{2}} \right\}$. From $a^2 + b^2 \leq (a+b)^2$,

$$\|\nabla U^n\|^2 \le C (\|\nabla U^0\| + 1)^2. \tag{56}$$

Hence,

$$\|\nabla U^n\| \le C\left(\|\nabla U^0\| + 1\right). \tag{57}$$

This completes the proof.

4 Existence-uniqueness of fully-discrete solution

In this section we prove the existence and uniqueness of fully-discrete solution of the problem (17). For this, we use following proposition which is a consequence of Brouwer fixed point theorem [22].

Proposition 1. Let H be a finite dimensional Hilbert space with scalar product (\cdot, \cdot) and norm $||\cdot||_H$. Let $S: H \to H$ be a continuous map with with following properties: there exist $\rho > 0$ such that

$$(S(v), v) > 0$$
 $\forall v \in H$ with $||v||_H = \rho$

Then, there exists an element $w \in H$ such that

$$S(w) = 0 ||w||_H \le \rho.$$

We also define

$$\tau = \max \left\{ \frac{1}{\sqrt[\alpha]{C_1 \Gamma(2-\alpha)}}, \sqrt[\alpha]{\frac{4 m_1}{(K R_1 + L)^2 \Gamma(2-\alpha)}} \right\}, \tag{58}$$

where C_1 is given in (41).

In the following we discuss the existence and uniqueness of the fully-discrete solution.

Theorem 4.1. Let $U^0, U^1, ..., U^{n-1}$ are given and $\max_{1 \le n \le N} \tau_n \le \tau$, then for all $1 \le n \le N$, there exists a unique solution $U^n \in X_h$ of (17).

Proof. Rewriting (17) as follows

$$(U^{n}, v_{h}) + \frac{1}{d_{n,1}} \left(-d_{n,n}U^{0} + \sum_{k=1}^{n-1} (d_{n,k+1} - d_{n,k})U^{n-k}, v_{h} \right) + \frac{\Gamma(2 - \alpha)}{d_{n,1}} a(l(U^{n})) (\nabla U^{n}, \nabla v_{h}) - \frac{\Gamma(2 - \alpha)}{d_{n,1}} (f(U^{n}), v_{h}) = 0.$$

$$(59)$$

Now, we define a map $G: X_h \longrightarrow X_h$ such that

$$(G(X^{n}), v_{h}) = (X^{n}, v_{h}) + \frac{1}{d_{n,1}} \left(-d_{n,n}U^{0} + \sum_{k=1}^{n-1} (d_{n,k+1} - d_{n,k})U^{n-k}, v_{h} \right) + \frac{\Gamma(2 - \alpha)}{d_{n,1}} a(l(X^{n})) (\nabla X^{n}, \nabla v_{h}) - \frac{\Gamma(2 - \alpha)}{d_{n,1}} (f(X^{n}), v_{h}).$$

$$(60)$$

Then the map G is continuous. By choosing $v_h = X^n$ in (60), we get

$$(G(X^{n}), X^{n}) = (X^{n}, X^{n}) + \frac{1}{d_{n,1}} \left(-d_{n,n}U^{0} + \sum_{k=1}^{n-1} (d_{n,k+1} - d_{n,k})U^{n-k}, X^{n} \right) - \frac{\Gamma(2 - \alpha)}{d_{n,1}} \left(f(X^{n}), X^{n} \right) + \frac{\Gamma(2 - \alpha)}{d_{n,1}} a(l(X^{n})) (\nabla X^{n}, \nabla X^{n}).$$
(61)

Applying the bound of a and Cauchy-Schwarz inequality in (61) we can arrive at

$$\left(G(X^{n}), X^{n}\right) \geq \|X^{n}\|^{2} + \frac{m_{1} \Gamma(2 - \alpha)}{d_{n,1}} \|\nabla X^{n}\|^{2} - \frac{\Gamma(2 - \alpha)}{d_{n,1}} \|f(X^{n})\| \|X^{n}\| - \frac{d_{n,n}}{d_{n,1}} \|U^{0}\| \|X^{n}\| - \frac{1}{d_{n,1}} \sum_{k=1}^{n-1} (d_{n,k} - d_{n,k+1}) \|U^{n-k}\| \|X^{n}\|.$$
(62)

Since $\frac{m_1 \Gamma(2-\alpha)}{d_{n,1}} \|\nabla X^n\|^2 \ge 0$, it follows that

$$\left(G(X^{n}), X^{n}\right) \geq \left(\|X^{n}\| - \frac{\Gamma(2-\alpha)}{d_{n,1}} \|f(X^{n})\| - \frac{d_{n,n}}{d_{n,1}} \|U^{0}\| - \frac{1}{d_{n,1}} \sum_{k=1}^{n-1} (d_{n,k} - d_{n,k+1}) \|U^{n-k}\| \right) \|X^{n}\|.$$
(63)

From (40) and (63), we get

$$\left(G(X^{n}), X^{n}\right) \geq \left\{ \left(1 - \frac{C_{1} \Gamma(2 - \alpha)}{d_{n,1}}\right) \|X^{n}\| - \frac{C_{1} \Gamma(2 - \alpha)}{d_{n,1}} - \frac{d_{n,n}}{d_{n,1}} \|U^{0}\| - \frac{1}{d_{n,1}} \sum_{k=1}^{n-1} (d_{n,k} - d_{n,k+1}) \|U^{n-k}\| \right\} \|X^{n}\|.$$
(64)

Thus, $(G(X^n), X^n) > 0$ if

$$\left(1 - \frac{C_1 \Gamma(2 - \alpha)}{d_{n,1}}\right) \|X^n\| - \frac{C_1 \Gamma(2 - \alpha)}{d_{n,1}} - \frac{d_{n,n}}{d_{n,1}} \|U^0\| - \frac{1}{d_{n,1}} \sum_{k=1}^{n-1} (d_{n,k} - d_{n,k+1}) \|U^{n-k}\| > 0.$$
(65)

Since $\max_{1 \le n \le N} \tau_n \le \tau$, $\left(1 - \frac{C_1 \Gamma(2-\alpha)}{d_{n,1}}\right) > 0$. Thus, $\exists X^n \in X_h$ such that

$$||X^{n}|| > \frac{1}{\left(1 - \frac{C_{1}\Gamma(2-\alpha)}{d_{n,1}}\right)} \left(\frac{C_{1}\Gamma(2-\alpha)}{d_{n,1}} + \frac{d_{n,n}}{d_{n,1}} ||U^{0}|| + \frac{1}{d_{n,1}} \sum_{k=1}^{n-1} (d_{n,k} - d_{n,k+1}) ||U^{n-k}|| \right),$$

$$(66)$$

Therefore, it is easy to see that $(G(X^n), X^n) > 0$, $\forall X^n \in X_h$ with $||X^n|| = \rho$, where

$$\rho = \frac{1}{\left(1 - \frac{C_1 \Gamma(2 - \alpha)}{d_{n,1}}\right)} \left(\frac{C_1 \Gamma(2 - \alpha)}{d_{n,1}} + \frac{d_{n,n}}{d_{n,1}} \|U^0\| + \frac{1}{d_{n,1}} \sum_{k=1}^{n-1} (d_{n,k} - d_{n,k+1}) \|U^{n-k}\|\right).$$

Thus, by Proposition 1, we can assure that (17) has a solution.

Now, we prove the uniqueness of solution. For this we assume that for given $U^0, U^1, ..., U^{n-1}$, there exist two solutions of (17), say U_1^n and U_2^n at time $t = t_n$. Throughout the proof, we denote U_1^n by U_1 and U_2^n by U_2 respectively. Let $U_1 - U_2 = r$. From (59) we can get

$$(U_{1} - U_{2}, v_{h}) + \frac{\Gamma(2 - \alpha)}{d_{n,1}} a(l(U_{1})) (\nabla U_{1}, \nabla v_{h}) - \frac{\Gamma(2 - \alpha)}{d_{n,1}} a(l(U_{2})) (\nabla U_{2}, \nabla v_{h})$$

$$= \frac{\Gamma(2 - \alpha)}{d_{n,1}} (f(U_{1}) - f(U_{2}), v_{h}).$$
(67)

By subtracting $\frac{\Gamma(2-\alpha)}{d_{n,1}} a(l(U_1)) (\nabla U_2, \nabla v_h)$ in both sides of (67), we get

$$(r, v_h) + \frac{\Gamma(2 - \alpha)}{d_{n,1}} a(l(U_1)) (\nabla r, \nabla v_h) = \frac{\Gamma(2 - \alpha)}{d_{n,1}} \left(a(l(U_2)) - a(l(U_1)) \right) (\nabla U_2, \nabla v_h) + \frac{\Gamma(2 - \alpha)}{d_{n,1}} \left(f(U_1) - f(U_2), v_h \right).$$
(68)

Now, we take $v_h = r$ in above equation to get

$$(r, r) + \frac{\Gamma(2 - \alpha)}{d_{n,1}} a(l(U_1)) (\nabla r, \nabla r) = \frac{\Gamma(2 - \alpha)}{d_{n,1}} \left(a(l(U_2)) - a(l(U_1)) \right) (\nabla U_2, \nabla r) + \frac{\Gamma(2 - \alpha)}{d_{n,1}} \left(f(U_1) - f(U_2), r \right).$$
(69)

Applying the bound of a and Cauchy-Schwarz inequality in (69), we have

$$||r||^{2} + \frac{m \Gamma(2 - \alpha)}{d_{n,1}} ||\nabla r||^{2} = \frac{\Gamma(2 - \alpha)}{d_{n,1}} |a(l(U_{2})) - a(l(U_{1}))| ||\nabla U_{2}|| ||\nabla r|| + \frac{\Gamma(2 - \alpha)}{d_{n,1}} ||f(U_{1}) - f(U_{2})|| ||r||.$$

$$(70)$$

Since a and f are Lipschitz continuous, we can get

$$|a(l(U_2)) - a(l(U_1))| \le L ||l(U_2) - l(U_1)|| \le L ||U_2 - U_1|| = L ||r||.$$
 (71)

$$|f(U_1) - f(U_2)| \le K \|U_1 - U_2\| = K \|r\|. \tag{72}$$

Also, from Theorem 3.2 one can get

$$\|\nabla U_2\| = R_1, \quad where \ R_1 = C(1 + \|U^0\|).$$
 (73)

Therefore, from (71), (72), (73) and Poincaré inequality, we get

$$||r||^{2} + \frac{m_{1} \Gamma(2-\alpha)}{d_{n,1}} ||\nabla r||^{2} \leq \left\{ \left(\frac{KR_{1} \Gamma(2-\alpha)}{d_{n,1}} + \frac{L \Gamma(2-\alpha)}{d_{n,1}} \right) ||r|| \right\} ||\nabla r||. \quad (74)$$

For a, b > 0, using the inequality $ab \le \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$ with $\epsilon = \frac{d_{n,1}}{2m_1\Gamma(2-\alpha)}$ in (74), we get

$$||r||^2 \le \frac{(KR_1 + L)^2 \Gamma(2 - \alpha)}{4 m_1 d_{n,1}} ||r||^2.$$
 (75)

This gives us,

$$\left(1 - \frac{(KR_1 + L)^2 \Gamma(2 - \alpha)}{4 m_1 d_{n,1}}\right) ||r||^2 \le 0.$$
(76)

Since $\max_{1 \le n \le N} \tau_n \le \tau$, $\left(1 - \frac{(KR_1 + L)^2 \Gamma(2 - \alpha)}{4 m_1 d_{n,1}}\right) > 0$. Using this condition in (76), we get

$$||r||^2 \le 0. \tag{77}$$

This shows that

$$U_1 = U_2. (78)$$

This completes the proof.

5 Error estimates

In this section, we derive a priori error estimate for fully-discrete solution U^n . For this, some additional regularity on solution u is required. Therefore, we assume that there exist constants R_2 , R_3 , R > 0 such that

$$\|_0^c D_{t_n}^{\alpha} u\|_{L^{\infty}(H^2(\Omega))} \le R_3, \|\Delta u\|_{L^{\infty}(L^2(\Omega))} \le R_2 \text{ and } \|\nabla u\|_{L^{\infty}(L^2(\Omega))} \le R.$$
 (79)

For our further analysis, we recall the definition of Ritz-projection and Discrete Laplacian operators.

Definition 5.1. [20] The Ritz-projection is a map $R_h: H_0^1(\Omega) \to X_h$ such that

$$(\nabla w, \nabla v) = (\nabla R_h w, \nabla v), \quad \forall w \in H_0^1(\Omega) \text{ and } \forall v \in X_h.$$
 (80)

It is easy to prove that R_h satisfies

$$\|\nabla R_h w\| \le R,\tag{81}$$

where R is given in (79).

Lemma 5.1. [21] There exists a positive constant C (independent of h) such that

$$\|w - R_h w\|_{L^2(\Omega)} + h \|\nabla (w - R_h w)\|_{L^2(\Omega)} \le Ch^2 \|\Delta w\|_{L^2(\Omega)}, \quad \forall w \in H^2 \cap H_0^1.$$
 (82)

Definition 5.2. [22] The discrete Laplacian is a map $\Delta_h: X_h \to X_h$ as

$$(\Delta_h u, v) = -(\nabla u, \nabla v), \quad \forall u, v \in X_h.$$
(83)

Now, with the help of the projection operator R_h , we split the error in two parts and it is given below.

$$u^{n} - U^{n} = u^{n} - R_{h}u^{n} + R_{h}u^{n} - U^{n} = \rho^{n} + \theta^{n},$$
(84)

where $u^n := u(t_n)$, $\rho^n := u^n - R_h u^n$ and $\theta^n := R_h u^n - U^n$.

Next, in the following theorem, we provide the convergence estimate for the fullydiscrete solution.

Theorem 5.1. Let u^n and U^n be the solution of (2) and (17) respectively, then

$$||u^n - U^n|| \le C \left(h^2 + N^{-\min\{2-\alpha, r\alpha\}}\right),$$
 (85)

$$\|\nabla(u^n - U^n)\| \le C\left(h + N^{-\min\{2-\alpha, r\alpha\}}\right),\tag{86}$$

where n = 1, 2, ..., N.

Proof. For simplicity, we write $a := a(l(u^n))$ and $a_h := a(l(U^n))$. For any $v_h \in X_h$ the estimate for θ^n is given by

$$(D_{N}^{\alpha}\theta^{n}, v_{h}) + a_{h} (\nabla \theta^{n}, \nabla v_{h})$$

$$= (D_{N}^{\alpha}R_{h}u^{n}, v_{h}) + a_{h} (\nabla R_{h}u^{n}, \nabla v_{h}) - (D_{N}^{\alpha}U^{n}, v_{h}) - a_{h} (\nabla U^{n}, \nabla v_{h})$$

$$= (D_{N}^{\alpha}R_{h}u^{n}, v_{h}) + a_{h} (\nabla u^{n}, \nabla v_{h}) - a (\nabla u^{n}, \nabla v_{h}) + a (\nabla u^{n}, \nabla v_{h}) - (f(U^{n}), v_{h})$$

$$= (D_{N}^{\alpha}R_{h}u^{n}, v_{h}) + (a_{h} - a) (\nabla u^{n}, \nabla v_{h}) + (f(u^{n}), v_{h}) - ({}_{0}^{c}D_{t_{n}}^{\alpha}u, v_{h}) - (f(U^{n}), v_{h})$$

$$= (D_{N}^{\alpha}R_{h}u^{n} - {}_{0}^{c}D_{t_{n}}^{\alpha}u, v_{h}) + (a_{h} - a) (\nabla R_{h}u^{n}, \nabla v_{h}) + (f(u^{n}) - f(U^{n}), v_{h}).$$
(87)

We choose $v_h = \theta^n$ in (87) to get

$$(D_N^{\alpha}\theta^n, \theta^n) + a_h (\nabla \theta^n, \nabla \theta^n) = \left(D_N^{\alpha} R_h u^n - {}_0^c D_{t_n}^{\alpha} u, \theta^n \right) + (a_h - a) (\nabla R_h u^n, \nabla \theta^n) + \left(f(u^n) - f(U^n), \theta^n \right).$$

$$(88)$$

An application of Cauchy-Schwarz inequality in (88) gives

$$(D_N^{\alpha}\theta^n, \theta^n) + a_h (\nabla \theta^n, \nabla \theta^n) \leq \|D_N^{\alpha} R_h u^n - {}_{0}^{c} D_{t_n}^{\alpha} u \| \|\theta^n\| + |a_h - a| \|\nabla R_h u^n\| \|\nabla \theta^n\| + \|f(u^n) - f(U^n)\| \|\theta^n\|.$$

$$(89)$$

By using the bound of a, triangle inequality, Poincaré inequality and (81) in (89), we have

$$(D_N^{\alpha}\theta^n, \theta^n) + m_1 \|\nabla \theta^n\|^2 \le \|D_N^{\alpha} R_h u^n - {}_0^c D_{t_n}^{\alpha} R_h u\| \|\theta^n\| + R \|a_h - a\| \|\nabla \theta^n\| + C_2 \|{}_0^c D_t^{\alpha} R_h u - {}_0^c D_t^{\alpha} u\| \|\nabla \theta^n\| + C_2 \|f(u^n) - f(U^n)\| \|\nabla \theta^n\|,$$

$$(90)$$

where C_2 is a constant which appears in Poincaré inequality.

For a, b > 0, using $ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$ (with $\epsilon = \frac{m_1}{3}$) in (90) to get

$$(D_{N}^{\alpha}\theta^{n}, \theta^{n}) + m_{1} (\nabla \theta^{n}, \nabla \theta^{n}) \leq \|D_{N}^{\alpha}R_{h}u^{n} - {}_{0}^{c}D_{t_{n}}^{\alpha}R_{h}u\| \|\theta^{n}\| + \frac{m_{1}}{2} \|\nabla \theta^{n}\|^{2} + \frac{3C_{5}}{2m_{1}} \Big(\|{}_{0}^{c}D_{t_{n}}^{\alpha}R_{h}u - {}_{0}^{c}D_{t_{n}}^{\alpha}u\|^{2} + |a_{h} - a|^{2} + \|f(u^{n}) - f(U^{n})\|^{2} \Big),$$

$$(91)$$

where $C_5 = max\{C_2^2, R^2\}$.

From (91), we have

$$(D_N^{\alpha}\theta^n, \theta^n) \leq \|D_N^{\alpha}R_h u^n - {}_{0}^{c}D_{t_n}^{\alpha}R_h u\| \|\theta^n\| + C_6 \Big(\|{}_{0}^{c}D_{t_n}^{\alpha}R_h u - {}_{0}^{c}D_{t_n}^{\alpha}u\|^2 + |a_h - a|^2 + \|f(u^n) - f(U^n)\|^2 \Big),$$

$$(92)$$

where C_6 is a constant depending on C_5 and m_1 . Note that

$$|a_h - a| = |a_h(l(U^n)) - a(l(u^n))| \le L|l(U^n) - l(u^n)| \le L||U^n - u^n||$$

$$\le L(||\rho^n|| + ||\theta^n||).$$
(93)

and

$$||f(u^n) - f(U^n)|| \le K ||u^n - U^n|| \le K (||\rho^n|| + ||\theta^n||).$$
(94)

Also from (12) and (82) one can get

$$||D_N^{\alpha} R_h u^n - {}_0^c D_{t_n}^{\alpha} R_h u|| \le C n^{-\min\{2-\alpha, r\alpha\}}, \tag{95}$$

$$\|_{0}^{c} D_{t_{n}}^{\alpha} R_{h} u - {}_{0}^{c} D_{t_{n}}^{\alpha} u\| \leq C h^{2} \|\Delta {}_{0}^{c} D_{t_{n}}^{\alpha} u\| \leq C_{7} h^{2}, \tag{96}$$

where constant C_7 is depending on R_3 . Also, from Lemma 3.4, we have

$$(D_N^{\alpha}\theta^n, \theta^n) \ge \frac{1}{2} D_N^{\alpha} \|\theta^n\|^2. \tag{97}$$

Using the values from (93) - (97) in (92) to get

$$D_{N}^{\alpha} \|\theta^{n}\|^{2} \leq 2 C n^{-\min\{2-\alpha, r\sigma\}} \|\theta^{n}\| + 2 C_{6} \left[(C_{7} h^{2})^{2} + (L^{2} + K^{2}) \left(\|\rho^{n}\| + \|\theta^{n}\| \right)^{2} \right]$$

$$\leq 2 C n^{-\min\{2-\alpha, r\sigma\}} \|\theta^{n}\| + 2 C_{6} (C_{7} h^{2})^{2} + 2 C_{6} (L^{2} + K^{2}) \|\rho^{n}\|^{2} + 2 C_{6} (L^{2} + K^{2}) \|\theta^{n}\|^{2}.$$

$$(98)$$

Using (82) in (98) to get

$$|D_N^{\alpha} \|\theta^n\|^2 \le C_8 \|\theta^n\|^2 + 2C n^{-\min\{2-\alpha, r\alpha\}} \|\theta^n\| + (C_9 h^2)^2, \tag{99}$$

where constant C_8 is depending on C_6 , L, K and constant C_9 is depending on C_6 , C_7 , L, K.

From Lemma 3.5 (with $v^n = \|\theta^n\|$, $\lambda_0 = C_8$, $\lambda_i = 0$, $\forall i = 1, 2, ..., (n-1)$, $\eta^n = C_9 h^2$, $\xi^n = 2 C n^{-min\{2-\alpha, r\alpha\}}$), we get

$$\|\theta^{n}\| \leq 2 E_{\alpha}(2 C_{8} t_{n}^{\alpha}) \left(\|\theta^{0}\| + 2 C \max_{1 \leq j \leq n} \sum_{s=1}^{j} p_{j-s}^{(j)} s^{-min\{2-\alpha, r\alpha\}} + \sqrt{\Gamma(1-\alpha)} \right)$$

$$\max_{1 \leq s \leq n} \left\{ t_{s}^{\frac{\alpha}{2}} C_{9} h^{2} \right\}.$$

$$(100)$$

Using Lemma 3.2 in (100) to obtain

$$\|\theta^n\| \le 2 E_{\alpha}(2 C_8 t_n^{\alpha}) \left(\|\theta^0\| + 2 C N^{-\min\{2-\alpha, r\alpha\}} + C_9 \sqrt{\Gamma(1-\alpha)} T^{\frac{\alpha}{2}} h^2 \right).$$
 (101)

Choosing $U^0 = R_h u^0$, we get $\|\theta^0\| = 0$. Hence, from (101), we have

$$\|\theta^n\| \le C_{10} \left(h^2 + N^{-\min\{2-\alpha, r\alpha\}}\right),$$
 (102)

where $C_{10} = 2 E_{\alpha}(2 C_8 t_n^{\alpha}) \max \left\{ 2 C, C_9 \sqrt{\Gamma(1-\alpha)} T^{\frac{\alpha}{2}} \right\}$. Thus,

$$||u^{n} - U^{n}|| \leq ||\rho^{n}|| + ||\theta^{n}||$$

$$\leq C h^{2} + C_{10} \left(h^{2} + N^{-\min\{2-\alpha, r\alpha\}}\right)$$

$$\leq C \left(h^{2} + N^{-\min\{2-\alpha, r\alpha\}}\right).$$
(103)

Now, we will derive the error estimate in H_0^1 -norm. Using the definition of Δ_h and R_h , we can rewrite the equation (87) as follows:

$$(D_N^{\alpha}\theta^n, v_h) - a_h (\Delta_h \theta^n, v_h) = (D_N^{\alpha} R_h u^n - {}_0^c D_{t_n}^{\alpha} u, v_h) + (a_h - a) (\nabla u^n, \nabla v_h) + (f(u^n) - f(U^n), v_h).$$

$$(104)$$

From Green's theorem, it follows that $(\nabla u^n, \nabla v_h) = -(\Delta u^n, v_h)$. Therefore, equation (104) takes the following form

$$(D_N^{\alpha}\theta^n, v_h) - a_h (\Delta_h \theta^n, v_h) = \left(D_N^{\alpha} R_h u^n - {}_0^c D_{t_n}^{\alpha} u, v_h \right) - (a_h - a) (\Delta u^n, v_h) + \left(f(u^n) - f(U^n), v_h \right).$$
(105)

Now, we take $v_h = -\Delta_h \theta^n$ in (105) to get

$$-\left(D_N^{\alpha}\theta^n, \Delta_h\theta^n\right) + a_h\left(\Delta_h\theta^n, \Delta_h\theta^n\right) = -\left(D_N^{\alpha}R_hu^n - {}_0^cD_{t_n}^{\alpha}u, \Delta_h\theta^n\right) + (a_h - a)$$

$$\left(\Delta u^n, \Delta_h\theta^n\right) - \left(f(u^n) - f(U^n), \Delta_h\theta^n\right).$$
(106)

Using the definition of Δ_h , we get

$$(\nabla D_N^{\alpha} \theta^n, \nabla \theta^n) + a_h \|\Delta_h \theta^n\|^2 = \left(\nabla (D_N^{\alpha} R_h u^n - {}_0^c D_{t_n}^{\alpha} u), \nabla \theta^n\right) + (a_h - a)$$

$$(\Delta u^n, \Delta_h \theta^n) - \left(f(u^n) - f(U^n), \Delta_h \theta^n\right).$$

$$(107)$$

Applying the bound of a, triangle inequality and Cauchy-Schwarz inequality in (107) to get

$$(\nabla D_{N}^{\alpha}\theta^{n}, \nabla \theta^{n}) + m_{1} \|\Delta_{h}\theta^{n}\|^{2} \leq \|\nabla (D_{N}^{\alpha}R_{h}u^{n} - {}_{0}^{c}D_{t_{n}}^{\alpha}R_{h}u)\| \|\nabla \theta^{n}\| + \|\nabla ({}_{0}^{c}D_{t_{n}}^{\alpha}R_{h}u - {}_{0}^{c}D_{t_{n}}^{\alpha}u)\| \|\nabla \theta^{n}\| + |a - a_{h}| \|\Delta u^{n}\| \|\Delta_{h}\theta^{n}\|$$

$$+ \|f(u^{n}) - f(U^{n})\| \|\Delta_{h}\theta^{n}\|.$$

$$(108)$$

By using (79) and $ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$ (with $\epsilon = 1$ in 1^{st} term and $\epsilon = \frac{m_1}{2}$ in 3^{rd} & 4^{th} term) in (108), we get

$$(\nabla D_{N}^{\alpha}\theta^{n}, \nabla \theta^{n}) + m_{1} \|\Delta_{h}\theta^{n}\|^{2} \leq \|\nabla (D_{N}^{\alpha}R_{h}u^{n} - {}_{0}^{c}D_{t_{n}}^{\alpha}R_{h}u)\| \|\nabla \theta^{n}\| + \frac{1}{2}\|\nabla \theta^{n}\|^{2} + \frac{1}{2}\|\nabla ({}_{0}^{c}D_{t_{n}}^{\alpha}R_{h}u - {}_{0}^{c}D_{t_{n}}^{\alpha}u)\|^{2} + \frac{R_{2}^{2}}{m_{1}}|a - a_{h}|^{2} + \frac{1}{m_{1}}\|f(u^{n}) - f(U^{n})\|^{2} + \frac{m_{1}}{2}\|\Delta_{h}\theta^{n}\|^{2}.$$

$$(109)$$

Therefore,

$$(D_{N}^{\alpha}(\nabla\theta^{n}), \nabla\theta^{n}) \leq \frac{1}{2} \|\nabla\theta^{n}\|^{2} + \|\nabla(D_{N}^{\alpha}R_{h}u^{n} - {}_{0}^{c}D_{t_{n}}^{\alpha}R_{h}u)\| \|\nabla\theta^{n}\| + \frac{R_{2}^{2}}{m_{1}}|a - a_{h}|^{2} + \frac{1}{2} \|\nabla({}_{0}^{c}D_{t_{n}}^{\alpha}R_{h}u - {}_{0}^{c}D_{t_{n}}^{\alpha}u)\|^{2} + \frac{1}{m_{1}} \|f(u^{n}) - f(U^{n})\|^{2}.$$

$$(110)$$

Note that from (12), we get

$$\|\nabla (D_N^{\alpha} R_h u^n - {}_0^c D_{t_n}^{\alpha} R_h u)\| \le C n^{-\min\{2-\alpha, r\alpha\}}.$$
(111)

Lipschitz continuity of a and f gives

$$|a_{h} - a| = |a_{h}(l(U^{n})) - a(l(u^{n}))| \le L|l(U^{n}) - l(u^{n})| \le L ||U^{n} - u^{n}||$$

$$\le L (||\rho^{n}|| + ||\theta^{n}||)$$

$$\le C_{2} L (||\nabla \rho^{n}|| + ||\nabla \theta^{n}||).$$
(112)

$$||f(u^n) - f(U^n)|| \le K ||u^n - U^n|| \le K (||\rho^n|| + ||\theta^n||) \le C_2 K (||\nabla \rho^n|| + ||\nabla \theta^n||).$$
(113)

From (82) one can get

$$\|\nabla({}_{0}^{c}D_{t_{n}}^{\alpha}R_{h}u - {}_{0}^{c}D_{t_{n}}^{\alpha}u)\| \le C h \|\Delta {}_{0}^{c}D_{t_{n}}^{\alpha}u\| \le C_{11} h, \tag{114}$$

where constant C_{11} is depending on R_3 . Also, from Lemma 3.4, we have

$$(D_N^{\alpha}(\nabla \theta^n), \theta^n) \ge \frac{1}{2} D_N^{\alpha} \|\nabla \theta^n\|^2.$$
 (115)

Using the values from (111) - (115) in (110) to get

$$D_N^{\alpha} \|\nabla \theta^n\|^2 \le \|\nabla \theta^n\|^2 + 2C n^{-\min\{2-\alpha, r\alpha\}} \|\nabla \theta^n\| + (C_{11} h)^2 + \left(\frac{C_2^2 L^2 R_2^2}{m_1} + \frac{C_2^2 K^2}{m_1}\right) \left(\|\nabla \rho^n\| + \|\nabla \theta^n\|\right)^2.$$

$$(116)$$

Since $\|\nabla \rho^n\|$, $\|\nabla \theta^n\| \ge 0$, we can find a constant C_{12} such that $(\|\nabla \rho^n\| + \|\nabla \theta^n\|)^2 \le C_{12} (\|\nabla \rho^n\|^2 + \|\nabla \theta^n\|^2)$ and from (82), $\|\nabla \rho^n\| \le Ch$. Therefore, from (116), we get

$$D_N^{\alpha} \|\nabla \theta^n\|^2 \le C_{13} \|\nabla \theta^n\|^2 + 2C n^{-\min\{2-\alpha, r\alpha\}} \|\nabla \theta^n\| + (C_{14} h)^2.$$
 (117)

where constant C_{13} is depending on L, K, R_2 , m_1 , C_2 and constant C_{14} is depending on L, K, R_2 , m_1 , C_2 and C_{11} .

An application of Lemma 3.5 (with $v^n = ||\nabla \theta^n||$, $\lambda_0 = C_{13}$, $\lambda_i = 0$, $\forall i = 1, 2, ..., (n - 1)$, $\xi^n = 2 C n^{-min\{2-\alpha, r\alpha\}}$, $\eta^n = C_{14} h$) gives

$$\|\nabla \theta^{n}\| \leq 2 E_{\alpha} (2 C_{13} t_{n}^{\frac{\alpha}{2}}) \left(\|\nabla \theta^{0}\| + 2 C \max_{1 \leq j \leq n} \sum_{s=1}^{j} p_{j-s}^{(j)} s^{-min\{2-\alpha, r\alpha\}} + \sqrt{\Gamma(1-\alpha)} \right)$$

$$\max_{1 \leq s \leq n} \left\{ t_{s}^{\frac{\alpha}{2}} C_{14} h \right\}.$$
(118)

From Lemma 3.2, we get

$$\|\nabla \theta^{n}\| \leq 2 E_{\alpha}(2 C_{13} t_{n}^{\alpha}) \left(\|\nabla \theta^{0}\| + 2 C N^{-\min\{2-\alpha, r\alpha\}} + C_{14} \sqrt{\Gamma(1-\alpha)} T^{\frac{\alpha}{2}} h \right).$$
(119)

Choosing $U^0 = R_h u^0$, to get $\|\nabla \theta^0\| = 0$. Hence, from (119), we have

$$\|\theta^n\| \le C_{15} \left(h + N^{-\min\{2-\alpha, r\alpha\}} \right),$$
 (120)

where $C_{15} = 2 E_{\alpha} (2 C_{13} t_n^{\alpha}) \max \left\{ 2 C, C_{14} \sqrt{\Gamma(1-\alpha)} T^{\frac{\alpha}{2}} \right\}$. Hence,

$$\|\nabla(u^{n} - U^{n})\| \leq \|\nabla\rho^{n}\| + \|\nabla\theta^{n}\|$$

$$\leq C h + C_{15} \left(h + N^{-\min\{2-\alpha, r\alpha\}}\right)$$

$$\leq C \left(h + N^{-\min\{2-\alpha, r\alpha\}}\right).$$
(121)

This completes the proof.

Corollary 5.1.1. The L1-FEM solution U^n satisfies:

$$||u^n - U^n||_1 \le C \left(h + N^{-\min\{2-\alpha, r\alpha\}}\right) \text{ for } n = 0, 1, ..., N.$$
 (122)

Proof. By using Poincaré inequality, we can write

$$||u^n - U^n||_1 \le C ||\nabla (u^n - U^n)||. \tag{123}$$

Now using the estimate (86) from Theorem 5.1, we get the desired result. \Box

Remark 5.1. For the time fractional partial differential equation (2a)- (2c), we have assumed certain regularity of solution u in Lemma 3.1 and in equation (79). The derivation of assumed regularity of solution u to our problem is still open.

6 Numerical experiments

In this section, we present three different numerical examples to conform our theoretical estimates. For calculating the error, we consider problems with known exact solutions which also satisfy the assumption of Lemma 3.1. In every example, we consider the time interval [0,1] and tolerance $\epsilon = 10^{-12}$ for stopping the iterations in Newton's method. Let N denote the number of sub-intervals in time direction and $(M_s + 1)$ be the number of node points in each spatial direction.

Example 1: Consider the time fractional non-local PDE

$${}_{0}^{c}D_{t}^{\alpha}u(x,t) - a(l(u)) \Delta u(x,t) = f(x,t,u), \text{ in } \Omega \times (0,1],$$

$$u(x,t) = 0 \text{ on } \partial\Omega, \ \forall t \in [0,1],$$

$$u(x,0) = 0 \text{ in } \Omega,$$

$$(124)$$

where $\Omega = (0, \pi)$, $a(x) = 3 + \sin x$ and we choose f(x, t, u) in such a way that the exact solution of given PDE be $u(x, t) = t^3 \sin x$.

To obtain the order of convergence in time, we fix $M_s = 1000$ and calculate the error for different values of N. Table 1 shows order of convergence in the temporal direction in L^{∞} norm using uniform mesh.

To obtain the convergence rate in spatial direction, we fix N=15000 and calculate the error for different values of M_s . The order of convergence in spatial direction in L^2 and H_0^1 norms are given in the Table 2 and Table 3, respectively. Through this example, we have shown that if solution u does not have initial singularity then there is no issue in getting optimal order of convergence in L^{∞} norm in temporal direction.

N	$\alpha = 0.4$		$\alpha = 0.5$		$\alpha = 0.7$	
	Error	OC	Error	OC	Error	OC
2^{6}	3.40E-04	1.5698	6.39E-04	1.4789	2.08E-03	1.2911
2^7	1.15E-04	1.5912	2.29E-04	1.4911	8.49E-04	1.2955
2^{8}	3.80E-05	1.6339	8.15E-05	1.5104	3.46E-04	1.300039266
2^{9}	1.23E-05	-	2.86E-05	-	1.40E-04	-

Table 1: Error and order of convergence in L^{∞} norm in temporal direction on uniform mesh for Example 1.

M_s	$\alpha = 0.4$		$\alpha = 0.5$		$\alpha = 0.7$	
	Error	OC	Error	OC	Error	OC
2^{4}	8.76E-03	2.0060	8.77E-03	2.0057	8.60E-03	2.0061
2^5	2.18E-03	2.0026	2.18E-03	2.0034	2.14E-03	2.0091
2^{6}	5.44E-04	2.0053	5.44E-04	2.0086	5.32E-04	2.0327
2^{7}	1.36E-04	-	1.35E-04	-	1.30E-04	-

Table 2: Error and order of convergence in L^2 norm in space for Example 1.

M_s	$\alpha = 0.4$		$\alpha = 0.5$		$\alpha = 0.7$	
	Error	OC	Error	OC	Error	OC
2^4	1.42E-01	0.9992	1.42E-01	0.9992	1.42E-01	0.9992
2^{5}	7.10E-02	0.9998	7.10E-02	0.9998	7.10E-02	0.9998
2^{6}	3.55E-02	0.9999	3.55E-02	0.9999	3.55E-02	0.9999
2^7	1.78E-02	-	1.78E-02	-	1.78E-02	-

Table 3: Error and order of convergence in H_0^1 norm in space for Example 1.

Example 2: Consider the time fractional non-local PDE

$${}_{0}^{c}D_{t}^{\alpha}u(x,t) - a(l(u)) \Delta u(x,t) = f(x,t,u), \text{ in } \Omega \times (0,1],$$

$$u(x,t) = 0 \text{ on } \partial\Omega, \ \forall t \in [0,1],$$

$$u(x,0) = 0 \text{ in } \Omega,$$

$$(125)$$

where $\Omega = (0, \pi)$, $a(x) = 3 + \sin x$ and we choose f(x, t, u) in such a way that the exact solution of given PDE be $u(x, t) = (t^3 + t^{\alpha}) \sin x$.

To obtain the order of convergence in temporal direction, we fix $M_s = 1000$ and calculate error for different values of N. Table 4 shows the order of convergence in the temporal direction in L^{∞} norm on uniform mesh. In Table 5, we provide the order of convergence in the temporal direction in L^{∞} norm on graded mesh with grading parameter $r = \frac{2-\alpha}{\alpha}$.

To obtain the convergence rate in spatial direction, we fix N = 15000 and calculate the error for different values of M_s . The order of convergence in spatial direction in L^2 and H_0^1 norms are given in the Table 6 and Table 7, respectively.

N	$\alpha = 0.4$		$\alpha = 0$	0.5	$\alpha = 0.7$	
	Error	OC	Error	OC	Error	OC
2^{6}	3.27E-02	0.2463	2.60E-02	0.3830	1.13E-02	0.6384
2^7	2.76E-02	0.2860	1.99E-02	0.4207	7.27E-03	0.6638
2^{8}	2.26E-02	0.3163	1.49E-02	0.4461	4.59E-03	0.6784
2^{9}	1.82E-02	-	1.09E-02	-	2.87E-03	-

Table 4: Error and order of convergence in L^{∞} norm in temporal direction on uniform mesh for Example 2.

N	$\alpha = 0.4$		$\alpha = 0$	0.5	$\alpha = 0.7$	
	Error	OC	Error	OC	Error	OC
2^{6}	3.77E-03	1.4771	4.21E-03	1.4157	6.10E-03	1.2649
2^7	1.36E-03	1.5210	1.58E-03	1.4498	2.54E-03	1.2815
2^{8}	4.72E-04	1.5530	5.78E-04	1.4735	1.04E-03	1.2921
2^{9}	1.61E-04	-	2.08E-04	-	4.27E-04	-

Table 5: Error and order of convergence in L^{∞} norm in temporal direction on graded mesh for Example 2.

M_s	$\alpha = 0.4$		$\alpha = 0.5$		$\alpha = 0.7$	
	Error	OC	Error	OC	Error	OC
2^4	4.03E-03	2.0014	3.94E-03	2.0018	3.77E-03	2.0041
2^5	1.01E-03	2.0018	9.85E-04	2.0030	9.40E-04	2.0113
2^{6}	2.51E-04	2.0063	2.46E-04	2.011	2.33E-04	2.0434
2^7	6.26E-05	-	6.10E-05	-	5.65E-05	-

Table 6: Error and order of convergence in L^2 norm in space for Example 2.

M_s	$\alpha = 0.4$		$\alpha = 0.5$		$\alpha = 0.7$	
	Error	OC	Error	OC	Error	OC
2^4	7.10E-02	0.9993	7.10E-02	0.9993	7.10E-02	0.9994
2^{5}	3.55E-02	0.9998	3.55E-02	0.9998	3.55E-02	0.9998
2^{6}	1.78E-02	0.9999	1.78E-02	0.9999	1.78E-02	0.9999
2^7	8.88E-03	-	8.88E-03	-	8.88E-03	-

Table 7: Error and order of convergence in H_0^1 norm in space for Example 2.

Example 3: Consider the time fractional non-local PDE

$${}_{0}^{c}D_{t}^{\alpha}u(x,t) - a(l(u)) \Delta u(x,t) = f(x,t,u), \text{ in } \Omega \times (0,1],$$

$$u(x,t) = 0 \text{ on } \partial\Omega, \ \forall t \in [0,1],$$

$$u(x,0) = 0 \text{ in } \Omega,$$

$$(126)$$

where $\Omega = (0,1) \times (0,1)$, $a(x) = 3 + \sin x$ and we choose f(x,t,u) in such a way that the exact solution of given PDE is $u(x,t) = (t^3 + t^{\alpha})(x - x^2)(y - y^2)$.

To obtain the order of convergence in temporal direction, we take different values of N and use the relation $M_s = \lfloor N^{\frac{2-\alpha}{2}} \rfloor$ for $\alpha = 0.5$, and , $M_s = 2 \lfloor N^{\frac{2-\alpha}{2}} \rfloor$ for $\alpha = 0.7$. In Table 8, we have given the order of convergence in the temporal direction in L^{∞} norm on graded mesh with grading parameter $r = \frac{2-\alpha}{\alpha}$.

To obtain the convergence rate in spatial direction, we take $N = \lfloor M_s^{\frac{2}{(2-\alpha)}} \rfloor$. The order of convergence in spatial direction in L^2 and H_0^1 norms are given in the Table 9 and Table 10, respectively.

The graph of exact and numerical solutions for $\alpha = 0.5$ are shown in Figure 1.

N	$\alpha = 0$	0.5	$\alpha = 0.7$		
	Error	OC	Error	OC	
2^{4}	3.16E-03	1.4909	1.39E-03	1.2815	
3^{4}	2.82E-04	1.4995	1.74E-04	1.3042	
4^{3}	5.01E-05	1.500	3.87E-05	1.3250	
5^{4}	1.31E-05	-	1.19E-05	-	

Table 8: Error and order of convergence in L^{∞} norm in time on graded mesh for Example 3.

M_s	$\alpha = 0$	0.5	$\alpha = 0.7$		
	Error	OC	Error	OC	
2^{2}	1.20E-02	1.9223	1.20E-02	1.9192	
2^{3}	3.16E-03	1.9808	3.16E-03	1.9797	
2^{4}	8.00E-04	1.9963	8.01E-04	1.9951	
2^{5}	2.00E-04	-	2.01E-04	-	

Table 9: Error and order of convergence in L^2 norm in space for Example 3.

M_s	$\alpha = 0$	0.5	$\alpha = 0.7$		
	Error	OC	Error	OC	
2^{2}	8.80E-02	0.9802	8.79E-02	0.9801	
2^{3}	4.46E-02	0.9950	4.46E-02	0.9950	
2^{4}	2.24E-02	0.9987	2.24E-02	0.9987	
2^{5}	1.12E-02	-	1.12E-02	-	

Table 10: Error and order of convergence in H_0^1 norm in space for Example 3.

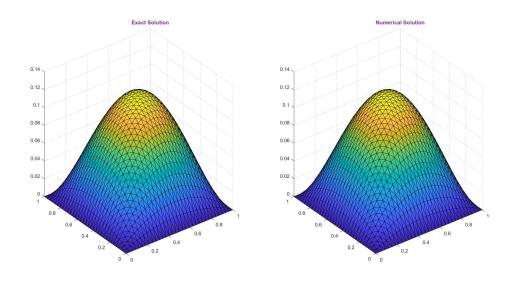


Figure 1: The exact and numerical solution at T=1, $\alpha=0.5$.

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