Signaling and Tacit Collusion in an Infinitely Repeated Prisoners' Dilemma*

Joseph E. Harrington, Jr.
Department of Economics
Johns Hopkins University
Baltimore, MD 21218
joe.harrington@jhu.edu

Wei Zhao
Department of Economics
Johns Hopkins University
Baltimore, MD 21218
weizhao@jhu.edu

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Abstract

In the context of an infinitely repeated Prisoners' Dilemma, we explore how cooperation is initiated when players signal and coordinate through their actions. There are two types of players - patient and impatient - and a player's type is private information. An impatient type is incapable of cooperative play, while if both players are patient types - and this is common knowledge - then they can cooperate with a grim trigger strategy. We find that the longer that players have gone without cooperating, the lower is the probability that they'll cooperate in the next period. While the probability of cooperation emerging is always positive, there is a positive probability that cooperation never occurs.

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1 Introduction

Antitrust and competition law has long recognized that collusion comes in two varieties: explicit and tacit. Explicit collusion involves express communication among the parties regarding the collusive agreement - what outcome is to be supported and how it is to be sustained. Tacit collusion is coordination without express communication. A common form of tacit collusion is indirect communication through price signaling: A firm raises its price with the hope that other firms will interpret this move as an invitation to collude and respond by matching the price increase. As a member of the 7th Circuit Court, Judge Richard Posner articulated such a mechanism in the High Fructose Corn Syrup decision:

> Section 1 of the Sherman Act forbids contracts, combinations, or conspiracies in restraint of trade. This statutory language is broad enough, as we noted in *JTC Petroleum Co. v. Piasa Motor Fuels, Inc* (1999), to encompass a purely tacit agreement to fix prices, that is, an agreement made without any actual communication among the parties to the agreement. If a firm raises price in the expectation that its competitors will do likewise, and they do, the firm's behavior can be conceptualized as the offer of a unilateral contract that the offerees accept by raising their prices.[1]

Of course, a firm raising its price in anticipation that it may be subsequently matched is taking a risk because rival firms may not respond in kind, either because they failed to properly interpret the price signal or deliberately chose not to collude. If the price rise is not matched then the firm will experience a decline in profit from a loss of demand. The prospect of such a signalling cost was well-recognized in the airlines industry where tacit collusion was implemented not with actual price increases but instead the announcement of future price increases which could be retracted (prior to any transactions taking place) in the event that rival firms did not respond with similar announcements (Borenstein, 2004). However, when such price announcements are unavailable as a signalling device, a firm must then consider the risky route of raising price without knowing how rivals will react. Of course, a firm always has the option of waiting on the hope that another firm will take the initiative of raising price. The trade-off from waiting is that it avoids the possible demand loss from raising price but could delay the time until a collusive outcome is reached.

The objective of this paper is to explore the emergence of tacit collusion when firms perceive themselves as facing a waiting game with regards to price signalling. The setting is an infinitely repeated two-player Prisoners' Dilemma under incomplete information. There are two player types. One type never colludes, while the other type has the capacity to collude and will surely do so once convinced its rival is also capable of colluding. As our approach will deploy the equilibrium framework, we will not be exploring the non-equilibrium process by which players settle upon a

[1] *In Re High Fructose Corn Syrup Antitrust Litigation Appeal of A & W Bottling Inc et al*, United States Court of Appeals, Seventh Circuit, 295 F3d 651, 2002; p. 2.

collusive equilibrium; players will always be playing according to some equilibrium. Tacit collusion in our setting refers to the coordination on *collusive prices* within the context of a particular equilibrium. To capture the uncertainty that a firm faces, we allow not just for uncertainty about the other firm's type - is my rival willing and able to collude? - but also uncertainty about what the other firm will do - even if my rival is willing and able to collude, will it take the lead or wait for me to make the first move and raise price? This latter uncertainty is modelled by characterizing an equilibrium in which collusive-type firms randomize between setting a low and a high price.

To be more concrete, it is not difficult to imagine the owners or managers of two gas stations at an intersection debating whether to post higher prices on its station's sign or instead deciding to "wait and see" what the other station's manager will do. Is the other station also contemplating a collusive price hike but similarly holding off raising price? Or is the other station oblivious to such reasoning and has no intent of trying to tacitly collude? As time moves on without any price hikes, a station manager adjusts her beliefs as to whether the other manager is "waiting" or "oblivious" and modifies her calculus accordingly whether or not to go ahead and raise price. Does waiting simply delay the ultimate arrival of collusion or could collusion never emerge?

It is important to emphasize that our objective is *not* to derive a Folk Theorem or, more generally, characterize the set of equilibrium payoffs. The focus is on behavior, rather than payoffs, and, more specifically, to explore the implications for the emergence of collusion when firms perceive themselves as in a waiting game when it comes to taking the lead in initiating collusion. For this reason, our analysis will examine a partial separating equilibrium that encompasses the waiting game feature, though we will also characterize a separating and a pooling equilibrium. The primary questions we will explore are: Is the likelihood of collusion declining over time? If so, does it converge to zero? If it converges to zero, does it occur asymptotically or in finite time? That is, does a sufficiently long string of failed attempts to collude result in a collusive type believing that it is so unlikely the other player is a collusive type that it gives up trying to collude? Or is collusion assured of eventually occurring?[2]

To address these questions, we focus on a class of equilibria that encompass two distinct phases: learning and collusion. In the learning phase, players are potentially signalling their types and seeking to initiate collusion. In the collusion phase, their types have been revealed and they cooperate in standard fashion using a grim trigger strategy. Our focus is on properties of the learning phase. We find that the probability of collusion emerging in any period is declining over time but is always positive; at no point are beliefs sufficiently pessimistic that collusive types give up trying to collude. While always positive, the probability of collusion emerging in the current period (given it has not yet occurred) converges to zero asymptotically. Furthermore, even

[2] As much of repeated games is about deriving Folk Theorems, it is important to recognize a distinct body of work for which the focus is on characterizing equilibrium behavior, rather than characterizing the set of equilibrium payoffs. This work is more applied in nature in that it is largely concerned with or motivated by observed behavior. An example is Harrington and Skrzypacz (2010) where conditions are derived for which an equilibrium exists that is consistent with collusive practices documented for several intermediate goods markets.

if both players are collusive types, the probability they never achieve the collusive outcome can be positive. Though collusive type players never give up trying to collude - in the sense that they always choose the collusive price with positive probability - they may never succeed in colluding.

While there is a huge body of work on the theory of collusion, none of it, to our knowledge, explores the emergence of collusion through means that can reasonably be interpreted as tacit.[3] Our model does, however, share some features with the literature on reputation in that it allows private information over a player's type and the space of types includes those which are committed to a particular strategy.[4] The seminal work of Kreps et al (1982) examines cooperation in a finitely repeated Prisoners' Dilemma where an "irrational" type might be endowed with tit-for-tat, while a "rational" type optimizes unconstrained. More recently, reputation research has considered an infinitely repeated game with commitment types with the typical research objective being to narrow down the set of equilibrium payoffs (compared to the usual Folk Theorem). When one player's type is private information, the issue is cast as whether equilibria with low payoffs for that player can be eliminated; see, for example, Cripps and Thomas (1997) and Cripps, Dekel, and Pesendorfer (2005). More recently, there has been research allowing both players to have private information; see, for example, Atakan and Ekmekci (2008).

Our model considers two-sided incomplete information in the infinitely repeated setting when the commitment type is myopic. It differs in several respects from previous work on reputation. Prior research for the infinitely repeated setting has not explored the Prisoners' Dilemma but rather other stage games including games of common interests, conflicting interests,[5] and strictly conflicting interests.[6] More importantly, the central issue in the reputation literature is about characterizing the set of equilibrium payoffs which, as noted above, is distinct from our objective. The task before us is not to limit the set of equilibria but rather to explore the dynamics of play for a particular class of equilibria. In our setting, a player ultimately wants to reveal it is a cooperative type but would like to do so only after the other player has done so. Thus, the issue is about the timing of building a reputation and whether that tendency to wait prevents cooperation from ever emerging. In this sense, our equilibrium has some commonality to the war of attrition characterized in Atakan and Ekmekci (2009) though they consider a different class of stage games.[7]

[3] Coordination within the context of a coordination game, rather than a game of conflict, is explored in Crawford and Haller (1990).

[4] For a review of some of the research on reputation, see Mailath and Samuelson (2006).

[5] The Stackelberg action for one player minimaxes the other player.

[6] Player 1's Stackelberg action along with player 2's best reply produces the highest stage game payoff for player 1 and the minimax payoff for player 2.

[7] In Atakan and Ekmekci (2009), the equilibrium is equivalent to a war of attrition as each player seeks to hold out revealing it is not committed to its Stackelberg action. In their setting, the player that concedes in the war of attrition increases its current period payoff relative to not conceding, but ends up with a lower future payoff than if its rival had conceded. In our setting, the player that concedes decreases its current period payoff, relative to its rival conceding, but suffers no disadvantage in terms of its future payoff from having conceded first. In our setting, waiting occurs in order to avoid a short-run cost fron conceding, while, in their setting, waiting occurs to influence the future

Though for complete information, a related mathematical structure to that explored here is Dixit and Shapiro (1985). They consider a repeated Battle of the Sexes game which can be interpreted as two players simultaneously deciding whether or not to enter a market. It is profitable for one and only one firm to enter. The stage game then has two asymmetric pure-strategy equilibria and one symmetric mixed-strategy equilibrium. In the repeated version, the dynamic equilibrium has randomization in each period with, effectively, the game terminating once there is entry. Farrell (1987) considers this structure when players can precede their actions with messages. One can consider our equilibrium as encompassing a waiting game for which the terminal payoff (received after firms' types are common knowledge) is either the present value of the collusive payoff (when both are collusive types) or the non-collusive payoff (when one or both are non-collusive types).

After describing the model in Section 2, we define in Section 3 a class of Markov Perfect Bayesian Equilibrium (MPBE) possessing distinct learning and collusion phases. Sections 4 and 5 consider particular MPBE for which the learning phase is non-trivial and derives properties relating to the likelihood of collusion emerging. In Section 6, additional results are derived for some examples. Concluding remarks are provided in Section 7, and all proofs are in the appendix.

2 Model

Consider a two-player Prisoners' Dilemma:

Prisoners' Dilemma

Player 1 \ Player 2	C	D
C	a, a	c, b
D	b, c	d, d

where C is interpreted as the high collusive price, and D as the low competitive price. Assume[8]

$$b > a > d \geq c.$$

and[9]

$$2a \geq b + c \geq a + d.$$

The first inequality is standard as it means the highest symmetric payoff has both players choosing C rather than taking turns cheating (that is, one player choosing D and the other choosing C).[10] The second inequality is new and is critical to our characterization. This assumption can be re-arranged to $b - a \geq d - c$, so that the

payoff.

[8] It is typical to assume $d > c$ but we allow $d = c$.

[9] Note that we cannot have $d = c$ and $b + c = a + d$ holding simultaneously as it would then imply $b = a$, which violates the assumption that $b > a$.

[10] The condition $2a \geq b + c$ is not necessary for our results but rather is to motivate the focus on players trying to sustain (C,C) in every period.

gain to playing D when the other player is expected to play C is at least as great as the gain to playing D when the other player is expected to play D. Let us show that this condition holds for both the Cournot and Bertrand oligopoly games.

Consider the symmetric Cournot quantity game with constant marginal cost c and inverse market demand for firm i of $\beta_0 - \beta_1 q_i - \beta_2 q_j$ where $\beta_0 > 0, \beta_1 \geq \beta_2 > 0$; thus, products can be differentiated. In mapping the Prisoners' Dilemma to this setting, action C corresponds to some low quantity q^l, and action D to some high quantity q^h. $b - a > d - c$ is then

$$\begin{aligned} & q^h \left[\beta_0 - \beta_1 q^h - \beta_2 q^l - c\right] - q^l \left[\beta_0 - (\beta_1 + \beta_2) q^l - c\right] \\ > \; & q^h \left[\beta_0 - (\beta_1 + \beta_2) q^h - c\right] - q^l \left[\beta_0 - \beta_1 q^l - \beta_2 q^h - c\right], \end{aligned}$$

which holds if and only if $q^h > q^l$. The Bertrand price game with homogeneous goods and constant marginal cost is, loosely speaking, the special case when

$$b = 2a, a > d = c = 0.$$

If both set the monopoly price then each earns a. Deviation from that outcome involves just undercutting the rival's price which means that the price-cost margin is approximately the same but sales are doubled so that the payoff is $2a$. Given the other firm prices at cost, pricing at cost as well yields a profit of zero (so, $d = 0$) as does pricing at the monopoly price (so, $c = 0$).[11]

Players are infinitely-lived and anticipate interacting in a Prisoners' Dilemma each period. If players have a common discount factor of δ, the grim trigger strategy is a subgame perfect equilibrium iff:

$$\delta > \frac{b-a}{b-d}. \tag{1}$$

To capture uncertainty on the part of a player as to whether the other player is willing to cooperate, it is assumed that a player's discount factor is private information. A player can be of two possible types. A player can be type L (for "long run") which means its discount factor is δ where $\delta > \frac{b-a}{b-d}$. Or a player can be type M (for "myopic") which means its discount factor is zero (though any value less than $\frac{b-a}{b-d}$ should suffice). Hence, type M players always choose D. A necessary condition for cooperative play to emerge and persist over time is then that both players are type L.

3 Class of Markov Perfect Bayesian Equilibria

There are potentially many equilibria to this game and we'll focus on what we believe is a natural class in which there is a learning phase and a collusion phase.[12] During

[11] The reference to "loosely speaking" is that this interpretation requires three prices - monopoly price, just below the monopoly price, and marginal cost - while the Prisoners' Dilemma has only two actions.

[12] Strategies are described only when a player is type L because, when type M, a player always chooses D.

the learning phase, players are exclusively trying to learn about the other player's type towards initiating collusion. This interpretation is made appropriate by focusing on strategies which depend only on beliefs as to the other player's type (as long as players' types are private information) and otherwise are independent of the history of play. When instead both players' types are public information, firms enter the collusion phase if they are both type L by adopting the grim trigger strategy for the remainder of the horizon. At that point, behavior depends on the history of play. Finally, there is the case when one player's type is revealed to be L and the other player's type is still private information. We will assume that both players (when they are type L) adopt the grim trigger strategy. As one player has revealed his type, the learning phase is over in which case it is natural that the player whose type has been revealed adopts a grim trigger strategy towards achieving collusion; and the other player's best response, if type L, will be to do the same.

To describe strategies during the learning phase, let α^t denote the probability that a player attaches to the other player being type L in period t. For the symmetric equilibria that we will characterize, α^t is common to both players as long as both players' types are private information. Since only type L players choose action C then if, on the equilibrium path, a player chooses C then the player must be type L. Hence, players' types are private information only as long as they have both chosen D. Given symmetric strategies (and symmetric initial beliefs), players have common beliefs regarding the other player's type, and these beliefs are common knowledge. Hence, α^t is not only the probability that player 1 attaches to player 2 being type L but is also player 1's point belief as to the probability that player 2 attaches to player 1 being type L, and so forth.

The solution concept is Markov Perfect Bayesian Equilibrium (MPBE), where a strategy is Markovian during the phase when players' types are not common knowledge. More specifically, if $\alpha^t \in (0,1)$ then a type L agent's period t play depends only on α^t and no other element of the history; a Markov strategy is then of the form, $q(\cdot) : [0,1] \rightarrow [0,1]$. As long as players' types are private information, beliefs are updated as follows. Suppose, in period t, $\alpha^t \in (0,1)$ and a type L player chooses C with probability $q^t \in (0,1)$. If a player was observed to choose D in period t then the other player updates using Bayes Rule:

$$\alpha^{t+1} = \frac{\alpha^t \left(1 - q^t\right)}{1 - \alpha^t q^t}. \tag{2}$$

α^1 is the common prior probability.

We can partition the class of MPBE according to q^1, the probability that a type L chooses C in the first period. Initially consider a strategy profile in which $q^1 = 1$; that is, a type L player chooses C for sure. The strategy is then separating which means that the learning phase is limited to the first period. If both players choose C in period 1 then it is common knowledge both are type L and they adopt the grim trigger strategy. If, say, player 1 chooses D then player 2 assigns probability zero to player 1 being type L in which case player 2 chooses D, whether of type L or M. Thus, one or both choosing D in period 1 results in both choosing D in all ensuing periods, in which case there is no collusion.

To verify this strategy profile is an equilibrium, we need to show that choosing C for sure in period 1 is optimal and, in response to both choosing C in period 1, it is optimal for players to adopt the grim trigger strategy. Regarding period 1, $q^1 = 1$ is optimal iff

$$\alpha^1 \left(\frac{a}{1-\delta}\right) + \left(1-\alpha^1\right)\left(c + \frac{\delta d}{1-\delta}\right) \geq \alpha^1 b + \left(1-\alpha^1\right) d + \frac{\delta d}{1-\delta} \Rightarrow \tag{3}$$

$$\alpha^1 \geq \frac{(1-\delta)(d-c)}{(1-\delta)(d-c) + \delta(a-d) - (1-\delta)(b-a)} \tag{4}$$

where (4) follows from (3) assuming the denominator is positive. (If the denominator is negative then (3) does not hold.) The denominator is positive and the RHS of (4) is less than one if and only if

$$\delta(a-d) - (1-\delta)(b-a) > 0 \Rightarrow \delta > \frac{b-a}{b-d},$$

which we assumed in (1) to ensure that collusion is feasible under complete information. Also note that if this condition is satisfied then, in response to both choosing C in period 1, it is optimal to adopt the grim trigger strategy for the remainder of the horizon. In sum, if players are sufficiently patient (as specified in (1)) and attach sufficient probability to the other player being type L (as specified in (4)) then, when both players are type L, they will choose action C in the first period and collusion will immediately ensue. For this equilibrium, the learning phase is trivial.

Next consider a MPBE in which $q^1 = 0$ so that type L players (as well as type M players) choose D in the first period. Since, by (2), $\alpha^2 = \alpha^1$ then, by the Markovian assumption, $q^2 = 0$. By induction, $q^t = 0$ for all t. This is a pooling equilibrium; it has no learning phase and firms never collude.

Finally, consider a MPBE in which $q^1 \in (0,1)$ so that a type L player assigns positive probability to both choosing C and D, so it is a partial separating equilibrium. In that we have already specified what happens when one or both players choose C (a player who chose C adopts the grim trigger strategy), let us explore the various possibilities when all previous play involves D having been chosen. There are three cases: i) $\exists T > 1$ such that $q^t \in (0,1)$ for all $t \in \{1,...,T-1\}$ and $q^T = 1$; ii) $\exists T > 1$ such that $q^t \in (0,1)$ for all $t \in \{1,...,T-1\}$ and $q^T = 0$; and iii) $q^t \in (0,1)$ for all t.[13]

Case (i) has firms randomizing until period T at which time (if both have always chosen D) they choose C for sure. Let us show that such behavior cannot be part of a MPBE. In period $T-1$ (assuming both players chose D over periods $1,...,T-2$), a type L firm is supposed to randomize in which case the payoffs from choosing C and

[13]With the Markovian assumption, note that case (i) means that if $\alpha = \alpha^T$ then $q = 1$, rather than stating that players set $q = 1$ in period T regardless of what their beliefs are. That is, on the equilibrium path, $q(\alpha) = 1$ for $\alpha = \alpha^T$ and $q(\alpha) \in (0,1)$ for all $\alpha \in \left\{\alpha^1, ..., \alpha^{T-1}\right\}$. This point analogously applies to case (ii).

D must be equal. The payoffs are

$$\text{Play C:} \qquad \alpha\left[q\left(\frac{a}{1-\delta}\right)+(1-q)\left(c+\frac{\delta a}{1-\delta}\right)\right]+(1-\alpha)\left(c+\delta c+\frac{\delta^2 d}{1-\delta}\right)$$

$$\text{Play D:} \qquad \alpha\left[q\left(b+\frac{\delta a}{1-\delta}\right)+(1-q)\left(d+\frac{\delta a}{1-\delta}\right)\right]+(1-\alpha)\left(d+\delta c+\frac{\delta^2 d}{1-\delta}\right)$$

and it is clear that D yields a strictly higher payoff than C regardless of q. The reasoning is simple and standard. The only way it can be optimal to choose C is that it somehow positively influences a player's future payoff. However, when the other player is type L, a player will receive $\frac{a}{1-\delta}$ in the future whether C or D is chosen in the current period; and if the other player is type M, $c+\frac{\delta d}{1-\delta}$ is received whether C or D is chosen. Thus, D is clearly preferred. There cannot then be a MPBE in which firms initially randomize and then adopt C for sure.

Turning to case (ii), players initially randomize and then (assuming it has been D all along) choose D for sure in period T. By our Markovian assumption, it also means choosing D in all ensuing periods. While such an equilibrium can be shown to exist by construction, this case is not very interesting for our purposes. One of our primary questions is determining whether players will eventually cooperate. By construction, this equilibrium provides a negative answer to that question by specifying that, after some series of periods in which D is chosen, players give up trying to collude and choose D for sure thereafter. What we cannot sort out with such an equilibrium is whether giving up collusion is arbitrary (each player chooses D for sure only because the other player does so) or is necessary (it is not an equilibrium for players to continue to randomize). This issue can be explored with the equilibria under case (iii).[14]

Case (iii) is when players randomize as long as D has always been chosen (and thus they are uncertain as to players' types). This is the equilibrium that will draw our attention for the remainder of the paper. It is worthy of analysis for several reasons. First, it is useful to know whether such an equilibrium exists or instead equilibria must be of the form in case (ii) in that beliefs eventually become sufficiently pessimistic that attempts at collusion must stop. We will see, in fact, that attempts to collude do not have to stop. Thus, the termination of learning in case (ii) is imposed arbitrarily. Second, if these equilibria do exist - players keep on trying to collude in the sense of choosing C with positive probability - there is the question of whether it implies that collusion will eventually occur for sure. The answer is not obvious as it depends on whether the probability of choosing C declines over time and the speed of decline. Third, the primary focus of the paper is on the learning phase which makes this equilibrium attractive because learning is not arbitrarily assumed to terminate in some period by firms forsaking the possibility of collusion by choosing D for sure (as with case (ii)). Instead, firms randomize as long as it is optimal to do so which continues to provide the opportunity to learn a rival's type.

[14] Case (ii) MPBE can be shown to exist by construction using backward induction from period T. In fact, the MPBE that we focus our attention is the case when $T=+\infty$ and is the limit of case (ii) MPBE as $T\rightarrow+\infty$.

4 Properties of a Markov Perfect Bayesian Equilibrium

In this section we explore some properties of a Markov Perfect Bayesian Equilibrium. Recall that a MPBE is partly described by: if $\alpha^t \in (0,1)$ then a type L agent's period t play depends only on α^t and no other element of the history, so it is of the form, $q(\cdot) : [0,1] \to [0,1]$. The particular class of MPBE we will explore are defined by the following properties. In period 1, choose C with probability $q\left(\alpha^1\right) \in (0,1)$. In period $t \geq 2$, if (D,D) in all previous periods then choose C with probability $q\left(\alpha^t\right) \in (0,1)$; and if (D,D) in periods $1, ..., t-2$ and not (D,D) in period t then choose C and adopt the grim trigger strategy. Recall that, as long as both players chose D, α^t evolves according to (2). Equilibrium conditions are of three types. First, conditions to ensure randomization is optimal when (D,D) has always been played. Second, given both players chose D up to the preceding period and then one player chose C and the other chose D in the preceding period, it is optimal for the player who chose C to do so again in the current period (it being the initial move for the grim trigger strategy). Third, in response to the history just described, it is optimal for the player who chose D to choose C (again, it being the initial move for the grim trigger strategy). The last scenario just requires optimality of the grim trigger strategy given the other player is type L and chooses the grim trigger strategy, which is satisfied iff (1) holds. The second case is distinct in that player 1 remains uncertain as to the other player's type. After dealing with the first set of conditions, we'll examine the second condition.[15]

Before tackling these conditions, a comment is order. In deriving equilibrium conditions, a player will go through the thought experiment of deviating from $q(\cdot)$. Note, however, that this does not upset the specification of common beliefs. For suppose player 1 deviates in period t by not choosing C with probability $q\left(\alpha^t\right)$. As each player expects the other to have chosen C with probability $q\left(\alpha^t\right)$, each player assigns probability $\frac{\alpha^t\left(1-q\left(\alpha^t\right)\right)}{1-\alpha^t q\left(\alpha^t\right)}$ to the other player being type L. While player 1 knows that player 2's beliefs about player 1's type are incorrect, that is irrelevant as all player 1 cares about is player 2's type and player 2's beliefs, both of which are summarized by $\frac{\alpha^t\left(1-q\left(\alpha^t\right)\right)}{1-\alpha^t q\left(\alpha^t\right)}$. Thus, $\frac{\alpha^t\left(1-q\left(\alpha^t\right)\right)}{1-\alpha^t q\left(\alpha^t\right)}$ remains the relevant state variable, even if a player deviates from equilibrium play.

Suppose both players' types are private information, so either it is period 1 or it is some future period but both players have thus far only chosen D. A player's expected payoff from choosing C is

$$W^C(\alpha) \equiv \alpha \left[q\left(\frac{a}{1-\delta}\right) + (1-q)\left(c + \frac{\delta a}{1-\delta}\right)\right] + (1-\alpha)\left(c + \delta c + \frac{\delta^2 d}{1-\delta}\right).$$

[15] As shown by, for example, Bhaskar (1998) and Bhaskar, Mailath, and Morris (2008), mixed strategy equilibria for an infinitely repeated game need not be purifiable, which, if that is the case, removes an important motivation for mixed strategy equilibria. The loss of purification is due to the loss of local uniqueness of Nash equilibrium. For example, Bhaskar (2000) derived a continuum of mixed strategy Nash equilibria for a repeated game, none of which were the limit of pure strategy equilibria of a perturbed game. This concern about purification could well provide a rationale for our focus on Markov equilibria. With MPBE, randomization only occurs when strategies condition on players' (common) belief over the other player's type. While we have not proven local uniqueness of such equilibria, it would be most surprising if that was not the case.

With probability α, the other player is type L and chooses C with probability q which results in cooperative payoff a being earned in the current and future periods; and chooses D with probability $1-q$ so that payoff c is earned in the current period and the cooperative payoff thereafter. Note that, regardless of the other player's action, if the other player is type L as well then both players adopt the grim trigger strategy thereafter so a is earned in the future. With probability $1-\alpha$, the other player is type M so D is chosen which results in a payoff of c in the current and subsequent period (as C is chosen in the next period as well on the hope that collusion will have been initiated) and the non-collusive payoff d thereafter. Simplifying this expression,

$$W^C(\alpha) = \alpha q\,(a-c) + (1+\delta)\,c + \delta\alpha\left[\frac{(a-d)}{1-\delta} + (d-c)\right] + \frac{\delta^2 d}{1-\delta}. \tag{5}$$

The expected payoff from choosing D is

$$\begin{aligned} W^D(\alpha) \;\;\equiv\;\; & \alpha\left[q\left(b+\frac{\delta a}{1-\delta}\right) + (1-q)\left(d+\delta V\left(\frac{\alpha\,(1-q)}{1-\alpha q}\right)\right)\right] \\ & +(1-\alpha)\left(d+\delta V\left(\frac{\alpha\,(1-q)}{1-\alpha q}\right)\right), \end{aligned}$$

which can be simplified to

$$W^D(\alpha) = \alpha q\left(b+\frac{\delta a}{1-\delta}\right) + (1-\alpha q)\left(d+\delta V\left(\frac{\alpha\,(1-q)}{1-\alpha q}\right)\right). \tag{6}$$

If, in equilibrium, $q\in(0,1)$ then the expressions in (5) and (6) must be the same:

$$\begin{aligned} & \alpha q\,(a-c) + (1+\delta)\,c + \delta\alpha\left[\frac{(a-d)}{1-\delta} + (d-c)\right] + \frac{\delta^2 d}{1-\delta} \\ = \;\; & \alpha q\left(b+\frac{\delta a}{1-\delta}\right) + (1-\alpha q)\left(d+\delta V\left(\frac{\alpha\,(1-q)}{1-\alpha q}\right)\right). \end{aligned}$$

Re-arranging gives us:

$$\alpha q = \frac{\delta\left[\frac{a}{1-\delta} - V\left(\frac{\alpha(1-q)}{1-\alpha q}\right)\right] - (1-\alpha)\,\frac{\delta(a-d)}{1-\delta} - \left[1+\delta\,(1-\alpha)\right](d-c)}{\delta\left(\frac{a}{1-\delta} - V\left(\frac{\alpha(1-q)}{1-\alpha q}\right)\right) + (b-a) - (d-c)}. \tag{7}$$

Define:

$$\underline{\alpha} \equiv \frac{\left(1-\delta^2\right)(d-c)}{\delta\left[(1-\delta)\,(a-c) + \delta\,(a-d)\right]} \in [0,1)\,,$$

where $\underline{\alpha} \geq 0$ follows from $d \geq c$ and $a > d$. In Theorem 1, we show that players randomize when $\alpha > \underline{\alpha}$.[16] To show $\underline{\alpha} < 1$, note that

$$\delta\left[(1-\delta)\,(a-c) + \delta\,(a-d)\right] > \left(1-\delta^2\right)(d-c) \Leftrightarrow \delta > \frac{d-c}{a-c}.$$

[16] For the case of $d=c$, they may also randomize when $\alpha = \underline{\alpha}$.

Given (1) holds, a sufficient condition for $\delta > \frac{d-c}{a-c}$ is

$$\frac{b-a}{b-d} \geq \frac{d-c}{a-c} \Leftrightarrow (b-a)(a-c) \geq (d-c)(b-d) \Leftrightarrow b+c \geq a+d$$

which is true by assumption. Thus, by our previous assumptions, $\underline{\alpha} \in [0,1)$.

Next consider the situation in which, prior to the previous period, (D,D) had always been played so that both players' types were private information, and, in the previous period, one player chose C and the other chose D. If both are type L, they adopt the grim trigger strategy. For the player who knows the other player's type (and thus knows he'll choose C), choosing C is optimal iff (1) holds. Now consider the player whose type has been revealed and remains uncertain as to the other player's type. If that player assigns probability α to the other player being type L then he prefers to choose C iff

$$\alpha\left(\frac{a}{1-\delta}\right) + (1-\alpha)\left(c + \frac{\delta d}{1-\delta}\right) \geq \alpha\left(b + \frac{\delta d}{1-\delta}\right) + (1-\alpha)\left(\frac{d}{1-\delta}\right)$$

which is equivalent to

$$\alpha \geq \frac{(1-\delta)(d-c)}{\delta(a-d) - (1-\delta)[(b-a)-(d-c)]} \equiv \alpha^*.$$

$\underline{\alpha} \geq \alpha^*$ iff:

$$\frac{\left(1-\delta^2\right)(d-c)}{\delta[(1-\delta)(a-c)+\delta(a-d)]} \geq \frac{(1-\delta)(d-c)}{\delta(a-d)-(1-\delta)[(b-a)-(d-c)]} \Rightarrow$$

$$\frac{1+\delta}{\delta[(1-\delta)(a-c)+\delta(a-d)]} \geq \frac{1}{\delta(a-d)-(1-\delta)[(b-a)-(d-c)]}. \tag{8}$$

Letting $\delta \to 1$, (8) holds iff

$$\frac{2}{a-d} \geq \frac{1}{a-d},$$

which is true. Thus, as long as $\alpha > \frac{\alpha(1-q(\alpha))}{1-\alpha q(\alpha)} > \underline{\alpha}$ and δ is sufficiently close to one then as soon as one player chooses C, both players, if they are type L, will optimally adopt the grim trigger strategy.

Theorem 1 states that there is a symmetric MPBE in which, as long as players' types are private information, a type L player randomizes between playing C and D when $\alpha > \underline{\alpha}$, and chooses D for sure when $\alpha \leq \underline{\alpha}$. When a player randomizes, C is chosen with probability $q(\alpha)$, as defined in (7). $V : [0,1] \to \Re$ denotes the MPBE value function for a type L player. Proofs are in the appendix.[17]

Theorem 1 *There exists $\widehat{\delta} \in (0,1)$ such that if $\delta > \widehat{\delta}$ then there is a symmetric Markov Perfect Bayesian Equilibrium $q(\cdot)$ such that*

$$q(\alpha)\begin{cases} = 0 & \text{if } \alpha \in (0,\underline{\alpha}] \\ \in (0,1) & \text{if } \alpha \in (\underline{\alpha},1] \end{cases}$$

[17]In Theorem 1, it can be show that $\widehat{\delta} = \frac{(d-c)+\sqrt{(d-c)^2+4(b-d)[(b-d)-(a-c)]}}{2(b-d)}$.

$$V(\alpha)\begin{cases} = \frac{d}{1-\delta} & \text{if } \alpha \in (0, \underline{\alpha}] \\ \in \left(\frac{d}{1-\delta}, \frac{a}{1-\delta}\right) & \text{if } \alpha \in (\underline{\alpha}, 1] \end{cases}$$

and $\lim_{\alpha \to 1} q(\alpha) < 1$.

The next result concerns the evolution of beliefs and behavior in response to a failure to cooperate, by which we mean both players have thus far always chosen D. Recall that if a player assigns probability α to the other player being type L then, after observing the other player choose D, the updated probability is $\frac{\alpha(1-q(\alpha))}{1-\alpha q(\alpha)}$ where $q(\alpha)$ is the equilibrium probability that a type L player chooses C given beliefs α. Further recall, from Theorem 1, that if $\alpha > \underline{\alpha}$ then $q(\alpha) > 0$.

Theorem 2 *If $q(\cdot)$ is a symmetric Markov Perfect Bayesian Equilibrium as described in Theorem 1 then: i) if $\alpha > \underline{\alpha}$ then $\frac{\alpha(1-q(\alpha))}{1-\alpha q(\alpha)} > \underline{\alpha}$; ii) if $\alpha^1 > \underline{\alpha}$ then $lim_{t\to\infty}\alpha^t = \underline{\alpha}$ and $q(\alpha^t) > 0$ for all t; iii) if $\underline{\alpha} > 0$ then $lim_{\alpha \downarrow \underline{\alpha}} q(\alpha) = 0$; and iv)* $\lim_{t\to\infty} \alpha^t q(\alpha^t) = 0$.

Theorem 2 shows that if $\alpha^1 > \underline{\alpha}$ then $\alpha^t > \underline{\alpha}$ for all t which then implies $q(\alpha^t) > 0$ for all t.[18] Therefore, no matter how long players have failed to cooperate, a type L player will continue to try to initiate cooperation (in the sense of assigning positive probability of choosing C). In other words, beliefs never become so pessimistic about the other player's willingness to cooperate that a player prefers to abandon any prospects of cooperation by playing D for sure. When $\underline{\alpha} > 0$, it is also the case that the probability of a player initiating cooperation converges to zero over time in response to the probability that the other player is type L converging to $\underline{\alpha}$ after a history of failed cooperation. Note that the probability of a type L player playing C must converge to zero as the probability of a player being type L approaches $\underline{\alpha}\,(>0)$ from above. If $q(\alpha)$ was instead bounded above zero then a sufficiently long sequence of playing D would have to result in a sufficiently small probability of the player being type L, which would contradict this probability being bounded below by $\underline{\alpha}$ (at least when $\underline{\alpha} > 0$). Finally, conditional on cooperation not yet having emerged, the probability assigned to a player initiating cooperation is $\alpha^t q(\alpha^t)$ in which case the probability that cooperation emerges out of period t is $1-\left(1-\alpha^t q(\alpha^t)\right)^2$. While this value is always positive - so collusion is always a possibility - it converges to zero in response to an ever-increasing sequence of failed attempts at collusion, in which case collusion eventually becomes very unlikely to emerge. Whether collusion emerges for sure is explored for the class of MPBE examined in the next section.

5 Affine Markov Perfect Bayesian Equilibrium

Consider the class of MPBE described in Theorem 1 and let us focus on those for which the value function is affine in α when players' types are private information.

[18] Note that this result is not obvious. If $\underline{\alpha} > 0$ then, in principle, $\alpha^t < \underline{\alpha}$ unless $q(\alpha) \to 0$ sufficiently fast as $\alpha \to \underline{\alpha}$.

Definition 3 *An affine Markov Perfect Bayesian Equilibrium is a MPBE (as described in Theorem 1) in which the value function is affine over* $\alpha \in [\underline{\alpha}, 1]$.

Theorem 4 *There exists a unique affine Markov Perfect Bayesian Equilibrium. The value function is*

$$V(\alpha) = \begin{cases} \frac{d}{1-\delta} & \text{if } \alpha \in [0, \underline{\alpha}] \\ x + y\alpha & \text{if } \alpha \in [\underline{\alpha}, 1] \end{cases} \tag{9}$$

where (x, y) *is the unique solution to:*

$$x + y \frac{\left(1-\delta^2\right)(d-c)}{\delta\left[(1-\delta)(a-c) + \delta(a-d)\right]} = \frac{d}{1-\delta} \tag{10}$$

$$x + y = \frac{2a\delta + (1-\delta)\left[(b-a) - (d-c) - \sqrt{\Omega}\right]}{2\delta(1-\delta)}. \tag{11}$$

and

$$\Omega \equiv \left[(b-a) - (d-c)\right]^2 + 4\delta(a-c)\left[(b-a) - (d-c)\right] + 4\delta(a-c)(d-c). \tag{12}$$

Furthermore, if $\alpha \in (\underline{\alpha}, 1]$ *then*

$$\begin{aligned} q(\alpha) \quad = \quad & \frac{\delta(a-d) + \delta(1-\delta)(d-c-y)}{(1-\delta)\left[(b-a)-(d-c)\right] + \delta a - \delta(1-\delta)(x+y)} \\ & + \left(\frac{1}{\alpha}\right)\left[\frac{\delta a - \delta(1-\delta)x - \delta(a-d) - \left(1-\delta^2\right)(d-c)}{(1-\delta)\left[(b-a)-(d-c)\right] + \delta a - \delta(1-\delta)(x+y)}\right] \end{aligned} \tag{13}$$

In the preceding section, we established that $\alpha^t q\left(\alpha^t\right)$ converges to zero and thus is eventually decreasing over time. For affine MPBE, we can now say that $\alpha^t q\left(\alpha^t\right)$ is monotonically declining over time, in which case the probability a player chooses C decreases with the length of time for which cooperative play has not yet occurred. It is also the case that a type L player's equilibrium value is decreasing with the likelihood assigned to players being type L.

Theorem 5 *If* $q(\cdot)$ *is defined by (13) then* $\alpha q(\alpha)$ *is increasing in* α *and* $V(\alpha)$ *is increasing in* α.

While $\alpha q(\alpha)$ is increasing in α, $q(\alpha)$ need not be increasing in α everywhere, though we know that eventually it must be increasing in α since it converges to zero (when $\underline{\alpha} > 0$). We next show that when $d > c$ then $q(\alpha)$ is decreasing over time as lower probability is attached to players being type L (given only D has been chosen thus far). However, when $d = c$ then $q(\alpha)$ is, interestingly, independent of a player's beliefs as to the other player's type and thus is constant over time. Though it is still the case that α^t is declining, a type L player maintains the same probability of acting cooperatively.

Theorem 6 *If $q(\cdot)$ is defined by (13) then, for $\alpha > \underline{\alpha}$: i) if $d > c$ then $q(\alpha)$ is increasing in α; and ii) if $d = c$ then $q(\alpha) = q'$ for some $q' \in (0,1)$.*

When $d = c$ - so a player is not harmed when choosing the cooperative action - the probability that a type L player chooses C is fixed at some positive value. Thus, if both players are type L then, almost surely, players will eventually achieve the collusive outcome. However, whether cooperative play ultimately emerges is not so clear when $d > c$ as then the probability of cooperation being initiated converges to zero. To examine this issue, define Q^T as the probability that players are still not colluding by the end of period T, conditional on both players being type L. If $q(\cdot)$ is a MPBE, Q^T is defined by

$$Q^T = \prod_{t=1}^{T} \left(1 - q^t\right)^2$$

where, given α^1, q^t is defined recursively by:

$$q^t = q\left(\alpha^t\right), t \geq 1; \; \alpha^t = \frac{\alpha^{t-1}\left(1 - q^{t-1}\right)}{1 - \alpha^{t-1} q^{t-1}}, t \geq 2.$$

The next result shows that, even when both players are type L, there is a positive probability that collusion never emerges even though they never give up trying (that is, they always choose C with positive probability).[19]

Theorem 7 *If $q(\alpha)$ is defined by (13) and $d > c$ then $\lim_{T\to\infty} Q^T > 0$.*

If both players are type L then, in any period, there is always a positive probability that one of them will choose the cooperative action and thereby result in the emergence of collusion. This property follows from $\alpha^t > \underline{\alpha}$ for all t; regardless of how long the other player has chosen D, a player assigns sufficient probability to its rival being type L that it is optimal to continue to try to cooperate (as reflected in choosing C with positive probability). For $\alpha^t > \underline{\alpha}\,(> 0)$, it must be the case that a long sequence of choosing D is not a sufficiently pessimistic signal that the other player is type L which can only be the case if, as $\alpha^t \to \underline{\alpha}$, the probability that a type L player chooses C converges sufficiently fast to zero. But, as shown in the previous result, this also has the implication that the probability that two type L players start colluding in period t is going to zero sufficiently fast, which means collusion is not assured. In short, even if both players are willing and able to cooperate, there is a positive probability that they never do so though they never give up trying.

6 Examples

In this section, we derive the affine MPBE from Theorem 4 for some examples. Example 1 is a case in which the probability of a player choosing the cooperative action is independent of α and, therefore, fixed over time. When players are more patient,

[19] Theorem 7 is true as long as $q(\alpha) = A + B\left(\frac{1}{\alpha}\right)$ for some A and B where $B < 0$ and $A + B < 1$.

we show that collusion is more likely to emerge. In Example 2, the probability a type L player chooses the cooperative action is increasing in the likelihood it assigns to the other player also being type L. In response to an ever-lengthening sequence of failed cooperation, the probability of cooperation emerging is declining. Furthermore, conditional on both players being type L, the probability that collusion never occurs is positive. Finally, Example 3 considers an asymmetric Prisoners' Dilemma in which the collusive outcome does not split the surplus equally. Surprisingly, greater asymmetry makes collusion more likely to emerge.

6.1 Example 1: Bertrand Price Game

Assume $b = 2a, d = c = 0$, and normalize so $a = 1$.

Bertrand Price Game

Player 2

Player 1		C	D
	C	$1, 1$	$0, 2$
	D	$2, 0$	$0, 0$

This case approximates the Bertrand price game in which, for example, market demand is perfectly inelastic at two units with a maximum willingness to pay of 1, and firms have zero marginal cost.

Since $d = c$, we know that $\underline{\alpha} = 0$ and the probability of a type L player cooperating is independent of α and thus constant over time. Equilibrium play and payoffs are described by[20]

$$\begin{aligned} q(\alpha) &= \frac{\sqrt{4\delta+1}-1}{\sqrt{4\delta+1}+1} \\ V(\alpha) &= \left(\frac{1+\delta-(1-\delta)\sqrt{4\delta+1}}{2\delta(1-\delta)}\right)\alpha. \end{aligned}$$

As one would expect, the probability of choosing C is higher when players are more patient:

$$\frac{\partial q}{\partial \delta} = \frac{4}{\left(\sqrt{4\delta+1}+1\right)^2\sqrt{4\delta+1}} > 0.$$

The probability that two type L players are colluding by period $T \geq 2$ is

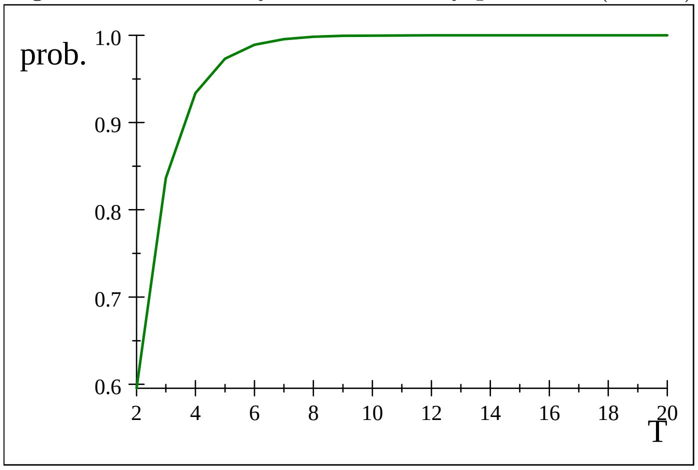
$$1 - \left[1 - \left(\frac{\sqrt{4\delta+1}-1}{\sqrt{4\delta+1}+1}\right)\right]^{2(T-1)}.$$

When $\delta = .9$, Figure 1 shows how the probability of collusion rises rapidly over time,

[20] Derivations for all examples are available on request.

and that it is quite close to one by period 10.

Figure 1: Probability of collusion by period T ($\delta = .9$)



6.2 Example 2: Bertrand Price Game with Relative Compensation

Let us modify the Bertrand price game so that managers - not owners - are repeatedly making price decisions and managerial compensation is based on relative performance. Specifically, a manager receives compensation equal to half of firm profit but, in the event that the other firm has higher profit, incurs a penalty equal to one-quarter of the rival firm's profit. The single-period payoff to a manager is then:

$$\text{Payoff of manager } i \text{ in period } t = \begin{cases} (1/2)\,\pi_i^t & \text{if } \pi_i^t \geq \pi_j^t \\ (1/2)\,\pi_i^t - (1/4)\,\pi_j^t & \text{if } \pi_i^t < \pi_j^t \end{cases}$$

where π_i^t is the period t profit of firm i. If market demand is perfectly inelastic at two units with a maximum willingness to pay of 2 (and zero marginal cost) then the managers' payoff matrix is represented by

Bertrand Price Game
with Relative Compensation

		Player 2	
		C	D
Player 1	C	$1,1$	$-1,2$
	D	$2,-1$	$0,0$

Equilibrium has:

$$q(\alpha) = \begin{cases} 0 & \text{if } \alpha \in \left[0, \frac{1-\delta^2}{2\delta-\delta^2}\right] \\ \frac{\left[\alpha\delta(2-\delta)-\left(1-\delta^2\right)\right]\left(\sqrt{2\delta}-1\right)}{\alpha\sqrt{2\delta}(2\delta-1)} & \text{if } \alpha \in \left(\frac{1-\delta^2}{2\delta-\delta^2}, 1\right] \end{cases}$$

$$V(\alpha) = \begin{cases} 0 & \text{if } \alpha \in \left[0, \frac{1-\delta^2}{2\delta-\delta^2}\right] \\ \frac{\left[\delta^2-1-\alpha\left(2\delta-\delta^2\right)\right]\left[\delta-(1-\delta)\sqrt{2\delta}\right]}{\delta(1-\delta)(2\delta-1)} & \text{if } \alpha \in \left(\frac{1-\delta^2}{2\delta-\delta^2}, 1\right] \end{cases}$$

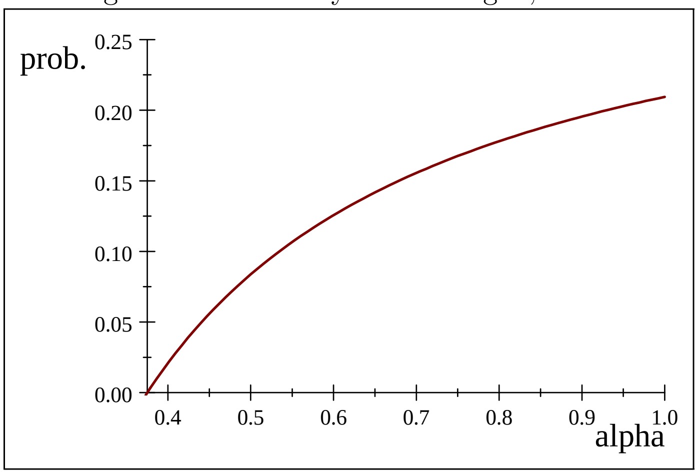
We know from Theorem 6 that, when $\alpha > \frac{1-\delta^2}{2\delta-\delta^2} \, (= \underline{\alpha})$, $q(\alpha)$ is increasing in α.

If $\delta = .8$ then $\underline{\alpha} = .375$ and, for $\alpha > .375$,

$$q(\alpha) \simeq .335 - \frac{.126}{\alpha},$$

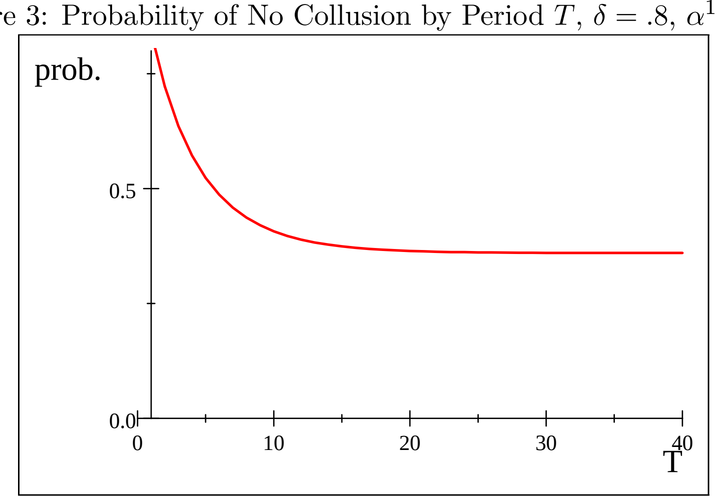
which is plotted in Figure 2. If players have thus far always played D then, in each player updating their beliefs as to the other player's type, α^t will fall over time which then induces type L players to choose C with a lower probability. If a string of (D,D) gets longer and longer, so that $\alpha^t \to \underline{\alpha}$, $q(\alpha) \to 0$ and does so at an increasingly fast rate; note that $q(\alpha)$ is strictly concave in α.

Figure 2: Probability of choosing C, $\delta = .8$



When a player initially assigns a 50% chance to its rival being type L, the probability that collusion has not been achieved by period T is shown in Figure 3. There is a 36% chance that collusion is never achieved.

Figure 3: Probability of No Collusion by Period T, $\delta = .8$, $\alpha^1 = 0.5$



6.3 Example 3: Asymmetric Bertrand Price Game

Consider the following generalization of Example 1 where the collusive outcome is now allowed to be asymmetric and $\gamma \in [1/2, 1)$.[21]

Asymmetric Bertrand Price Game

		Player 2	
		Cooperate	*Defect*
Player 1	*Cooperate*	$\gamma, 1-\gamma$	$0, 1$
	Defect	$1, 0$	$0, 0$

The collusive outcome gives player 1 a market share of γ which is at least 1/2. There is an affine MPBE with

$$\begin{aligned}
q_1 &= \frac{\sqrt{\gamma(\gamma + 4\delta(1-\gamma))} - \gamma}{\sqrt{\gamma(\gamma + 4\delta(1-\gamma))} + \gamma} \\
q_2 &= \frac{\sqrt{(1-\gamma)(1-\gamma+4\delta\gamma)} - (1-\gamma)}{\sqrt{(1-\gamma)(1-\gamma+4\delta\gamma)} + (1-\gamma)} \\
V_2(\alpha_1) &= \left[\frac{(\gamma - \delta + 3\delta(1-\gamma)) - (1-\delta)\sqrt{\gamma(\gamma + 4\delta(1-\gamma))}}{2(1-\delta)\delta}\right]\alpha_1 \\
V_1(\alpha_2) &= \left[\frac{(1-\delta-\gamma+3\delta\gamma) - (1-\delta)\sqrt{(1-\gamma)(1-\gamma+4\delta\gamma)}}{2(1-\delta)\delta}\right]\alpha_2.
\end{aligned}$$

As with Example 1, q_1 and q_2 do not depend on α. One can prove that q_1 is decreasing in γ and increasing in δ, and q_2 is increasing in γ and δ.

It might be expected that the player with the higher share of collusive profit would play C with a higher probability. However, when the share of collusive profit for player 1 (γ) is larger, the probability of playing C is actually higher for player 2 and lower for player 1. Since player 1 gains more by achieving cooperative play when γ is bigger, player 2 must be more likely to play C if player 1 is to be indifferent between playing C and D; and recall that D is more attractive when the other player is more likely to initiate cooperation. The player who benefits more from colluding is then less likely to take the first move in cooperating.

To explore the effect of asymmetry on the likelihood of collusion, consider the probability that collusion is initiated in any period:

$$1 - (1-q_1)(1-q_2) = 1 - \frac{4}{\left(\sqrt{\frac{\gamma+4\delta(1-\gamma)}{\gamma}} + 1\right)\left(\sqrt{\frac{1-\gamma+4\delta\gamma}{1-\gamma}} + 1\right)}.$$

It is straightforward to show that it is increasing in γ,

$$\frac{\partial\left[1-(1-q_1)(1-q_2)\right]}{\partial\gamma} > 0, \tag{14}$$

[21] A preliminary analysis suggests that many of the results in Sections 3 and 4 can be extended to when the Prisoners' Dilemma is asymmetric.

so collusion is more likely when the collusive outcome is more skewed to favor one firm.

As the equilibrium condition for the grim trigger strategy is $\delta \geq \gamma$, increasing asymmetry by raising γ makes collusion more difficult in the sense that the minimum discount factor is higher. However, conditional on the collusive outcome being sustainable, asymmetry reduces the expected time until collusion is achieved, as reflected in (14). In fact, as asymmetry becomes extreme, collusion is achieved immediately.[22]

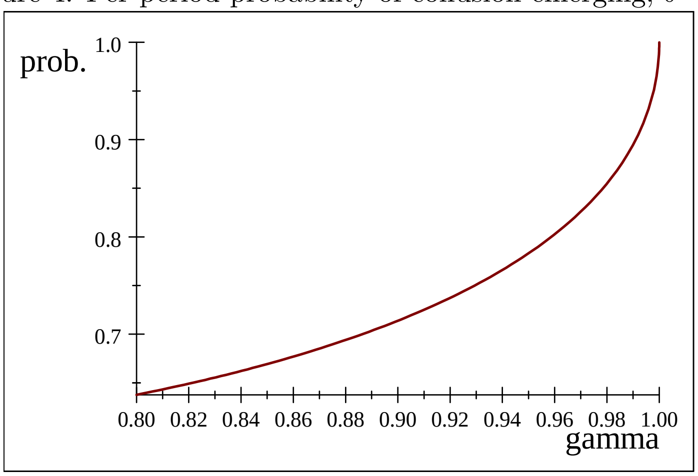
$$\begin{aligned}
\lim_{\gamma \to 1} q_1(\alpha_1) &= \frac{\sqrt{\gamma(\gamma + 4\delta(1-\gamma))} - \gamma}{\sqrt{\gamma(\gamma + 4\delta(1-\gamma))} + \gamma} = 0 \\
\lim_{\gamma \to 1} q_2(\alpha_2) &= \lim_{\gamma \to 1} \frac{\sqrt{(1-\gamma)(1-\gamma+4\delta\gamma)} - (1-\gamma)}{\sqrt{(1-\gamma)(1-\gamma+4\delta\gamma)} + (1-\gamma)} = \lim_{\gamma \to 1} \frac{\sqrt{\frac{1-\gamma+4\delta\gamma}{1-\gamma}} - 1}{\sqrt{\frac{1-\gamma+4\delta\gamma}{1-\gamma}} + 1} = 1
\end{aligned}$$

Therefore,

$$\lim_{\gamma \to 1} 1 - (1 - q_1)(1 - q_2) = 1.$$

For when $\delta = .8$, Figure 4 depicts the relationship between the asymmetry of the collusive outcome and the probability of collusion emerging, given it has not yet happened.

Figure 4: Per period probability of collusion emerging, $\delta = .8$



7 Concluding Remarks

In practice, communication - either express or implicit - is essential to collusion. This we know from both experimental work and the many documented episodes of cartels. Communication can manifest itself in two ways - exchange of information and exchange of intentions. There is a limited amount of work in oligopoly theory on

[22] Keep in mind that as we let $\gamma \to 1$, we must have $\delta \to 1$ so that $\delta \geq \gamma$ is satisfied.

collusion and the exchange of information. In Athey and Bagwell (2001, 2008), firms have private information about their cost and exchange (costless) messages about cost, while in Hanazono and Yang (2007) and Gerlach (2009), firms have private signals on demand and seek to share that information. Then there is work in which sales or some other endogenous variable is private information and firms exchange messages for monitoring purposes; see Aoyagi (2002), Chan and Zhang (2009), and Harrington and Skrzypacz (2010).[23] Communication may also be used to resolve strategic uncertainty; specifically, in order to coordinate a move from a non-collusive to a collusive equilibrium. Here, intentions rather than hard information is being communicated.

Within the context of the equilibrium paradigm, the current paper sought to make progress on the tacit signalling of the intention to collude. In a sense, signalling in our model is part information (regarding a player's type) and part intentions (regarding cooperative play). Let us summarize our main findings. If the initial probability that players are capable of colluding is sufficiently high then, in any period, there is always the prospect of collusion emerging; no matter how long is there a history of failed collusion, beliefs as to players being cooperative types remain sufficiently high that it is worthwhile for them to continue to try to cooperate. This does not imply, however, that collusion is assured. For a wide class of situations, there is a positive probability that collusion never emerges. Players never give up trying to collude but they may also never succeed.

In terms of future work, one research direction is to allow a player's type to change over time, rather than remain fixed forever.[24] When a cooperative type raises price and does not receive a favorable response, it'll infer that its rival is an uncooperative type. In that case, it might be inclined to try again later on the hope that the rival's type has changed. But it may also be the case that a player who has previously failed to respond in kind to an invitation to collude will see itself as having the onus to initiate cooperation (in the event that its type changes) because its rival believes it is an uncooperative type. Now suppose players are currently engaged in cooperative play. A deviation by a player is part of equilibrium play and signals a change in a player's type to being uncooperative. Assuming persistence in types, the punishment of the deviator would have a certain credibility (beyond simply being an equilibrium) in that the other player believes there is little point in trying to cooperate. Indeed, non-cooperation may be the unique equilibrium. All this could put the burden on the deviator to re-initiate cooperation. Even this cursory analysis suggests that a rich set of behavior could arise from allowing types to evolve stochastically over time.

[23] There is also an extensive game theory literature on the issue of private monitoring. See Compte (1998), Kandori and Matsushima (1998), Kandori (2002), Zheng (2008), and Obara (2009)

[24] Recent work by Escobar and Toikka (2009) provides a foundation for such an analysis.

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8 Appendix: Proofs

Proof of Theorem 1. Let us first establish the stated properties on $q(\cdot)$. A player strictly prefers D to C iff:

$$\begin{aligned} & \alpha q \left(b + \frac{\delta a}{1-\delta} \right) + (1-\alpha q) \left(d + \delta V \left(\frac{\alpha(1-q)}{1-\alpha q} \right) \right) \\ > \ & \alpha q \left(\frac{a}{1-\delta} \right) + \alpha (1-q) \left(c + \frac{\delta a}{1-\delta} \right) + (1-\alpha) \left(c + \delta c + \frac{\delta^2 d}{1-\delta} \right). \end{aligned} \tag{15}$$

Note that $V(\alpha)$ has a lower bound of $\frac{d}{1-\delta}$ - as a player can assure itself of a payoff of at least $\frac{d}{1-\delta}$ by always choosing D - which then implies $V\left(\frac{\alpha(1-q)}{1-\alpha q}\right) \geq \frac{d}{1-\delta}$. Thus, a sufficient condition for (15) involves substituting $\frac{d}{1-\delta}$ for $V\left(\frac{\alpha(1-q)}{1-\alpha q}\right)$:

$$\begin{aligned} & \alpha q \left(b + \frac{\delta a}{1-\delta} \right) + (1-\alpha q) \left(d + \delta \left(\frac{d}{1-\delta} \right) \right) \\ > \ & \alpha q \left(\frac{a}{1-\delta} \right) + \alpha (1-q) \left(c + \frac{\delta a}{1-\delta} \right) + (1-\alpha) \left(c + \delta c + \frac{\delta^2 d}{1-\delta} \right) \end{aligned} \tag{16}$$

or

$$\begin{aligned} & \alpha q \left(b + \frac{\delta a}{1-\delta} \right) + (1-\alpha q) \left(\frac{d}{1-\delta} \right) - \alpha q \left(\frac{a}{1-\delta} \right) \\ & -\alpha (1-q) \left(c + \frac{\delta a}{1-\delta} \right) - (1-\alpha) \left(c + \delta c + \frac{\delta^2 d}{1-\delta} \right) \\ > \ & 0. \end{aligned} \tag{17}$$

Take the derivative of the LHS of (17) with respect to q:

$$\begin{aligned} & \alpha \left(b + \frac{\delta a}{1-\delta} \right) - \alpha \left(\frac{d}{1-\delta} \right) - \alpha \left(\frac{a}{1-\delta} \right) + \alpha \left(c + \frac{\delta a}{1-\delta} \right) \\ = \ & \alpha \left[(b-a) - (d-c) \right] + \alpha \delta \left(\frac{a-d}{1-\delta} \right) > 0, \end{aligned} \tag{18}$$

since $b - a \geq d - c$ and $a - d > 0$. Hence, the difference between the payoff to D and the payoff to C is minimized when $q = 0$. Thus, D is surely strictly preferred to C if (16) holds when $q = 0$:

$$\frac{d}{1-\delta} > \alpha \left(c + \frac{\delta a}{1-\delta} \right) + (1-\alpha) \left(c + \delta c + \frac{\delta^2 d}{1-\delta} \right) \tag{19}$$

$$
\begin{aligned}
\frac{d}{1-\delta} &> c+\alpha\left(\frac{\delta a}{1-\delta}\right)+(1-\alpha)\left(\delta c+\frac{\delta^2 d}{1-\delta}\right) \\
\frac{d}{1-\delta}-\delta c-\frac{\delta^2 d}{1-\delta}-c &> \alpha\left(\frac{\delta a}{1-\delta}-\delta c-\frac{\delta^2 d}{1-\delta}\right) \\
(1+\delta)(d-c) &> \delta\alpha\left(a-c+\frac{\delta(a-d)}{1-\delta}\right) \\
\alpha &< \frac{(1+\delta)(d-c)}{\delta\left(a-c+\frac{\delta(a-d)}{1-\delta}\right)}=\frac{\left(1-\delta^2\right)(d-c)}{\delta\left[(1-\delta)(a-c)+\delta(a-d)\right]}\ (\equiv \underline{\alpha})
\end{aligned}
$$

Thus, if $\alpha < \underline{\alpha}$ then, in equilibrium, $q(\alpha)=0$.

To prove that $q(\underline{\alpha})=0$, suppose not. It follows from $q(\underline{\alpha})>0$ that

$$
\frac{\underline{\alpha}(1-q(\underline{\alpha}))}{1-\underline{\alpha}q(\underline{\alpha})}<\underline{\alpha}.
$$

The preceding analysis showed $q(\alpha)=0\ \forall\alpha<\underline{\alpha}$ and since $q=0$ implies

$$
\frac{\alpha(1-q)}{1-\alpha q}=\alpha
$$

then, by stationary, $q^t=0\ \forall t\geq t'$ when $\alpha^{t'}<\underline{\alpha}$. Hence,

$$
V\left(\frac{\underline{\alpha}(1-q(\underline{\alpha}))}{1-\underline{\alpha}q(\underline{\alpha})}\right)=\frac{d}{1-\delta}. \tag{20}
$$

For $q(\underline{\alpha})>0$, the expected payoff from choosing C must be at least as great as that from choosing D:

$$
\begin{aligned}
&\alpha q\left(\frac{a}{1-\delta}\right)+\alpha(1-q)\left(c+\frac{\delta a}{1-\delta}\right)+(1-\alpha)\left(c+\delta c+\frac{\delta^2 d}{1-\delta}\right) \qquad (21)\\
\geq\ &\alpha q\left(b+\frac{\delta a}{1-\delta}\right)+(1-\alpha q)\left(\frac{d}{1-\delta}\right),
\end{aligned}
$$

where we used (20). However, note that the expressions in (21) are the same as those in (16). By our previous analysis, if $\alpha=\underline{\alpha}$ then (16) holds with equality when $q=0$ and with strict inequality when $q>0$. We conclude that (21) and $q(\underline{\alpha})>0$ are inconsistent and, therefore, $q(\underline{\alpha})=0$.

Finally, let us prove that if $\alpha\in(\underline{\alpha},1]$ then $q(\alpha)\in(0,1)$ and $\lim_{\alpha\to 1}q(\alpha)<1$. To show that $q(\alpha)>0$, suppose not so $\exists\alpha'>\underline{\alpha}$ such that $q(\alpha')=0$. By the preceding logic, $V(\alpha')=\frac{d}{1-\delta}$. In that case, the payoff to D is at least as great as that from C iff (19) holds with a weak inequality, but the previous analysis showed that is the case iff $\alpha\leq\underline{\alpha}$. Therefore, if $\alpha>\underline{\alpha}$ then $q(\alpha)>0$. To show that $q(\alpha)<1$, evaluate the payoffs from C and from D as $q\to 1:$

$$
\begin{aligned}
\text{Play C} &: \quad \alpha\left(\frac{a}{1-\delta}\right)+(1-\alpha)\left(c+\delta c+\frac{\delta^2 d}{1-\delta}\right)\\
\text{Play D} &: \quad \alpha\left(b+\frac{\delta a}{1-\delta}\right)+(1-\alpha)\left(d+\frac{\delta d}{1-\delta}\right)
\end{aligned}
$$

Since choosing D yields a strictly higher payoff $\forall \alpha \in (0,1]$, it follows that $q(\alpha)$ must be bounded below 1 $\forall \alpha \in [0,1]$. Therefore, $q(\alpha) < 1\ \forall \alpha \in (0,1]$ and $\lim_{\alpha \to 1} q(\alpha) < 1$.

To complete the proof, let us show the properties on $V(\cdot)$ are true, given the properties on $q(\cdot)$ hold. First note that, in equilibrium, $V : [0,1] \to \left[\frac{d}{1-\delta}, \frac{a}{1-\delta}\right]$, as $V(\alpha)$ has a lower bound of $\frac{d}{1-\delta}$ and $\frac{a}{1-\delta}$ is an upper bound because the highest average symmetric payoff is a. If $q(\alpha) = 0$ then type L players play D for sure in the current period and since $\frac{\alpha(1-q(\alpha))}{1-\alpha q} = \alpha$ then the same is true for all ensuing periods; hence, by stationarity, if $q(\alpha) = 0$ then $V(\alpha) = \frac{d}{1-\delta}$. To show that $V(\alpha) \in \left(\frac{d}{1-\delta}, \frac{a}{1-\delta}\right)$ when $\alpha \in (\underline{\alpha}, 1)$, note that $q(\alpha) \in (0,1)$ implies $V(\alpha) = W^C(\alpha) = W^D(\alpha)$. $\frac{d}{1-\delta}$ is a lower bound on $V(\alpha)$ for all α since at least that value can be achieved by choosing D in every period. Using the payoff from choosing D, we have:

$$\begin{aligned} V(\alpha) &= \alpha q \left(b + \frac{\delta a}{1-\delta}\right) + (1-\alpha q)\left(d + \delta V\left(\frac{\alpha(1-q)}{1-\alpha q}\right)\right) \\ &\geq \alpha q \left(b + \frac{\delta a}{1-\delta}\right) + (1-\alpha q)\left(d + \delta \frac{d}{1-\delta}\right) \\ &> \frac{d}{1-\delta} + \alpha q \left(\frac{a-d}{1-\delta}\right) > \frac{d}{1-\delta} \end{aligned}$$

since $b > a > d$. Using the payoff from choosing C, we have:

$$\begin{aligned} V(\alpha) &= \alpha q \left(\frac{a}{1-\delta}\right) + \alpha(1-q)\left(c + \frac{\delta a}{1-\delta}\right) + (1-\alpha)\left(c + \delta c + \frac{\delta^2 d}{1-\delta}\right) \\ &= \frac{a}{1-\delta} - \alpha(1-q)(a-c) - (1-\alpha)\left(\frac{a}{1-\delta} - c - \delta c - \frac{\delta^2 d}{1-\delta}\right) < \frac{a}{1-\delta}, \end{aligned}$$

since $a > c, d$. This establishes the properties on $V(\cdot)$. ■

Proof of Theorem 2. To show that $\alpha > \underline{\alpha}$ implies $\frac{\alpha(1-q(\alpha))}{1-\alpha q(\alpha)} > \underline{\alpha}$, suppose not so that $\exists \alpha' > \underline{\alpha}$ such that $\frac{\alpha'(1-q(\alpha'))}{1-\alpha' q(\alpha')} \leq \underline{\alpha}$. By the proof of Theorem 1, $V\left(\frac{\alpha'(1-q(\alpha'))}{1-\alpha' q(\alpha')}\right) = \frac{d}{1-\delta}$ and, from (7), we have:

$$\alpha' q(\alpha') = \frac{\delta\left(\frac{a-d}{1-\delta}\right) - [1 + \delta(1-\alpha')](d-c) - (1-\alpha')\frac{\delta(a-d)}{1-\delta}}{\delta\left(\frac{a-d}{1-\delta}\right) + (b-a) - (d-c)} \tag{22}$$

We've made the supposition

$$\underline{\alpha} \geq \frac{\alpha'(1-q(\alpha'))}{1-\alpha' q^*(\alpha')}$$

which is equivalent to

$$\alpha' q(\alpha') \geq \frac{\alpha' - \underline{\alpha}}{1-\underline{\alpha}}. \tag{23}$$

Substitute (22) into (23):

$$\frac{\alpha'\delta\left(\frac{a-d}{1-\delta}\right)-[1+\delta(1-\alpha')](d-c)}{\delta\left(\frac{a-d}{1-\delta}\right)+(b-a)-(d-c)} \geq \frac{\alpha'-\frac{\left(1-\delta^2\right)(d-c)}{\delta[(1-\delta)(a-c)+\delta(a-d)]}}{1-\frac{\left(1-\delta^2\right)(d-c)}{\delta[(1-\delta)(a-c)+\delta(a-d)]}} \Rightarrow$$

$$\begin{aligned} & \frac{\alpha'\delta(a-d)-(1-\delta)(d-c)-\delta(1-\delta)(1-\alpha)(d-c)}{\delta(a-d)+(1-\delta)[(b-a)-(d-c)]} \\ \geq\ & \frac{\alpha\delta(1-\delta)(a-c)+\alpha\delta^2(a-d)-\left(1-\delta^2\right)(d-c)}{\delta[(1-\delta)(a-c)+\delta(a-d)]-\left(1-\delta^2\right)(d-c)} \end{aligned}$$

As the numerators are equal and positive (since it equals $\alpha'-\underline{\alpha}$) then the inequality holds iff

$$\begin{aligned} \delta[(1-\delta)(a-c)+\delta(a-d)]-\left(1-\delta^2\right)(d-c) &\geq \delta(a-d)+(1-\delta)[(b-a)-(d-c)] \Rightarrow \\ 0 &\geq (1-\delta)(b-a) \end{aligned}$$

which is not true. Hence, $\nexists\alpha'>\underline{\alpha}$ such that $\frac{\alpha'(1-q(\alpha'))}{1-\alpha' q(\alpha')}\leq\underline{\alpha}$ which means if $\alpha'>\underline{\alpha}$ then $\frac{\alpha'(1-q(\alpha'))}{1-\alpha' q(\alpha')}>\underline{\alpha}$.

Next consider: if $\alpha^1>\underline{\alpha}$ then $\lim_{t\to\infty}\alpha^t=\underline{\alpha}$. By Bayes rule,

$$\alpha^{t+1}=\alpha^t\left(\frac{1-q^t}{1-\alpha^t q^t}\right)\Rightarrow\alpha^{t+1}\leq\alpha^t.$$

By part (i) of this theorem, if $\alpha^1>\underline{\alpha}$ then $\underline{\alpha}$ is a lower bound of the sequence $\{\alpha^t\}$. Hence, $\{\alpha^t\}$ has a limit and it is sufficient to show that $\underline{\alpha}$ is the infimum of $\{\alpha^t\}$. Suppose not, and let $\alpha'>\underline{\alpha}$ be the infimum of $\{\alpha^t\}$. Then as $\alpha^t\to\alpha'$, $\alpha^{t+1}\to\alpha^t$, which indicates $q^t\to 0$. As $q^t\to 0$, $V(\alpha^{t+1})\to\frac{d}{1-\delta}$. But we know from the proof of Theorem 1 that the payoff to D is the same as the payoff from C iff $\alpha^t\to\underline{\alpha}$, which contradicts $\alpha^t\to\alpha'$ and $\alpha'>\underline{\alpha}$. Therefore, $\lim_{t\to\infty}\alpha^t=\underline{\alpha}$, for $\alpha^1>\underline{\alpha}$.

That $\alpha^1>\underline{\alpha}$ implies $q\left(\alpha^t\right)>0\ \forall t$ immediately follows from $\alpha^t>\underline{\alpha}\ \forall t$ and Theorem 1.

Next let us show that $\lim_{\alpha\downarrow\underline{\alpha}}q(\alpha)=0$ when $\underline{\alpha}>0$. It has already been proven: if $\alpha^1>\underline{\alpha}$ then $\lim_{t\to\infty}\alpha^t=\underline{\alpha}$. Therefore,

$$\lim_{\alpha\downarrow\underline{\alpha}}\frac{\alpha(1-q(\alpha))}{1-\alpha q(\alpha)}=\underline{\alpha}\,(>0),$$

which implies $\lim_{\alpha\downarrow\underline{\alpha}}q(\alpha)=0$.

Finally, it is easy to prove $\lim_{t\to\infty}\alpha^t q\left(\alpha^t\right)=0$. If $\alpha^1\leq\underline{\alpha}$ then $q\left(\alpha^t\right)=0\forall t$ and therefore $\lim_{t\to\infty}\alpha^t q\left(\alpha^t\right)=0$. If $\alpha^1>\underline{\alpha}>0$ then, by the other results of Theorem 2, $\lim_{t\to\infty}\alpha^t=\underline{\alpha}$ and $\lim_{\alpha\downarrow\underline{\alpha}}q(\alpha)=0$ which implies $\lim_{t\to\infty}\alpha^t q\left(\alpha^t\right)=0$. If $\alpha^1>\underline{\alpha}=0$ then $\lim_{t\to\infty}\alpha^t=0$ which implies $\lim_{t\to\infty}\alpha^t q\left(\alpha^t\right)=0$. ∎

Proof of Theorem 4. Re-arranging (7), an equilibrium $q(\cdot)$ is defined by

$$\begin{aligned} & \alpha q\left[(b-a)-(d-c)\right]+(1-\alpha)\frac{\delta(a-d)}{1-\delta}+(d-c)+\delta(1-\alpha)(d-c) \quad (24)\\ = & \ \delta(1-\alpha q)\left[\frac{a}{1-\delta}-V\left(\frac{\alpha(1-q)}{1-\alpha q}\right)\right] \end{aligned}$$

Conjecturing that the value function is linear in α,

$$V(\alpha)=x+y\alpha, \tag{25}$$

substitute (25) into (24).

$$\begin{aligned} & \alpha q\left[(b-a)-(d-c)\right]+(1-\alpha)\frac{\delta(a-d)}{1-\delta}+(d-c)+\delta(1-\alpha)(d-c) \quad (26)\\ = & \ \delta(1-\alpha q)\left[\frac{a}{1-\delta}-x-y\left(\frac{\alpha(1-q)}{1-\alpha q}\right)\right]\Rightarrow \end{aligned}$$

$$\begin{aligned} \alpha q = & \ \alpha\left[\frac{\delta(a-d)+\delta(1-\delta)(d-c-y)}{(1-\delta)\left[(b-a)-(d-c)\right]+\delta a-\delta(1-\delta)(x+y)}\right] \quad (27)\\ & +\frac{\delta a-\delta(1-\delta)x-\delta(a-d)-\left(1-\delta^2\right)(d-c)}{(1-\delta)\left[(b-a)-(d-c)\right]+\delta a-\delta(1-\delta)(x+y)} \end{aligned}$$

Thus, αq is affine in α if the value function is affine in α. As a player is indifferent between playing C and D, the value can be given by the payoff to choosing C for sure:

$$V(\alpha)=\alpha q(a-c)+\frac{\alpha\delta(a-d)}{1-\delta}+c+\frac{\delta d}{1-\delta}-\delta(1-\alpha)(d-c).$$

The value function is affine in αq and, since αq is affine in α, $V(\alpha)$ is affine in α.

The next step is to show that there exist unique values for x and y. Using the payoff to playing C, in equilibrium the value function equals:

$$\begin{aligned} V(\alpha) = & \ \alpha q(a-c)+c+\frac{\alpha\delta(a-d)}{1-\delta}+\frac{\delta d}{1-\delta}-\delta(1-\alpha)(d-c)\\ = & \ \alpha\left[\frac{\delta(a-d)+\delta(1-\delta)(d-c-y)}{(1-\delta)\left[(b-a)-(d-c)\right]+\delta a-\delta(1-\delta)(x+y)}\right](a-c)\\ & +\left[\frac{\delta a-\delta(1-\delta)x-\delta(a-d)-\left(1-\delta^2\right)(d-c)}{(1-\delta)\left[(b-a)-(d-c)\right]+\delta a-\delta(1-\delta)(x+y)}\right](a-c)\\ & +c+\frac{\alpha\delta(a-d)}{1-\delta}+\frac{\delta d}{1-\delta}-\delta(1-\alpha)(d-c) \end{aligned}$$

$$\begin{aligned} &= \alpha \left[\frac{\delta(a-c)\left[(a-d)+(1-\delta)(d-c-y)\right]}{(1-\delta)\left[(b-a)-(d-c)\right]+\delta a-\delta(1-\delta)(x+y)} + \right. \\ &\quad \frac{\delta(a-d)}{1-\delta}+\delta(d-c) \\ &\quad +(a-c)\left[\frac{\delta a-\delta(1-\delta)x-\delta(a-d)-\left(1-\delta^2\right)(d-c)}{(1-\delta)\left[(b-a)-(d-c)\right]+\delta a-\delta(1-\delta)(x+y)}\right] \\ &\quad +c+\frac{\delta d}{1-\delta}-\delta(d-c) \end{aligned} \tag{28}$$

Equating coefficients between (25) and (28), we have

$$\begin{aligned} x &= (a-c)\left[\frac{\delta a-\delta(1-\delta)x-\delta(a-d)-\left(1-\delta^2\right)(d-c)}{(1-\delta)\left[(b-a)-(d-c)\right]+\delta a-\delta(1-\delta)(x+y)}\right] \\ &\quad +c+\frac{\delta d}{1-\delta}-\delta(d-c) \end{aligned} \tag{29}$$

$$y=\frac{\delta(a-c)\left[(a-d)+(1-\delta)(d-c-y)\right]}{(1-\delta)\left[(b-a)-(d-c)\right]+\delta a-\delta(1-\delta)(x+y)}+\frac{\delta(a-d)}{1-\delta}+\delta(d-c) \tag{30}$$

To show that there is a unique solution to (29)-(30), define $z \equiv x+y$ and note that:

$$z=x+y=V(1)=W^C(1)=Q(a-c)+\frac{\delta(a-d)}{1-\delta}+c+\frac{\delta d}{1-\delta},$$

where $Q=q(1)$. Simplifying the preceding equation gives:

$$z=Q(a-c)+\frac{\delta a}{1-\delta}+c. \tag{31}$$

If we can show that there exists a unique $Q \in (0,1)$ satisfying the equilibrium condition (26) when $\alpha=1$, then $z=x+y=V(1)$ is unique.

Evaluating (26) at $\alpha=1$, we have:

$$\begin{aligned} Q\left[(b-a)-(d-c)\right]+(d-c) &= \delta(1-Q)\left[\frac{a}{1-\delta}-x-y\left(\frac{1-Q}{1-Q}\right)\right] \\ Q\left[(b-a)-(d-c)\right]+(d-c) &= \delta(1-Q)\left[\frac{a}{1-\delta}-z\right] \\ Q\left[(b-a)-(d-c)\right]+(d-c) &= \delta(1-Q)\left[\frac{a}{1-\delta}-\left(Q(a-c)+\frac{\delta a}{1-\delta}+c\right)\right] \\ Q\left[(b-a)-(d-c)\right]+(d-c) &= \delta(1-Q)^2(a-c), \end{aligned}$$

and re-arranging gives us

$$\delta(a-c)Q^2-\left[2\delta(a-c)+(b-a)-(d-c)\right]Q+\left[\delta(a-c)-(d-c)\right]=0.$$

This quadratic has two solutions:

$$Q=\frac{2\delta(a-c)+(b-a)-(d-c)\pm\sqrt{\Omega}}{2\delta(a-c)},$$

where

$$\Omega \equiv [2\delta(a-c)+(b-a)-(d-c)]^2 - 4\delta(a-c)[\delta(a-c)-(d-c)] \qquad (32)$$

$$\begin{aligned}
&= 4\delta^2(a-c)^2+[(b-a)-(d-c)]^2+4\delta(a-c)[(b-a)-(d-c)] \\
&\quad -4\delta^2(a-c)^2+4\delta(a-c)(d-c) \\
&= [(b-a)-(d-c)]^2+4\delta(a-c)[(b-a)-(d-c)]+4\delta(a-c)(d-c) \\
&> 0
\end{aligned}$$

since $a > c$, $d \geq c$ and $b + c \geq a + d$; and recall that the assumption $b > a$ implies $d = c$ and $b + c = a + d$ cannot both hold. Hence, the two solutions are real. Next note that the bigger root exceeds one:

$$Q^b = 1 + \frac{(b-a)-(d-c)+\sqrt{\Omega}}{2\delta(a-c)} > 1.$$

Thus, we only need to show that the smaller root falls in $(0, 1)$.

$$Q^s = 1 + \frac{(b-a)-(d-c)-\sqrt{\Omega}}{2\delta(a-c)} < 1$$

if and only if

$$\begin{aligned}
(b-a)-(d-c) &< \sqrt{\Omega} \Leftrightarrow [(b-a)-(d-c)]^2 < \Omega \Leftrightarrow \\
[(b-a)-(d-c)]^2 &< [(b-a)-(d-c)]^2+4\delta(a-c)[(b-a)-(d-c)]+4\delta(a-c)(d-c),
\end{aligned}$$

which is equivalent to

$$4\delta(a-c)[(b-a)-(d-c)]+4\delta(a-c)(d-c) > 0,$$

and, therefore, $Q^s < 1$. $Q^s > 0$ if and only if

$$\begin{aligned}
2\delta(a-c)+(b-a)-(d-c) &> \sqrt{\Omega} \\
[2\delta(a-c)+(b-a)-(d-c)]^2 &> \Omega.
\end{aligned}$$

From (32), the preceding condition is equivalent to

$$4\delta(a-c)[\delta(a-c)-(d-c)] > 0,$$

which holds since

$$\delta(a-c)-(d-c) > 0 \Leftrightarrow \delta > \frac{d-c}{a-c}.$$

The last property follows from $\delta > \frac{b-a}{b-d} \geq \frac{d-c}{a-c}$.

There then exists a unique $Q \in (0,1)$, and $z = x + y = V(1)$ is unique since it is linear in Q. In addition, plugging Q^s in (31) gives

$$\begin{aligned} z &= \frac{2\delta(a-c) + (b-a) - (d-c) - \sqrt{\Omega}}{2\delta} + \frac{\delta a}{1-\delta} + c \\ &= \frac{a}{1-\delta} + \frac{(b-a) - (d-c) - \sqrt{\Omega}}{2\delta} \\ &= \frac{2a\delta + (1-\delta)\left[(b-a) - (d-c) - \sqrt{\Omega}\right]}{2\delta(1-\delta)}. \end{aligned}$$

To close the model, use the initial condition

$$V(\underline{\alpha}) = \frac{d}{1-\delta},$$

which takes the form:

$$x = \frac{d}{1-\delta} - y\frac{\left(1-\delta^2\right)(d-c)}{\delta\left[(1-\delta)(a-c) + \delta(a-d)\right]}.$$

x^* is then the unique solution to

$$x^* = \frac{d}{1-\delta} - (z - x^*)\frac{\left(1-\delta^2\right)(d-c)}{\delta\left[(1-\delta)(a-c) + \delta(a-d)\right]},$$

and y^* is the unique solution to: $y^* = z - x^*$. This completes the proof that there is a unique affine MPBE. Finally, solving for q from (27) gives us (13). ■

Proof of Theorem 5. Since the equilibrium probability of choosing C is

$$\begin{aligned} \alpha q(\alpha) &= \alpha\left[\frac{\delta(a-d) + \delta(1-\delta)(d-c-y)}{(1-\delta)\left[(b-a)-(d-c)\right] + \delta a - \delta(1-\delta)(x+y)}\right] \\ &+ \left[\frac{\delta a - \delta(1-\delta)x - \delta(a-d) - \left(1-\delta^2\right)(d-c)}{(1-\delta)\left[(b-a)-(d-c)\right] + \delta a - \delta(1-\delta)(x+y)}\right] \end{aligned}$$

then $\alpha q(\alpha)$ is increasing in α iff

$$\frac{\delta(a-d) + \delta(1-\delta)(d-c-y)}{(1-\delta)\left[(b-a)-(d-c)\right] + \delta a - \delta(1-\delta)(x+y)} > 0. \tag{33}$$

By assumption

$$(b-a) - (d-c) \geq 0,$$

and $V(1) < \frac{a}{1-\delta}$ implies

$$\frac{a}{1-\delta} - (x+y) > 0. \tag{34}$$

Thus, (33) is true iff the numerator is positive:

$$\begin{aligned} (a-d) + (1-\delta)(d-c-y) &> 0 \\ \frac{(a-d)}{1-\delta} + (d-c) &> y. \end{aligned} \tag{35}$$

Suppose (35) was not true. From (30), we have

$$y=\frac{\delta\left(a-c\right)\left[\left(a-d\right)+\left(1-\delta\right)\left(d-c-y\right)\right]}{\left(1-\delta\right)\left[\left(b-a\right)-\left(d-c\right)\right]+\delta a-\delta\left(1-\delta\right)\left(x+y\right)}+\frac{\delta\left(a-d\right)}{1-\delta}+\delta\left(d-c\right). \quad (36)$$

If (35) is not true then the first term of (36) is non-positive, but then (36) implies

$$y\leq\delta\left[\frac{\left(a-d\right)}{1-\delta}+\left(d-c\right)\right]<\frac{\left(a-d\right)}{1-\delta}+\left(d-c\right)$$

which contradicts the supposition that (35) is not true. From this contradiction, we conclude (35) and thus $\alpha q\left(\alpha\right)$ is increasing in α.

To show that $V\left(\alpha\right)$ is increasing in α, recall that

$$V\left(\alpha\right)=\alpha q\left(\alpha\right)\left(a-c\right)+\frac{\alpha\delta\left(a-d\right)}{1-\delta}+c+\frac{\delta d}{1-\delta}-\delta\left(1-\alpha\right)\left(d-c\right).$$

That $\alpha q\left(\alpha\right)$ is increasing in α delivers the result. ■

Proof of Theorem 6. For $\alpha\leq\underline{\alpha}$, $q\left(\alpha\right)=0$, so it is non-decreasing in α for $\alpha\in[0,\underline{\alpha}]$. From hereon, suppose $\alpha>\underline{\alpha}$ so that

$$\begin{aligned} q\left(\alpha\right) &= \frac{\delta\left(a-d\right)+\delta\left(1-\delta\right)\left(d-c-y\right)}{\left(1-\delta\right)\left[\left(b-a\right)-\left(d-c\right)\right]+\delta a-\delta\left(1-\delta\right)\left(x+y\right)} \\ &\quad +\left(\frac{1}{\alpha}\right)\left[\frac{\delta a-\delta\left(1-\delta\right)x-\delta\left(a-d\right)-\left(1-\delta^{2}\right)\left(d-c\right)}{\left(1-\delta\right)\left[\left(b-a\right)-\left(d-c\right)\right]+\delta a-\delta\left(1-\delta\right)\left(x+y\right)}\right]\end{aligned}$$

Thus, $q\left(\alpha\right)$ is increasing in α iff

$$\left[\frac{\delta a-\delta\left(1-\delta\right)x-\delta\left(a-d\right)-\left(1-\delta^{2}\right)\left(d-c\right)}{\left(1-\delta\right)\left[\left(b-a\right)-\left(d-c\right)\right]+\delta a-\delta\left(1-\delta\right)\left(x+y\right)}\right]<0 \quad (37)$$

The denominator of the LHS of (37) is positive because $b-a\geq d-c$ by assumption and

$$\frac{a}{1-\delta}>x+y$$

as shown in (34). Thus, (37) is true iff the numerator is negative:

$$\delta a-\delta\left(1-\delta\right)x-\delta\left(a-d\right)-\left(1-\delta^{2}\right)\left(d-c\right)<0. \quad (38)$$

Suppose (38) was not true. From (29), we would then have

$$x\geq c+\frac{\delta d}{1-\delta}-\delta\left(d-c\right)$$

which implies

$$\begin{aligned} &\delta a-\delta\left(1-\delta\right)x-\delta\left(a-d\right)-\left(1-\delta^{2}\right)\left(d-c\right) \quad (39)\\ \leq\ &\delta a-\delta\left(1-\delta\right)\left(c+\frac{\delta d}{1-\delta}-\delta\left(d-c\right)\right)-\delta\left(a-d\right)-\left(1-\delta^{2}\right)\left(d-c\right)\end{aligned}$$

By rearranging terms, the RHS of (39) is equivalent to

$$-(1-\delta)(1-\delta^2)(d-c) \tag{40}$$

which is negative iff $d > c$. Hence, the LHS of (38) is negative for $d > c$, which contradicts the supposition that (38) is not true. From this contradiction, we conclude (38) is true for $d > c$. Namely, $q(\alpha)$ is increasing in α for $d > c$.

If $d = c$, (40) implies

$$\delta a - \delta(1-\delta)x - \delta(a-d) - \left(1-\delta^2\right)(d-c) = 0 \Rightarrow \frac{\partial q(\alpha)}{\partial \alpha} = 0.$$

■

Proof of Theorem 7. First note that if $\alpha^1 \leq \underline{\alpha}$ then $q^t = 0\ \forall t$ in which case $Q^T = 1$. From hereon, assume $\alpha^1 \in (\underline{\alpha}, 1)$. If $d > c$ then, with the affine MPBE,

$$q(\alpha) = A + B\left(\frac{1}{\alpha}\right)$$

for some A and B where $B < 0$ and $A + B < 1$. Then

$$\alpha^t = \frac{\alpha^{t-1}\left(1-q^{t-1}\right)}{1-\alpha^{t-1}q^{t-1}} = \frac{\alpha^{t-1}\left(1-A-\frac{B}{\alpha^{t-1}}\right)}{1-\alpha^{t-1}\left(A+\frac{B}{\alpha^{t-1}}\right)}$$

$$q^t = A + B\left(\frac{1}{\alpha^t}\right) = A + B\left(\frac{1-\alpha^{t-1}\left(A+\frac{B}{\alpha^{t-1}}\right)}{\alpha^{t-1}\left(1-A-\frac{B}{\alpha^{t-1}}\right)}\right) \tag{41}$$

Since $B \neq 0$, we can invert

$$q^{t-1} = A + B\left(\frac{1}{\alpha^{t-1}}\right)$$

to derive

$$\alpha^{t-1} = \frac{B}{q^{t-1}-A}.$$

Insert this expression in (41),

$$q^t = A + B\left(\frac{1-\left(\frac{B}{q^{t-1}-A}\right)\left(A+\frac{B}{\frac{B}{q^{t-1}-A}}\right)}{\left(\frac{B}{q^{t-1}-A}\right)\left(1-A-\frac{B}{\frac{B}{q^{t-1}-A}}\right)}\right) = q^{t-1}\left(\frac{1-A-B}{1-q^{t-1}}\right) \tag{42}$$

By $B < 0$ and $\alpha^t < 1$, we have

$$A + B > A + B\left(\frac{1}{\alpha^t}\right) = q^t,\ \forall\ t.$$

By $B < 0$ and that α^t decreasing over time, we have that q^t decreasing over time. Hence,

$$1 - q^1 \leq 1 - q^{t-1}, \ \forall\ t > 2,$$

Therefore,

$$q^t \leq \left(\frac{1-A-B}{1-q^1}\right) q^{t-1}.$$

As this holds for all t, it implies

$$q^t \leq \left(\frac{1-A-B}{1-q^1}\right)^{t-1} q^1 = \nu^{t-1} q,$$

where $\nu \equiv \left(\frac{1-A-B}{1-q^1}\right) \in (0,1)$. Hence,

$$\prod_{t=1}^{T} \left(1-q^t\right)^2 > \left[\prod_{t=1}^{T} \left(1-\nu^{t-1} q\right)\right]^2.$$

To prove this theorem, it is then sufficient to show

$$\lim_{T\to\infty} \prod_{t=1}^{T} \left(1-\nu^{t-1} q\right) > 0,$$

which is equivalent to

$$\lim_{T\to\infty} \prod_{t=1}^{T} \left(1-\nu^{t} q\right) > 0,$$

which, because $q \in (0,1)$, is true if

$$\lim_{T\to\infty} \prod_{t=1}^{T} \left(1-\nu^{t}\right) > 0,$$

which is equivalent to

$$\sum_{t=1}^{\infty} \log\left(1-\nu^t\right) > -\infty.$$

Since $\nu \in (0,1)$ then

$$\begin{aligned} & \sum_{t=1}^{\infty} \log\left(1-\nu^t\right) \\ = \ & -\left[\left(\nu + \frac{\nu^2}{2} + \frac{\nu^3}{3} + ...\right) + \left(\nu^2 + \frac{\nu^4}{2} + \frac{\nu^6}{3} + ...\right) + ... + \left(\nu^t + \frac{\nu^{2t}}{2} + \frac{\nu^{3t}}{3} + ...\right) + ...\right] \end{aligned}$$

Some manipulation yields the desired result:

$$\begin{aligned}
& -\left[\left(\nu+\frac{\nu^2}{2}+\frac{\nu^3}{3}+...\right)+\left(\nu^2+\frac{\nu^4}{2}+\frac{\nu^6}{3}+...\right)+...+\left(\nu^t+\frac{\nu^{2t}}{2}+\frac{\nu^{3t}}{3}+...\right)+...\right] \\
= & -\left[\left(\nu+\nu^2+\nu^3+...\right)+\frac{1}{2}\left(\nu^2+\nu^4+\nu^6+...\right)+\frac{1}{3}\left(\nu^3+\nu^6+\nu^9+...\right)+...\right] \\
= & -\left(\frac{\nu}{1-\nu}+\frac{1}{2}\frac{\nu^2}{1-\nu^2}+\frac{1}{3}\frac{\nu^3}{1-\nu^3}+...\right)=-\frac{\nu}{1-\nu}\left(1+\frac{1}{2}\frac{\nu}{1+\nu}+\frac{1}{3}\frac{\nu^2}{1+\nu+\nu^2}+...\right) \\
\geq & -\frac{\nu}{1-\nu}\left(1+\nu+\nu^2+...\right)=-\frac{\nu}{(1-\nu)^2}>-\infty
\end{aligned}$$

■