Rationing problems with ex-ante conditions

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Abstract

An extension of the standard rationing model is introduced. Agents are not only identified by their respective claims to some amount of a scarce resource, but also by some exogenous ex-ante conditions (initial stock of resource or net worth of agents, for instance), other than claims. Within this framework, we define a generalization of the constrained equal awards rule and provide two different characterizations of this generalized rule. Finally, we use the corresponding dual properties to characterize a generalization of the constrained equal losses rule.

Keywords: rationing, equal awards rule, equal losses rule, claims problem, ex-ante conditions
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1. Introduction

A standard rationing problem is an allocation problem in which each individual in a group of agents has a claim to a quantity of some (perfectly divisible) resource (e.g., money) and the available amount of this resource is insufficient to satisfy all claims. Assignment of taxes, bankruptcy situations and the distribution of emergency supplies are examples of such rationing problems, which have been widely studied in the literature.[1] Since ancient times, several solutions to this simple problem have been proposed (see, for instance, Aumann and Maschler, 1985; O'Neill, 1982), based mainly on equalizing gains or losses from claims, or by using a proportional yardstick.

Standard rationing analysis considers claims to be the only relevant information affecting the final distribution. Recently, several authors have studied complex rationing situations in which not only claims, but also individual rights or other entitlements, affect

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[1] These problems are also known in the literature as problems of adjudicating conflicting claims (see, for instance, the surveys undertaken by Thomson, 2003, 2015).

the final distribution. Hougaard et al. (2012, 2013a,b) and Pulido et al. (2002, 2008) introduce baselines based on past experience or on exogenous entitlements in order to refine the claims of agents. Indeed, Hougaard et al. (2013a) consider baselines as consolidated rights represented by positive numbers. The authors propose that agents be first assigned their baselines truncated by their claims before allocating the resulting deficit, or surplus, using a standard rationing rule in which the claims are the truncated baselines (in the case of a deficit) or the gap between each claim and its respective truncated baseline (in the case of a surplus).

In the above models, the baselines can be interpreted as objective evaluations of the agents' real needs that usually differ from their claims. They can also be understood as a tentative allocation, which become upper or lower bounds for the final distribution depending on whether they are feasible or not. In the present chapter, we consider exogenous information (namely ex-ante conditions) other than claims, but from a completely distinct point of view from that taken by the baseline interpretation. The ex-ante condition of an agent reflects his initial stock or endowment[2] of the corresponding resource. Hence, in contrast to baselines, ex-ante conditions are not tentative allocations, but rather they aim to reveal inequalities between agents that might suggest payoff compensations in favour of some agents and to the detriment of some others. The following examples seek to clarify this point.

Imagine there are n agents and each agent i has an initial stock of resource; let us denote this by $\delta_i \geq 0$. Furthermore, let us suppose that there is scarcity and that the available amount $r > 0$ of resource to be currently distributed does not cover the agents' claims. In this chapter, we propose giving priority to an agent with a small stock with respect to that of another agent by compensating as much as possible the gap between their initial stocks. Consider, for instance, the distribution of irrigation water among a group of farmers during a drought. Imagine that each farmer has a reservoir in which to collect rainwater, but the current levels (stock of water) of their reservoirs are not all equal. Even in the case that the area under crop of each farmer is equal, the distribution of water should be affected by inequalities between the farmers' water reserves.

Another situation in which ex-ante conditions between agents arise is in the distribution of grants or subsidies by a public institution. Often the distribution process takes into account the net worth of agents in order to make a fairer allocation. Notice that this net worth might be positive or negative (if debts are larger than assets). A real example of an allocation problem that considers ex-ante conditions is the distribution of scholarships, where allocation criteria are often related to family income. In this situation, it seems unfair to treat agents with different family income equally, even in the case in which their claims be equal.

[2]This endowment can be positive (in most situations) but it might be negative (if, for example, we are distributing money and the net worth of an agent is negative).

In this chapter, we propose a generalization of two well-known rules defined for standard rationing problems: the constrained equal awards (CEA) and the constrained equal losses (CEL) rules. We name these generalized rules as the generalized equal awards (GEA) and the generalized equal losses (GEL) rules, respectively. The two rules are the dual of each other in a proper sense. Obviously, the generalizations are consistent with the CEA and the CEL rules respectively, when ex-ante conditions are equal for all agents. Having defined the rules, two characterizations of the GEA rule are provided. The first adapts and extends to the new framework the characterization of the CEA rule given by Herrero and Villar (2001). The second is based on new and specific axioms for the ex-ante conditions model. Based on the corresponding dual properties, we also obtain two characterizations of the GEL rule.

The remainder of this chapter is organized as follows. In Section 2, we introduce the main notations, we describe a rationing problem with ex-ante conditions and we define the GEA and the GEL rules. In Section 3, we carry out the axiomatic analysis of the GEA rule and in Section 4 we use the duality relations between rules and properties to characterize the GEL rule. In Section 5, we conclude.

2. Rationing problems with ex-ante conditions and rules

We first introduce some notations and recall the definition of a standard rationing problem. Let us denote by $\mathbb{N}$ the set of natural numbers that we identify with the universe of potential agents, and by $\mathcal{N}$ the family of all finite subsets of $\mathbb{N}$. Let $S \in \mathcal{N}$, we denote by s the cardinality of S. Given a finite subset of agents $N = \{1, 2, \ldots, n\} \in \mathcal{N}$, a *standard rationing problem* for N is to distribute $r \geq 0$ among these n agents with claims $c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^N_+$. It is assumed that $r \leq \sum_{i \in N} c_i$ since otherwise no rationing problem exists. We denote a standard rationing problem by the pair $(r, c) \in \mathbb{R}_+ \times \mathbb{R}^N_+$.

A feasible allocation for (r, c) is represented by a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^N$ such that $0 \leq x_i \leq c_i$ and $\sum_{i \in N} x_i = r$, where x_i represents the payoff to agent $i \in N$. A rationing rule associates a unique allocation to each standard rationing problem. As previously mentioned in the Introduction, two well-known rules are the *constrained equal awards* (CEA) and the *constrained equal losses* (CEL). The CEA rule aims to equalize gains and the CEL rule aims to equalize losses from claims.

Definition 1. (CEA). *For any standard rationing problem $(r, c) \in \mathbb{R}_+ \times \mathbb{R}^N_+$ the CEA rule is defined as*

$$CEA_i(r, c) = \min\{c_i, \lambda\}, \text{ for all } i \in N,$$

where $\lambda \in \mathbb{R}_+$ satisfies $\sum_{i \in N} \min\{c_i, \lambda\} = r$.

Definition 2. (CEL). *For any standard rationing problem $(r, c) \in \mathbb{R}_+ \times \mathbb{R}^N_+$ the CEL rule is defined as*

$$CEL_i(r, c) = \max\{0, c_i - \lambda\}, \text{ for all } i \in N,$$

where $\lambda \in \mathbb{R}_+$ *satisfies* $\sum_{i \in N} \max\{0, c_i - \lambda\} = r$.

The aim of a rationing problem with ex-ante conditions is to distribute an amount of a scarce resource fairly taking into account the inequalities in the ex-ante conditions.

Definition 3. *Let* $N \in \mathcal{N}$ *be a finite subset of agents. A* rationing problem with ex-ante conditions *for* N *is a triple* (r, c, δ)*, where* $r \in \mathbb{R}_+$ *is the amount of resource,* $c \in \mathbb{R}^N_+$ *is the vector of claims, such that* $r \leq \sum_{i \in N} c_i$*, and* $\delta \in \mathbb{R}^N$ *is the vector of ex-ante conditions.*

We denote by $\mathcal{RC}^N$ the set of all rationing problems with ex-ante conditions and agent set N, and by $\mathcal{RC} = \cup_{N \in \mathcal{N}} \mathcal{RC}^N$ the family of all rationing problems with ex-ante conditions.

The definition of an allocation rule for these problems does not differ essentially from the standard definition.

Definition 4. *A* generalized rationing rule *is a function* F *that associates to each rationing problem with ex-ante conditions* $(r, c, \delta) \in \mathcal{RC}^N$*, where* $N \in \mathcal{N}$*, a unique allocation* $x = F(r, c, \delta) = (F_1(r, c, \delta), F_2(r, c, \delta), \ldots, F_n(r, c, \delta)) \in \mathbb{R}^N_+$ *such that*

- $\sum_{i \in N} x_i = r$ *(efficiency) and*
- $0 \leq x_i \leq c_i$*, for all* $i \in N$ *(claims boundedness).*

Next, we extend the CEA rule to this new framework.

Definition 5. (Generalized equal awards rule, GEA)*. For any* $(r, c, \delta) \in \mathcal{RC}^N$*, where* $N \in \mathcal{N}$*, the* GEA *rule is defined as*[3]

$$GEA_i(r, c, \delta) := \min\{c_i, (\lambda - \delta_i)_+\}, \text{ for all } i \in N,$$

where $\lambda \in \mathbb{R}$ *satisfies* $\sum_{i \in N} GEA_i(r, c, \delta) = r$.

Notice that the GEA rule is well defined. Indeed, by applying Bolzano's Theorem to the continuous function $\varphi(\lambda) = \sum_{i \in N} \varphi_i(\lambda) = \sum_{i \in N} \min\{c_i, (\lambda - \delta_i)_+\}$, the existence of a value λ, such that $\varphi(\lambda) = r$, is guaranteed since

$$\varphi\Big(\min_{i \in N}\{\delta_i\}\Big) = 0 \leq r \leq \varphi\Big(\max_{i \in N}\{c_i + \delta_i\}\Big) = \sum_{i \in N} c_i.$$

Moreover, let us suppose that there exist $\lambda, \lambda' \in \mathbb{R}$, with $\lambda < \lambda'$, such that $\varphi(\lambda) = \varphi(\lambda') = r$. As the reader may verify, $\varphi_k(\lambda)$ is a non-decreasing function for all $k \in N$. Hence, we have that $\varphi_k(\lambda) \leq \varphi_k(\lambda')$ for all $k \in N$. Therefore, we obtain $r = \sum_{k \in N} \varphi_k(\lambda) \leq \sum_{k \in N} \varphi_k(\lambda') = r$ and thus $\varphi_k(\lambda) = \varphi_k(\lambda')$ for all $k \in N$. We conclude that the solution is unique and so it is well defined for all problems. Let us illustrate the application of the rule with an example.

[3] Henceforth, we use the following notation: for all $a \in \mathbb{R}$, $(a)_+ = \max\{0, a\}$.

Example 1. *Consider the three-person rationing problem with ex-ante conditions*

$$(r,(c_1,c_2,c_3),(\delta_1,\delta_2,\delta_3)) = (3,(2.5,3,2.5),(0,1.5,4.5)).$$

The allocation assigned by the GEA *rule is* $GEA(r,c,\delta) = (2.25, 0.75, 0)$ *where* λ *takes the value* 2.25 *in the formula, as the reader may verify. Inspired by the hydraulic representation of rationing rules given by Kaminski (2000) (see Figure 1), a dynamic interpretation of how this rule assigns gains is as follows.*

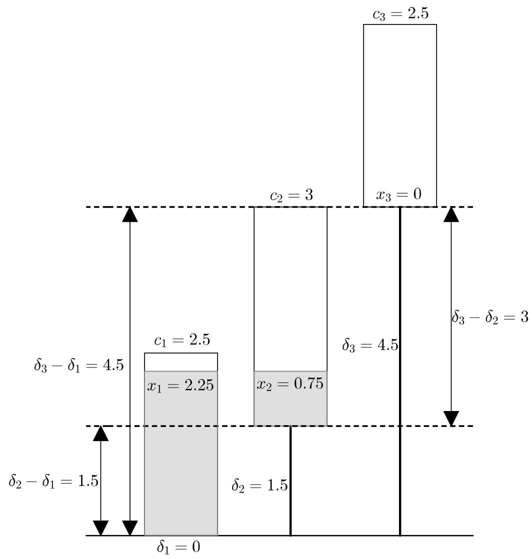

Figure 1: **Equalizing awards with ex-ante conditions.**

Agent 1*, the one with the lowest ex-ante condition, is the first agent to be assigned gains. Thus, agent* 1 *receives* $\delta_2 - \delta_1 = 1.5$ *units of resource in order to offset the inequality in the ex-ante conditions with respect to the agent with the second lowest ex-ante condition. At this point, there are still* 1.5 *units left to be allocated. Agents* 1 *and* 2 *share this amount equally (*0.75 *units each) and agent* 3 *receives nothing. This holds since neither agent* 1 *nor agent* 2 *has been fully compensated with respect to agent* 3*. We finally obtain the distribution* $(2.25, 0.75, 0)$*.*

Let us remark that the values of the ex-ante conditions are not allocated. Indeed, what is relevant is not the numerical value of an agent's ex-ante condition, but the difference between its value and the respective values of the ex-ante condition of the other agents. Specifically, as the above example shows, bilateral compensations are induced by the inequalities in the ex-ante conditions between any pair of agents. With these allocations, notice that agents 1 and 3, with the same claim, do not receive equal amounts. Moreover, agent 1, in spite of having a smaller claim than that of agent 2, receives a larger payoff and, thus, the rule does not satisfy the classical property of *order preservation in awards* (see

Thomson, 2015). However, the GEA rule satisfies an adaptation of this property that we call *order preservation in awards with ex-ante conditions*. A rule F satisfies this property if, for any pair of agents $i, j \in N$, if $c_i \leq c_j$ and $\delta_i \geq \delta_j$, then $F_i(r, c, \delta) \leq F_j(r, c, \delta)$. That is, if the agent (in the pair) with the largest stock also has the smallest claim, then this agent should receive the smallest award.

Obviously, the GEA rule generalizes the CEA rule. In other words, the allocation assigned by the GEA rule when applied to a problem without inequalities in ex-ante conditions coincides with the allocation of the CEA rule applied to the corresponding standard rationing problem (without ex-ante conditions), that is, if $\delta = (\alpha, \alpha, \ldots, \alpha) \in \mathbb{R}^N$, then $GEA(r, c, \delta) = CEA(r, c)$.

In standard rationing problems, the CEA rule seeks to minimize the differences between the payoffs awarded to agents. Therefore, if there is a difference between the payoffs of two agents $i, j \in N$ with $i \neq j$ it is because the agent with the smallest payoff has received all his claim: that is, if $CEA_i(r, c) < CEA_j(r, c)$, then $CEA_i(r, c) = c_i$. This principle can be extended to rationing problems with ex-ante conditions by minimizing the differences between the payoffs plus the corresponding ex-ante condition of agents as shown by Proposition 1. This feature of the GEA rule is crucial to prove Theorems 1 and 2. The proof of the next proposition can be found in the Appendix.

Proposition 1. *Let $(r, c, \delta) \in \mathcal{RC}^N$, $N \in \mathcal{N}$, and let $x^* \in \mathbb{R}^N_+$ be such that $x^*_i \leq c_i$, for all $i \in N$, and $\sum_{i \in N} x^*_i = r$. The following statements are equivalent:*

1. $x^* = GEA(r, c, \delta)$.
2. *For all $i, j \in N$ with $i \neq j$, if $x^*_i + \delta_i < x^*_j + \delta_j$, then either $x^*_j = 0$, or $x^*_i = c_i$.*

Now, we extend the idea of equalizing losses to rationing problems with ex-ante conditions. An agent's loss is the difference between his claim and his assigned payoff. If an agent has a better ex-ante condition than another, then he may suffer a higher loss than that suffered by this other agent. We define the *generalized equal losses rule* as follows:

Definition 6. (Generalized equal losses rule, GEL). *For any $(r, c, \delta) \in \mathcal{RC}^N$, where $N \in \mathcal{N}$, the GEL rule is defined as*

$$GEL_i(r, c, \delta) := \max\left\{0, c_i - (\lambda + \delta_i)_+\right\}, \text{ for all } i \in N,$$

where $\lambda \in \mathbb{R}$ satisfies $\sum_{i \in N} GEL_i(r, c, \delta) = r$.

The GEL rule assigns losses in an egalitarian way, but taking into account that no agent can receive a negative payoff and that the differences between ex-ante conditions might induce bilateral compensations of losses between agents. The reader may verify that the GEL rule is well defined by using similar arguments to those for the case of the GEA rule.

Analogous to the case of equalizing awards, the GEL rule generalizes the CEL rule; that is, if $\delta = (\alpha, \alpha, \ldots, \alpha) \in \mathbb{R}^N$, then $GEL(r, c, \delta) = CEL(r, c)$. Let us illustrate the application of the GEL rule with an example.

Example 2. *Consider the rationing problem with ex-ante conditions given in Example 1, $(r, c, \delta) = (3, (2.5, 3, 2.5), (0, 1.5, 4.5))$. The allocation assigned by the GEL rule is $GEL(r, c, \delta) = (2, 1, 0)$, where $\lambda = 0.5$. A dynamic interpretation of how this rule assigns losses is as follows. Notice that the total loss is $c_1 + c_2 + c_3 - r = 5$. Agent 3 is the first agent to be assigned losses since he has the largest ex-ante condition. In the first step, this agent suffers the maximum loss, i.e. all his claim, since the amount that he claims is not enough to compensate the difference between his own ex-ante condition and the second highest ex-ante condition, i.e. $c_3 = 2.5 < \delta_3 - \delta_2 = 3$. At this point there are still 2.5 units of losses left to be allocated. In the next step, 1.5 units of losses are assigned to agent 2 in order to compensate (fully) the difference between ex-ante conditions, i.e. $\delta_2 - \delta_1 = 1.5$. Finally, the remaining unit of loss is equally divided between both agents. Therefore, the losses allocation is $(0.5, 2, 2.5)$ and so the assigned payoff vector is $(c_1 - 0.5, c_2 - 2, c_3 - 2.5) = (2, 1, 0)$.*

Observe that, agent 3, in spite of having a smaller claim than that of agent 2, is assigned with a larger loss and thus, the rule does not satisfy the classical property of *order preservation in losses* (see Thomson, 2015). However, the GEL rule satisfies an adapted property to this framework that we call *order preservation in losses with ex-ante conditions*. A rule F satisfies this property if, for any pair of agents $i, j \in N$, if $c_i \leq c_j$ and $\delta_i \leq \delta_j$, then $c_i - F_i(r, c, \delta) \leq c_j - F_j(r, c, \delta)$. That is, if the agent (in the pair) with the smallest stock also has the smallest claim, then this agent should be assigned with the smallest loss.[4]

3. Axiomatic characterizations of the GEA rule

In this section we provide two characterizations of the GEA rule: the first extends a well-known characterization of the CEA rule; the second is new and proposes specific properties for this model.

The CEA and the CEL rules (for standard rationing problems) have been characterized in several studies (see the surveys undertaken by Thomson, 2003, 2015). Herrero and Villar (2001) characterize the CEA rule by means of three axioms: *consistency*,

[4]The GEA and the GEL rules satisfy the adapted properties of order preservation in awards and losses with ex-ante conditions, as the reader may check.

path-independence and *exemption*. In this section, we characterize the GEA rule by drawing on these axioms. Specifically, we adapt the properties of consistency and path-independence, and we introduce a new property, *ex-ante exemption*.

Path-independence states that if we apply a rule to a problem but resource availability diminishes suddenly, the new allocation obtained by applying the same rule again (to the new amount and with the original claims) is equal to that obtained when using the previous allocation as claims. This property was first suggested by Plott (1973) for choice functions and by Kalai (1977) in the theory of axiomatic bargaining. Moreover, the property was originally introduced in the context of standard rationing problems by Moulin (1987).

Definition 7. *A generalized rationing rule F satisfies* path-independence *if for all $N \in \mathcal{N}$ and all $(r, c, \delta) \in \mathcal{RC}^N$ with $\sum_{i \in N} c_i \geq r' \geq r$ it holds*

$$F(r, c, \delta) = F(r, F(r', c, \delta), \delta).$$

Because of claim boundedness (see Definition 4), if a rule satisfies path-independence, then it is monotonic with respect to r. That is, for all $N \in \mathcal{N}$, all $c \in \mathbb{R}^N_+$ and all r, r' :

$$\{r \leq r' \leq \sum_{i \in N} c_i\} \Rightarrow \{F(r, c, \delta) \leq F(r', c, \delta)\}. \tag{1}$$

This property is known as *resource monotonicity.*

Consistency is a property that requires that when we re-evaluate the resource allocation within a subgroup of agents using the same rule, the allocation does not change. This property was first introduced in the standard rationing context by Aumann and Maschler (1985). In order to gain more insights of the consistency principle see also Thomson (2012). To define this property, we use the following notation. Given a vector $x \in \mathbb{R}^N$ and a subset $S \subseteq N$, we denote by $x_{|S} \in \mathbb{R}^S$ the vector x restricted to the members of S.

Definition 8. *A generalized rationing rule F is* consistent *if for all $N \in \mathcal{N}$, all $(r, c, \delta) \in \mathcal{RC}^N$ and all $T \subseteq N$, $T \neq \emptyset$, it holds*

$$F(r, c, \delta)_{|T} = F\Big(r - \sum_{i \in N \setminus T} F_i(r, c, \delta), c_{|T}, \delta_{|T}\Big).$$

Before defining ex-ante exemption, let us remark that in the standard rationing framework, exemption is a property that ensures that an agent with a small enough claim will not suffer from rationing. Specifically, for the two-person case $N = \{i, j\}$, a solution $(x_i, x_j) = F(r, (c_i, c_j))$ satisfies exemption if $x_k = c_k$ whenever $c_k \leq \frac{r}{2}$ for some $k \in N$.

The application of exemption to our framework needs to take into account ex-ante conditions, and only applies to two-person problems. Ex-ante exemption states that an agent with a small enough maximum final stock (initial stock plus the claim truncated by the amount of the resource) must not be rationed.

Definition 9. *A generalized rationing rule F satisfies* ex-ante exemption *if for any two-person rationing problem with ex-ante conditions $(r, c, \delta) \in \mathcal{RC}^N$, with $N = \{i, j\}$, it holds that*

$$\textit{if } \min\{r, c_i\} + \delta_i \leq \frac{r + \delta_i + \delta_j}{2} \textit{ then } F_i(r, c, \delta) = \min\{r, c_i\}.$$

Notice that if there are no ex-ante inequalities between agents ($\delta_i = \delta_j$), then this property is equivalent to the classical exemption property for the two-person case.

The next proposition states that the GEA rule satisfies all these properties. The rather technical proof is provided in the Appendix.

Proposition 2. *The GEA rule satisfies path-independence, consistency and ex-ante exemption.*

The next theorem says that the only rule satisfying all these properties is the GEA rule. The proof can be found in the Appendix.

Theorem 1. *A rule F on $\mathcal{RC}$ satisfies path-independence, ex-ante exemption and consistency if and only if F is the GEA rule.*

The properties in Theorem 1 are logically independent, as the reader may check in Examples 3, 4 and 5 provided in the Appendix.

Now, we undertake another characterization for this rule, based on specific properties for the ex-ante condition framework, namely ex-ante fairness and transfer composition. Let us define these properties.

Ex-ante fairness is applied to any pair of agents that exhibits differences in ex-ante conditions. It states that if the available amount of resource is not large enough to compensate the poorest agent in the pair (either with his full claim or at least with the difference between their respective ex-ante conditions), then the richest agent must receive nothing. This property guarantees that social inequalities will not increase.

Definition 10. *A generalized rationing rule F satisfies* ex-ante fairness *if for all $N \in \mathcal{N}$, all $(r, c, \delta) \in \mathcal{RC}^N$ and all $i, j \in N$, $i \neq j$, it holds that*

$$\textit{if } r \leq \min\{\delta_j - \delta_i, c_i\} \textit{ then } F_j(r, c, \delta) = 0.$$

Transfer composition states that the result of directly allocating the available amount of resource is the same as that achieved when first distributing a smaller amount and, after that, distributing the remaining quantity in a new problem in which the claim of each agent is diminished by the amount initially received and the ex-ante condition is augmented by the same amount. Part of the claim is received as payoff in the first allocation and transferred as stock in the second problem.

Definition 11. *A generalized rationing rule F satisfies* transfer composition *if for all $N \in \mathcal{N}$, all $(r, c, \delta) \in \mathcal{RC}^N$ and all $r_1, r_2 \in \mathbb{R}_+$ such that $r_1 + r_2 = r$, it holds*

$$F(r, c, \delta) = F(r_1, c, \delta) + F(r_2, c - F(r_1, c, \delta), \delta + F(r_1, c, \delta)).$$

The GEA rule satisfies both ex-ante fairness and transfer composition. The proof of this result can be found in the Appendix.

Proposition 3. *The GEA rule satisfies transfer composition and ex-ante fairness.*

In fact, these two properties characterize the GEA rule.

Theorem 2. *A rule F on $\mathcal{RC}$ satisfies ex-ante fairness and transfer composition if and only if F is the GEA rule.*

Proof. By Proposition 3, we know that the GEA rule satisfies ex-ante fairness and transfer composition. Next, we concentrate on proving the uniqueness of the rule. Let F be a rule satisfying these properties, but suppose on the contrary that $F \neq GEA$. Hence, there exists a rationing problem with ex-ante conditions $(r, c, \delta) \in \mathcal{RC}^N$ such that $x = F(r, c, \delta) \neq GEA(r, c, \delta)$. Then, by Proposition 1, there exist $i, j \in N$ such that $x_i + \delta_i < x_j + \delta_j$ with $x_i < c_i$ and $x_j > 0$.

Let us remark that transfer composition implies resource monotonicity (see (1)). As a consequence, it follows that F is a continuous and increasing function in r. Thus, for all $r' \in [0, r]$, we have that $x \geq F(r', c, \delta)$. Take $\alpha^* \in (0, r]$ such that $F_j(\alpha^*, c, \delta) = x_j$ and $F_j(\alpha, c, \delta) < x_j$ for all $\alpha \in [0, \alpha^*)$. Moreover, let $\widehat{\alpha} \in (0, \alpha^*)$ such that

$$0 < \alpha^* - \widehat{\alpha} \leq \min\left\{\frac{x_j + \delta_j - (x_i + \delta_i)}{2}, c_i - x_i\right\}. \tag{2}$$

Notice that $\alpha^* - \widehat{\alpha} < r$. Let us denote $z^* = F(\alpha^*, c, \delta)$ and $\widehat{z} = F(\widehat{\alpha}, c, \delta)$. By transfer composition, we have that

$$z^* = \widehat{z} + F(\alpha^* - \widehat{\alpha}, c - \widehat{z}, \delta + \widehat{z}). \tag{3}$$

Let us denote $z' = F(\alpha^* - \widehat{\alpha}, c - \widehat{z}, \delta + \widehat{z})$. Taking into account the definition of α^*, expression (3) and since $\widehat{\alpha} < \alpha^*$, we obtain

$$x \geq z^* \geq \widehat{z} \text{ and, in particular, } x_j = z_j^* > \widehat{z}_j. \tag{4}$$

Making use of (2) and (4), we have that

$$\begin{aligned} 2 \cdot (\alpha^* - \widehat{\alpha}) &\leq x_j + \delta_j - (x_i + \delta_i) \leq z_j^* + \delta_j - (z_i^* + \delta_i) \\ &= \widehat{z}_j + \delta_j - (\widehat{z}_i + \delta_i) + (z_j^* - \widehat{z}_j) - (z_i^* - \widehat{z}_i) \\ &\leq \widehat{z}_j + \delta_j - (\widehat{z}_i + \delta_i) + \textstyle\sum_{k \in N}(z_k^* - \widehat{z}_k) \\ &= \widehat{z}_j + \delta_j - (\widehat{z}_i + \delta_i) + \alpha^* - \widehat{\alpha}, \end{aligned}$$

which implies $\alpha^* - \widehat{\alpha} \leq (\widehat{z}_j + \delta_j) - (\widehat{z}_i + \delta_i)$. Moreover, by (2) and (4), we have that $\alpha^* - \widehat{\alpha} \leq c_i - x_i \leq c_i - z_i^* \leq c_i - \widehat{z}_i$. Therefore, $\alpha^* - \widehat{\alpha} \leq \min\{(\widehat{z}_j + \delta_j) - (\widehat{z}_i + \delta_i), c_i - \widehat{z}_i\}$. Then, by ex-ante fairness, it holds that $z'_j = 0$. However, by (3) and (4), we reach a contradiction since $x_j = z_j^* = \widehat{z}_j + z'_j = \widehat{z}_j < x_j$.

Therefore, we conclude that $F = GEA$ and thus the GEA is the unique rule that satisfies ex-ante fairness and transfer composition. □

The properties in Theorem 2 are logically independent. The rule F^1 defined as $F^1(r, c, \delta) := CEA(r, c)$ satisfies transfer-composition but not ex-ante fairness. The priority rule with respect to ex-ante conditions F^2 satisfies ex-ante fairness but not transfer composition. This rule is defined as follows. Let $\{N_1, N_2, \ldots, N_m\}$ be a partition of the set N such that: (i) for all $p \in \{1, \ldots, m\}$ and all $i, j \in N_p$, $\delta_i = \delta_j$; (ii) for all $p \in \{1, \ldots, m-1\}$, all $i \in N_p$ and all $j \in N_{p+1}$, $\delta_i < \delta_j$. That is, we divide N in m groups by the increasing value of ex-ante conditions. Then, if $k \in \{1, \ldots, m\}$ is such that $\sum_{p=1}^{k-1} \sum_{j \in N_p} c_j < r \leq \sum_{p=1}^{k} \sum_{j \in N_p} c_j$ then

$$F_i^2(r, c, \delta) := \begin{cases} c_i & \text{if } i \in \bigcup_{p=1}^{k-1} N_p, \\ GEA_i\Big(r - \sum_{p=1}^{k-1} \sum_{j \in N_p} c_j, c_{|N_k}, \delta_{|N_k}\Big) & \text{if } i \in N_k, \\ 0 & \text{else.} \end{cases}$$

It is interesting to point out that the GEA rule combines the principle of equality, represented by F^1 and the idea of giving priority to agents with worse ex-ante conditions, represented by F^2.

Remark 1. *Since the GEA rule satisfies consistency, it follows that ex-ante fairness and transfer composition imply consistency.*

4. Dual results: characterizations of the GEL rule

In this section, we characterize the GEL rule by using the duality approach (introduced by Aumann and Maschler, 1985) and the duality relations between rules and properties (developed by Herrero and Villar, 2001). These relations can also be extended to our model.

In the standard framework, two rules are the dual of each other if one rule distributes the total gain r, in the same way as the other rule distributes the total loss $\ell = \sum_{i \in N} c_i - r$. The idea of duality can be adapted to our approach but taking into account that the vector δ becomes $-\delta$ when passing from the primal problem (r, c, δ) to the dual problem $(\ell, c, -\delta)$: formally, for all $(r, c, \delta) \in \mathcal{RC}^N$, F^* and F are dual if $F^*(r, c, \delta) = c - F(\ell, c, -\delta)$. In this way, it can be verified that the GEA rule and the GEL rule are the dual of each other (for a detailed proof see Proposition 4 in the Appendix).

On the other hand, two properties are dual if whenever a rule satisfies one of them, its dual satisfies the other. The dual of ex-ante exemption is *ex-ante exclusion*: for any two-person problem (r, c, δ) with $N = \{i, j\}$, if $\min\{\ell, c_i\} - \delta_i \leq \frac{\ell-\delta_i-\delta_j}{2}$, then $F_i(r, c, \delta) = (r - c_j)_+$. Parallel to standard rationing problems, the dual of path-independence is *composition*: for all $r_1, r_2 \in \mathbb{R}_+$ such that $r_1 + r_2 = r$, $F(r, c, \delta) = F(r_1, c, \delta) + F(r_2, c - F(r_1, c, \delta), \delta)$. The dual of ex-ante fairness is *ex-ante fairness**: for any pair of agents $i, j \in N$ that exhibits differences in their initial stock, i.e. $\delta_j \geq \delta_i$, if $\ell \leq \min\{\delta_j - \delta_i, c_j\}$, then $F_i(r, c, \delta) = c_i$. The dual of transfer composition is *transfer path-independence*: for any $r, r' \in \mathbb{R}_+$ such that $\sum_{i\in N} c_i \geq r' \geq r$, $F(r, c, \delta) = F\Big(r, F(r', c, \delta), \delta - \Big(c - F(r', c, \delta)\Big)\Big)$. Finally, the dual of consistency is itself. The proofs of these duality relations between the aforementioned properties are provided in Propositions 5, 6, 7 and 8, respectively, in the Appendix.

Taking these relations into account, the dual of Theorem 1 says that the GEL rule is the only rule satisfying *composition*, *ex-ante exclusion* and *consistency*. Finally, the dual of Theorem 2 says that the GEL rule is the only rule satisfying *ex-ante fairness** and *transfer path-independence*.

5. Conclusions

We have presented an extension of the standard rationing model. The aim of this extension is to take into account ex-ante inequalities between agents involved in the rationing process and try to compensate them for these inequalities. Two of the principal rationing rules (equal gains and equal losses) have been generalized and characterized within this new framework.

As previously mentioned in the Introduction, Hougaard et al. (2013a) propose an extension of the standard rationing model but from a different point of view. They consider a vector of baselines $b = (b_i)_{i=1,\ldots,n}$, where b_i is interpreted as a tentative allocation for agent i. These authors use the CEA rule in the baseline model as follows:

$$\widetilde{CEA}(r, c, b) = \begin{cases} t(c, b) + CEA\Big(r - \sum_{i\in N} t_i(c, b), c - t(c, b)\Big) & \text{if } \sum_{i\in N} t_i(c, b) \leq r \\ t(c, b) - CEA\Big(\sum_{i\in N} t_i(c, b) - r, t(c, b)\Big) & \text{if } \sum_{i\in N} t_i(c, b) > r \end{cases},$$

where $t(c, b) = (t_i(c, b))_{i\in N} = (\min\{c_i, b_i\})_{i\in N}$ denotes the truncated baseline vector. That is, the allocation is made in a two-step process: first, truncated baselines are assigned and, after that, the surplus or the deficit with respect to the available amount of resource is shared equally.

We would like to point out that baselines and ex-ante conditions are of a completely different nature and cannot be directly identified with each other. In contrast to the baselines that are preassigned, the stocks of resource or ex-ante conditions are not redistributed in any case. Thus, the final stock of any agent (initial stock plus the amount received) cannot be smaller than his initial stock (ex-ante condition). In Hougaard et al.,'s model, an agent's baseline is just an objective evaluation of his actual needs. Indeed, the agent may be awarded an allocation above or below his baseline (baselines act as bounds). However, when using the extension of the CEA rule, there is a link that allows a problem with baselines to be reinterpreted as a problem with ex-ante conditions. If we take $\delta^* = -t(c, b)$, then $\widetilde{CEA}(r, c, b) = GEA(r, c, \delta^*)$. Notice that the truncated baselines are embedded in our model as debts to agents and, thus, they are represented by a negative value. The other way around, that is, defining a problem with baselines based upon a problem with ex-ante conditions such that the allocations in both models coincide, is not possible in a non-trivial way.[5]

Our model can also be viewed as a situation in which a certain priority is given to some agents and where asymmetric allocations arise. Indeed, the model we introduce allows us to combine full and partial priority between agents.[6] Asymmetric allocations have previously been analysed in Moulin (2000) and in Hokari and Thomson (2003). Moulin assigns weights to agents and distributes awards or losses (up to the value of the claims) proportionally with respect to the weights. He also combines these weighted solutions with full priority rules. In our approach, the asymmetries are induced by the ex-ante conditions but not by the rules we apply which preserve the idea of equal (gains or losses) distribution.

Future research might, first, usefully adapt some characterizations of the CEA and the CEL rules provided in the literature (see Thomson, 2003, 2015) to our framework. Second, we believe our model can be applied to allocate resources in other contexts, for instance, those in which the same group of agents faces a sequence of rationing problems at different periods of time. The distribution in the current period is influenced by the amount received in previous periods, which can be considered as an ex-ante condition for the current rationing problem. Third, inequalities in the ex-ante conditions might also be useful to analyse taxation problems when differences in the net wealth of agents are relevant in the final allocation of taxes. Finally, two important rationing rules have yet to be analysed under our new framework: the Talmud

[5]Notice that if we consider the three-person problem $(r, c, \delta) = (2.5, (2, 1, 1), (0, 2, 3))$, then $GEA(r, c, \delta) = (2, 0.5, 0)$. For this problem, the reader may verify that the only way to define a problem with baselines $(2.5, (2, 1, 1), b)$ such that $\widetilde{CEA}(2.5, (2, 1, 1), b) = (2, 0.5, 0)$ is by taking $t(c, b) = GEA(r, c, \delta)$, which implies knowing beforehand the allocation proposed by the GEA. Even in the variant of the model proposed by the same authors (Hougaard et al., 2013b), the only way to define compatible baselines for this problem is also to take the trivial option $b = (2, 0.5, 0) = GEA(r, c, \delta)$.

Furthermore, the same example can be used to show that the GEL allocation cannot be reached by the baseline extension of the CEL rule.

[6]Kaminski (2006) considers priority in case of bankruptcy, assigning to different categories of claimants lexicographic full priorities. Furthermore, there is an extensive literature on bankruptcy laws discussing the insertion of partial priority in bankruptcy codes (e.g., Bebchuk and Fried, 1996; Bergström et al., 2004; Warren, 1997).

rule and the proportional rule. Both solutions are self-dual rules. Self-duality establishes a symmetry principle in the behaviour of the rule when distributing awards and losses. A natural proposal to generalize the Talmud rule is, for all $(r, c, \delta) \in \mathcal{RC}^N$, $GT(r, c, \delta) = GEA\left(\min\left\{r, \frac{\sum_{i\in N} c_i}{2}\right\}, \frac{c}{2}, \delta\right) + GEL\left(\max\left\{0, r - \frac{\sum_{i\in N} c_i}{2}\right\}, \frac{c}{2}, \delta\right)$. This is a self-dual[7] rule and it obviously generalizes the Talmud rule.[8] However, it is not so clear that the extension of the proportional solution to the ex-ante condition framework would likewise maintain the self-duality property.

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Appendix

Proof of Proposition 1 $1 \Rightarrow 2$) Let us suppose that $x^* = GEA(r, c, \delta)$ and there exist $i, j \in N$, such that $x_i^* + \delta_i < x_j^* + \delta_j$ but $x_j^* > 0$ and $x_i^* < c_i$. Hence, $\lambda - \delta_j > 0$, $x_i^* = (\lambda - \delta_i)_+$, and so

$$\begin{aligned} x_i^* + \delta_i &= (\lambda - \delta_i)_+ + \delta_i \geq \lambda \geq \min\{c_j + \delta_j, \lambda\} \\ &= \min\{c_j, \lambda - \delta_j\} + \delta_j = \min\{c_j, (\lambda - \delta_j)_+\} + \delta_j = x_j^* + \delta_j. \end{aligned}$$

Hence, we reach a contradiction with the hypothesis $x_i^* + \delta_i < x_j^* + \delta_j$ and we conclude that either $x_i^* = c_i$, or $x_j^* = 0$.
$2 \Rightarrow 1$) Let us suppose that for all $i, j \in N$ with $x_i^* + \delta_i < x_j^* + \delta_j$, it holds that either $x_j^* = 0$, or $x_i^* = c_i$, but $x^* \neq GEA(r, c, \delta)$. Then, by efficiency, there exist $i, j \in N$ such that

$$0 \leq x_i^* < GEA_i(r, c, \delta) \leq c_i \text{ and } c_j \geq x_j^* > GEA_j(r, c, \delta) \geq 0. \tag{5}$$

This means that $x_i^* < c_i$, $\lambda - \delta_i > 0$ and $(\lambda - \delta_j)_+ < c_j$. However,

$$\begin{aligned} x_j^* + \delta_j &> GEA_j(r, c, \delta) + \delta_j = (\lambda - \delta_j)_+ + \delta_j \geq \lambda \geq \min\{c_i + \delta_i, \lambda\} \\ &= \min\{c_i, \lambda - \delta_i\} + \delta_i = GEA_i(r, c, \delta) + \delta_i > x_i^* + \delta_i. \end{aligned}$$

[7] The proof can be found in Proposition 9 in the Appendix.

[8] The allocation assigned by the GT rule when applied to a problem without inequalities in the ex-ante conditions coincides with the allocation of the Talmud rule applied to the corresponding standard rationing problem (without ex-ante conditions).

By assumption, it should hold that either $x_j^* = 0$, or $x_i^* = c_i$, but this contradicts (5). Hence we conclude that $x^* = GEA(r, c, \delta)$.

$\square$

Proof of Proposition 2 First, we prove path-independence. If $r = r'$, the result is straightforward. If $r < r'$, we claim that

$$GEA(r, c, \delta) = GEA(r, GEA(r', c, \delta), \delta).$$

By definition, and for all $i \in N$, we have

$$\begin{aligned}
&GEA_i(r, c, \delta) = \min\{c_i, (\lambda - \delta_i)_+\} \text{ with } \textstyle\sum_{k\in N} GEA_k(r, c, \delta) = r,\\
&GEA_i(r', c, \delta) = \min\{c_i, (\lambda' - \delta_i)_+\} \text{ with } \textstyle\sum_{k\in N} GEA_k(r', c, \delta) = r' \text{ and}\\
&GEA_i(r, GEA(r', c, \delta), \delta) = \min\{\min\{c_i, (\lambda' - \delta_i)_+\}, (\lambda'' - \delta_i)_+\}\\
&\text{with } \textstyle\sum_{k\in N} GEA_k(r, GEA(r', c, \delta), \delta) = r.
\end{aligned}$$

First, we show

$$\lambda < \lambda'. \tag{6}$$

Suppose on the contrary, that $\lambda \geq \lambda'$. Then, for all $i \in N$,

$$GEA_i(r, c, \delta) = \min\{c_i, (\lambda - \delta_i)_+\} \geq \min\{c_i, (\lambda' - \delta_i)_+\} = GEA_i(r', c, \delta).$$

Summing up all the above inequalities, we obtain

$$r = \sum_{i\in N} GEA_i(r, c, \delta) \geq \sum_{i\in N} GEA_i(r', c, \delta) = r',$$

which contradicts $r < r'$.

Let us suppose now that $GEA(r, c, \delta) \neq GEA(r, GEA(r', c, \delta), \delta)$. Then, by efficiency of the GEA rule, there exist $i^* \in N$ and $j^* \in N$ such that

$$\begin{aligned}
&GEA_{i^*}(r, c, \delta) < GEA_{i^*}(r, GEA(r', c, \delta), \delta) \text{ and}\\
&GEA_{j^*}(r, c, \delta) > GEA_{j^*}(r, GEA(r', c, \delta), \delta).
\end{aligned} \tag{7}$$

Then, we have

$$\begin{aligned}
GEA_{i^*}(r, c, \delta) \quad &= \min\{c_{i^*}, (\lambda - \delta_{i^*})_+\}\\
&< \min\{\min\{c_{i^*}, (\lambda' - \delta_{i^*})_+\}, (\lambda'' - \delta_{i^*})_+\}\\
&= GEA_{i^*}(r, GEA(r', c, \delta), \delta) \leq c_{i^*},
\end{aligned} \tag{8}$$

which leads to $\min\{c_{i^*}, (\lambda - \delta_{i^*})_+\} = (\lambda - \delta_{i^*})_+$. Taking this into account, and substituting in (8), we have

$$(\lambda - \delta_{i^*})_+ < \min\{\min\{c_{i^*}, (\lambda' - \delta_{i^*})_+\}, (\lambda'' - \delta_{i^*})_+\} \leq (\lambda'' - \delta_{i^*})_+.$$

Hence, $\lambda - \delta_{i^*} \leq (\lambda - \delta_{i^*})_+ < (\lambda'' - \delta_{i^*})_+ = \lambda'' - \delta_{i^*}$ which implies

$$\lambda < \lambda''. \tag{9}$$

Combining (6) and (9) we obtain, for all $j \in N \setminus \{i^*\}$,

$$\begin{aligned} GEA_j(r, c, \delta) &= \min\{c_j, (\lambda - \delta_j)_+\} \\ &\leq \min\{c_j, \min\{(\lambda' - \delta_j)_+, (\lambda'' - \delta_j)_+\}\} \\ &= \min\{\min\{c_j, (\lambda' - \delta_j)_+\}, (\lambda'' - \delta_j)_+\} \\ &= GEA_j(r, GEA(r', c, \delta), \delta). \end{aligned}$$

However, this contradicts (7) and we obtain

$$GEA(r, c, \delta) = GEA(r, GEA(r', c, \delta), \delta),$$

which proves that the GEA rule satisfies path-independence.
Next, we prove consistency. Let $(r, c, \delta) \in \mathcal{RC}^N$ and $T \subsetneq N$, with $T \neq \emptyset$. Let us denote $x^* = GEA(r, c, \delta)$. By Proposition 1 it holds that, for all $i, j \in T$ with $i \neq j$, if $x_i^* + \delta_i < x_j^* + \delta_j$, then either $x_j^* = 0$, or $x_i^* = c_i$. Since $x^*_{|T}$ is feasible in the reduced problem $(r - \sum_{i \in N \setminus T} x_i^*, c_{|T}, \delta_{|T})$ and again by Proposition 1, we conclude that $x^*_{|T} = GEA(r - \sum_{i \in N \setminus T} x_i^*, c_{|T}, \delta_{|T})$ which proves consistency.
Finally, we prove ex-ante exemption. If $r = 0$, the result is straightforward. Let $(r, c, \delta) \in \mathcal{RC}^{\{i,j\}}$, $r > 0$, be a two-person rationing problem with ex-ante conditions and let $x^* = GEA(r, c, \delta)$. Suppose on the contrary, that w.l.o.g., $\min\{r, c_i\} \leq \frac{r - (\delta_i - \delta_j)}{2}$ but $x_i^* < \min\{r, c_i\}$. Hence, by efficiency, $x_j^* = r - x_i^* > 0$.

We consider two cases:

Case 1: $r \leq c_i$. In this case $r \leq \frac{r - (\delta_i - \delta_j)}{2}$, or, equivalently,

$$r + \delta_i \leq \delta_j \text{ and thus } \delta_j \geq \delta_i. \tag{10}$$

Moreover, since $x^* = GEA(r, c, \delta)$ and $x_i^* < c_i$, we have $x_i^* = \min\{c_i, (\lambda - \delta_i)_+\} = (\lambda - \delta_i)_+ = \lambda - \delta_i$, since, otherwise, from (10) $0 > \lambda - \delta_i \geq \lambda - \delta_j$, and then $x_j^* = 0$, which implies a contradiction.

On the other hand, since $x^* = GEA(r, c, \delta)$ and $x_j^* > 0$, we get

$$0 < x_j^* = \min\{c_j, (\lambda - \delta_j)_+\} = \min\{c_j, \lambda - \delta_j\} \leq \lambda - \delta_j.$$

However, if $\lambda - \delta_j > 0$ we would have that, by (10), $\lambda > \delta_j \geq r + \delta_i$ and thus $r < \lambda - \delta_i = x_i^*$ which is a contradiction.

Case 2: $r > c_i$. In this case, by hypothesis, we get

$$c_i \leq \frac{r - (\delta_i - \delta_j)}{2}. \tag{11}$$

Since we are assuming that $x_i^* < c_i < r$, we have $x_i^* = \min\{c_i, (\lambda - \delta_i)_+\} = (\lambda - \delta_i)_+$. If $\lambda - \delta_i \geq 0$, then $r = x_i^* + x_j^* = \lambda - \delta_i + x_j^* \leq \lambda - \delta_i + \lambda - \delta_j$, where the last inequality follows from $0 < x_j^* = \min\{c_j, (\lambda - \delta_j)_+\} = \min\{c_j, \lambda - \delta_j\}$. Using this inequality in (11), we get $c_i \leq \lambda - \delta_i$, which implies that $x_i^* = c_i$, in contradiction with our hypothesis. On the other hand, if $\lambda - \delta_i < 0$, then $x_i^* = 0$ and $r = x_j^* \leq \lambda - \delta_j$. Hence $r + \delta_j \leq \lambda$ and so, by substitution in (11), we get $c_i \leq \frac{\lambda - \delta_i}{2} < 0$, which is a contradiction.

We conclude that the GEA rule satisfies ex-ante exemption.

□

Proof of Theorem 1 By Proposition 2, we know that the GEA rule satisfies path-independence, consistency and ex-ante exemption. We now concentrate on proving the uniqueness of the rule. Let F be a rule satisfying these properties. If $|N| = 1$, it is straightforward. Consider now the two-person case $N = \{1, 2\}$ and $(r, c, \delta) \in \mathcal{RC}^{\{1,2\}}$. Let us suppose that, w.l.o.g., $\delta_1 \leq \delta_2$ and denote $x^* = (x_1^*, x_2^*) = F(r, c, \delta)$. We consider three cases:

Case 1: $r \leq \delta_2 - \delta_1$. Then,

$$\min\{r, c_1\} \leq r = \frac{r}{2} + \frac{r}{2} \leq \frac{r - (\delta_1 - \delta_2)}{2}.$$

Hence, $\min\{r, c_1\} + \delta_1 \leq \frac{r + \delta_1 + \delta_2}{2}$, and thus, by ex-ante exemption, we have that $x_1^* = \min\{r, c_1\}$ and $x_2^* = (r - c_1)_+$, and the solution F is uniquely determined.

Case 2: $r > \delta_2 - \delta_1 \geq c_1$. Then,

$$\min\{r, c_1\} = c_1 \leq \delta_2 - \delta_1 = \frac{\delta_2 - \delta_1}{2} + \frac{\delta_2 - \delta_1}{2} < \frac{r - (\delta_1 - \delta_2)}{2}.$$

Hence, by ex-ante exemption, we have that $x_1^* = \min\{r, c_1\} = c_1$ and $x_2^* = r - c_1$, and the solution F is also uniquely determined.

Case 3: $r > \delta_2 - \delta_1$ and $c_1 > \delta_2 - \delta_1$. We consider two subcases.

Subcase 3a: $c_1 + \delta_1 = c_2 + \delta_2$. Since $r > \delta_2 - \delta_1$, we claim that $x_1^* + \delta_1 = x_2^* + \delta_2$. First, suppose on the contrary that

$$x_1^* + \delta_1 < x_2^* + \delta_2. \tag{12}$$

From (12), it comes that $x_1^* + \delta_1 < \frac{x_1^* + \delta_1 + x_2^* + \delta_2}{2} = \frac{r + \delta_1 + \delta_2}{2}$ and thus

$$x_1^* = F_1(r, c, \delta) < \frac{r + \delta_2 - \delta_1}{2}. \tag{13}$$

Now, let us prove that there exists $r' > r$ such that $F_1(r', c, \delta) = \frac{r + \delta_2 - \delta_1}{2}$. Notice that $\frac{r + \delta_2 - \delta_1}{2} > 0$ since $x_1^* \geq 0$. Moreover, $\frac{r + \delta_2 - \delta_1}{2} \leq c_1$ since $c_1 + \delta_1 = c_2 + \delta_2$. Since F satisfies path-independence it also satisfies resource monotonicity (see (1)). Hence, F is a continuous and increasing function in r. Therefore, by continuity, since $F_1(0, c, \delta) = 0$, $F_1(c_1 + c_2, c, \delta) = c_1$ and F is an increasing function in r, there exists $r' \in [0, c_1 + c_2]$

such that $F_1(r', c, \delta) = \frac{r+\delta_2-\delta_1}{2}$. Now, by (13), we have $F_1(r, c, \delta) < F_1(r', c, \delta)$. Hence, by resource monotonicity, we conclude $r' > r$.

Next, let us denote $x' = F(r', c, \delta)$. Notice that $\min\{r, x'_1\} \leq x'_1 = \frac{r-(\delta_1-\delta_2)}{2}$ which implies, by ex-ante exemption applied to the problem (r, x', δ), that $F_1(r, x', \delta) = \min\{r, x'_1\} = \min\{r, \frac{r+\delta_2-\delta_1}{2}\} = \frac{r+\delta_2-\delta_1}{2}$, where the last equality follows from $r > \delta_2 - \delta_1$. Finally, by path-independence, we obtain

$$x^* = F(r, c, \delta) = F(r, F(r', c, \delta), \delta) = F(r, x', \delta) = \left(\frac{r+\delta_2-\delta_1}{2}, \frac{r+\delta_1-\delta_2}{2}\right).$$

We conclude that $x_1^* + \delta_1 = \frac{r+\delta_1+\delta_2}{2} = x_2^* + \delta_2$ reaching a contradiction with (12). In case $x_1^* + \delta_1 > x_2^* + \delta_2$ the proof also follows the same argument to reach a contradiction. Hence, the proof of the claim is done and, thus, $x_1^* + \delta_1 = x_2^* + \delta_2$. Finally, taking into account that $x_1^* + x_2^* = r$, we conclude that the solution F is uniquely determined.

Subcase 3b: $c_1 + \delta_1 \neq c_2 + \delta_2$. First, if $\min\{r, c_1\} + \delta_1 \leq \frac{r+\delta_1+\delta_2}{2}$, then by ex-ante exemption $x_1^* = \min\{r, c_1\}$ and $x_2^* = (r-c_1)_+$, and the solution F is uniquely determined. Similarly, if $\min\{r, c_2\} + \delta_2 \leq \frac{r+\delta_2+\delta_1}{2}$, then by ex-ante exemption $x_2^* = \min\{r, c_2\}$ and $x_1^* = (r - c_2)_+$, and the solution F is uniquely determined. Otherwise,

$$\min\{r, c_i\} + \delta_i > \frac{r+\delta_1+\delta_2}{2}, \text{ for all } i \in \{1, 2\}. \tag{14}$$

By the hypothesis of Subcase 3b

$$c_i + \delta_i < c_j + \delta_j, \text{ where } i, j \in \{1, 2\} \text{ with } i \neq j. \tag{15}$$

Now we claim that for $r' = 2c_i + \delta_i - \delta_j$, we have that $x' = F(r', c, \delta)$ is such that $x'_i = c_i$ and $x'_j = c_i + \delta_i - \delta_j$. To verify this, first notice that, by (15), $r' < c_i + c_j$. Moreover, we show that $c_i + \delta_i - \delta_j \geq 0$. Suppose on the contrary that $c_i < \delta_j - \delta_i$. If $i = 1$ and $j = 2$, we obtain a contradiction with the hypothesis of Case 3; if $i = 2$ and $j = 1$ then $c_2 < \delta_1 - \delta_2 \leq 0$, getting again a contradiction. Notice that the second inequality follows from the assumption $\delta_1 \leq \delta_2$. Now, since $c_i + \delta_i - \delta_j \geq 0$, we have

$$\min\{r', c_i\} + \delta_i = \min\{2c_i + \delta_i - \delta_j, c_i\} + \delta_i = c_i + \delta_i = \frac{r' + \delta_i + \delta_j}{2},$$

and so $\min\{r', c_i\} = \frac{r'-(\delta_i-\delta_j)}{2} = c_i$. Hence, by ex-ante exemption, we have that $x'_i = c_i$ and, by efficiency, $x'_j = r' - x'_i = c_i + \delta_i - \delta_j$, and the proof of the claim is done.

On the other hand, $r' = 2c_i + \delta_i - \delta_j \geq 2\min\{r, c_i\} + \delta_i - \delta_j > r$, where the last inequality follows from (14). Therefore, by path-independence, we obtain

$$F(r, c, \delta) = F(r, F(r', c, \delta), \delta) = F(r, x', \delta).$$

Finally, since $x'_j + \delta_j = c_i + \delta_i = x'_i + \delta_i$ and $r > \delta_2 - \delta_1$, where the inequality comes from the hypothesis of Case 3, applying an analogous reasoning to that of Subcase 3a to the problem (r, x', δ) we obtain

$$F_i(r, c, \delta) + \delta_i = F_i(r, x', \delta) + \delta_i = F_j(r, x', \delta) + \delta_j = F_j(r, c, \delta) + \delta_j,$$

where the first and the last equalities come from path-independence. Hence, by efficiency, the solution F is uniquely determined. Therefore, we conclude that, for the two-person case, the GEA rule is the unique rule that satisfies path-independence and ex-ante exemption.

Let $|N| \geq 3$ and suppose that F and F' satisfy the three properties, but $F \neq F'$. Hence, there exists $(r, c, \delta) \in \mathcal{RC}^N$ such that $x = F(r, c, \delta) \neq F'(r, c, \delta) = x'$. This means that there exist $i, j \in N$ such that $x_i > x'_i$, $x_j < x'_j$ and, w.l.o.g., $x_i + x_j \leq x'_i + x'_j$. However, since F and F' are consistent,

$$\begin{aligned}(x_i, x_j) &= F(r - \textstyle\sum_{k \in N \setminus \{i,j\}} x_k, (c_i, c_j), (\delta_i, \delta_j)) \text{ and} \\ (x'_i, x'_j) &= F'(r - \textstyle\sum_{k \in N \setminus \{i,j\}} x'_k, (c_i, c_j), (\delta_i, \delta_j)).\end{aligned}$$

Since $F = F'$ for the two-person case and path-independence implies resource monotonicity, we have that

$$\begin{aligned}(x'_i, x'_j) &= F'(x'_i + x'_j, (c_i, c_j)(\delta_i, \delta_j)) = F(x'_i + x'_j, (c_i, c_j)(\delta_i, \delta_j)) \\ &\geq F(x_i + x_j, (c_i, c_j)(\delta_i, \delta_j)) = (x_i, x_j),\end{aligned}$$

in contradiction with $x_i > x'_i$. Hence, we conclude that $F = F' = GEA$.

$\square$

Example 3. *A rule F that satisfies consistency and path-independence but does not satisfy ex-ante exemption. Let F be a generalized rationing rule defined as follows, for all $(r, c, \delta) \in \mathcal{RC}^N$, $N \in \mathcal{N}$, we have*

$$F(r, c, \delta) = GEA\,(r, c, \mathbf{0})\,.$$

$\diamondsuit$

Example 4. *A rule F that satisfies consistency and ex-ante exemption but does not satisfy path-independence. Let $(r, c, \delta) \in \mathcal{RC}^N$, $N \in \mathcal{N}$, and let us denote by $\widehat{c}_i = \min\{r, c_i\}$ the truncated claim of agent $i \in N$. Up to reordering agents, there exist natural numbers $k_1, k_2, \ldots, k_m$ such that $k_1 + k_2 + \ldots + k_m = n$ and*

$$\begin{aligned}\widehat{c}_1 + \delta_1 &= \widehat{c}_2 + \delta_2 = \ldots = \widehat{c}_{k_1} + \delta_{k_1} \\ &< \widehat{c}_{k_1+1} + \delta_{k_1+1} = \widehat{c}_{k_1+2} + \delta_{k_1+2} = \ldots = \widehat{c}_{k_1+k_2} + \delta_{k_1+k_2} \\ &< \widehat{c}_{k_1+k_2+1} + \delta_{k_1+k_2+1} = \ldots = \widehat{c}_{k_1+k_2+k_3} + \delta_{k_1+k_2+k_3} \\ &\vdots \\ &< \widehat{c}_{k_1+\ldots+k_{m-1}+1} + \delta_{k_1+\ldots+k_{m-1}+1} = \ldots = \widehat{c}_{k_1+\ldots+k_m} + \delta_{k_1+\ldots+k_m}.\end{aligned}$$

Notice that we have divided agents in m groups according to the value $\widehat{c}_i + \delta_i$, where this value is constant within groups and strictly increasing across groups. Let us denote each group by $N_1 = \{i \in N : 1 \leq i \leq k_1\}$ and $N_t = \{i \in N : k_1 + \ldots + k_{t-1} + 1 \leq i \leq k_1 + \ldots + k_t\}$, for all $t \in \{2, \ldots, m\}$. Then, we can define recursively an allocation rule by assigning payoffs to the members of each group as follows.

Step 1 $\big($*group* $N_1\big)$*:*

If $\sum_{i \in N_1} c_i \geq r$ then $x_i = GEA_i(r, c_{|N_1}, \delta_{|N_1})$, for all $i \in N_1$, and $x_i = 0$, otherwise. Stop.

If not, $\sum_{i \in N_1} c_i < r$, we assign $x_i = c_i$, for all $i \in N_1$ and we proceed to the next step.

Step t $\big(2 \leq t \leq m$, *groups* N_2 *to* $N_m\big)$*:*

If $\displaystyle\sum_{i \in N_t} c_i \geq r - \sum_{\substack{i \in N_j \\ j=1,\ldots,t-1}} c_i$ *then* $x_i = GEA_i\left(r - \displaystyle\sum_{\substack{k \in N_j \\ j=1,\ldots,t-1}} c_k, c_{|N_t}, \delta_{|N_t}\right)$, *for all* $i \in N_t$, *and* $x_i = 0$, *for all* $i \in N_k$ *with* $k = t+1, t+2, \ldots, m$. *Stop.*

If not, $\displaystyle\sum_{i \in N_t} c_i < r - \sum_{\substack{i \in N_j \\ j=1,\ldots,t-1}} c_i$, *we assign* $x_i = c_i$, *for all* $i \in N_t$ *and we proceed to the next step.*

$\diamondsuit$

Example 5. *A rule F that satisfies ex-ante exemption and path independence but it is not consistent. Let $N \in \mathbb{N}$ with $|N| \geq 3$. Define*[9] *$N_1 = \{i, j\} \subseteq N$ such that $i < k$ and $j < k$ for all $k \in N \setminus \{i, j\}$ and $N_2 = N \setminus N_1$. Let $C_{N_1} = c_i + c_j$, $C_{N_2} = \sum_{k \in N_2} c_k$, $\Delta_{N_1} = \delta_i + \delta_j$, and $\Delta_{N_2} = \sum_{k \in N_2} \delta_k$. Next, let us denote by $z = (z_1, z_2)$ the allocation obtained by applying the GEA rule to the two-subgroup problem; that is*

$$z := (z_1, z_2) = GEA\left(r, (C_{N_1}, C_{N_2}), (\Delta_{N_1}, \Delta_{N_2})\right).$$

Then, define F as follows: if $|N| \leq 2$, $F(r, c, \delta) = GEA(r, c, \delta)$; if $|N| \geq 3$

$$F_k(r, c, \delta) := \begin{cases} GEA_k\left(z_1, (c_i, c_j), (\delta_i, \delta_j)\right) & \text{if } k \in N_1, \\ GEA_k\left(z_2, (c_k)_{k \in N_2}, (\delta_k)_{k \in N_2}\right) & \text{if } k \in N_2. \end{cases}$$

$\diamondsuit$

Proof of Proposition 3 First, we prove transfer composition. If $r = r_1$, the result is straightforward. If $r_1 < r$ and $r_1 + r_2 = r$, we claim that $x = x' + x''$, where $x = GEA(r, c, \delta)$, $x' = GEA(r_1, c, \delta)$ and $x'' = GEA(r_2, c - x', \delta + x')$. By definition, and

[9]That is, N_1 is formed by the two agents associated to the smallest natural numbers in N.

for all $i \in N$, we have

$$\begin{aligned}
&x_i = \min\{c_i, (\lambda - \delta_i)_+\} \text{ with } \textstyle\sum_{k\in N} x_k = r,\\
&x'_i = \min\{c_i, (\lambda' - \delta_i)_+\} \text{ with } \textstyle\sum_{k\in N} x'_k = r_1 \text{ and}\\
&x''_i = \min\left\{c_i - \min\{c_i, (\lambda' - \delta_i)_+\}, \left(\lambda'' - \delta_i - \min\{c_i, (\lambda' - \delta_i)_+\}\right)_+\right\}\\
&\text{with } \textstyle\sum_{k\in N} x''_k = r_2.
\end{aligned}$$

Moreover, notice that

$$x'_i + x''_i = \min\left\{c_i, \max\left\{\lambda'' - \delta_i, \min\{c_i, (\lambda' - \delta_i)_+\}\right\}\right\}. \tag{16}$$

Next, we show

$$\lambda > \lambda'. \tag{17}$$

Suppose on the contrary that $\lambda \leq \lambda'$. Then, for all $i \in N$,

$$x_i = \min\{c_i, (\lambda - \delta_i)_+\} \leq \min\{c_i, (\lambda' - \delta_i)_+\} = x'_i.$$

Summing up all the above inequalities, we obtain

$$r = \sum_{i\in N} x_i \leq \sum_{i\in N} x'_i = r_1,$$

which contradicts $r_1 < r$.

Let us suppose on the contrary that the GEA rule does not satisfy transfer composition, that is, $x \neq x' + x''$. Then, by efficiency of the GEA rule, there exist $i^* \in N$ and $j^* \in N$ such that

$$x_{i^*} < x'_{i^*} + x''_{i^*} \text{ and } x_{j^*} > x'_{j^*} + x''_{j^*}. \tag{18}$$

Then, by (16), we have

$$\begin{aligned}
x_{i^*} &= \min\{c_{i^*}, (\lambda - \delta_{i^*})_+\}\\
&< \min\left\{c_{i^*}, \max\left\{\lambda'' - \delta_{i^*}, \min\{c_{i^*}, (\lambda' - \delta_{i^*})_+\}\right\}\right\}\\
&= x'_{i^*} + x''_{i^*} \leq c_{i^*},
\end{aligned} \tag{19}$$

which leads to $x_{i^*} = \min\{c_{i^*}, (\lambda - \delta_{i^*})_+\} = (\lambda - \delta_{i^*})_+$. Taking this into account, and substituting in (19), we have

$$\begin{aligned}
x_{i^*} = (\lambda - \delta_{i^*})_+ &< \min\left\{c_{i^*}, \max\left\{\lambda'' - \delta_{i^*}, \min\{c_{i^*}, (\lambda' - \delta_{i^*})_+\}\right\}\right\}\\
&\leq \max\left\{\lambda'' - \delta_{i^*}, \min\{c_{i^*}, (\lambda' - \delta_{i^*})_+\}\right\}\\
&\leq \max\left\{\lambda'' - \delta_{i^*}, (\lambda' - \delta_{i^*})_+\right\}.
\end{aligned} \tag{20}$$

Next, we show that

$$\lambda'' > \lambda. \tag{21}$$

Otherwise, $\lambda'' \leq \lambda$ and thus, by (17), we have that

$$\max\left\{\lambda'' - \delta_{i^*}, (\lambda' - \delta_{i^*})_+\right\} \leq (\lambda - \delta_{i^*})_+ = x_{i^*},$$

getting a contradiction with (20). Now, by (21) and (16), we obtain that, for all $j \in N \setminus \{i^*\}$,

$$\begin{aligned} x_j &= \min\{c_j, (\lambda - \delta_j)_+\} \\ &\leq \min\left\{c_j, \max\left\{\lambda'' - \delta_j, \min\{c_j, (\lambda' - \delta_j)_+\}\right\}\right\} = x'_j + x''_j. \end{aligned}$$

However, this contradicts (18) and we conclude $x = x' + x''$, which proves that the GEA rule satisfies transfer composition.

Next, we prove ex-ante fairness. If $r = 0$, the result is straightforward. Let $(r, c, \delta) \in \mathcal{RC}^N$, $r > 0$ and let $x = GEA(r, c, \delta)$. Suppose on the contrary that there exist $i, j \in N$ such that $r \leq \min\{\delta_j - \delta_i, c_i\}$ but $x_j > 0$. Hence, by efficiency of the GEA rule, we obtain that $x_i < c_i$, and thus, since $x_j > 0$,

$$\delta_j - \delta_i \geq \min\{\delta_j - \delta_i, c_i\} \geq r = \sum_{k \in N} x_k \geq x_i + x_j > x_i - x_j,$$

we conclude $x_i + \delta_i < x_j + \delta_j$ with $x_j > 0$ and $x_i < c_i$ getting a contradiction with Proposition 1. Therefore, we conclude that the GEA rule satisfies ex-ante fairness.

□

Proposition 4. *The GEA and the GEL are the dual rules of each other.*

Proof. Let us first prove $GEA(r, c, \delta) = c - GEL(\ell, c, -\delta)$. For all $i \in N$,

$$GEA_i(r, c, \delta) = \min\{c_i, (\lambda - \delta_i)_+\} = c_i - \max\{0, c_i - (\lambda - \delta_i)_+\}. \tag{22}$$

By (22), $\sum_{i \in N} GEA_i(r, c, \delta) = \sum_{i \in N} c_i - \sum_{i \in N} \max\{0, c_i - (\lambda - \delta_i)_+\}$ and thus, $\sum_{i \in N} \max\{0, c_i - (\lambda - \delta_i)_+\} = \sum_{i \in N} c_i - r = \ell$. Hence, $\max\{0, c_i - (\lambda - \delta_i)_+\} = GEL_i(\ell, c, -\delta)$. Next we prove $GEL(r, c, \delta) = c - GEA(\ell, c, -\delta)$. For all $i \in N$,

$$GEL_i(r, c, \delta) = \max\{0, c_i - (\lambda + \delta_i)_+\} = c_i - \min\{c_i, (\lambda + \delta_i)_+\}. \tag{23}$$

By (23), $\sum_{i \in N} GEL_i(r, c, \delta) = \sum_{i \in N} c_i - \sum_{i \in N} \min\{c_i, (\lambda + \delta_i)_+\}$ and
$\sum_{i \in N} \min\{c_i, (\lambda + \delta_i)_+\} = \sum_{i \in N} c_i - r = \ell$. Hence, $\min\{c_i, (\lambda + \delta_i)_+\} = GEA_i(\ell, c, -\delta)$.
□

Proposition 5. *Ex-ante exemption and ex-ante exclusion are dual properties.*

Proof. Let $(r, c, \delta) \in \mathcal{RC}^{\{1,2\}}$ be a two-person rationing problem with ex-ante conditions and let us suppose that F and F^* are dual rules, that is, $F^*(r, c, \delta) = c - F(\ell, c, -\delta)$.

Hence, we claim that if F satisfies ex-ante exemption, then F^* satisfies ex-ante exclusion. To verify this, suppose, w.l.o.g., that, for the problem (r, c, δ), we have

$$\min\{\ell, c_1\} - \delta_1 \leq \frac{\ell - \delta_1 - \delta_2}{2}. \tag{24}$$

Notice that (24) is the same condition as that used in the definition of ex-ante exemption when we apply rule F to the problem $(\ell, c, -\delta)$. Hence, since F satisfies ex-ante exemption and by (24), we have

$$\begin{aligned} F_1^*(r, c, \delta) &= c_1 - F_1(\ell, c, -\delta) = c_1 - \min\{c_1, \ell\} = \max\{0, c_1 - \ell\} \\ &= \max\{0, c_1 - (c_1 + c_2 - r)\} = (r - c_2)_+, \end{aligned}$$

which proves that F^* satisfies ex-ante exclusion.

Similarly, we claim that if F satisfies ex-ante exclusion, then F^* satisfies ex-ante exemption. Let us suppose, w.l.o.g., that for the problem (r, c, δ), we have

$$\min\{r, c_1\} + \delta_1 \leq \frac{r + \delta_1 + \delta_2}{2}. \tag{25}$$

Notice that (25) is the same condition as that used in the definition of ex-ante exclusion when we apply rule F to the problem $(\ell, c, -\delta)$. Hence, since F satisfies ex-ante exclusion, we have that

$$\begin{aligned} F_1^*(r, c, \delta) &= c_1 - F_1(\ell, c, -\delta) = c_1 - (\ell - c_2)_+ \\ &= c_1 - \max\{0, \ell - c_2\} = \min\{c_1, c_1 + c_2 - \ell\} = \min\{c_1, r\}, \end{aligned}$$

which proves that F^* satisfies ex-ante exemption. □

Proposition 6. *Path-independence and composition are dual properties.*

Proof. Let us suppose that F and $F^*$are dual rules, that is, $F^*(r, c, \delta) = c - F(\ell, c, -\delta)$. We claim that if F satisfies composition, then F^* satisfies path-independence. To verify this, let $r \geq r_1 \geq 0$ and define $r_2 = r - r_1$ and $\ell_1 = \sum_{i \in N} c_i - r_1$. Hence,

$$\ell = \sum_{i \in N} c_i - r = \ell_1 - r_2, \text{ and so } \ell_1 \geq \ell. \tag{26}$$

On the one hand, we have

$$\begin{aligned} F^*(r_1, c, \delta) = c - F(\ell_1, c, -\delta) &= c - (F(\ell, c, -\delta) + F(r_2, c - F(\ell, c, -\delta), -\delta)) \\ &= F^*(r, c, \delta) - F(r_2, c - F(\ell, c, -\delta), -\delta), \end{aligned} \tag{27}$$

where the first and the last equalities follow from the definition of dual rule, and the remaining equality follows from the composition property of F and (26).

By definition of dual rule, we have

$$\begin{aligned} F^*(r_1, F^*(r,c,\delta),\delta) &= F^*(r,c,\delta) - F(r-r_1, F^*(r,c,\delta), -\delta) \\ &= F^*(r,c,\delta) - F(r_2, c - F(\ell, c, -\delta), -\delta). \end{aligned} \tag{28}$$

Thus, taken into account (27) and (28), we conclude that F^* satisfies path-independence.

Similarly, we claim that if F satisfies path-independence, then F^* satisfies composition. To verify this, let $r_1 + r_2 = r$, where $r_1, r_2 \in \mathbb{R}_+$ and $\ell_1 = \sum_{i\in N} c_i - r_1$. Notice that $\ell_1 \geq \ell$. By path-independence and by definition of dual rule, we have

$$F(\ell, c, -\delta) = F(\ell, F(\ell_1, c, -\delta), -\delta) = F(\ell_1, c, -\delta) - F^*(r_2, F(\ell_1, c, -\delta), \delta). \tag{29}$$

Then, by definition of dual rule and by (29), we have

$$\begin{aligned} F^*(r,c,\delta) &= c - F(\ell, c, -\delta) = c - (F(\ell_1, c, -\delta) - F^*(r_2, F(\ell_1, c, -\delta), \delta)) \\ &= F^*(r_1, c, \delta) + F^*(r_2, F(\ell_1, c, -\delta), \delta) \\ &= F^*(r_1, c, \delta) + F^*(r_2, c - F^*(r_1, c, \delta), \delta). \end{aligned} \tag{30}$$

Therefore, F^* satisfies composition. □

Proposition 7. *Ex-ante fairness and ex-ante fairness* are dual properties.*

Proof. Let $(r, c, \delta) \in \mathcal{RC}^N$ and suppose that F and F^* are dual rules, that is, $F^*(r,c,\delta) = c - F(\ell, c, -\delta)$. We claim that if F satisfies ex-ante fairness, then F^* satisfies ex-ante fairness*. To verify this, suppose that, given (r, c, δ) there exist $i, j \in N$ such that

$$\ell \leq \min\{\delta_j - \delta_i, c_j\}. \tag{31}$$

Notice that (31) is the same condition as the one used in the definition of ex-ante fairness when we apply rule F to the problem $(\ell, c, -\delta)$. Hence, since F satisfies ex-ante fairness and by (31), we have

$$F_i^*(r,c,\delta) = c_i - F_i(\ell, c, -\delta) = c_i - 0 = c_i,$$

which proves that F^* satisfies ex-ante fairness*.

Similarly, we claim that if F satisfies ex-ante fairness*, then F^* satisfies ex-ante fairness. Let us suppose that, given (r, c, δ) there exist $i, j \in N$ such that

$$r \leq \min\{\delta_j - \delta_i, c_i\}. \tag{32}$$

Notice that (32) is the same condition as the one used in the definition of ex-ante fairness* when we apply rule F to the problem $(\ell, c, -\delta)$. Hence, since F satisfies ex-ante fairness* and by (32), we have

$$F_j^*(r,c,\delta) = c_j - F_j(\ell, c, -\delta) = c_j - c_j = 0,$$

which proves that F^* satisfies ex-ante fairness.

□

Proposition 8. *Transfer composition and trasnfer path-independence are dual properties.*

Proof. The proof follows the same guidelines of the proof of Proposition 6. Just replace expression (30) by $F^*(r,c,\delta) = F^*(r_1,c,\delta) + F^*(r_2, c - F^*(r_1,c,\delta), \delta + F^*(r_1,c,\delta))$. $\square$

Proposition 9. *The GT rule is self-dual.*

Proof. First of all, let us recall that F and F^* are dual rules if, for all $(r,c,\delta) \in \mathcal{RC}^N$, $F(r,c,\delta) = c - F^*(\ell, c, -\delta)$. A self-dual rule is one with $F = F^*$. Next, we show that the GT rule is self-dual, i.e. $GT(r,c,\delta) = c - GT(\ell,c,-\delta)$. We consider two cases:

Case 1: $r < \frac{\sum_{i\in N} c_i}{2}$. Hence, for all $i \in N$,

$$\begin{aligned} GT_i(r,c,\delta) &= GEA_i\left(r, \tfrac{c}{2}, \delta\right) + GEL_i\left(0, \tfrac{c}{2}, \delta\right) = GEA_i\left(r, \tfrac{c}{2}, \delta\right) \\ &= \min\{\tfrac{c_i}{2}, (\lambda - \delta_i)_+\} = \tfrac{c_i}{2} + \min\{0, (\lambda-\delta_i)_+ - \tfrac{c_i}{2}\} \\ &= \tfrac{c_i}{2} - \max\{0, \tfrac{c_i}{2} - (\lambda - \delta_i)_+\}. \end{aligned} \tag{33}$$

On the other hand, by Case 1, $\ell = \sum_{i\in N} c_i - r > \frac{\sum_{i\in N} c_i}{2}$. Hence, for all $i \in N$,

$$\begin{aligned} c_i - GT_i(\ell, c, -\delta) &= c_i - \Big(GEA_i\big(\min\{\ell, \tfrac{\sum_{i\in N} c_i}{2}\}, \tfrac{c}{2}, -\delta\big) \\ &+ GEL_i\big(\max\{0, \ell - \tfrac{\sum_{i\in N} c_i}{2}\}, \tfrac{c}{2}, -\delta\big)\Big) \\ &= c_i - GEA_i\left(\tfrac{\sum_{i\in N} c_i}{2}, \tfrac{c}{2}, -\delta\right) \\ &- GEL_i\left(\ell - \tfrac{\sum_{i\in N} c_i}{2}, \tfrac{c}{2}, -\delta\right) \\ &= \tfrac{c_i}{2} - \max\{0, \tfrac{c_i}{2} - (\lambda - \delta_i)_+\}. \end{aligned} \tag{34}$$

Therefore, by (33) and (34), $GT(r,c,\delta) = c - GT(\ell,c,-\delta)$, and thus, Case 1 is done.

Case 2: $r \geq \frac{\sum_{i\in N} c_i}{2}$. Hence, for all $i \in N$,

$$\begin{aligned} GT_i(r,c,\delta) &= GEA_i\left(\tfrac{\sum_{i\in N} c_i}{2}, \tfrac{c}{2}, \delta\right) + GEL_i\left(r - \tfrac{\sum_{i\in N} c_i}{2}, \tfrac{c}{2}, \delta\right) \\ &= \tfrac{c_i}{2} + \max\{0, \tfrac{c_i}{2} - (\lambda + \delta_i)_+\} = c_i + \max\{\tfrac{-c_i}{2}, -(\lambda+\delta_i)_+\} \\ &= c_i - \min\{\tfrac{c_i}{2}, (\lambda + \delta_i)_+\}. \end{aligned} \tag{35}$$

On the other hand, by Case 2, $\ell = \sum_{i\in N} c_i - r \leq \frac{\sum_{i\in N} c_i}{2}$. Hence, for all $i \in N$,

$$\begin{aligned} c_i - GT_i(\ell, c, -\delta) &= c_i - \Big(GEA_i\big(\min\{\ell, \tfrac{\sum_{i\in N} c_i}{2}\}, \tfrac{c}{2}, -\delta\big) \\ &+ GEL_i\big(\max\{0, \ell - \tfrac{\sum_{i\in N} c_i}{2}\}, \tfrac{c}{2}, -\delta\big)\Big) \\ &= c_i - GEA_i\left(\ell, \tfrac{c}{2}, -\delta\right) = c_i - \min\{\tfrac{c_i}{2}, (\lambda + \delta_i)_+\}. \end{aligned} \tag{36}$$

Therefore, by (35) and (36), $GT(r,c,\delta) = c - GT(\ell,c,-\delta)$, and thus, Case 2 is done. Then, we conclude that the GT rule is self-dual. $\square$

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