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Households, Markets and Public Choice

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Abstract

We develop a model that combines competitive exchange of private commodities across endogenously formed groups with public good provision and global collective decisions. There is a tension between local and global collective decisions. In particular, we show that group formation and collective decisions on public goods may destabilize each other, even if there exist favorable conditions for matching on the one hand, and for global collective decisions on the other hand. We establish sufficient conditions for the existence of competitive equilibria with endogenous household formation and public choice.

JEL-Code: D100, D510, D620, D700, H200, H410.

Keywords: household formation, matching, general equilibrium, public goods, public choice, median voter theorem.

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1 Introduction

Motivation

Most of the work on general equilibrium models with multi-member households is predicated on two interrelated premises. First, the allocation of resources is impacted by the prevailing household structure, that is the partition of the population into households. Second, formation and stability of households and the resulting household structure are influenced by economic conditions and the anticipated allocative implications of household membership.

Previous investigations (Gersbach and Haller (2010, 2011)) have considered consumers who care about their own private consumption, and possibly the identity and private consumption of other members of their own household. Such "household-specific preferences" allow for consumption externalities within households and for local public goods for the household. They disallow consumption externalities across households and global public goods. Here, we extend the model by introducing global public goods that may affect the entire population. The externalities caused by private good consumption are still confined to members of the same household.

Public goods are ubiquitous. Classical examples such as defense and security provided by the military and police remain important. Recently, other public goods such as mitigating climate change or financial stability have been at the center of public and scientific debates. Integrating the provision of public goods into general equilibrium theory poses, however, several well known challenges. It ultimately necessitates the existence of governmental authorities and the introduction of constitutional rules. First, public or collective provision of public goods requires an authority with the power to tax people. Second, rules for collective decision-making on public good provision involving all citizens have to be specified. Third, one has to decide whether non-discriminatory constitutional clauses for taxation ought to be introduced.

A comprehensive general equilibrium model with public goods provision, household formation and household decisions poses a novel challenge: it exhibits a tension between local and global collective decisions.

Model and Results

We incorporate public good provision and a constitutionally founded state into the general equilibrium models with group formation developed in Gersbach and Haller (2011). Like for local collective decisions at the group level, we adopt a flexible approach to global collective decisions on public good provision. In particular, we require that equilibrium public good bundles and a financing scheme cannot be improved upon by a coalition of a certain size. Special cases are simple majority or super-majority coalitions. We impose a uniform tax rule to fund public good provision. That is, public goods are financed by a linear tax on the wealth of individuals. In Gersbach and Haller (2011), we obtain several equilibrium notions distinguished by the stability requirements for households. These notions can be extended by adding *"public choice"*. Thus, we will focus on "competitive equilibria with endogenous household formation and *public choice*". We are going to impose the most stringent stability requirement for households: that no group of consumers can benefit from forming a new household.

Our main contributions are as follows. First, we develop a consistent formal structure to integrate decisions to form, join or exit groups; local collective decisions on consumption in such groups; competitive exchange across endogenously formed groups; and global collective decisions on public good provision across all citizens.

Second, we establish sufficient conditions for the existence of competitive equilibria with endogenous household formation and public choice. We demonstrate that existence may depend on whether household members, acting as voters, take into account that global collective decisions may affect prices for private commodities.

Third, we show that paradoxically, a priori stable matchings and stable collective decisions (Condorcet winners) may destabilize each other when they are integrated. The noteworthy feature of this result is that favorable conditions for matching and global collective decisions in the form of the simple majority rule prevail. In particular, for any given tax rate, a stable matching, i.e., a stable household structure exists. In addition, global collective decisions have Condorcet winners for any conceivable household structure. Despite these features, however, there are circumstances in which no stable matching exists. In other words, stable matchings and stable outcomes of voting according to the simple majority rule cannot be achieved simultaneously.

The paper is related to several strands of literature — in addition to our own work on general equilibrium models with multi-member households.

Relation to the Literature

In Gersbach and Haller (2011), we explore the potential tension between endogenous household formation and consumption decisions of households on the one hand and competitive markets for private goods on the other hand. In that analysis, group and consumption externalities within households play a crucial role. Now the emphasis shifts to the tension between local and global collective decisions. For that purpose, externalities play still an important role like in the illustrative Example 1 and for Proposition 3, but receive a less prominent treatment in most of the paper.

One of our main results shows that public good provision implemented via collective decisions on taxation can destabilize any household structure — even if a stable matching exists given any conceivable collective decision. This kind of non-existence is absent from the matching literature because there, global collective decisions are typically irrelevant. See the work on hedonic coalitions (e.g., Banerjee, Konishi and Sönmez (2001), Bogomolnaia and Jackson (2002), Greenberg (1978)), matching (e.g., Alkan (1988), Gale and Shapley (1962), Roth and Sotomayor (1990)), assignment games (e.g., Roth and Sotomayor (1990), Shapley and Shubik (1972)), and multilateral bargaining (e.g., Bennett (1988, 1997), Crawford and Rochford (1986), Rochford (1984)). Consequently, that literature does not observe the tension between matching and global collective decisions.

In the tradition of a large literature on public good provision and in particular the work of Guesnerie and Oddou (1979, 1981) and Buchanan and Yoon (2004), which we will discuss below, we assume the presence of tax rules. While we focus on linear wealth taxes, the essential constraint is that the tax schedule is given when public good provision takes place. We refer to Gersbach, Hahn and Imhof (2013) for the justification why such tax rules should be separated from the actual decision on public good provision.

The theory of second-best taxation in the context of local public good provision and linear wealth taxation has been significantly developed by Guesnerie and Oddou (1979, 1981) and Westhoff (1977). They consider coalition formation in which a coalition cannot benefit from public goods produced by other coalitions. They show that it may not be socially desirable that the grand coalition forms and that stable coalition structures may exist in which each coalition chooses a particular level of public goods and a corresponding tax rate. Our model shares the perspective that local externalities will typically lead to the partition of the society in groups. It differs, however, in a variety of crucial aspects. First, apart from local externalities there are also global public goods. No citizen can be excluded from the benefits when a certain level of the public good is provided. Second, in our model the local externalities and global public goods can be present simultaneously. [1] Third, groups or coalitions trade through competitive markets. As a consequence, equilibrium notions and their properties differ significantly

[1] Insofar, our approach also differs from traditional public economics and public finance that look at taxes or public goods for the polity at large, without the formation of local jurisdictions.

from the cited second-best literature.

Another strand of literature that works with tax rules is exemplified in Buchanan and Yoon (2004). When all individuals have to pay the same tax rate, majoritarian institutions and the prospect for individual membership in more than one decision authority limit exploitation of the tax base, and thus limit the tragedy of the fiscal commons. In our context, overexploitation of the tax base by majoritarian institutions does not occur as tax rules prevail and tax revenues are solely used for financing of public goods.

The paper is organized as follows. In the next section we introduce the model. The concept of competitive equilibrium with public choice and rational price expectations is introduced and investigated in section 3. In section 4, we define the concept of competitive equilibrium with public choice and myopic price expectations. We study the relationship between the two equilibrium concepts. Section 5 concludes. An appendix contains the proof of Proposition 3.

2 Consumer Characteristics and Allocations

In this section, we outline the basic structure of the model. It consists of decision units (consumers, households, and the government), the objects of decisions (private and public goods, households), decision criteria (preferences), and outcomes (allocations). Essentially, we integrate public good provision, global collective decisions and (in a very rudimentary form) the state into the general equilibrium model of Gersbach and Haller (2011).

Consumers and Household Structures. We consider a finite population of **consumers**, represented by a set $I = \{1, \dots, N\}$. A generic consumer is denoted i or j. The population I is partitioned into households, i.e., the population I is subdivided into a partition P of non-empty subsets referred to as households. For a consumer $i \in I$, $P(i)$ denotes the unique element of P (unique household in P) to which i belongs. If a partition P consists of H households, we frequently label them $h = 1, \dots, H$, provided this causes no confusion.

A (potential) household or group of consumers is any non-empty subset h of the population I. A generic household is denoted h. $\mathcal{H} = \{h \subseteq I | h \neq \emptyset\}$ denotes the set of all potential households. For $i \in I$, $\mathcal{H}_i = \{h \subseteq I | i \in h\}$ denotes the set of all potential households which have i as a member.

We call any partition P of I a **household structure in** I. We treat the household structure as an object of endogenous choice. Households are formed so that some household structure P is ultimately realized. Consequently, our **consumer allocation space** is $\mathcal{P}$, the set of all household structures in I.

Relative to a household structure P, we use the following terminology regarding $i \in I$ and $h \subseteq I$, $h \neq \emptyset$:

"household h exists" or "household h is formed"	iff $h \in P$;
"i belongs to h" or "individual i is a member of household h"	iff $i \in h$.

Private Commodities. There exists a finite number $\ell \geq 1$ of private commodities. Thus the private commodity space is $\mathbb{R}^\ell$. Private commodities are denoted by superscripts $k = 1, \ldots, \ell$. Each private commodity is formally treated as a private good, possibly with externalities in consumption. That is, private commodities are rival in consumption and a particular household's ownership excludes other households from consumption. Consumer $i \in I$ has a private consumption set $X_i = \mathbb{R}^\ell_+$ so that the **private commodity allocation space** is $\mathcal{X} \equiv \prod_{j \in I} X_j$. Generic elements of $\mathcal{X}$ are denoted $\mathbf{x} = (x_i)_{i \in I}$, $\mathbf{y} = (y_i)_{i \in I}$ with $x_i = (x_i^1, ..., x_i^\ell)$, $y_i = (y_i^1, ..., y_i^\ell)$. For a potential household $h \subseteq I$, $h \neq \emptyset$, we set $\mathcal{X}_h = \prod_{i \in h} X_i$, the consumption set for household h. $\mathcal{X}_h$ has generic elements $\mathbf{x_h} = (x_i)_{i \in h}$. If $\mathbf{x} = (x_i)_{i \in I} \in \mathcal{X}$ is a private commodity allocation, then consumption for household h is the restriction of $\mathbf{x} = (x_i)_{i \in I}$ to h, $\mathbf{x_h} = (x_i)_{i \in h}$.

Endowments with Private Commodities. For a potential household $h \subseteq I, h \neq \emptyset$, its **endowment** is a private commodity bundle $\omega_h \in \mathbb{R}^\ell$ given by the sum of the endowments of all participating individuals: $\omega_h = \sum_{i \in h} \omega_{\{i\}}$ where $\omega_{\{i\}}$ is the endowment when individual i forms a single-person household. The **social endowment with private commodities** is given as

$$\omega_S \equiv \sum_{h \in P} \omega_h = \sum_{i \in I} \omega_{\{i\}}. \tag{1}$$

Note that the social endowment is independent of the household structure.

Public Goods. There exists a finite number $q \geq 1$ of **public goods**. These goods are non-excludable and non-rivalrous in consumption. The public good space is $\mathbb{R}^q$. The consumption set for public goods is $\mathbb{R}^q_+$. A generic bundle of public goods is denoted by $g = (g_1, ..., g_q) \in \mathbb{R}^q_+$ where for $m = 1, ..., q$, the amount of public good m is denoted by g_m.

Provision of Public Goods. Public goods are produced by the government with the use of private commodities. In particular,

$$g_m \leq F_m(v_m)$$

where $v_m = (v_m^1, ..., v_m^\ell) \in \mathbb{R}_+^\ell$ is the vector of private commodities used to produce the amount g_m of the public good m. The production functions $F_m : \mathbb{R}_+^\ell \to \mathbb{R}_+$ $(m = 1, ..., q)$ are assumed to be continuous and concave.

We introduce the following notation when only one public good is produced, i.e., $g \in \mathbb{R}_+$:

$$\begin{aligned} g^{max}(p,t) = \quad & \max F(v) \\ & s.t.\ v \in \mathbb{R}_+^\ell; \\ & \quad pv = tp\omega_S. \end{aligned}$$

$g^{max}(p,t)$ is the maximal level of the public good that can be afforded when prices p prevail in the market for private commodities and nominal endowments are taxed at the rate t. $g^{max}(p,t)$ always exists if F is continuous.

Allocations. An **allocation** is a triple $(\mathbf{x}, g; P) \in \mathcal{X} \times \mathbb{R}_+^q \times \mathcal{P}$ specifying an allocation bundle of private commodities, a bundle of public goods, and household membership of each consumer. We call an allocation $(\mathbf{x}, g; P) \in \mathcal{X} \times \mathbb{R}_+^q \times \mathcal{P}$ **feasible** if there exist $v_m \in \mathbb{R}_+^\ell$ $(m = 1, ..., q)$ such that

$$\sum_{i \in I} x_i + \sum_{m=1}^{q} v_m = \omega_S; \tag{2}$$

$$g_m = F_m(v_m) \text{ for } m = 1, ..., q. \tag{3}$$

After the specification of individual preferences, by means of utility representations, an allocation determines the welfare of all members of society. In the current model, feasibility of an allocation does not depend on the household structure while its desirability may well depend on it.

Consumer Preferences. In principle, a consumer might have preferences on the allocation space $\mathcal{X} \times \mathbb{R}_+^q \times \mathcal{P}$ and care about each and every detail of an allocation. But we shall restrict our analysis to situations of **household-specific preferences**, expanding the corresponding notion in Gersbach and Haller (2011) to economies with public goods. Consumers with such preferences care about public goods. They do not care about consumption of private commodities and household composition beyond the boundaries of their own household. That is, given a particular household structure, an individual is indifferent with respect to the affiliation and consumption of individuals belonging to other households.

We are going to make the **Assumption of Household-Specific Preferences (HSP)** throughout this paper. We represent these preferences by utility functions. To this end, let us denote $\mathcal{X}^* = \bigcup_{h \in \mathcal{H}} \mathcal{X}_h$ and define $\mathcal{A}_i = \{(\mathbf{x_h}; h) \in$

$\mathcal{X}^* \times \mathcal{H} : h \in \mathcal{H}_i,\, \mathbf{x_h} \in \mathcal{X}_h\}$ and $\mathcal{C}_i = \{(\mathbf{x_h}, g; h) \in \mathcal{X}^* \times \mathbb{R}^q_+ \times \mathcal{H} : h \in \mathcal{H}_i,\, \mathbf{x_h} \in \mathcal{X}_h,\, g \in \mathbb{R}^q_+\}$ for $i \in I$. We assume that each individual $i \in I$ has a UTILITY REPRESENTATION $U_i : \mathcal{C}_i \to \mathbb{R}$.

The assumption HSP has been extensively justified in Gersbach and Haller (2011). Among other things, it allows for **local public goods** within households. Private commodity k virtually constitutes a local public good for household h if for all members $i \in h$, U_i does not depend on individual consumption, but only on the aggregate consumption $\sum_{i \in h} x_i^k$ of good k by the household members.

For later use, it proves useful to distinguish several special cases of HSP:

(PGE) Pure Group Externalities: For each consumer i, there exist functions $U_i^C : X_i \times \mathbb{R}^q_+ \to \mathbb{R}$ and $U_i^G : \mathcal{H}_i \to \mathbb{R}$ such that $U_i(\mathbf{x_h}, g; h) = U_i^C(x_i, g) + U_i^G(h)$ for $g \in \mathbb{R}^q_+, h \in \mathcal{H}_i, \mathbf{x_h} \in \mathcal{X}_h$.

(SEP) Separable Preferences: For each consumer i, there exist functions $U_i^c : X_i \to \mathbb{R}$, $V_i^c : \mathbb{R}^q_+ \to \mathbb{R}$ and $U_i^G : \mathcal{H}_i \to \mathbb{R}$ such that $U_i(\mathbf{x_h}, g; h) = U_i^c(x_i) + V_i^c(g) + U_i^G(h)$ for $g \in \mathbb{R}^q_+, h \in \mathcal{H}_i, \mathbf{x_h} \in \mathcal{X}_h$.

(SSP) Semi-Separable Preferences: For each consumer i, there exist functions $\widehat{U}_i : \mathcal{A}_i \to \mathbb{R}$ and $\widehat{V}_i : \mathbb{R}^q_+ \times \mathcal{H}_i \to \mathbb{R}$ such that $U_i(\mathbf{x_h}, g; h) = \widehat{U}_i(\mathbf{x_h}; h) + \widehat{V}_i(g; h)$ for $g \in \mathbb{R}^q_+, h \in \mathcal{H}_i, \mathbf{x_h} \in \mathcal{X}_h$.

Obviously, SEP is a special case of PGE and of SSP.

Pareto Optimal Allocations. Pareto optimality is defined in terms of the welfare of individual consumers. Formally, a feasible allocation allocation $(\mathbf{x}, g; P)$ is **Pareto optimal** if there is no feasible allocation $(\mathbf{x}', g'; P')$ that makes some consumer better off and nobody worse off, that is, $U_i(\mathbf{x}'_{\mathbf{P'(i)}}, g'; P'(i)) > U_i(\mathbf{x_{P(i)}}, g; P(i))$ for some $i \in I$ and $U_i(\mathbf{x}'_{\mathbf{P'(i)}}, g'; P'(i)) \geq U_i(\mathbf{x_{P(i)}}, g; P(i))$ for all $i \in I$.

3 Equilibrium with Public Choice and Rational Price Expectations

In order to formulate an equilibrium of the economic system, several aspects have to be considered. At the conceptual level, we have to integrate household formation, collective decisions by households, market clearing, and collective decisions on public good provision by the entire polity. In particular, we have to combine local collective decisions (at the household level) and global collective decisions on public good provision. There are several ways to formulate the latter. Global collective decisions concern the bundle of public goods the society is offering to its members and the way the expenditures are financed. Here we present a simple variant thereof. It requires that at the going market prices and at the given household structure, a proposed bundle of public goods g cannot be improved by a coalition of households subject to two constraints: First, public good provision is financed by linear income taxes at the household level. Second, the coalition consists of households in the existing household structure and comprises at least a set of individuals of size n ($1 \leq n \leq N$). The case $n = \frac{N+1}{2}$ when N is uneven corresponds to the requirement that g is a Condorcet winner. For $n = N$, the requirement says that a change of the bundle of public goods has to be a Pareto improvement.

We define an equilibrium as a price system and a tax rate together with a household structure and a feasible resource allocation with private commodities and public good production such that:

- a household chooses collectively an efficient consumption schedule for its members, subject to the household disposable income;
- markets clear;
- no group of individual can form a new household and choose a feasible allocation at the going prices and the going tax rate which makes some member of the newly formed household better off and no member worse off;
- the budget of the public sector is balanced;
- no subset of households, which comprises at least n individuals, can propose an alternative bundle of public goods and a corresponding tax rate that will balance the public budget and make all individuals in these households weakly better off and at least one individual strictly better off at the going market prices.

The combination of all conditions defines a competitive equilibrium with new household formation and public choice. To fix ideas, let us consider a first numerical example.

Example 1. There are one private good and one public good. It takes g units of the private good to produce the quantity g of the public good. There are two types of consumers, 30 consumers of “type A” and 20 consumers of “type B”. Each consumer is endowed with 1/2 unit of the private good. If endowments are taxed at rate $t \in [0,1]$, then the amount of the public good provided is $t \cdot \omega_S = 25t$ and a household h of size $|h|$ has aggregate consumption $\overline{x}_h = (1-t)\omega_h = (1-t)|h|/2$. Define $u_i(g) = 0$ for consumers i of type A and $u_i(g) = g/(1+g)$ for consumers i of type B. Consumer i has

utility representation $U_i(\mathbf{x_h}, g; h) = \overline{x}_h + u_i(g)$
for $g \geq 0, h \in \mathcal{H}_i, |h| \in \{1,2\}, \mathbf{x_h} \in \mathcal{X}_h$ and
utility representation $U_i(\mathbf{x_h}, g; h) = \frac{1}{|h|}(\overline{x}_h + u_i(g))$
for $g \geq 0, h \in \mathcal{H}_i, |h| \geq 3, \mathbf{x_h} \in \mathcal{X}_h$.

Note that preferences satisfy SSP. Notice further that a household’s aggregate private goods consumption $\overline{x}_h$ can be viewed as consumption of a local public good for the household. With these particular preferences, for any given tax rate t, a consumer achieves her maximimal utility, $1 - t + u_i(25t)$, as a member of a two-person household, regardless with whom she is matched. Hence in equilibrium, the population will be partitioned into 25 two-person households. For a type A consumer, $1-t+u_i(25t) = 1-t$ which is maximized at $t = 0$. For a type B consumer, $1-t+u_i(25t) = 1-t+(25t)/(1+25t)$ which is maximized at $t^* = 0.16$. Given that the majority of consumers is of type A, one would expect that $t = 0$ is adopted. However, if changing the prevailing tax rate takes a coalition Ω of households, comprising at least 26 consumers none of which is made worse off by the change, then the situation becomes more intricate and multiple equilibrium tax rates emerge. If consumers sort themselves into homogeneous households, where every consumer is matched with someone of the same type, then indeed, $t = 0$ will be adopted. Consider instead a partition consisting of 20 mixed 2-person households with one person of each type and 5 homogeneous households with 2 persons of type A. Then a coalition of sufficient size must include some mixed households. For any prevailing tax rate $t \in [0, t^*]$, a change of the tax rate would hurt at least one member of any mixed household and, consequently, t and the particular household structure are consistent with equilibrium. This demonstrates the interplay — and potential tension — between group formation driven by group and consumption externalities (local public goods) and the determinants of the amount of public goods. With different specifications of the u_i, the choice of tax rate might influence household formation. $\square$

3.1 Basic Definitions

In order to define the equilibrium concept formally, we fix n at some level and we consider a household $h \in P$ and a price system $p \in \mathbb{R}^{\ell}$. For $\mathbf{x_h} = (x_i)_{i \in h} \in \mathcal{X}_h$,

$$p * \mathbf{x_h} \equiv p \cdot \left(\sum_{i \in h} x_i \right)$$

denotes the expenditure of household h on household consumption plan $\mathbf{x_h}$ at the price system p. As p and $\mathbf{x_h}$ are of different dimension for multi-member households, we use the $*$-product in lieu of the familiar inner product. Then h's **budget set** is defined as

$$B_h(p,t) = \{\mathbf{x_h} \in \mathcal{X}_h \ : \ p * \mathbf{x_h} \leq (1-t) p \cdot \omega_h\}$$

where t is the tax rate on the nominal value of the endowments. We next define the household's **efficient budget set** $EB_h(p,g,t)$ that depends on p, t and g, the given bundle of public goods. $EB_h(p,g,t)$ consists of the $\mathbf{x_h} \in B_h(p,t)$ with the property that there is no $\mathbf{y_h} \in B_h(p,t)$ such that

$$U_i(\mathbf{y_h}, g; h) \geq U_i(\mathbf{x_h}, g; h) \ \text{ for all } i \in h;$$
$$U_i(\mathbf{y_h}, g; h) > U_i(\mathbf{x_h}, g; h) \ \text{ for some } i \in h.$$

We further define a **state** of the economy as a tuple $(p, \mathbf{x}, g, t; P)$ such that $p \in \mathbb{R}^{\ell}$ is a price system, $t \in [0,1]$ is a tax rate and $(\mathbf{x}, g; P) \in \mathcal{X} \times \mathbb{R}^q_+ \times \mathcal{P}$ is an allocation, i.e., $\mathbf{x} = (x_i)_{i \in I}$ is an allocation of private commodities, $g = (g_1, ..., g_q)$ a bundle of public goods and P is an allocation of consumers (a household structure, that is a partition of the population into households). Following Gersbach and Haller (2011), we say that in state $(p, \mathbf{x}, g, t; P)$,

(a) consumer i can benefit from exit, if $P(i) \neq \{i\}$ and there exists $y_i \in B_{\{i\}}(p,t)$ such that $U_i(y_i, g; \{i\}) > U_i(\mathbf{x_{P(i)}}, g; P(i))$;

(b) consumer i can benefit from joining another household h', if $h' \in P$, $h' \neq P(i)$ and there exists $\mathbf{y_{h' \cup \{i\}}} \in B_{h' \cup \{i\}}(p,t)$ such that $U_j(\mathbf{y_{h' \cup \{i\}}}, g; h' \cup \{i\}) \geq U_j(\mathbf{x_{P(j)}}, g; P(j))$ for all $j \in h'$ and $U_i(\mathbf{y_{h' \cup \{i\}}}, g; h' \cup \{i\}) > U_i(\mathbf{x_{P(i)}}, g; P(i))$ for i.

(c) a group of consumers can benefit from forming a new household h', if $h' \notin P$ and there exists $\mathbf{y_h} \in B_{h'}(p,t)$ such that $U_j(\mathbf{y_{h'}}, g; h') \geq U_j(\mathbf{x_{P(j)}}, g; P(j))$ for all $j \in h'$ and $U_j(\mathbf{y_{h'}}, g; h') > U_j(\mathbf{x_{P(j)}}, g; P(j))$ for some $j \in h'$.

Notice that if no group of consumers can benefit from forming a new household, then no consumer can benefit from exit or joining another household.

3.2 Competitive Equilibrium with Given Household Structure and Tax Rate

We first consider price systems that clear the market for private commodities when the household structure and the tax rate are exogenously given.

Definition. A triple $(p, \mathbf{x}, g)$ consisting of a price system p, an allocation of private commodities $\mathbf{x}$ and a bundle of public goods g is a **competitive equilibrium given household structure** P **and tax rate** t if there exist $v_1, \ldots, v_q \in \mathbb{R}^\ell_+$ such that

1. $\mathbf{x_h} \in EB_h(p, g, t)$ for all $h \in P$.
2. $\sum_{i \in I} x_i + \sum_{m=1}^q v_m = \omega_S$.
3. $g_m = F_m(v_m)$ for $m = 1, \ldots, q$.
4. $\sum_{m=1}^q pv_m \leq tp \sum_{h \in P} \omega_h$.

Proposition 1 *Let $P \in \mathcal{P}$ and $t \in [0, 1]$. Suppose that $q = 1$, that is there is a single public good with production function F. Further suppose that*

(i) *F is continuous, strictly increasing and concave;*

(ii) *each consumer i has semi-separable preferences with utility representation $U_i(\mathbf{x_h}, g; h) = \widehat{U}_i(\mathbf{x_h}; h) + \widehat{V}_i(g; h)$ for $g \in \mathbb{R}^q_+, h \in \mathcal{H}_i, \mathbf{x_h} \in \mathcal{X}_h$;*

(iii) *each function $\widehat{U}_i$ is continuous and concave, strictly increasing in x_i and exhibits non-negative externalities.*

Then there exists a competitive equilibrium given household structure P and tax rate t.

Proof. We add an additional consumer to the economy, whose endowment is generated by taxing the other consumers. Specifically, we consider a pure exchange economy with consumers $i = 1, \ldots, N + 1$ and given household structure

$P' = P \cup \{\{N+1\}\}$. Households $h \in P$ have endowments $(1-t) \cdot \omega_h$. Household $h = \{N+1\}$ has endowment $t \cdot \omega_S$, where ω_S is the social endowment of the economy with N consumers. Consumers $i = 1, \ldots, N$ have semi-separable preferences with utility representations U_i as hypothesized. The additional single consumer $N+1$ is only interested in the consumption of private goods, with utility representation F. This agent chooses an input combination v from its budget set that produces the maximum amount of public good. The $(N+1)$-person economy satisfies the assumptions of Corollary 1 of Sato (2009), which asserts the existence of a competitive equilibrium of such an economy. q.e.d.

Under the assumptions of Proposition 1, a competitive equilibrium allocation of the artificial pure exchange economy constructed in the proof is Pareto-optimal for that economy. This follows from Corollary 1 of Haller (2000). Hence the associated competitive equilibrium allocation of the actual economy has corresponding constrained optimality properties: With the household structure P and tax rate t in place, there is no feasible allocation where some consumers's utility from private good consumption is higher without lowering someone else's utility from private good consumption or lowering the amount of the public good. However, that does not imply that the overall allocation is Pareto optimal. Typically, it is not. Take for instance $\ell = 2$ and absence of consumption externalities so that $\widehat{U}_i(\cdot, h)$ only depends on x_i. Under standard assumptions then, each consumer's marginal rate of substitution MRS_i and the technical rate of substitution TRS satisfy $MRS_i = TRS$ at the equilibrium allocation of the artificial economy, while Pareto optimality for the actual economy requires $\sum_i MRS_i = TRS$.

3.3 Competitive Equilibrium with Endogenous Household Structure and Tax Rate

We now define an equilibrium with new household formation and public choice where every component of the equilibrium state is endogenous. Moreover, consumers have rational price expectations: If a group of consumers suggests an alternative public policy, consisting of a tax rate and a bundle of public goods, it anticipates how this change would affect equilibrium prices for private goods.

Definition. A state $(p, \mathbf{x}, g, t; P)$ is a **competitive equilibrium with new household formation, public choice and rational price expectations** if it satisfies the following conditions:

1. $(p, \mathbf{x}, g)$ is a competitive equilibrium given household structure P and tax rate t.

2. No group of consumers can benefit from forming a new household.

3. There is no subcoalition Ω of households, i.e. $\Omega \subseteq P$, with at least n individuals that can propose an alternative public good bundle and financing scheme (g', t') such that for some price system p' and some allocation of private goods $\mathbf{x}'$,

   3.a. $(p', \mathbf{x}', g')$ is a competitive equilibrium given household structure P and tax rate t';

   3.b. $U_i(\mathbf{x_h}', g'; h) \geq U_i(\mathbf{x_h}, g; h)$ for all $i \in h \in \Omega$;

   3.c. $U_i(\mathbf{x_h}', g'; h) > U_i(\mathbf{x_h}, g; h)$ for some $i \in h \in \Omega$.

The competitive equilibrium with new household formation, public choice and rational price expectations assumes that households proposing an alternative scheme (g', t') recognize that an alternative bundle of public goods may require a different tax rate to balance the budget. Moreover, the coalition Ω correctly anticipates how the different tax rate and public good bundle affect market clearing prices for private commodities.

In the next proposition, we provide sufficient conditions for the existence of competitive equilibria with new household formation, public choice and rational price expectations. We are going to assume separable preferences. In particular, an individual i ranks the households h it may belong to according to the utilities $U_i^G(h)$. We say that **a household structure P consists of preferred households for all consumers** if $U_i^G(P(i)) \geq U_i^G(h)$ for all $i \in I, h \in \mathcal{H}_i$. In general, a household structure with that property need not exist. In a number of interesting cases, however, it does exist: If every individual strictly prefers to remain single, then $P = \{\{i\} | \, i \in I\}$ will do. Another example would be the case $N = 2R$ where for $r = 1, \ldots, R$, $i = 2r - 1$ strictly prefers to be paired with $j = 2r$ and the latter strictly prefers to be paired with the former. Then $P = \{\{2r-1, 2r\} | \, r = 1, \ldots, R\}$ will do.

Proposition 2 *Suppose there is a single public good and preferences are separable, that is $U_i(\mathbf{x_h}, g; h) = U_i^c(x_i) + V_i^c(g) + U_i^G(h)$ for all $h \in \mathcal{H}_i, \mathbf{x_h} \in \mathcal{X}_h$, $g \in \mathbb{R}_+$. Suppose further:*

(i) *If $t \in [0,1]$ is the tax rate on the nominal value of endowments, then up to price normalization, there exists a unique competitive or Walrasian equilibrium $(p(t), \mathbf{x}(t), v(t))$ of the $(N+1)$-person pure exchange economy represented by $((U_i^c, (1-t)\omega_{\{i\}})_{i \in I}; (F, t\omega_S))$. Let $g(t) = F(v(t))$.*

(ii) *The preferences given by* $U_i^c(x_i(t)) + V_i^c(g(t)))$ *— with* $(\mathbf{x}(t), g(t))$ *from (i) — are single-peaked with respect to* t.

(iii) *All* U_i^c, $i \in I$, *satisfy local non-satiation.*

(iv) *All* V_i^c, $i \in I$, *are strictly increasing.*

(v) *There exists a household structure* P *that consists of preferred households for all consumers.*

(vi) $n \geq (N+1)/2$.

Then there exists an equilibrium with new household formation, public choice and rational price expectations.

Proof. Like in the proof of Proposition 1, the $N+1^{st}$ consumer of the pure exchange economy in (i) determines the level of public good $g(t)$, choosing an input combination $v(t)$. By (i) and (iv), the public good level $g(t)$ is optimal given the tax rate t. By (i) and (ii), there exists for all individuals i a tax rate denoted by $\hat{t}_i$ with the property

$$\hat{t}_i = \arg\max_{t\in[0,1]} \Big\{U_i^c(x_i(t)) + V_i^c(g(t))\Big\}.$$

The tax rate $\hat{t}_i$ is the most preferred one by individual i. We order the individuals according to their ideal tax rate $\hat{t}_i$ and we denote by $\hat{t}_m$ the median agent's optimal tax rate.

We claim that $(p(\hat{t}_m), \mathbf{x}(\hat{t}_m), g(\hat{t}_m), \hat{t}_m; P)$ is a competitive equilibrium with new household formation, public choice and rational price expectations.

First of all, $(p(\hat{t}_m), \mathbf{x}(\hat{t}_m), g(\hat{t}_m))$ is a competitive equilibrium given household structure P and tax rate $\hat{t}_m$. By (i), $\sum_i x_i(\hat{t}_m) + v(\hat{t}_m) = \omega_S$, $g(\hat{t}_m) = F(v(\hat{t}_m))$, and $p(\hat{t}_m)\cdot v(\hat{t}_m) \leq p(\hat{t}_m)\cdot \hat{t}_m \cdot \omega_S$. Furthermore, if for some $h \in P$, $\mathbf{x_h}(\hat{t}_m) \notin EB_h(p(\hat{t}_m), g(\hat{t}_m), \hat{t}_m)$, then there exists $\mathbf{x_h} \in B_h(p(\hat{t}_m), \hat{t}_m)$ such that $U_i^c(x_i) \geq U_i^c(x_i(\hat{t}_m))$ for all $i \in h$, with at least one strict inequality. Because of (i) and (iii), this implies $p(\hat{t}_m) * \mathbf{x_h} > p(\hat{t}_m)\omega_h$, contradicting $\mathbf{x_h} \in B_h(p(\hat{t}_m), \hat{t}_m)$. Therefore, $\mathbf{x_h}(\hat{t}_m) \in EB_h(p(\hat{t}_m), g(\hat{t}_m), \hat{t}_m)$ has to hold for all $h \in P$. This completes the proof that $(p(\hat{t}_m), \mathbf{x}(\hat{t}_m), g(\hat{t}_m))$ is a competitive equilibrium given household structure P and tax rate $\hat{t}_m$.

Second, no group of consumers can benefit from forming a new household. Namely, consider a group $h \notin P$. Then by (v), for each $j \in h$, $U_j^G(P(j)) \geq U_j^G(h)$. If $\mathbf{y_h} \in \mathcal{X}_h$ such that $U_j(\mathbf{y_h}, g(\hat{t}_m); h) > U_j(\mathbf{x_{P(j)}}(\hat{t}_m), g(\hat{t}_m); P(j))$ for all $j \in h$, then $U_j^c(y_j) > U_j^c(x_j(t))$ and, consequently $p(\hat{t}_m)y_j > p(\hat{t}_m)(1-\hat{t}_m)\omega_{\{j\}}$ has to hold for all $j \in h$. But then $\mathbf{y_h} \notin B_h(p(\hat{t}_m), \hat{t}_m)$. In case $U_j(\mathbf{y_h}, g(\hat{t}_m); h) \geq U_j(\mathbf{x_{P(j)}}(\hat{t}_m), g(\hat{t}_m); P(j))$ for all $j \in h$, with strict inequality for at least one $j \in h$, we can invoke (iii) to conclude that $p(\hat{t}_m)y_j \geq p(\hat{t}_m)(1-\hat{t}_m)\omega_{\{j\}}$ has to hold for all $j \in h$, with at least one strict inequality. Then $\mathbf{y_h} \notin B_h(p(\hat{t}_m), \hat{t}_m)$ again. This shows that h cannot benefit from forming a new household.

Third, there is no coalition Ω of households in P with at least $(N+1)/2$ individuals that can propose an alternative public good bundle and financing scheme (g', t') such that for some price system p' and some allocation of private goods $\mathbf{x}'$, the analogue of 3.a.–3.c. holds. For consider a coalition Ω of households with at least $(N+1)/2$ consumers. Suppose a tax $t \neq \hat{t}_m$ is put in place. Then the utility a consumer i obtains from the new equilibrium consumption is $U_i^c(x_i(t)) + V_i^c(g(t))$. There is at least one consumer i in the coalition with $\hat{t}_i \leq \hat{t}_m$ who would be opposed to a tax increase. There is also at least one consumer j in the coalition with $\hat{t}_j \geq \hat{t}_m$ who would be against a tax decrease. It follows that coalition Ω cannot propose an alternative public good bundle and financing scheme $(g(t), t)$ such that $U_i(\mathbf{x_h}(t), g(t); h) \geq U_i(\mathbf{x_h}(\hat{t}_m), g(\hat{t}_m); h)$ for all $i \in h \in \Omega$ and $U_i(\mathbf{x_h}(t), g(t); h) > U_i(\mathbf{x_h}(\hat{t}_m), g(\hat{t}_m); h)$ for some $i \in h \in \Omega$ hold.

This completes the proof that $(p(\hat{t}_m), \mathbf{x}(\hat{t}_m), g(\hat{t}_m), \hat{t}_m; P)$ is a competitive equilibrium with new household formation, public choice and rational price expectations.

q.e.d.

The proof of Proposition 2 encompasses a median voter result. To illustrate and explore the scope of the proposition, we examine the following example.

Example 2. Let $\ell = 2, q = 1$. Assume separable preferences and let P be a household structure that consists of preferred households for all consumers. Let $U_i^c(x_i^1, x_i^2) = \ln x_i^1 + \ln x_i^2$ for all $i \in I, (x_i^1, x_1^2) \in X_i$, $V_i^c(g) = \lambda_i \ln g$ for all $i \in I$, $g \geq 0$, and let $\omega_{\{i\}} = (1,1)$ for $i \in I$. We assume heterogeneity — or diversity if you want — with respect to preference for public good consumption. Specifically, we assume $0 < \lambda_i < \lambda_j$ for $i, j \in I, i < j$. Further let $F(v^1, v^2) = v^1 v^2$ for $(v^1, v^2) \in \mathbb{R}_+^2$.

Then it is readily verified that the hypotheses of Proposition 2 are satisfied. For $t \in [0,1]$, $(p(t), \mathbf{x}(t), g(t), t; P) = ((1,1), ((1-t, 1-t), \ldots, (1-t, 1-t)), t^2N^2, t; P)$ is up to price normalization the unique competitive equilibrium (with new household formation). We obtain for $i \in I$, $U_i^c(x_i(t)) + V_i^c(t^2N^2) = 2\ln(1-t) + 2\lambda_i \ln t + 2\lambda_i \ln N$ and, consequently, $\hat{t}_i = \lambda_i/(1+\lambda_i)$. Hence a median consumer i_m with respect to the canonical order on I is a median agent with respect to the ideal tax rate so that $\hat{t}_m = \hat{t}_{i_m}$. $\square$

While the assumption of separable preferences proves very convenient, it is by no means crucial for the existence of equilibrium as the next example demonstrates.

Example 3. Suppose $\ell = 2, q = 1$. Assume pure group externalities and a household structure P that consists of preferred households for all consumers. Let $U_i(\mathbf{x_h}, g; h) = \min\{x_i^1 x_i^2, g\} + U_i^G(h)$ for all $i \in I$, $h \in \mathcal{H}_i$, $(x_i^1, x_i^2) \in X_i, g \geq 0$ and let $\omega_{\{i\}} = (1, 1)$ for $i \in I$. Further let $F(v^1, v^2) = v^2$ for $(v^1, v^2) \in \mathbb{R}_+^2$.

Put $p_1 + p_2 = 1$. For $t \in [0, 1]$ and $i \in I$ set

$$\begin{aligned} x_i^1 &= \frac{1-t}{2p_1}; \\ x_i^2 &= \frac{1-t}{2p_2}; \\ v^2 &= \frac{Nt}{p_2}; \\ v^2 &= x_i^1 x_i^2. \end{aligned}$$

Market clearing requires $x_i^1 = 1$ and $N x_i^2 + v^2 = N$. The system of equations has a unique solution $t^* = \dfrac{1}{2N+1}, p_1^* = \dfrac{N}{2N+1}, p_2^* = \dfrac{N+1}{2N+1}$, etc. It satisfies

$$x_i^1 x_i^2 = g.$$

At any tax rate $t \neq t^*$, $x_i^1 x_i^2$ or g is smaller than its equilibrium value and, consequently, U_i is smaller than its equilibrium value for some member i of each household. Hence t^*, p_1^*, p_2^*, ..., P constitute a competitive equilibrium with new household formation, public choice and rational price expectations. $\square$

The example violates separability of preferences, but still exhibits pure group externalities. Without the latter, existence of equilibrium need not obtain:

Proposition 3 *There exists an economy with $\ell = 1$, $q = 1$, $N = 3$ and semi-separable utility representations of the form*

$$U_i(\mathbf{x_h}, g; h) = \mu_h \ln x_i + V_i^c(g) + U_i^G(h) \ \text{ with } \mu_h > 0 \tag{4}$$

for $h \in \mathcal{H}_i$, $\mathbf{x_h} = (x_i)_{i \in h} \in \mathcal{X}_h$, $g \in \mathbb{R}_+^q$, that does not have a competitive equilibrium with new household formation, public choice and rational price expectations.

The proof by means of an elaborate example is given in the appendix. In that example, we demonstrate that public good provision combined with global collective decisions regarding taxation can destabilize any household structure. This can occur even if there exist favorable conditions for matching on the one hand, and for global collective decisions on the other hand. First, for a given tax rate and amount of public good, a stable matching (household structure) exists. Second, global collective decisions have Condorcet winners for any conceivable household structure. Third, consumer preferences are separable with respect to the public good and exhibit no externalities with respect to private consumption. However, the semi-separable form (4) fails to satisfy pure group preferences.

4 Equilibrium with Public Choice and Myopic Price Expectations

Conceivably, when a group of consumers contemplates an alternative public policy, consisting of a tax rate and a bundle of public goods, it may be unaware of or disregard the impact of such a change on equilibrium prices for private goods. Assuming that the equilibrium prices for private goods remain unchanged gives rise to a different equilibrium concept and possibly different equilibrium outcomes.

Definition. A state $(p, \mathbf{x}, g, t; P)$ is a **competitive equilibrium with new household formation, public choice and myopic price expectations** if it satisfies the following conditions:

1. $(p, \mathbf{x}, g)$ is a competitive equilibrium given household structure P and tax rate t.

2. No group of consumers can benefit from forming a new household.

4. There is no subcoalition Ω of households, i.e. $\Omega \subseteq P$, with at least n individuals that can propose an alternative public good bundle and financing scheme (g', t') such that for some input bundles $v'_1, \ldots, v'_q$ and some consumption plans $\mathbf{x_h}' \in \mathcal{X}_h$, $h \in \Omega$,

   4.a. $\mathbf{x_h}' \in B_h(p, t')$ for all $h \in \Omega$;

   4.b. $U_i(\mathbf{x_h}', g'; h) \geq U_i(\mathbf{x_h}, g; h)$ for all $i \in h \in \Omega$;

   4.c. $U_i(\mathbf{x_h}', g'; h) > U_i(\mathbf{x_h}, g; h)$ for some $i \in h \in \Omega$;

   4.d. $\sum_{m=1}^{q} pv'_m \leq t'p \sum_{h \in P} \omega_h$;

   4.e. $g'_m = F_m(v'_m) \quad \text{for} \quad m = 1, ..., q$.

Under certain circumstances, rational price expectations and myopic price expectations lead to the same equilibrium outcomes. One would expect that conditions 3 and 4 lead to identical competitive equilibrium states with public choice if (g,t) does not affect the competitive equilibrium prices for private goods at all. To be more specific, we say that **public choice is non-distortionary** if for any two competitive equilibria $(p,\mathbf{x},g,t;P)$ and $(p',\mathbf{x}',g',t';P)$ with the same household structure P, the market price systems p and p' are equal up to normalization, that is, there exists $\lambda > 0$ such that $p' = \lambda p$.

Proposition 4 *Suppose that public choice is non-distortionary. Then a competitive equilibrium with new household formation, public choice and myopic price expectations is also a competitive equilibrium with new household formation, public choice and rational price expectations.*

Proof. Assume public choice to be non-distortionary. Let $(p,\mathbf{x},g,t;P)$ be a competitive equilibrium with new household formation, public choice and myopic price expectations. If a subcoalition Ω of households with at least n individuals proposes an alternative public good bundle and financing scheme (g',t') with an ensuing equilibrium state $(p',\mathbf{x}',g',t';P)$, then because of non-distortionary public choice, $(p,\mathbf{x}',g',t';P)$ is a competitive equilibrium as well. Therefore, 4.a. holds, that is, $\mathbf{x_h}' \in B_h(p,t')$ for all $h \in \Omega$, and the input bundles satisfy 4.d. and 4.e. But then the combination of 4.b. and 4.c. cannot hold, since $(p,\mathbf{x},g,t;P)$ is an equilibrium with new household formation, public choice and myopic price expectations. Consequently, the combination of 3.b. and 3.c. does not hold. This shows that $(p,\mathbf{x},g,t;P)$ is a competitive equilibrium with new household formation, public choice and rational price expectations. q.e.d.

A special case of non-distortionary public choice is the case $\ell = 1$. In that case, we can set $p = 1$ and the conditions $x_i = (1-t)\omega_i$, $\sum_m v_m = t\omega_S$ yield market clearing. Under the additional assumption of the absence of consumption externalities, i.e., utility representations of the form $U_i(\mathbf{x_h},g;h) = \mathfrak{U}_i(x_i,g;h)$ for member i of household h, the converse of Proposition 4 holds:

Proposition 5 *Suppose $\ell = 1$ and absence of consumption externalities, with utility representations $U_i(\mathbf{x_h},g;h) = \mathfrak{U}_i(x_i,g;h)$ for $i \in h$. Suppose further that each $\mathfrak{U}_i(x_i,g;h)$ is strictly increasing in x_i. Then a competitive equilibrium with new household formation, public choice and rational price expectations is also a competitive equilibrium with new household formation, public choice and myopic price expectations.*

Proof. Assume $\ell = 1$ and preferences as hypothesized. Let $(p, \mathbf{x}, g, t; P)$ be a competitive equilibrium with new household formation, public choice and rational price expectations. Suppose there exists a subcoalition Ω of households with at least n individuals that proposes an alternative public good bundle and financing scheme (g', t') such that for some input bundles $v'_1, \ldots, v'_q$ and some consumption plans $\mathbf{x_h}' \in \mathcal{X}_h$, $h \in \Omega$, conditions 4.a.–4.e. hold.

Since $\ell = 1$ and the $\mathfrak{U}_i(x_i, g; h)$ are strictly increasing in private consumption, we may assume $p = 1$. The assumed monotonicity of preferences further implies that 4.a.–4.e. continue to hold if for all $i \in h \in \Omega$, x'_i is replaced by $x''_i = x'_i + ((1-t)\omega_h - \sum_{j \in h} x'_j)/|h|$ so that the budget constraints in 4.a. are binding. Moreover, set $x''_i = (1-t)\omega_i$ for all consumers outside the coalition. Then $(p, \mathbf{x}'', g')$ is a competitive equilibrium given household structure P and tax rate t' and the analogues of 3.b. and 3.c. hold, contradicting the fact that $(p, \mathbf{x}, g, t; P)$ is a competitive equilibrium with new household formation, public choice and rational price expectations. It follows that there does not exist a subcoalition as stated. Hence $(p, \mathbf{x}, g, t; P)$ is a competitive equilibrium with new household formation, public choice and myopic price expectations. q.e.d.

Under the hypothesis of Proposition 3, non-existence of a competitive equilibrium with new household formation, public choice and rational price expectations is equivalent to non-existence of a competitive equilibrium with new household formation, public choice and myopic price expectations, by Propositions 4 and 5.

In general, neither equilibrium concept implies the other as the following two examples show.

Example 4. This example has a competitive equilibrium with new household formation, public choice and rational price expectations that is not a competitive equilibrium with new household formation, public choice and myopic price expectations.

Let $N = 5, \ell = 2, q = 1$. Assume separable preferences and let P be a household structure that consists of preferred households for all consumers. Let $U_i^c(x_i^1, x_i^2) = x_i^1 x_i^2$ for all $i \in I, (x_i^1, x_i^2) \in X_i$, $V_i^c(g) = g$ for all $i \in I, g \geq 0$, and let $\omega_{\{i\}} = (1, 1)$ for $i \in I$. Further let $F(v^1, v^2) = v^1$ for $(v^1, v^2) \in \mathbb{R}_+^2$.

In a competitive equilibrium, $p_1 \geq 0$ and $p_2 \geq 0$, with at least one price different from 0. Therefore, we can normalize prices so that $p_1 + p_2 = 1$. Then for $t \in [0, 1]$:

$p(t) = ((1+t)/2, (1-t)/2)$, $g(t) = g^{max}(p(t), t) = 5t/p_1(t)$, and

$$x_i(t) = \left(\frac{1-t}{1+t}, 1 \right),$$

for $i \in I$. In the corresponding competitive equilibrium with new household formation (with household structure P), each consumer's utility from consumption is

$$x_i^1(t) x_i^2(t) + g(t) = \frac{1-t}{1+t} + \frac{5t}{(1+t)/2} = 1 + \frac{8t}{1+t}.$$

Next consider $t = 1$. $p(1) = (1, 0)$, $x_i(1) = (0, 1)$ for $i \in I$ and $g(1) = g^{max}(p(1), 1) = 5$ constitute a competitive equilibrium with new household formation (with household structure P), given the public choice $t = 1$ and $g(1) = g^{max}(p(1), 1) = 5$. A consumer's utility from consumption equals 5. Utility comparisons show that each consumer i has single-peaked preferences with respect t, with the peak at $\hat{t}_i = 1$. By essentially the same argument as in the proof of Proposition 1, it follows that $(p(1), \mathbf{x}(1), 5, 1; P)$ constitutes a competitive equilibrium with new household formation, public choice and rational price expectations.

Now let Ω comprise all households in P and consider a tax rate $t = 1 - \tau$ with τ very small but positive. On the one hand, at the prevailing price system $p(1)$, the maximal affordable amount of public good is $g^{max}(p(1), 1-\tau) = 5(1-\tau)$. On the other hand, at the prevailing price system, each consumer now has income $\tau > 0$ and can afford the amount τ of good 1 and an unlimited amount of good 2. Hence all consumers could be made better off if the competitive equilibrium prices were not affected by the reduction in the tax rate. Hence $(p(1), \mathbf{x}(1), 5, 1; P)$ is not an equilibrium with new household formation, public choice and myopic price expectations. $\square$

Example 5. This example possesses a competitive equilibrium with new household formation, public choice and myopic price expectations that is not a competitive equilibrium with new household formation, public choice and rational price expectations.

Let $N = 3, \ell = 2, q = 1$. Suppose that preferences are separable, that is consumer i has a utility representation of the form $U_i(\mathbf{x_h}, g; h) = U_i^c(x_i) + V_i^c(g) + U_i^G(h)$ for all $h \in \mathcal{H}_i, \mathbf{x_h} \in \mathcal{X}_h$, $g \in \mathbb{R}_+$. Specifically, we assume:

- $U_i^G \equiv 0$ for all $i \in I$. Hence household formation proves irrelevant. In the sequel, we assume that the household structure $P = \{\{1\}, \{2\}, \{3\}\}$ prevails.

- $U_i^c(x_i^1, x_i^2) = \min\{x_i^1, x_i^2\}$ for $i = 1, 2$, $(x_i^1, x_i^2) \in X_i$.
- $U_3^c(x_3^1, x_3^2) = x_3^1$ for $(x_3^1, x_3^2) \in X_3$.
- $\omega_{\{1\}} = \omega_{\{2\}} = (1, 1)$, $\omega_{\{3\}} = (2, 1)$.
- $F(v^1, v^2) = \min\{v^1, v^2\}$ for $(v^1, v^2) \in \mathbb{R}_+^2$.
- $V_i^c(g) = \lambda_i g$ with $\lambda_i > 0$ for all $i \in I, g \geq 0$.

Then equilibrium prices have to satisfy $(p_1, p_2) \geq 0$, $(p_1, p_2) \neq 0$, and we can work with the price normalization $p_1 + p_2 = 1$.

Next consider a tax rate $t \in [0, 1]$ and corresponding market clearing prices. We distinguish three cases.

CASE 1: $t = 1$. In that case, no market clearing price system exists.

CASE 2: $t \in [1/2, 1)$. In that case, consumers $i = 1, 2$ have income $m_i = 1-t$ and demand $x_i^1 = x_i^2 = 1 - t$, with resulting utility from private good consumption $U_i^c(x_i^1, x_i^2) = 1 - t$. Consumer 3 has income $m_3 = 1 - t + (1 - t)p_1$ and demand $x_3^1 = (1-t)(1+1/p_1), x_3^2 = 0$, with resulting utility from private good consumption $U_3^c(x_3^1, x_3^2) = (1 - t)(1 + 1/p_1)$. Tax revenue is $m = 3t + p_1 t$, yielding $v^1 = v^2 = g = m = 3t + p_1 t$. Market clearing for good 2 amounts to $2 + (1 + p_1)t = 3$ or $1 + p_1 = 1/t$ or $p_1 = 1/t - 1$. At this price, both markets are cleared.

CASE 3: $t \in [0, 1/2)$. In this case, $(p_1, p_2) = (1, 0)$ is the market clearing price system.

Now let $t = 2/3$. Then $1/t = 3/2$ so that the market clearing price system is $(p_1, p_2) = (1/2, 1/2)$. If this price system is taken for granted, but t is treated as variable, then the following utilities would obtain:

$U_i = 1 - t + \lambda_i 3.5t$ for $i = 1, 2$ and

$U_3 = 3(1 - t) + \lambda_3 3.5t$.

In the particular case $\lambda_1 = \lambda_2 = 2/7$, $\lambda_3 = 6/7$, it follows that $U_1 = U_2 = 1$ and $U_3 = 3$ whatever t. This shows that for the particular values of the λ_i and any $n \geq 1$, the state $(p, \mathbf{x}, g, t; P)$ is a competitive equilibrium with new household formation, public choice and myopic price expectations, where $p = (1/2, 1/2), \mathbf{x} = ((1/3, 1/3), (1/3, 1/3), (1, 0)), g = 7/3, t = 2/3$, and $P = \{\{1\}, \{2\}, \{3\}\}$.

Now let us assume instead that consumers anticipate the equilibrium price change caused by a change in t. In CASE 2, consumers 1 and 2 (with the particular $\lambda_i = 2/7$), can expect an equilibrium utility of $U_i = 1 - t + \frac{2}{7}(3t + p_1 t) =$

$1 - t + \frac{2}{7}(2t + 1) = \frac{9}{7} - \frac{3}{7}t$, using $1 + p_1 = 1/t$. Therefore, consumers 1 and 2 can expect a higher utility at $t' = 7/12$ than at $t = 2/3$. This shows that for $n = 2$, the state $(p, \mathbf{x}, g, t; P)$ is not a competitive equilibrium with new household formation, public choice and rational price expectations. □

5 Conclusion

We have developed a general framework to integrate four allocation mechanisms: group formation, local collective decisions, competitive exchange across groups and global collective decisions on public good provision. The integration of these allocation mechanisms reveals tensions between the stability of matching and voting. Nevertheless, median-voter type results and stable matchings emerge under suitable conditions, like in Proposition 2 and Example 2. Numerous issues deserve further scrutiny and the present set-up may inspire further work on how markets, household formation and democracy interact and impact each other. We provide two promising examples for such extensions.

First, while we have identified situations for which household formation and collective decisions destabilize each other, there may be circumstances where the opposite occurs. For instance, household formation may overcome inefficiencies resulting from a lack of markets that otherwise would have to be taken care of by the state. One example are human capital investments, for which credit markets do not function well as the borrower's pledgeable future income is limited. Another example are risks that cannot or can only partially be insured through markets such as depreciation of skills or defaults on loans. In such circumstances, household formation can overcome some of these inefficiencies. For example, members can provide loans to other members and provide insurance for each other by pooling their resources and monitoring and enforcing promises differently than in anonymous markets. In such a situation, household formation essentially substitutes for government supply of such services. While government supply is subject to instability of collective decisions, household formation substituting government intervention may increase the stability of society.

Second, besides the allocation of public goods and its financing, the government is also involved in redistributing income. That is, part of the government revenues is not used to produce public goods but is redistributed to subgroups of households. How such redistribution activities would affect household stability is unknown and left to future research.

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Appendix: Proof of Proposition 3

Consider an economy with $\ell = 1$, $q = 1$, $N = 3$, individual endowments $\omega_{\{i\}} > 0$, and semi-separable utility representations of the form

$$U_i(\mathbf{x_h}, g; h) = \mu_h \ln x_i + V_i^c(g) + U_i^G(h) \text{ with } \mu_h > 0 \tag{4}$$

for $h \in \mathcal{H}_i$, $\mathbf{x_h} = (x_i)_{i \in h} \in \mathcal{X}_h$, $g \in \mathbb{R}_+^q$, where x_i is the quantity of the private commodity consumed by agent i. Specifically, we assume

$$\begin{aligned} U_i^G(h) &= \Delta_i && \text{for } h = \{1,2\},\ i \in h; \\ U_i^G(h) &= 0 && \text{for } h = \{1\}, \{2\}, \{3\}; \\ U_i^G(h) &= -A && \text{for } h = \{1,3\}, \{2,3\}, \{1,2,3\}; \\ V_i^c(g) &= g && \text{for } i = 1,2,3. \end{aligned}$$

The parameters μ_h satisfy

$$0 < \mu_1 < \mu_{\{1,2\}} < \mu_2 = \mu_3 < \omega_S \text{ where } \mu_i := \mu_{\{i\}} \text{ for } i = 1,2,3.$$

The values of Δ_1 and Δ_2 may be positive or negative. Furthermore, we assume $A > 0$ sufficiently large so that $h = \{1, 2\}$ is the only multi-member household with a chance to be formed. We note that in the example, group membership can affect the welfare of consumer $i \in h$ in two ways, directly through the additive term $U_i^G(h)$ and indirectly through the marginal utility of private good consumption, μ_h/x_i.

Finally,

$$g = v,$$

where v is the amount of private commodities employed in the production of the public good. Public good provision is determined by simple majority voting on the tax rate t with which individuals' or households' endowments are taxed. Hence, $g = v \leq t\omega_S$ and $\sum_{i \in h} x_i = (1-t)\omega_h$. Without loss of generality we can focus on combinations (g, t) of public good provision and tax for which $g = t\omega_S$ holds. Otherwise, there would be unanimous agreement to increase public good provision until tax revenues are exhausted.

We are going to show that there exist parameter constellations for which no competitive equilibrium with new household formation, public choice and myopic price expectations exists. By Proposition 5, for those parameters, a competitive equilibrium with new household formation, public choice and rational price expectations does not exist either.

Household Decisions

We start by examining the decision problems of individuals and potential households. For ease of presentation, we normalize the price p of the private commodity to 1. A single household facing the tax rate t derives utility

$$U_i(t) := U_i((1-t)\omega_{\{i\}}, t\omega_S; \{i\}) = \mu_i \ln\left[(1-t)\omega_{\{i\}}\right] + t\omega_S.$$

We observe that the tax rate maximizing the utility of the single consumer i, denoted by t_i^*, is given by

$$t_i^* = 1 - \frac{\mu_i}{\omega_S} \in (0,1).$$

Moreover, $U_i'(t) > 0$ for $t < t_i^*$ and $U_i'(t) < 0$ for $t > t_i^*$. Thus, the preferences of single households are single-peaked.

We next investigate the local collective decision in household $\{1,2\}$ given some tax rate t and weights α and $1-\alpha$ for the utility of individual 1 and individual 2 in the household's welfare function. In the sequel we denote the specific household $\{1,2\}$ by $\widehat{h}$. The household's decision problem is described as follows:

$$\max_{\{x_1,x_2\}} \{\alpha\{\mu_{\widehat{h}} \ln x_1 + \Delta_1 + g\} + (1-\alpha)\{\mu_{\widehat{h}} \ln x_2 + \Delta_2 + g\}\}$$

s.t. $x_1 + x_2 = (1-t)(\omega_{\{1\}} + \omega_{\{2\}}) = (1-t)\omega_{\widehat{h}}$.

After substituting $x_2 = (1-t)\omega_{\widehat{h}} - x_1$, the first-order condition of the household's optimization problem amounts to

$$\alpha\mu_{\widehat{h}}\frac{1}{x_1} - (1-\alpha)\mu_{\widehat{h}}\frac{1}{(1-t)\omega_{\widehat{h}} - x_1} = 0$$

which yields

$$\begin{aligned} x_1 &= \alpha(1-t)\omega_{\widehat{h}}, \\ x_2 &= (1-\alpha)(1-t)\omega_{\widehat{h}}. \end{aligned}$$

The indirect utilities of household members as functions of the tax rate and household weights are given by

$$\begin{aligned} U_1^*(t;\widehat{h}) &= \mu_{\widehat{h}} \ln(\alpha(1-t)\omega_{\widehat{h}}) + \Delta_1 + t\omega_S, \\ U_2^*(t;\widehat{h}) &= \mu_{\widehat{h}} \ln((1-\alpha)(1-t)\omega_{\widehat{h}}) + \Delta_2 + t\omega_S. \end{aligned}$$

Again, we observe that their utility functions are single-peaked and the optimal tax rate for both individuals is

$$t_{\widehat{h}}^* = 1 - \frac{\mu_{\widehat{h}}}{\omega_S}.$$

Equilibrium Candidates

We look first at the household structure $P_1 = \{\{1,2\},\{3\}\}$.

As both members of household $\{1,2\}$ have $t^*_{\widehat{h}} = 1 - \frac{\mu_{\widehat{h}}}{\omega_S}$ as their bliss tax rate and thus constitute the median voter, the tax rate $t^*_{\widehat{h}}$ is the Condorcet winner, i.e. it obtains a majority of votes in all pairwise majority decisions against any other tax rate.

However, individuals 1 and 2 will only stay in household $\{1,2\}$ if they cannot fare better as singletons. Exit at the tax rate $t^*_{\widehat{h}}$ is not profitable if

$$\begin{aligned}
\mu_{\widehat{h}} \ln(\alpha(1-t^*_{\widehat{h}})\omega_{\widehat{h}}) + \Delta_1 + t^*_{\widehat{h}}\omega_S &\geq \mu_1 \ln((1-t^*_{\widehat{h}})\omega_{\{1\}}) + t^*_{\widehat{h}}\omega_S, \\
\mu_{\widehat{h}} \ln((1-\alpha)(1-t^*_{\widehat{h}})\omega_{\widehat{h}}) + \Delta_2 + t^*_{\widehat{h}}\omega_S &\geq \mu_2 \ln((1-t^*_{\widehat{h}})\omega_{\{2\}}) + t^*_{\widehat{h}}\omega_S
\end{aligned}$$

which yields

$$\mu_{\widehat{h}} \ln \alpha \geq \mu_1 \ln\left(\frac{\mu_{\widehat{h}}\omega_{\{1\}}}{\omega_S}\right) - \mu_{\widehat{h}} \ln\left(\frac{\mu_{\widehat{h}}\omega_{\widehat{h}}}{\omega_S}\right) - \Delta_1, \tag{5}$$

$$\mu_{\widehat{h}} \ln(1-\alpha) \geq \mu_2 \ln\left(\frac{\mu_{\widehat{h}}\omega_{\{2\}}}{\omega_S}\right) - \mu_{\widehat{h}} \ln\left(\frac{\mu_{\widehat{h}}\omega_{\widehat{h}}}{\omega_S}\right) - \Delta_2. \tag{6}$$

If and only if there exists a value $\alpha \in [0,1]$ that satisfies both equations, an equilibrium with the household structure P_1 exists.

We next examine possible equilibria with the household structure $P_2 = \{\{1\},\{2\},\{3\}\}$.

Due to our assumption $0 < \mu_1 < \mu_2 = \mu_3$, the tax rate $t^*_2 = t^*_3 = 1 - \frac{\mu_2}{\omega_S}$ is the most preferred tax rate of the median voters and thus the Condorcet winner.

At the tax rate t^*_2, individuals 1 and 2 could form the household $\widehat{h} = \{1,2\}$. Household formation is not profitable if for every welfare weight α:

$$\begin{aligned}
\mu_{\widehat{h}} \ln\left(\alpha\frac{\mu_2\omega_{\widehat{h}}}{\omega_S}\right) + \Delta_1 &\leq \mu_1 \ln\left(\frac{\mu_2\omega_{\{1\}}}{\omega_S}\right), \\
\mu_{\widehat{h}} \ln\left((1-\alpha)\frac{\mu_2\omega_{\widehat{h}}}{\omega_S}\right) + \Delta_2 &\leq \mu_2 \ln\left(\frac{\mu_2\omega_{\{2\}}}{\omega_S}\right)
\end{aligned}$$

and equivalently

$$\mu_{\widehat{h}} \ln \alpha \leq \mu_1 \ln\left(\frac{\mu_2\omega_{\{1\}}}{\omega_S}\right) - \mu_{\widehat{h}} \ln\left(\frac{\mu_2\omega_{\widehat{h}}}{\omega_S}\right) - \Delta_1, \tag{7}$$

$$\mu_{\widehat{h}} \ln(1-\alpha) \leq \mu_2 \ln\left(\frac{\mu_2\omega_{\{2\}}}{\omega_S}\right) - \mu_{\widehat{h}} \ln\left(\frac{\mu_2\omega_{\widehat{h}}}{\omega_S}\right) - \Delta_2. \tag{8}$$

Non-Existence of Equilibria

We next show that there exist parameter constellations for which no equilibrium exists. We proceed in several steps.

Step 1: We start from a set of parameters such that conditions (5) and (6) both hold with equality for some value $\widehat{\alpha} \in (0,1)$, $\widehat{\alpha} \approx 1$. This is always possible as the number of parameters is larger than 2, Δ_1 and Δ_2 can be freely chosen and A can be chosen such that no other household structures than P_1 and P_2 are candidates for equilibria. We observe that the no-exit conditions cannot hold simultaneously for any other value $\alpha \neq \widehat{\alpha}$.

Step 2: Let us choose $\widetilde{\Delta}_1 = \Delta_1 - \varepsilon_1$ for some arbitrarily small ε_1 with $\varepsilon_1 > 0$. Then, both conditions (5) and (6) cannot hold simultaneously when Δ_1 is replaced by $\widetilde{\Delta}_1 = \Delta_1 - \varepsilon_1$ for any value of $\alpha \in [0,1]$ and thus an equilibrium with $P_1 = \{\{1,2\},\{3\}\}$ cannot exist.

Step 3: We next show that also an equilibrium with $P_2 = \{\{1\},\{2\},\{3\}\}$ may not exist for the values $\widetilde{\Delta}_1$ and Δ_2 of the direct group externalities.

From Step 1 and Step 2 we obtain:

$$\begin{aligned}
\mu_{\widehat{h}} \ln \widehat{\alpha} &= \mu_1 \ln \mu_{\widehat{h}} - \mu_{\widehat{h}} \ln \mu_{\widehat{h}} + \mu_1 \ln\left(\frac{\omega_{\{1\}}}{\omega_S}\right) - \mu_{\widehat{h}} \ln\left(\frac{\omega_{\widehat{h}}}{\omega_S}\right) - \widetilde{\Delta}_1 - \varepsilon_1, \\
\mu_{\widehat{h}} \ln(1-\widehat{\alpha}) &= \mu_2 \ln \mu_{\widehat{h}} - \mu_{\widehat{h}} \ln \mu_{\widehat{h}} + \mu_2 \ln\left(\frac{\omega_{\{2\}}}{\omega_S}\right) - \mu_{\widehat{h}} \ln\left(\frac{\omega_{\widehat{h}}}{\omega_S}\right) - \Delta_2.
\end{aligned}$$

Hence, conditions (7) and (8) can be written as:

$$\begin{aligned}
\mu_{\widehat{h}} \ln \alpha - \mu_1 \ln \mu_2 + \mu_{\widehat{h}} \ln \mu_2 &\leq \mu_{\widehat{h}} \ln \widehat{\alpha} - \mu_1 \ln \mu_{\widehat{h}} + \mu_{\widehat{h}} \ln \mu_{\widehat{h}} + \varepsilon_1, \\
\mu_{\widehat{h}} \ln(1-\alpha) - \mu_2 \ln \mu_2 + \mu_{\widehat{h}} \ln \mu_2 &\leq \mu_{\widehat{h}} \ln(1-\widehat{\alpha}) - \mu_2 \ln \mu_{\widehat{h}} + \mu_{\widehat{h}} \ln \mu_{\widehat{h}}.
\end{aligned}$$

Hence, in order to show that an equilibrium with $P_2 = \{\{1\},\{2\},\{3\}\}$ does not exist, it is sufficient to show that there exists $\alpha \in [0,1]$ such that

$$\begin{aligned}
\mu_{\widehat{h}} \ln\left(\frac{\widehat{\alpha}}{\alpha}\right) + \mu_1 \ln\left(\frac{\mu_2}{\mu_{\widehat{h}}}\right) - \mu_{\widehat{h}} \ln\left(\frac{\mu_2}{\mu_{\widehat{h}}}\right) + \varepsilon_1 &< 0, \\
\mu_{\widehat{h}} \ln\left(\frac{1-\widehat{\alpha}}{1-\alpha}\right) + \mu_2 \ln\left(\frac{\mu_2}{\mu_{\widehat{h}}}\right) - \mu_{\widehat{h}} \ln\left(\frac{\mu_2}{\mu_{\widehat{h}}}\right) &< 0.
\end{aligned}$$

If a value α exists that fulfills both inequalities, both individuals $i = 1$ and $i = 2$ have a strict incentive to form household $\widehat{h} = \{1,2\}$ at the tax rate t_2^*. We choose $\mu_2 = \mu_{\widehat{h}} + \varepsilon_2$ for some arbitrarily small ε_2 ($\varepsilon_2 > 0$). Then, the conditions become

$$\begin{aligned} \mu_{\widehat{h}} \ln\left(\frac{\widehat{\alpha}}{\alpha}\right) - (\mu_{\widehat{h}} - \mu_1) \ln\left(1 + \frac{\varepsilon_2}{\mu_{\widehat{h}}}\right) + \varepsilon_1 &< 0, \\ \mu_{\widehat{h}} \ln\left(\frac{1-\widehat{\alpha}}{1-\alpha}\right) + \varepsilon_2 \ln\left(1 + \frac{\varepsilon_2}{\mu_{\widehat{h}}}\right) &< 0. \end{aligned}$$

For sufficiently small $\dfrac{\varepsilon_2}{\mu_{\widehat{h}}}$, these two conditions are implied by the following two conditions:

$$\mu_{\widehat{h}} \ln\left(\frac{\widehat{\alpha}}{\alpha}\right) - (\mu_{\widehat{h}} - \mu_1) \left(\frac{\varepsilon_2}{\mu_{\widehat{h}}}\right)^2 + \varepsilon_1 < 0, \tag{9}$$

$$\mu_{\widehat{h}} \ln\left(\frac{1-\widehat{\alpha}}{1-\alpha}\right) + \frac{\varepsilon_2^2}{\mu_{\widehat{h}}} < 0. \tag{10}$$

The reason is that $\eta > \ln(1+\eta) > \eta^2$ for $\eta \in (0, 1/4)$. Now choose $\mu_{\widehat{h}} = 1$, $\mu_1 = 1/16$, $\widehat{\alpha} = 9/10$, $\varepsilon_2^2 = 1/40$, $\alpha = \widehat{\alpha} - (1/4) \cdot \varepsilon_2^2 > 4/5$, $\varepsilon_1 = (1/8) \cdot \varepsilon_2^2$. Then

$$\begin{aligned} & \mu_{\widehat{h}} \ln\left(\frac{\widehat{\alpha}}{\alpha}\right) - (\mu_{\widehat{h}} - \mu_1) \left(\frac{\varepsilon_2}{\mu_{\widehat{h}}}\right)^2 + \varepsilon_1 \\ = \; & \ln(\widehat{\alpha}) - \ln(\alpha) - (15/16) \cdot \varepsilon_2^2 + (1/8) \cdot \varepsilon_2^2 \\ < \; & \ln'(4/5) \cdot (\widehat{\alpha} - \alpha) - (15/16) \cdot \varepsilon_2^2 + (1/8) \cdot \varepsilon_2^2 \\ = \; & (5/4) \cdot (1/4) \cdot \varepsilon_2^2 - (15/16) \cdot \varepsilon_2^2 + (1/8) \cdot \varepsilon_2^2 \\ = \; & -(1/2) \cdot \varepsilon_2^2 < 0, \end{aligned}$$

hence (9), and

$$\begin{aligned} & \mu_{\widehat{h}} \ln\left(\frac{1-\widehat{\alpha}}{1-\alpha}\right) + \frac{\varepsilon_2^2}{\mu_{\widehat{h}}} \\ = \; & \ln(1-\widehat{\alpha}) - \ln(1-\alpha) + \varepsilon_2^2 \\ < \; & \ln'(1-\widehat{\alpha}) \cdot ((1-\widehat{\alpha}) - (1-\alpha)) + \varepsilon_2^2 \\ = \; & 10 \cdot (-(1/4) \cdot \varepsilon_2^2) + \varepsilon_2^2 = -\tfrac{3}{2} \cdot \varepsilon_2^2 < 0, \end{aligned}$$

hence (10).

q.e.d.