Evaluation of decision power in multi-dimensional rules

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Abstract

This work deals with the evaluation of decision power in Multi-dimensional rules. Courtin and Laruelle [2020] introduced a decision process that specifies the collective acceptance or rejection of a proposal with several dimensions. The decision process is modeled as follows: (i) There are several individuals. (ii) There are several dimensions. (iii) Each of the individuals expresses a binary choice (”Yes” or ”No”) on each dimension. (iv) A decision process maps each choice to a final binary decision (”Yes” or ”No”). We extend and characterize six well-known power indices within this context: the Shapley-Shubik index (Shapley and Shubik [1954]), the Banzhaf index (Banzhaf [1965]), the Public good index (Holler [1982]), the Null individual free index (Alonso-Meijide et al. [2011]), the Shift index (Alonso-Meijide and Freixas [2010]) and the Deegan-Packel index (Deegan and Packel [1978]).

KEYWORDS: Multi-dimensional rules; Power index; Shapley-Shubik index; Banzhaf index; Public good index; Deegan-Packel index.

JEL Classification: C71, D71

1 Introduction

At university, various subjects are taught throughout different periods. A student in a Master degree in economics (four periods) will take several tests, mostly in Microeconomics, Macroeconomics, Industrial organization, Game theory... The students'

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success depends mainly on their various test scores. The final results are also correlated with the students' progress from one year of study to the next. Can a period be compensated by another one? Are there any compulsory tests? Are the subjects (periods) equally weighted? In a parliament, when a bill is under discussion, the members have to express their choices on various sections of the bill. The decision process is key in the final enforcement of the bill. Are some sections more "important" than others? Do all the sections have to be adopted in order for the bill to be adopted? In different fields, the decision process is not a one-dimensional one[1] since individuals express their choices on many questions simultaneously.

In this paper, we follow the model of Courtin and Laruelle [2020] who introduced multi-dimensional rules. The decision process is modeled as follows: (i) There are several individuals. (ii) There are several dimensions. (iii) Each individuals expresses a binary choice (1 or 0) on each dimension. (iv) A decision process maps each choice to a final binary decision (1 or 0) . The university example can be modeled by the following multi-dimensional rules: (i) the individuals are the subject which are supposed to be all taught in the different periods, (ii) the dimensions are the different periods, (iii) the result of the test in each subject and each period gives us the binary choice, (iv) the student does or does not obtain her degree. The associated multi-dimensional rules of the parliament example are such that: (i) the individuals are the members of parliament, (ii) the dimensions are the different sections of the bill, (iii) the vote of each member on each section gives us the binary choice, (iv) the output is dichotomous, i.e. the bill is or is not passed.

As a real-life example, one can present the parliamentary vote of the French government budget. The vote is organized in two steps: expenses are first approved or rejected ministry after ministry, the search for a majority agreement on the overall expenses and tax revenues being made afterwards. Here the individuals are the members of the parliament, the dimensions are the different ministry, the dichotomous nature of choice refers to a "yes-no" position on each partial budget. The decision process is the following: as a first step, the parliament's members make their global opinion on each partial budget (dimension) by simple majority and, in the second step, the global opinions of the parliaments members are aggregated by simple majority and the government budget is or is not adopted.

Courtin and Laruelle [2020] studied some of the properties of multi-dimensional

[1] see Andjiga et al. [2003] and Laruelle and Valenciano [2008] for a detailed description of these one-dimensional processes.

rules, such as separability and weightedness. Separability refers to rules that can be decomposed by unidimensional rules, whereas weightedness concerns rules where each binary choice can be weighted.

Our aim, in this paper, is different. We focus on how a given individual can affect the final result. Given a student's passing grades rule, how is "game theory" an important subject? Is a given member of parliament more influential than another? To answer these questions we use the tools that are power indices. Power indices evaluate the decision power of a given individual or a given dimension. In a single-dimensional context, Holler [1982] divided the different power indices into two categories: private and public. The common factor in the first category is that, once formed, the value of a coalition of individuals is shared among the coalition members even though it was created collectively. The amount of power received by individuals of a given coalition can be different. The second category assumes that the value of the coalition cannot be seen as a "private good" but as a "public good". This means that once a coalition is achieved, all the individuals must be treated equally regardless of their own or of others' manner of contribution (with the exception of those who do not contribute). In other words, individuals can have different contributions but share the same amount of power.

In this paper we extend these two categories to multi-dimensional rules. In particular, we consider two well-known private indices, the Shapley-Shubik index (Shapley and Shubik [1954]) and the Banzhaf index (Banzhaf [1965]). We also take into consideration three public indices: the Public good index (Holler [1982]), the Null individual free index (Alonso-Meijide et al. [2011]) and the Shift index (Alonso-Meijide and Freixas [2010]). We also extend the Deegan-Packel index (Deegan and Packel [1978]) which is an index behind the private indices and the public indices. A full characterization of each of these six power indices is provided. Our characterization results can be seen as straightforward extensions of characterization results for the one-dimensional framework. Since the beginning of the use of game theory to study the distribution of power in voting systems, a wide collection of studies providing different games, different power indices notions have been developed. We understand our approach as a complementary step in this research line.

This paper is structured as follows. Section 2 introduces the general framework of multi-dimensional rules. Section 3 defines and characterizes the Shapley-Shubik index and the Banzhaf index for multi-dimensional rules, while the Deegan-Packel index and the public index for multi-dimensional rules category is discussed in Section 4. Section 5 concludes.

2 General Framework

Let $N = \{1,...,n\}$ be a set of n individuals (generic individuals will be denoted by i, j, k), and let $M = \{1,...,m\}$ be a set of m dimensions (generic dimensions will be denoted by a, b, c). Each individual faces a binary choice and casts a positive or a negative decision on each dimension. Let $s_i^a \in \{0,1\}$ be the decision cast by individual i on dimension a with $s_i^a = 1(0)$ when i casts a positive (negative) decision. Let $\mathbb{Z}_n^m$ denote the set of $n \times m$ matrices of 0 and 1. A configuration is a matrix $\mathbf{S} \in \mathbb{Z}_n^m$ where the element of row i and column a is s_i^a:

$$\mathbf{S} = \begin{pmatrix} s_1^1 & & s_1^m \\ & & \\ & & \\ s_n^1 & & s_n^m \end{pmatrix}.$$

We denote by s the number of positive votes in the configuration $\mathbf{S}$. Let $\mathbf{S}$ and $\mathbf{T}$ be two configurations then, we will write $\mathbf{S} \leq \mathbf{T}$ if $s_i^a \leq t_i^a$ for any $i \in N$, $a \in M$. Matrix $\mathbf{I}_n^m$ is the $n \times m$ matrix of 1, and $\mathbf{O}_n^m$ is the $n \times m$ matrix of 0.

A multi-dimensional rule specifies the configurations of $\mathbb{Z}_n^m$ that lead to a positive final decision and those that lead to a negative one.

Definition 1. *A multi-dimensional rule with n voters and m dimensions is a function*

$$\begin{aligned} \mathbf{w}: \quad & \mathbb{Z}_n^m \to \{0,1\} \\ & \mathbf{S} \longmapsto \mathbf{w}(\mathbf{S}) \end{aligned}$$

with: $\mathbf{w}(\mathbf{I}_n^m) = 1$; $\mathbf{w}(\mathbf{O}_n^m) = 0$; $\mathbf{w}(\mathbf{S}) = 1 \Rightarrow \mathbf{w}(\mathbf{T}) = 1$ *for all* $\mathbf{S} \leq \mathbf{T}$.

We denote by $\mathbb{W}_n^m$ the set of all multi-dimensional rules. We refer to the configurations that lead to a positive outcome as winning configurations, the other ones being referred to as losing configurations. The set of winning configurations for a rule $\mathbf{w}$ is denoted by $\mathcal{W}(\mathbf{w})$. A configuration $\mathbf{S}$ is minimal winning if $\mathbf{w}(\mathbf{S}) = 1$ and for all configurations $\mathbf{T} \neq \mathbf{S}$ with $\mathbf{T} \leq \mathbf{S}$ we have $\mathbf{w}(\mathbf{T}) = 0$. Note that a given multi-dimensional rule $\mathbf{w}$ can also be represented by its set of minimal winning configurations, denoted $\mathcal{M}(\mathbf{w})$.

Given two rules $\mathbf{w}, \bar{\mathbf{w}} \in \mathbb{W}_n^m$, we have:

1. rule $\mathbf{w} \vee \bar{\mathbf{w}}$ (*max rule*) defined as $(\mathbf{w} \vee \bar{\mathbf{w}})(\mathbf{S}) = max\{\mathbf{w}(\mathbf{S}), \bar{\mathbf{w}}(\mathbf{S})\}$
2. rule $\mathbf{w} \wedge \bar{\mathbf{w}}$ (*min rule*) defined as $(\mathbf{w} \wedge \bar{\mathbf{w}})(\mathbf{S}) = min\{\mathbf{w}(\mathbf{S}), \bar{\mathbf{w}}(\mathbf{S})\}$

Note that $\mathcal{W}(\mathbf{w} \vee \bar{\mathbf{w}}) = \mathcal{W}(\mathbf{w}) \cup \mathcal{W}(\bar{\mathbf{w}})$ and $\mathcal{W}(\mathbf{w} \wedge \bar{\mathbf{w}}) = \mathcal{W}(\mathbf{w}) \cap \mathcal{W}(\bar{\mathbf{w}})$

One can also present a specific class of multi-dimensional rules, the one of multi-dimensional unanimity rules.

Definition 2. *For all configurations $\mathbf{S} \neq \mathbf{O}_n^m$, the multi-dimensional unanimity rule $\mathbf{u_S}$ with n players is defined as*

$$\mathbf{u_S}(\mathbf{T}) = \begin{cases} 1 & if\, \mathbf{S} \leqslant \mathbf{T} \\ 0 & otherwise \end{cases}$$

Note that whenever $s = 1$, $\mathbf{u_S}$ is called dictatorial.

The following proposition shows that multi-dimensional rules can be derived from a combination of multi-dimensional unanimity rules.

Proposition 1. *Let $\mathbf{w}$ be a multi-dimensional rule with $\mathcal{M}(\mathbf{w}) = \{\mathbf{S_1}, ..., \mathbf{S_k}\}$ the set of minimal winning configurations. $\mathbf{w}$ can be expressed as $\mathbf{w} = u_{\mathbf{S_1}} \vee ... \vee u_{\mathbf{S_k}}$.*

Proof. Consider a configuration $\mathbf{T} \in \mathbb{Z}_n^m$.

Suppose that $\mathbf{w}(\mathbf{T}) = 1$ then, there exists at least one minimal winning configuration $\mathbf{S_l} \leqslant \mathbf{T}$. Therefore, $\mathbf{u_{S_l}}(\mathbf{T}) = 1$ and $\mathbf{w}(\mathbf{T}) = u_{\mathbf{S_1}}(\mathbf{T}) \vee ... \vee u_{\mathbf{S_k}}(\mathbf{T}) = 1$

Suppose that $\mathbf{w}(\mathbf{T}) = 0$ then, for all $l = 1, ..., k$, $\mathbf{S_l} \not\leqslant \mathbf{T}$ which implies that $\mathbf{u_{S_l}}(\mathbf{T}) = 0$ for all $l = 1, ..., k$. Therefore $\mathbf{w}(\mathbf{T}) = u_{\mathbf{S_1}}(\mathbf{T}) \vee ... \vee u_{\mathbf{S_k}}(\mathbf{T}) = 0$ □

Before presenting our main results, we introduce the notion of power indices.

Definition 3. *Let $\mathbf{w}$ be a multi-dimensional rule, a power index is a mapping ϕ which assigns to the multi-dimensional rule $\mathbf{w}$ a vector $\phi(\mathbf{w}) = (\phi_1(\mathbf{w}), ..., \phi_n(\mathbf{w}))$.*

For a given individual i, $\phi_i(\mathbf{w})$ can be seen as the power/influence of i over the rule $\mathbf{w}$. We make the choice of presenting the evaluation of power from the individual perspective. However, the power indices present in this paper will also be relevant in evaluating the power for each dimension.

We called i^a a representative which is designated by each individual i for each dimension a. Thus, there are $n.m$ representatives. The set of all the representatives will be denoted by N^*. We define $\phi_{i^a}(\mathbf{w})$ as the power index of the representative i^a of individual i in dimension a. Since we assume that all the representatives of an individual of any dimension are *a priori* equally powerful, we are $\phi_i(\mathbf{w}) = \sum_{a=1}^{m} \phi_{i^a}(\mathbf{w})$. This is a particular class of power indices, other less restrictive classes could be explored in further research.

Finally, a representative i^a is called decisive in a configuration $\mathbf{S}$ if $[\mathbf{w}(\mathbf{S}) - \mathbf{w}(\mathbf{S} - i^a)] = 1$ with $\mathbf{S} - i^a = (\mathbf{T} \in \mathbb{Z}_n^m : t_i^a = 0, s_i^a = 1 \text{ and } t_j^a = s_j^a \text{ for all } a \text{ and for all } j \neq i)$ [2]. We denote by $D_i^a(\mathbf{w})$ the number of configurations where i^a is decisive.

3 Private index for multi-dimensional rules

In this section, we extend the classical Shapley-Shubik index and the Banzhaf index to multi-dimensional rules. As previously mentioned, in a single-dimension context, for these two indices, the power can be inequitably distributed in a given coalition, unlike the public good indices. Another similarity of these two indices, is that they only consider the formation of winning coalitions as relevant. The benefit of a winning coalition is then assigned to decisive individuals. A decisive individual changes a coalition from a losing one to a winning one. The more an individual is decisive, the bigger her influence and the greater the power she derives. We apply this approach to define two new indices for multi-dimensional rules and we outline an axiomatic approach for these two indices.

3.1 Shapley-Shubik index

In the same spirit as Shapley and Shubik [1954], the Shapley-Shubik index for multi-dimensional rules is based on the notion of permutation.

1. We suppose that all of the representatives of a player are randomly queuing up to enter a room to make a decision. All the permutations of these representatives are possible, *i.e* there are $(n.m)!$ ways in which the representatives can be ordered in the queue.

2. When a representative i^a enters the room, she expresses the binary choice of the individual i she represents for this dimension a.

[2]We also have $\mathbf{S} - \{i^a, j^b\} = (\mathbf{T} \in \mathbb{Z}^{n \times m} : t_i^a = t_j^b = 0, s_i^a = s_j^b = 1 \text{ and } t_k^c = s_k^c \text{ for all } k^c \neq i^a, j^b)$.

3. When a representative of individual i for a dimension a enters and expresses a positive vote which allows the configuration to be a winning one, this representative is then decisive.

4. The representatives who already expressed a positive vote before the decisive representative do not have enough votes to reach a decision ($(s-1)!$ orderings), and the representatives who come after the decisive representative do not really count ($(n.m-s)!$ orderings)

5. Thus, each representative receives an amount of power depending on the number of times she is decisive, when all the orders in the queue have the same probability.

6. Finally, the Shapley-Shubik index of an individual i is simply the sum of the power received by each representative of this individual.

Definition 4. *Let* $\mathbf{w}$ *be a multi-dimensional rule, the Shapley-Shubik index for a player* $i \in N$ *is given by:*

$$SH_i(\mathbf{w}) = \sum_{a \in M} \overline{SH}_{i^a}$$

with

$$\overline{SH}_{i^a}(\mathbf{w}) = \sum_{\mathbf{S} \in \mathbb{Z}_n^m} \frac{(s-1)!(n.m-s)!}{(n.m)!} [\mathbf{w}(\mathbf{S}) - \mathbf{w}(\mathbf{S} - i^a)]$$

We assume here that no representative of an individual i has more weight than any other. A given representative can bring more power than another solely on account of the framework of the multi-dimensional rule. We also notice that for a single dimension ($|m| = 1$), the Shapley-Shubik index for a multi-dimensional rule is equivalent to the Shapley-Shubik index introduced by Shapley and Shubik [1954].

Example 1. *Consider a university program in which three subjects ($n = 3$) are taught over two periods ($m = 2$). The process of going on to the next year is the following: for the first subject, the first-period test is compulsory and at least two positive tests in the second period are needed. More formally,* $\mathbf{w} : \mathbb{Z}_2^3 \rightarrow \{0, 1\}$ *with*

$$\mathbf{w}(\mathbf{S}) = \begin{cases} 1 \textit{ if } s_1^1 = 1 \textit{ and } s_i^2 = 1 \textit{ for at least two } i \\ 0 \textit{ otherwise.} \end{cases}$$

That is,

$$\mathcal{M}(\mathbf{w}) = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}.$$

For period 1 ($a = 1$), we have $\overline{SH}_{1^1}(\mathbf{w}) = \frac{1}{2}$ and $\overline{SH}_{2^1}(\mathbf{w}) = \overline{SH}_{3^1}(\mathbf{w}) = 0$.

For period 2 ($a = 2$), we have $\overline{SH}_{1^2}(\mathbf{w}) = \overline{SH}_{2^2}(\mathbf{w}) = \overline{SH}_{3^2}(\mathbf{w}) = \frac{1}{6}$.

Then $SH_1(\mathbf{w}) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$ and $SH_2(\mathbf{w}) = SH_3(\mathbf{w}) = \frac{1}{6}$.

The first subject has more impact than the other ones since it is compulsory for the first period.

On the efficiency property

Axiom 1. *(Efficiency)*
For all $\mathbf{w} \in \mathbb{W}_n^m$,

$$\sum_{i=1}^{n} \phi_i(\mathbf{w}) = \sum_{i=1}^{n} \sum_{a=1}^{m} \phi_{i^a}(\mathbf{w}) = 1$$

Axiom 1 states that the total aggregated power of the individual is the same in any multi-dimensional rule, and also equals one.

On the null individual property

An individual is *a null individual* if the decision she makes never makes any difference. More formally, for all $\mathbf{S} \in \mathbb{Z}_n^m$, let us define $\bar{\mathbf{S}} \in \mathbb{Z}_n^m$ with $\bar{s}_i^a = 0$ for all $a \in M$ and $\bar{s}_j^a = s_j^a$ for all $a \in M$ and for all $j \neq i$. i is a null individual if $\mathbf{S} \in \mathcal{W}(\mathbf{w}) \Leftrightarrow \bar{\mathbf{S}} \in \mathcal{W}(\mathbf{w})$. Note also that we call a representative i^a a null representative if, in any minimal winning configuration, she always casts a null vote; *i.e.* if $s_{i^a} = 0$ for all $\mathbf{S} \in \mathcal{M}(\mathbf{w})$. Thus an individual is *a null individual* if all of her representatives are null representatives.

Axiom 2. *(Null individual)*
For all $\mathbf{w} \in \mathbb{W}_n^m$, and any individual $i \in N$, if i is a null individual for rule $\mathbf{w}$ then,

$$\phi_i(\mathbf{w}) = 0.$$

For a null individual i, since all its representatives are null representatives, an implication of this axiom is that $\phi_{i^a}(\mathbf{w}) = 0$ for each $a \in M$.

According to Axiom 2, an individual or a representative whose action has no influence on the final decision, has a null power.

On the anonymity property

For all $\mathbf{S}$, let $\tilde{\mathbf{S}}_{i\leftrightarrow j}^{a\leftrightarrow b} \in \mathbb{Z}_n^m$ denotes a configuration where representative i^a and representative j^b permute their decisions: $\tilde{s}_k^c = s_k^c$ for any $k \neq i, j$ and for any $c \neq a, b$ and $\tilde{s}_i^a = s_j^b$ and $\tilde{s}_j^b = s_i^a$.
Let π be any permutation of the set N^* then, the multi-dimensional rule $\pi\mathbf{w}$ is given by $\pi\mathbf{w}(\mathbf{S}) = \mathbf{w}(\pi\mathbf{S})$ where $\pi\mathbf{S} = \left\{\tilde{\mathbf{S}}_{i\leftrightarrow j}^{a\leftrightarrow b} \in \mathbb{Z}_n^m \text{ for any } a, b \text{ and for any } i, j\right\}$.

Axiom 3. *(Anonymity)*
Let $\mathbf{w} \in \mathbb{W}_n^m$. For any permutation π of N^*, and any representative $i^a \in N^*$,

$$\phi_{\pi(i^a)}(\pi\mathbf{w}) = \phi_{i^a}(\mathbf{w})$$

Axiom 3 says that the amount that a representative receives does not depend on her label or relative position in N^*.

On the transfer property

We extend to multi-dimensional rules the transfer axiom introduced by Dubey and Shapley [1979] which reflects a principle of linearity.

Axiom 4. *(Transfer)*
For all $\mathbf{w}$, $\bar{\mathbf{w}} \in \mathbb{W}_n^m$

$$\phi(\mathbf{w}) + \phi(\bar{\mathbf{w}}) = \phi(\mathbf{w} \vee \bar{\mathbf{w}}) + \phi(\mathbf{w} \wedge \bar{\mathbf{w}})$$

When considering two multi-dimensional rules, Axiom 4 means that the total influence of an individual is simply the sum of influences availables in the min rule and in the max rule.

The Shapley-Shubik index for multi-dimensional rules is the only index that simultaneously satisfies all four previous axioms.

Theorem 1. *Let ϕ be an index for multi-dimensional rules. Then, ϕ satisfies Axiom 1, Axiom 2, Axiom 3 and Axiom 4 if and only if $\phi = SH$.*

Before introducing the proof of Theorem 1, we present the following Lemma.

Lemma 1. *Let $\mathbf{w}$ be a multi-dimensional rule and let $\mathbf{u_S}$ be the corresponding multi-dimensional unanimity rule. If ϕ satisfies Axiom 1, Axiom 2 and Axiom 3 then, $\phi_{i^a}(\mathbf{u_S})$ is unique and for any $i \in N$ and any $a \in M$ is given by*

$$\phi_{i^a}(\mathbf{u_S}) = \begin{cases} \frac{1}{s} \ if \ s_i^a \neq 0 \\ 0 \ otherwise \end{cases}$$

Proof. Let $\mathbf{w}$ be a multi-dimensional rule and let $\mathbf{u_S}$ a unanimity multi-dimensional rule associated to a configuration $\mathbf{S}$.

Every individual i such that $s_i^a = 0$ for all $a \in M$ is a null individual in $\mathbf{u_S}$. Thus each of her representative i^a is a null representative in $\mathbf{u_S}$. Therefore, by Axiom 2, $\phi_{i^a}(\mathbf{u_S}) = 0$ for such i and each $a \in M$.

If π is the permutation that interchanges i^a and j^b, for some $i, j \in N$ and some $a, b \in N$ such that $s_i^a = s_j^b = 1$, and the other votes remains unchanged then, $\pi\mathbf{u_S} = \mathbf{u_S}$ and thus, by Axiom 3, it follows that $\phi_{j^b}(\pi\mathbf{u_S}) = \phi_{j^b}(\mathbf{u_S}) = \phi_{i^a}(\mathbf{u_S})$.

Therefore, ϕ_i^a is uniquely determined if ϕ exists. Using Axiom 1, for each $a \in M$ and for each $i \in N$, $\phi_{i^a}(\mathbf{u_S}) = \begin{cases} \frac{1}{s} \text{ if } s_i^a \neq 0 \\ 0 \text{ otherwise} \end{cases}$

□

We can now present the proof of Theorem 1.

Proof. Theorem 1

Uniqueness

Let $\mathbf{w}$ be a multi-dimensional rule and $\mathcal{M}(\mathbf{w}) = \{\mathbf{S_1}, \ldots, \mathbf{S_k}\}$ be the set of minimal winning configurations. For all configurations $\mathbf{S_l}$, we define the unanimity multi-dimensional rule $\mathbf{u_{S_l}}$ by $\mathbf{u_{S_l}}(\mathbf{T}) = \begin{cases} 1 & \text{if } \mathbf{S_l} \in \mathbf{T} \\ 0 & otherwise \end{cases}$. By Proposition 1 $\mathbf{w}$ can be written as $\mathbf{w} = u_{\mathbf{S_1}} \vee \ldots \vee u_{\mathbf{S_k}}$.

In order to prove the uniqueness of ϕ , an induction on the number of minimal winning configurations and on their cardinality is performed.

Let $\eta(\mathbf{w}) = \min\{q \in Z^+ |\text{there exists a minimal winning configuration } \mathbf{S} \text{ such that } s = q\}$ and let $\bar{\eta}(\mathbf{w})$ be the number of minimal winning configurations $\mathbf{S}$ such that $s = \eta(\mathbf{w})$

.

Step 1: $\eta(\mathbf{w}) = n.m$

For $\eta(\mathbf{w}) = n.m$, the only winning configuration for the rule $\mathbf{w}$ is $\mathbf{I}_n^m$ then, $\mathbf{I}_n^m$ is the unique minimal winning configuration and, therefore, $\mathbf{w} = \mathbf{u}_{\mathbf{I}_n^m}$. By Lemma 1, $\phi_i^a(\mathbf{u}_{\mathbf{I}_n^m}) = \frac{1}{n.m}$ which implies that $\phi(\mathbf{w})$ is obviously unique.

Step 2: $\eta(\mathbf{w}) < n.m$

In order to prove that $\phi_i(\mathbf{w})$ is unique when there are minimal winning coalitions with $s < n.m$, we consider two cases.

Case A: Suppose that $\phi(\mathbf{w})$ is unique for all $\mathbf{w}$ such that $\eta(\mathbf{w}) > q$ and $\bar{\eta}(\mathbf{w}) = 1$. We will see that it is also unique for $\mathbf{w}$ when $\eta(\mathbf{w}) = q$ and $\bar{\eta}(\mathbf{w}) = 1$. Let $\mathbf{S_q}$ be such a configuration with cardinality q.

a) If $\mathbf{S_q}$ is the only minimal winning configuration then, $\mathbf{w} = \mathbf{u_{S_q}}$. By Lemma 1, we know that for any $i \in N$ and any $a \in M$, $\phi_{i^a}(\mathbf{u_{S_q}}) = \phi_{i^a}(\mathbf{w}) = \begin{cases} \frac{1}{s_q} \text{ if } s_{q_i}^{a} = 1 \\ 0 \text{ otherwise} \end{cases}$ which implies that $\phi(\mathbf{w})$ is uniquely determined.

b) If there are any other minimal winning configurations $S_1, ..., S_k$ such that $s_l > q$ for all $1 \leq l \leq k$ then, by Proposition 1, $\mathbf{w} = u_{\mathbf{S_1}} \vee ... \vee u_{\mathbf{S_k}} \vee u_{\mathbf{S_q}}$ which can also be written $\mathbf{w} = \mathbf{w}' \vee u_{\mathbf{S_q}}$ with $\mathbf{w}' = u_{\mathbf{S_1}} \vee ... \vee u_{\mathbf{S_k}}$. It follows that $\eta(\mathbf{w}') > q$ then, $\phi(\mathbf{w}')$ is unique by the induction hypothesis.

Now consider $\mathbf{w}' \wedge u_{\mathbf{S_q}}$. By the definition of $\wedge$, it is obvious that $\eta(\mathbf{w}' \wedge u_{\mathbf{S_q}}) > q$. Therefore, $\phi(\mathbf{w}' \wedge u_{\mathbf{S_q}})$ is also unique by the induction hypothesis.

Applying Axiom 4 leads to $\phi(\mathbf{w}) = \phi(\mathbf{w}' \vee u_{\mathbf{S_q}}) = \phi(\mathbf{w}') + \phi(u_{\mathbf{S_q}}) - \phi(\mathbf{w}' \wedge u_{\mathbf{S_q}})$. Since all the right terms are unique then, $\phi(\mathbf{w})$ is unique.

Case B: Suppose that $\phi(\mathbf{w})$ is unique for all $\mathbf{w}$ such that $[\eta(\mathbf{w}) > q]$ or $[\eta(\mathbf{w}) = q$ and $\bar{\eta}(\mathbf{w}) = h, h \neq 1]$. We will show that it is also unique for $\mathbf{w}$ when $\eta(\mathbf{w}) = q$ and $\bar{\eta}(\mathbf{w}) = h + 1$.

Let S_1, ..., S_{h+1} such that $s_l = q$ for all $1 \leq l \leq h+1$ and let T_1, ..., T_p be other minimal winning configurations with $t_l > q$ for all $1 \leq l \leq p$.

By Proposition 1, $\mathbf{w} = u_{\mathbf{T_1}} \vee ... \vee u_{\mathbf{T_p}} \vee u_{\mathbf{S_1}} \vee ... \vee u_{\mathbf{S_h}} \vee u_{\mathbf{S_{h+1}}}$ which can also be expressed as $\mathbf{w} = \mathbf{w}'' \vee u_{\mathbf{S}_{h+1}}$ with $\mathbf{w}'' = u_{\mathbf{T_1}} \vee ... \vee u_{\mathbf{T_p}} \vee u_{\mathbf{S_1}} \vee ... \vee u_{\mathbf{S_h}}$.

From the definition of $\vee$, we determine that there are h minimal winning configurations with cardinalities equal to q for $\mathbf{w}''$, *i.e.* $\eta(\mathbf{w}'') = q$ and $\bar{\eta}(\mathbf{w}'') = h$ meaning that $\phi(\mathbf{w}'')$ is unique.

Similarly, from the definition of $\wedge$, it appears that $\eta(\mathbf{w}'' \wedge u_{\mathbf{S}_{h+1}}) > q$ and then, $\phi(\mathbf{w}'' \wedge u_{\mathbf{S}_{h+1}})$ is unique. By Axiom 4, we have $\phi(\mathbf{w}) = \phi(\mathbf{w}'' \vee u_{\mathbf{S_{h+1}}}) = \phi(\mathbf{w}'') + \phi(\mathbf{S_{h+1}}) - \phi(\mathbf{w}'' \wedge u_{\mathbf{S_{h+1}}})$. Since all three vectors on the right-hand side are unique, so is $\phi(\mathbf{w})$.

Therefore, we have proved that $\phi(\mathbf{w})$ is unique for any feasible number of, *i.e* for all $\mathbf{w} \in \mathbb{W}^{n \times m}$

Existence

It is obvious that SH satisfies the efficiency, the null player, the symmetry and the transfer axioms.

$\square$

3.2 Banzhaf index

Unlike the Shapley-Shubik index, the Banzhaf index for multi-dimensional rules depends on the number of combinations, rather than the number of permutations of the representative players. When a representative i^a expresses a positive decision which allows for a configuration to be winning, then this representative is decisive. Thus, each representative receives a share of power based on the number of times she is decisive over the total number of configurations where she expresses a positive decision ($2^{m.n-1}$ configurations where i^a expresses a positive decision). The Banzhaf index of individual i is the sum of the power received by each of her representatives.

Definition 5. *Let* $\mathbf{w}$ *be a multi-dimensional rule, the Banzhaf index for a player* $i \in N$ *is given by :*

$$BZ_i(\mathbf{w}) = \sum_{a \in M} \overline{BZ}_{i^a}$$

with

$$\overline{BZ}_{i^a}(\mathbf{w}) = \sum_{\mathbf{S} \in \mathbb{Z}_n^m} \frac{1}{2^{n.m-1}} [\mathbf{w}(\mathbf{S}) - \mathbf{w}(\mathbf{S} - i^a)] = \frac{D_i^a(\mathbf{w})}{2^{n.m-1}}$$

Note that for $|m| = 1$, the Banzhaf index for multi-dimensional rules is equivalent to the Banzhaf index for a simple rule (one dimension).

Example 2. *We compute the Banzhaf index for the previous example.*
For $a = 1$, *we have* $\overline{BZ}_{1^1}(\mathbf{w}) = \frac{15}{32}$ *and* $\overline{BZ}_{2^1}(\mathbf{w}) = \overline{BZ}_{3^1}(\mathbf{w}) = 0$.
For $a = 2$, *we have* $\overline{BZ}_{1^2}(\mathbf{w}) = \overline{BZ}_{2^2}(\mathbf{w}) = \bar{B}Z_{3^2}(\mathbf{w}) = \frac{8}{32}$.
Then $BZ_1(\mathbf{w}) = \frac{15}{32} + \frac{8}{32} = \frac{23}{32}$ *and* $BZ_2(\mathbf{w}) = BZ_3(\mathbf{w}) = \frac{8}{32}$

Since the Banzhaf index does not satisfy the efficiency axiom, we introduce a new axiom.

On the efficiency* property

Axiom 5. *(Efficiency*)*
For all $\mathbf{w} \in \mathbb{W}_n^m$,

$$\sum_{i=1}^{n} \phi_i(\mathbf{w}) = \sum_{i=1}^{n} \sum_{a=1}^{m} \phi_{i^a}(\mathbf{w}) = \frac{1}{2^{n.m-1}} \sum_{i=1}^{n} \sum_{a=1}^{m} D_i^a(\mathbf{w}) = \frac{\overline{D_i^a(\mathbf{w})}}{2^{n.m-1}}$$

with $\overline{D_i^a(\mathbf{w})}$ the total number of decisive representative individuals.

According to Axiom 5 the total influence is distributed among the individuals according to the total number of decisive representative individuals.

Theorem 2. *Let ϕ be an index for multi-dimensional rules. Then, ϕ satisfies Axiom 2, Axiom 3, Axiom 4 and Axiom 5 if and only if $\phi = BZ$.*

Lemma 2. *Let $\mathbf{w}$ be a multi-dimensional rule and let $\mathbf{u_S}$ be the corresponding multi-dimensional unanimity rule. If ϕ satisfies Axiom 2, Axiom 3, and Axiom 5 then, $\phi_i^a(\mathbf{u_S})$ is unique and for any $i \in N$ and any $a \in M$ is given by*

$$\phi_{i^a}(\mathbf{u_S}) = \begin{cases} \frac{1}{2^{s-1}} \textit{ if } s_i^a \neq 0 \\ 0 \textit{ otherwise} \end{cases}$$

Proof. The beginning of the proof is similar to the proof of Lemma 1 and omitted. Only the last sentence of the proof of Lemma 1 is replaced by: Using Axiom 5, for each $a \in M$ and for each $i \in N$, $\phi_{i^a}(\mathbf{u_S}) = \begin{cases} \frac{\overline{D_i^a(\mathbf{w})}}{s.2^{n.m-1}} = \frac{2^{n.m-s}}{2^{n.m-1}} = \frac{1}{2^{s-1}} \\ 0 \text{ otherwise} \end{cases}$.

□

Proof. Theorem 2

Lemma 1 will be replaced by Lemma 2 in the proof. The rest of the proof is similar to that of Theorem 1 and omitted.

□

In one dimensional context, Lehrer [1988] introduces the two-efficiency property in order to characterize the Banzhaf index.

On the two efficiency property

To extend this property in the context of multi-dimensional rule one need to introduce two new rules.

Let $S = \{i^a \in N^*, s_i^a = 1\}$ be a set of representatives derived from a configuration $\mathbf{S} \in \mathbb{Z}_n^m$, and $S \subseteq N^*$. Let Z_{n^*} be the set of all set of representatives. A representative rule with n^* representatives is a function

$$\begin{aligned} w : \quad & Z_{n^*} \to \{0,1\} \\ & S \longmapsto w(S) \end{aligned}$$

such that for all S in N^* derived from $\mathbf{S}$, $w(S) = \mathbf{w}(\mathbf{S})$.
For i^a and j^b in N^*, $i^a \neq j^b$, let $w_{i^a j^b}$ the amalgamated rule obtained from w when representative i^a and j^b amalgamate in a new representative $p = \{i^a, j^b\}$, *i.e.* $N^*_{i^a,j^b} = \{N^* \setminus \{i^a, j^b\}) \cup \{p\}\}$ and for every $S \subseteq N^*_{i^a,j^b}$

$$w_{i^a j^b}(S) = \begin{cases} w(S) \text{ if } p \notin S \\ w((S \setminus p) \cup \{i^a, j^b\}) \text{ otherwise} \end{cases}$$

Axiom 6. *(Two-efficiency)*
For all $\mathbf{w} \in \mathbb{W}^m_n$ and pair of representative i^a, j^b in N^*

$$\phi_{i^a}(\mathbf{w}) + \phi_{j^b}(\mathbf{w}) = \phi_p(w_{i^a j^b})$$

with $\phi_p(w_{i^a j^b})$ the power index of p in the game $w_{i^a j^b}$.

Axiom 6 describes how the power should behave when two representatives of the original game are not allowed to act in-dependently anymore. Indeed, this axiom state that the power of the representatives should be immune against such changes.

Theorem 3. *Let ϕ be an index for multi-dimensional rules. Then, ϕ satisfies Axiom 2, Axiom 3, Axiom 4, and Axiom 6 if and only if $\phi = BZ$.*

Lemma 3. *Let $\mathbf{w}$ be a multi-dimensional rule and let $\mathbf{u_S}$ be the corresponding multi-dimensional unanimity rule. If ϕ satisfies Axiom 2, Axiom 3, Axiom 4, and Axiom 6 then, $\phi^a_i(\mathbf{u_S})$ is unique and for any $i \in N$ and any $a \in M$ is given by*

$$\phi_{i^a}(\mathbf{u_S}) = \begin{cases} \frac{1}{2^{s-1}} \textit{ if } s^a_i \neq 0 \\ 0 \textit{ otherwise} \end{cases}$$

Proof. Let ϕ satisfies Axiom 2, Axiom 3, Axiom 4, and Axiom 6. We proceed by two inductions, first on the number of representatives $|N^*|$ and second on the number s of positive votes on a configuration $\mathbf{S} \in \mathbb{Z}^m_n$.

Suppose first that ϕ is unique for all unanimity multi-dimensional rule with $|N^*| \leq nm$ and all $s \leq nm$. Secondly suppose that ϕ is unique for all unanimity multi-dimensional rule with $|N^*| = nm + 1$,(denoted $\mathbf{u}^{+1}_{\mathbf{S}}$) and $s \leq k$, and determined for any $i \in N$ and for any $a \in M$ by

$$\phi^a_i(\mathbf{u}^{+1}_{\mathbf{S}}) = \begin{cases} \frac{1}{2^{s-1}} \text{ if } s^a_i \neq 0 \\ 0 \text{ otherwise} \end{cases}$$

We will see that ϕ is also determined for all unanimity multi-dimensional rule with $|N^*| = nm+1$ and $s = k+1$. Let $\bar{\mathbf{S}}$ be its corresponding minimal winning configuration and assume that for representative i^a, $\bar{s}^a_i = 1$, and let $\tilde{\mathbf{S}} = \bar{\mathbf{S}} - i^a$.

By Transfer, $\phi(\mathbf{u}^{+1}_{\tilde{\mathbf{S}}} \vee \mathbf{u}^{+1}_{\mathbf{i^a}}) + \phi(\mathbf{u}^{+1}_{\tilde{\mathbf{S}}} \wedge \mathbf{u}^{+1}_{\mathbf{i^a}}). = \phi(\mathbf{u}^{+1}_{\tilde{\mathbf{S}}}) + \phi(\mathbf{u}^{+1}_{\mathbf{i^a}})$. Note that $(\mathbf{u}^{+1}_{\tilde{\mathbf{S}}} \wedge \mathbf{u}^{+1}_{\mathbf{i^a}}) = \mathbf{u}^{+1}_{\bar{\mathbf{S}}}$.

By null representative and anonymity there are constants x, y, z such that for each $a \in M$, and each $j \in N$, $\phi^a_j(\mathbf{u}^{+1}_{\bar{\mathbf{S}}}) = \begin{cases} x \text{ if } \bar{s}^a_j \neq 0 \\ 0 \text{ otherwise} \end{cases}$ and $\phi^a_j(\mathbf{u}^{+1}_{\tilde{\mathbf{S}}} \vee \mathbf{u}^{+1}_{\mathbf{i^a}}) = \begin{cases} y \text{ if } \tilde{s}^a_j \neq 0, j \neq i \\ z \text{ if } \tilde{s}^a_j \neq 0, j = i \\ 0 \text{ otherwise} \end{cases}$

By the induction hypothesis and the two previous equations we have $x + y = \frac{1}{2^{\tilde{s}-1}} = \frac{1}{2^{k-1}}$ and by null representative $x + z = 1$.

We apply now two-efficiency. First take two representatives i^a and j^b from $\bar{S}$ the representative set obtained from $\bar{\mathbf{S}}$ and amalgamate them to one representative. The amalgamated rule obtained is $w_{i^a j^b} = \mathbf{u}_{\tilde{\mathbf{S}}}$, (after identifying j^b, where $\tilde{s}^b_j = 1$ with the new representative). Hence by two-efficiency $\phi^a_i(\mathbf{u}^{+1}_{\bar{\mathbf{S}}}) + \phi^b_j(\mathbf{u}^{+1}_{\bar{\mathbf{S}}}) = \phi_{j^b}(\mathbf{u}_{\tilde{\mathbf{S}}})$, then $2x = \frac{1}{2^{\tilde{s}-1}} = \frac{1}{2^{k-1}}$.

Now in the game $(\mathbf{u}^{+1}_{\tilde{\mathbf{S}}} \vee \mathbf{u}^{+1}_{\mathbf{i^a}})$, amalgamate the representative i^a with any representative j^b from $\tilde{S}$ the representative set obtained from $\tilde{\mathbf{S}}$. The amalgamated rule obtain is dictatorial then $z + y = 1$.

Therefore we have $2x + y + z = 1 + \frac{1}{2^{k-1}}$ All the previous inequalities permit us to conclude that $2x = \frac{1}{2^{k-1}}$, and therefore $x = \frac{1}{2^k}$ which determine ϕ^a_i on $\mathbf{u}^{+1}_{\bar{\mathbf{S}}}$.

□

Proof. Theorem 3

Existence

BZ satisfies the null player, the anonymity and the transfer axioms. Let us prove that BZ satisfies the two-effciciency axiom.

Let $\mathbf{w}$ be a multi-dimensional rule, then by the definition of $\overline{BZ}$ we have

$$= 2^{nm-1}[\overline{BZ}_{i^a}(\mathbf{w}) + \overline{BZ}_{j^b}(\mathbf{w})]$$

$$= \sum_{\mathbf{S} \in \mathbb{Z}^m_n} [\mathbf{w}(\mathbf{S}) - \mathbf{w}(\mathbf{S} - i^a)] + \sum_{\mathbf{S} \in \mathbb{Z}^{n \times m},} [\mathbf{w}(\mathbf{S}) - \mathbf{w}(\mathbf{S} - j^b)]$$

$$= \sum_{\mathbf{S} \in \mathbb{Z}^m_n, s_{j^b}=1} [\mathbf{w}(\mathbf{S}) - \mathbf{w}(\mathbf{S} - i^a) + \mathbf{w}(\mathbf{S} - j^b) - \mathbf{w}(\mathbf{S} - \{i^a, j^b\})] + \sum_{\mathbf{S} \in \mathbb{Z}^{n \times m}, s_{i^a}=1} [\mathbf{w}(\mathbf{S}) - \mathbf{w}(\mathbf{S} - j^b) + \mathbf{w}(\mathbf{S} - i^a) - \mathbf{w}(\mathbf{S} - \{i^a, j^b\})] \, .$$

$$= 2 \sum_{\mathbf{S} \in \mathbb{Z}^m_n, s^a_i = s^b_j = 1} [\mathbf{w}(\mathbf{S}) - \mathbf{w}(\mathbf{S} - \{i^a, j^b\})]$$

$= 2.2^{nm-2}\overline{BZ}_p(w_{i^a,j^b})= 2^{nm-1}\overline{BZ}_p(w_{i^a,j^b})$ for $p = \{i^a, j^b\}$ which concludes the proof.

Uniqueness

Lemma 1 will be replaced by Lemma 3 in the proof. The rest of the proof is similar to that of Theorem 1 and omitted.

□

4 Public index for multi-dimensional rules

For a single-dimension context, in Barry [1980] and Holler [1982], it has been argued that private indices are inadequate measures since a coalition formation is not about sharing benefits but about achieving a decision result which cannot be obtained by separate individuals. Therefore, the category of public indices assigns the same amount of power to the coalition members of a coalition that was created collectively, unlike the private indices in which inequalities between members can arise. The question is not to distinguish between the individuals in a coalition but rather to determine which winning coalitions take into account the evaluation of power. In the one-dimension literature, three types of winning coalitions have been considered: minimal (Holler [1982]), shift (Alonso-Meijide et al. [2011]) and null individual free (Alonso-Meijide and Freixas [2010]).

4.1 Public good index

Holler [1982] introduced the "Public good" index by assuming that only the minimal winning coalitions are relevant in evaluating the power. A winning coalition is a minimal winning coalition when the removal of any of its individuals would prevent the coalition from being a winning one. Intuitively, an individual, who is not needed to raise the result above a certain threshold, has no influence on the final decision. Furthermore, she has no incentive to be rewarded. We can extend this argument to multi-dimensional rules.

Definition 6. *Let* $\mathbf{w}$ *be a multi-dimensional rule, the Public good index for a player* $i \in N$ *is given by:*

$$PG_i(\mathbf{w}) = \sum_{a\in M} \overline{PG}_{i^a}$$

with

$$\overline{PG}_{i^a}(\mathbf{w}) = \frac{|\mathcal{M}_{i^a}(\mathbf{w})|}{\sum_{i^a \in N^*} |\mathcal{M}_{i^a}(\mathbf{w})|}.$$

with $\mathcal{M}_{i^a}(\mathbf{w})$ *being the set of minimal winning configurations where representative* i^a *makes a positive vote.*

The *PG* for multi-dimensional rules is then based on the following principles:

1. The configurations that are not minimal are irrelevant when it comes to measuring power. Indeed, in some winning configurations, a representative might feel unhappy because she thinks that she could obtain more power by excluding some other representative.

2. Since all the representatives belong to a configuration, not adding a second principle would lead to giving equal power to all the representatives regardless of the rule. We choose to give power only to representatives who express a positive vote, following Holler [1982]' approach. Each of them receives one unit of power.

3. A given representative will be rewarded according to the total number of minimal winning configurations she belongs to, normalized by the sum of these numbers for all representatives.

4. The sum of all the power received by the representatives of a given individual gives us the final power of this individual.

Example 3. *We apply the* PG *for multi-dimensional rules to the first example.*

For $a = 1$, *we have* $\overline{PG}_{1^1}(\mathbf{w}) = \frac{3}{9}$ *and* $\overline{PG}_{2^1}(\mathbf{w}) = \overline{PG}_{3^1}(\mathbf{w}) = 0$.

For $a = 2$, *we have* $\overline{PG}_{1^2}(\mathbf{w}) = \overline{PG}_{2^2}(\mathbf{w}) = \overline{PG}_{3^2}(\mathbf{w}) = \frac{2}{9}$.

Then, $PGI_1(\mathbf{w}) = \frac{3}{9} + \frac{2}{9} = \frac{5}{9}$ *and* $PGI_2(\mathbf{w}) = PGI_3(\mathbf{w}) = \frac{2}{9}$

Subjects 2 *and* 3 *are more influential than with Shapley since they are rewarded by belonging to minimal winning configurations.*

It is obvious that the *PG* for multi-dimensional rules satisfies the efficiency, the null individual and the anonymity properties, but not the transfer property. We extend the Merger axiom (see Holler [1982]) to multi-dimensional rules in order to fully characterize this index.

On the merger property

Let $\mathbf{w}$ and $\bar{\mathbf{w}}$ be two multi-dimensional rules. The rules $\mathbf{w}$ and $\bar{\mathbf{w}}$ are said to be mergeable if, for all $\mathbf{S} \in \mathcal{M}(\mathbf{w})$, it holds that $\mathbf{S} \notin \mathcal{W}(\bar{\mathbf{w}})$ and for all $\mathbf{S} \in \mathcal{M}(\bar{\mathbf{w}})$ it holds that $\mathbf{S} \notin \mathcal{W}(\mathbf{w})$. In words, $\mathbf{w}$ and $\bar{\mathbf{w}}$ are mergeable if each minimal winning configuration related to one of the two rules is also a minimal winning configuration for the max rule $\mathbf{w} \vee \bar{\mathbf{w}}$.

Axiom 7. *(Merger)*
Let $\mathbf{w}$, $\bar{\mathbf{w}} \in \mathbb{W}_n^m$, $\mathbf{w}$ *and* $\bar{\mathbf{w}}$ *mergeable implies that for all* $i^a \in N*$,

$$\phi_{i^a}(\mathbf{w} \vee \bar{\mathbf{w}}) = \frac{\sum\limits_{i^a \in N^*} |\mathcal{M}_{i^a}(\mathbf{w})|}{\sum\limits_{i^a \in N^*} |\mathcal{M}_{i^a}(\mathbf{w} \vee \bar{\mathbf{w}})|} \phi_{i^a}(\mathbf{w}) + \frac{\sum\limits_{i^a \in N^*} |\mathcal{M}_{i^a}(\bar{\mathbf{w}})|}{\sum\limits_{i^a \in N^*} |\mathcal{M}_{i^a}(\mathbf{w} \vee \bar{\mathbf{w}})|} \phi_{i^a}(\bar{\mathbf{w}})$$

Axiom 7 states that the amount of power of a representative in a union rule is a weighted mean of its power in the two component rules, where the weights come from the number of minimal winning configurations in each component rule.

Theorem 4. *Let ϕ be an index for multi-dimensional rules. Then ϕ satisfies Axiom 1, Axiom 2, Axiom 3 and Axiom 7 if and only if $\phi = PG$.*

Proof. **Uniqueness**

Let ϕ be an index that satisfies Axiom 1, Axiom 2, Axiom 3, and Axiom 7 . We shall prove that $\phi = PG$. In order to do so, let $\mathbf{w}$ be a multi-dimensional rule. We will proceed by induction on the number of minimal winning configurations of $\mathbf{w}$.

Let us first assume that $|\mathcal{M}(\mathbf{w})| = 1$. $\mathbf{S}$ is the only minimal winning configuration then, $\mathbf{w} = \mathbf{u_S}$. By Lemma 1, we know that for any $i \in N$ and any $a \in M$, $\phi_{i^a}(\mathbf{u_S}) = \phi_{i^a}(\mathbf{w}) = \begin{cases} \frac{1}{s} \text{ if } s_i^a = 1 \\ 0 \text{ otherwise} \end{cases}$, which is equivalent to $\overline{PG}_{i^a}(\mathbf{w})$. Thus, $PGI_i(\mathbf{w}) = \phi_i(\mathbf{w})$.

Let $k > 1$ and assume that $PGI_{i^a}(\mathbf{w}) = \phi_{i^a}(\mathbf{w})$ holds for all rules $\mathbf{w}$ for which there are at most $k-1$ minimal winning configurations. Consider a rule $\mathbf{w}$ with k minimal winning configurations, say $\mathbf{S_1}, \ldots, \mathbf{S_k}$. The rules $\mathbf{w_1} \vee \ldots \vee \mathbf{w_{k-1}}$ and $\mathbf{w_k}$ are mergeable, where $\mathbf{w_l}$, $\mathbf{l} = \mathbf{1}, \ldots \mathbf{k}$ is the multi-dimensional rule with the single minimal winning configuration $\mathbf{S_l}$. Furthermore, $\mathbf{w} = [\mathbf{w_1} \vee \ldots \vee \mathbf{w_{k-1}}] \vee \mathbf{w_k}$ and it follows, from Axiom 6 that

$$
\begin{aligned}
\phi_{i^a}(\mathbf{w}) &= \frac{\sum\limits_{i^a\in N^*}|\mathcal{M}_{i^a}(\mathbf{w_1}\vee\ldots\vee\mathbf{w_{k-1}})|}{\sum\limits_{i^a\in N^*}|\mathcal{M}_{i^a}(\mathbf{w})|}\phi_{i^a}(\mathbf{w_1}\vee\ldots\vee\mathbf{w_{k-1}}) + \frac{\sum_{i^a\in N^*}|\mathcal{M}_{i^a}(\mathbf{w_k})|}{\sum\limits_{i^a\in N^*}|\mathcal{M}_{i^a}(\mathbf{w})|}\phi_{i^a}(\mathbf{w_k})\\
&= \frac{\sum\limits_{i^a\in N^*}|\mathcal{M}_{i^a}(\mathbf{w_1}\vee\ldots\vee\mathbf{w_{k-1}})|}{\sum\limits_{i^a\in N^*}|\mathcal{M}_{i^a}(\mathbf{w})|}\overline{PG}_{i^a}(\mathbf{w_1}\vee\ldots\vee\mathbf{w_{k-1}}) + \frac{\sum\limits_{i^a\in N^*}|\mathcal{M}_{i^a}(\mathbf{w_k})|}{\sum\limits_{i^a\in N^*}|\mathcal{M}_{i^a}(\mathbf{w})|}\overline{PG}_{i^a}(\mathbf{w_k})\\
&\quad \text{(thanks to the induction hypothesis)}\\
&= \overline{PG}_{i^a}([\mathbf{w_1}\vee\ldots\vee\mathbf{w_{k-1}}]\vee\mathbf{w_k}) \text{ (because } \overline{PG} \text{ satisfies Axiom 7)}\\
&= \overline{PG}_{i^a}(\mathbf{w})
\end{aligned}
$$

In conclusion, for all multi-dimensional rules $\mathbf{w}$ and all players $i^a \in N^*$, $\phi_{i^a}(\mathbf{w}) = \overline{PG}_{i^a}(\mathbf{w})$. Therefore, $\phi_i(\mathbf{w}) = PGI_i(\mathbf{w})$ for all $i \in N$.

Existence

It is obvious that PG satisfies Axiom 1, Axiom 2 and Axiom 3,

Let us prove that PG satisfies Axiom 7 . Let $\mathbf{w}$ and $\bar{\mathbf{w}}$ be two multi-dimensional rules in $\mathbb{W}_n^m$ and let $i \in N$. If $\mathbf{w}$ and $\bar{\mathbf{w}}$ are mergeable then, $\mathcal{M}_{i^a}(\mathbf{w}\vee\bar{\mathbf{w}}) = \mathcal{M}_{i^a}(\mathbf{w}) \cup \mathcal{M}_{i^a}(\bar{\mathbf{w}})$ for each representative i^a of individual i, and $\mathcal{M}_{i^a}(\mathbf{w}) \cap \mathcal{M}_{i^a}(\bar{\mathbf{w}}) = \emptyset$, which implies that $|\mathcal{M}_{i^a}(\mathbf{w}\vee\bar{\mathbf{w}})| = |\mathcal{M}_{i^a}(\mathbf{w})| + |\mathcal{M}_{i^a}(\bar{\mathbf{w}})|$. Therefore,

$$
\begin{aligned}
\overline{PG}_{i^a}(\mathbf{w}\vee\bar{\mathbf{w}}) = \frac{|\mathcal{M}_{i^a}(\mathbf{w}\vee\bar{\mathbf{w}})|}{\sum\limits_{i^a\in N^*}|\mathcal{M}_{i^a}(\mathbf{w}\vee\bar{\mathbf{w}})|} &= \frac{|\mathcal{M}_{i^a}(\mathbf{w})|+|\mathcal{M}_{i^a}(\bar{\mathbf{w}})|}{\sum\limits_{i^a\in N^*}|\mathcal{M}_{i^a}(\mathbf{w}\vee\bar{\mathbf{w}})|}\\
&= \frac{|\mathcal{M}_{i^a}(\mathbf{w})|}{\sum\limits_{i^a\in N^*}|\mathcal{M}_{i^a}(\mathbf{w}\vee\bar{\mathbf{w}})|} + \frac{|\mathcal{M}_{i^a}(\bar{\mathbf{w}})|}{\sum\limits_{i^a\in N^*}|\mathcal{M}_{i^a}(\mathbf{w}\vee\bar{\mathbf{w}})|}\\
&= \frac{\overline{PG}_{i^a}(\mathbf{w}).\sum_{i^a\in N^*}|\mathcal{M}_{i^a}(\mathbf{w})|}{\sum\limits_{i^a\in N^*}|\mathcal{M}_{i^a}(\mathbf{w}\vee\bar{\mathbf{w}})|} + \frac{\overline{PG}_{i^a}(\bar{\mathbf{w}}).\sum\limits_{i^a\in N^*}|\mathcal{M}_{i^a}(\bar{\mathbf{w}})|}{\sum\limits_{i^a\in N^*}|\mathcal{M}_{i^a}(\mathbf{w}\vee\bar{\mathbf{w}})|}\\
&= \frac{\sum\limits_{i^a\in N^*}|\mathcal{M}_{i^a}(\mathbf{w})|}{\sum\limits_{i^a\in N^*}|\mathcal{M}_{i^a}(\mathbf{w}\vee\bar{\mathbf{w}})|}\overline{PG}_{i^a}(\mathbf{w}) + \frac{\sum\limits_{i^a\in N^*}|\mathcal{M}_{i^a}(\bar{\mathbf{w}})|}{\sum\limits_{i^a\in N^*}|\mathcal{M}_{i^a}(\mathbf{w}\vee\bar{\mathbf{w}})|}\overline{PG}_{i^a}(\bar{\mathbf{w}})
\end{aligned}
$$

□

When there is only one dimension, Alonso-Meijide et al. [2008] charaterizes the *PG* index by the *PG* minimal monotonicity property, which can be extended to the multi-dimensional case in the following way.

On *PG*-minimal monotonicity property

Axiom 8. (PG-*minimal monotonicity)*
Let $\mathbf{w}$, $\bar{\mathbf{w}} \in \mathbb{W}_n^m$, *for all representatives* $i^a \in N^*$ *such that* $\mathcal{M}_{i^a}(\bar{\mathbf{w}}) \subseteq \mathcal{M}_{i^a}(\mathbf{w})$,

$$\phi_{i^a}(\mathbf{w}) \sum_{i^a \in N^*} |\mathcal{M}_{i^a}(\mathbf{w})| \geq \phi_{i^a}(\bar{\mathbf{w}}) \sum_{i^a \in N^*} |\mathcal{M}_{i^a}(\bar{\mathbf{w}})|$$

According to Axiom 8, if the set of minimal winning configurations including a representative i^a in rule $\bar{\mathbf{w}}$ is a subset of the set of minimal winning configurations including the same representative i^a in another multi-dimensional rule $\mathbf{w}$ then, the power of i^a in $\mathbf{w}$ is no lesser than the power of i^a in $\bar{\mathbf{w}}$. In other words, if i^a improves her position for a multi-dimensional rule, her influence must increase.

Theorem 5. *Let ϕ be an index for multi-dimensional rules. Then, ϕ satisfies Axiom 1, Axiom 2, Axiom 3 and Axiom 8 if and only if $\phi = PG$.*

Proof. **Uniqueness**

Let ϕ be an index that satisfies Axiom 1, Axiom 2, Axiom 3 and Axiom 8. Let $\mathbf{w}$ be a multi-dimensional rule. Again, we will proceed by induction on the number of minimal winning configurations of $\mathbf{w}$.

If $|\mathcal{M}(\mathbf{w})| = 1$, $\phi_{i^a}(V) = \overline{PG}_{i^a}(\mathbf{w})$.

Now let us assume that $PGI_{i^a}(\mathbf{w}) = \phi_{i^a}(\mathbf{w})$ holds for all rules $\mathbf{w}$ for which there are at most $k-1$ minimal winning configurations and consider a rule $\mathbf{w}$ with $k > 1$ minimal winning configurations, say $\mathbf{S_1}, ..., \mathbf{S_k}$. Take $L = \{i^a \in N^* | s_{l\,i^a} = 1 \text{ for all } l = 1, ..., k\}$.

First case: $i^a \notin L$

For each i^a, let us define the multi-dimensional rule $\mathbf{w_0}$ by $\mathcal{M}(\mathbf{w_0}) = \mathcal{M}_{i^a}(\mathbf{w})$. Then, by definition of $\mathbf{w_0}$, $\mathcal{M}_{i^a}(\mathbf{w_0}) = \mathcal{M}_{i^a}(\mathbf{w})$, applying Axiom 8 leads twice to $\phi_{i^a}(\mathbf{w_0}) \sum_{i^a \in N^*} |\mathcal{M}_{i^a}(\mathbf{w_0})| = \phi_{i^a}(\mathbf{w}) \sum_{i^a \in N^*} |\mathcal{M}_{i^a}(\mathbf{w})|$. Since $i^a \notin L$, the multi-dimensional rule $\mathbf{w_0}$ has less than k minimal winning configurations. Thus, $\phi_{i^a}(\mathbf{w_0}) = \overline{PG}_{i^a}(\mathbf{w_0})$ due to the induction hypothesis. On the other hand, as PG satisfies Axiom 8, we have $\overline{PG}_{i^a}(\mathbf{w_0}) \sum_{i^a \in N^*} |\mathcal{M}_{i^a}(\mathbf{w_0})| = \overline{PG}_{i^a}(\mathbf{w}) \sum_{i^a \in N^*} |\mathcal{M}_{i^a}(\mathbf{w})|$.

We conclude that $\phi_{i^a}(\mathbf{w}) \sum_{i^a \in N^*} |\mathcal{M}_{i^a}(\mathbf{w})| = \phi_{i^a}(\mathbf{w_0}) \sum_{i^a \in N^*} |\mathcal{M}_{i^a}(\mathbf{w_0})| = \overline{PG}_{i^a}(\mathbf{w_0}) \sum_{i^a \in N^*} |\mathcal{M}_{i^a}(\mathbf{w_0})| = \overline{PG}_{i^a}(\mathbf{w}) \sum_{i^a \in N^*} |\mathcal{M}_{i^a}(\mathbf{w})|$ which implies $\phi_{i^a}(\mathbf{w}) = \overline{PG}_{i^a}(\mathbf{w})$.

Second case: $i^a \in L$

By Axiom 3 , $\phi_{i^a}(\mathbf{w}) = \phi_{j^b}(\mathbf{w})$ and $\overline{PG}_{i^a}(\mathbf{w}) = \overline{PG}_{j^b}(\mathbf{w})$ for all $i^a, j^b \in L$; and, furthermore, for all $i^a \notin L$, $\phi_{i^a}(\mathbf{w}) = \overline{PG}_{i^a}(\mathbf{w})$. By Axiom 1 (of ϕ and PG), we have: $\sum_{i^a \in L} \phi_{i^a}(\mathbf{w}) + \sum_{i^a \notin L} \phi_{i^a}(\mathbf{w}) = 1 = \sum_{i^a \in L} \overline{PG}_{i^a}(\mathbf{w}) + \sum_{i^a \notin L} \overline{PG}_{i^a}(\mathbf{w})$. This implies that $\sum_{i^a \in L} \phi_{i^a}(\mathbf{w}) = \sum_{i^a \in L} \overline{PG}_{i^a}(\mathbf{w})$, that is, $|L|.\phi_{i^a}(\mathbf{w}) = |L|.\overline{PG}_{i^a}(\mathbf{w})$ and finally, $\phi_{i^a}(\mathbf{w}) = \overline{PG}_{i^a}(\mathbf{w})$.

Existence

We know that the PG index satisfies Axiom 1, Axiom 2 and Axiom 3. In addition, PG satisfies Axiom 8 because if i^a is a representative such that $\mathcal{M}_{i^a}(\bar{\mathbf{w}}) \subseteq \mathcal{M}_{i^a}(\mathbf{w})$, then $|\mathcal{M}_{i^a}(\bar{\mathbf{w}})| \leq |\mathcal{M}_{i^a}(\mathbf{w})|$ which means that $\overline{PG}_{i^a}(\bar{\mathbf{w}}) \sum_{i^a \in N^*} |\mathcal{M}_{i^a}(\bar{\mathbf{w}})| \leq \overline{PG}_{i^a}(\mathbf{w}) \sum_{i^a \in N^*} |\mathcal{M}_{i^a}(\mathbf{w})|$ □

4.2 Null individual free index

Alonso-Meijide et al. [2011] extends the PG index when considering a set of coalitions bigger than the set of minimal winning coalitions and smaller than the set of winning coalitions. This set is based on the notion of a null individual. One can extend their index as follows. A winning configuration $\mathbf{S}$ is a null individual free winning configuration if there are no null representatives i^a casting a positive vote. More formally, the set of null individual free winning configurations is given by $\mathcal{NIF}(\mathbf{w}) = \{\mathbf{S} \in \mathcal{W} \mid s_{i^a} = 0$ for all null representatives $i^a\}$. The set of null individual free winning configurations in which i^a casts a positive vote is denoted by $\mathcal{NIF}_{i^a}(\mathbf{w}) = \{\mathbf{S} \in \mathcal{NIF}(\mathbf{w}) \mid s_{i^a} = 1\}$.

Definition 7. *Let $\mathbf{w}$ be a multi-dimensional rule, the Null individual Free Index index for a player $i \in N$ is given by:*

$$NIF_i(\mathbf{w}) = \sum_{a \in M} \overline{NIF}_{i^a}(\mathbf{w})$$

with

$$\overline{NIF}_{i^a}(\mathbf{w}) = \frac{|\mathcal{NIF}_{i^a}(\mathbf{w})|}{\sum_{i^a \in N^*} |\mathcal{NIF}_{i^a}(\mathbf{w})|}.$$

The main idea behind this index is to exclude winning configurations with null representatives. This index is less restrictive than the PG index which also excludes configurations with surplus representatives (representatives not needed to pass the final decision).

Example 4. *Once again, we apply our index to the first example. As a reminder,*

$$\mathcal{M}(\mathbf{w}) = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}.$$

Representatives 2^1 *and* 3^1 *are null representatives. Therefore,*

$$\mathcal{NIF}(\mathbf{w}) = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}.$$

For $a = 1$, *we have* $\overline{NIF}_{1^1}(\mathbf{w}) = \frac{4}{13}$ *and* $\overline{PG}_{2^1}(\mathbf{w}) = \overline{NIF}_{3^1}(\mathbf{w}) = 0$.
For $a = 2$, *we have* $\overline{NIF}_{1^2}(\mathbf{w}) = \overline{PG}_{2^2}(\mathbf{w}) = \overline{PG}_{3^2}(\mathbf{w}) = \frac{3}{13}$.
Then $NIF_1(\mathbf{w}) = \frac{4}{13} + \frac{3}{13} = \frac{7}{13}$ *and* $NIF_2(\mathbf{w}) = NIF_3(\mathbf{w}) = \frac{3}{13}$

In order to characterize this new index, one can replace the axiom of *PG* minimal monotonicity with the axiom of null individual free minimal monotonicity .

On the *NIF*-minimal monotonicity property

Axiom 9. *(*NIF*-minimal monotonicity)*
Let $\mathbf{w}, \bar{\mathbf{w}} \in \mathbb{W}_n^m$, *for all representatives* $i^a \in N^*$ *such that* $\mathcal{NIF}_{i^a}(\bar{\mathbf{w}}) \subseteq \mathcal{NIF}_{i^a}(\mathbf{w})$,

$$\phi_{i^a}(\mathbf{w}) \sum_{i^a \in N^*} |\mathcal{NIF}_{i^a}(\mathbf{w})| \geq \phi_{i^a}(\bar{\mathbf{w}}) \sum_{i^a \in N^*} |\mathcal{NIF}_{i^a}(\bar{\mathbf{w}})|$$

Similarly to Axiom 8, Axiom 9 implies that when a representative improves her position for a multi-dimensional rule, her influence must increase.

Theorem 6. *Let* ϕ *be an index for multi-dimensional rules. Then,* ϕ *satisfies Axiom 1, Axiom 2, Axiom 3 and Axiom 9, if and only if* $\phi = NIF$.

Proof. The proof follows immediately from a similar reasoning to the one used in the proof of Theorem 5. □

4.3 Shift index

Alonso-Meijide and Freixas [2010] argues that not all minimal winning coalitions will be formed in one dimension, and uses a smaller set of minimal winning coalitions - the shift coalitions. The idea is to consider winning coalitions where there are no surplus individuals but also no individuals who can be replaced by a weaker individual. The notion of a weaker individual is based on the notion of the desirability relation (Isbell [1958]).

We follow two principles to extend Alonso-Meijide and Freixas [2010]'s results to multi-dimensional rules. Firstly, we assume that every representative wishes to form part of a minimal winning configuration by giving a positive vote. Secondly, every representative prefers to partner with weaker players, as long as the configuration is winning. Therefore, we consider a smaller set than the set of minimal winning configurations: the shift minimal configurations.
To introduce these special configurations, one needs to introduce the notion of desirability for multi-dimensional rules as an extension of Isbell [1958]'s relation. In one dimensional context the desirability relation consists in ranking players with respect to how much influential they are. It is a preordering on the set of players. Instead of comparing players influence with a cardinal approach, the desirability relation permits an ordinal comparison. A given player who is never needed in any minimal winning coalition may be regarded as not being of any influence at all. On the contrary, if that player is indispensable to every minimal winning coalition, we may think that he is very influential. Most often, it is within these two extreme limits that the majority of players lie.

Desirability relation for multi-dimensional rules

In a multi-dimensional rule $\mathbf{w}$, representative i^a is said to be more desirable than representative j^b, denoted by $i^a \succ j^b$ if the following two conditions are fulfilled: i) for every configuration $\mathbf{S}$ such that $s_i^a = 0$ and $s_j^b = 0$, $\mathbf{S} + \mathbf{I}_j^b \in \mathcal{W}(\mathbf{w}) \Rightarrow \mathbf{S} + \mathbf{I}_i^a \in \mathcal{W}(\mathbf{w})$ with $\mathbf{S} + \mathbf{I}_i^a = \{\mathbf{T} \in \mathbb{Z}_n^m : t_i^a = 1, t_k^a = s_k^a$ for all a and for all $k \neq i\}$ and $\mathbf{S} + \mathbf{I}_j^b = \{\mathbf{T} \in \mathbb{Z}_n^m :$ $t_j^b = 1, t_k^b = s_k^b$ for all b and for all $k \neq j\}$; ii) there exists some configuration $\mathbf{T}$ such that $t_i^a = 0$ and $t_j^b = 0$, such that $\mathbf{T} + \mathbf{I}_i^a \in \mathcal{W}(\mathbf{w})$ and $\mathbf{T} + \mathbf{I}_j^b \notin \mathcal{W}(\mathbf{w})$. Individuals i^a and j^b are said to be equally desirable, denoted by $i^a \sim j^b$ if: for any configuration $\mathbf{S}$ such that $s_i^a = 0$ and $s_j^b = 0$, $\mathbf{S} + \mathbf{I}_i^a \in \mathcal{W}(\mathbf{w}) \Longleftrightarrow \mathbf{S} + \mathbf{I}_j^b \in \mathcal{W}(\mathbf{w})$.

The desirability relation denoted by $\succeq$ is defined on N^* as follows: $i^a \succeq j^j$ if $i^a \succ j^b$ or $i^a \sim j^b$.

Shift index for multi-dimensional rules

Let a rule $\mathbf{w}$ and its desirability relation $\succeq$. A configuration $\mathbf{S} \in \mathcal{M}(\mathbf{w})$ is shift minimal if for every i^a and j^b such that $s_i^a = 1$, $s_j^b = 0$ and $i^a \succ j^b$, it holds that $\bar{\mathbf{S}} \notin \mathcal{W}(\mathbf{w})$ with $\bar{\mathbf{S}} = \{\mathbf{T} \in \mathbb{Z}_n^m : t_i^a = 0, t_j^b = 1, t_k^a = s_k^a$ for all a and for all $k \neq i, j\}$. The set of shift minimal configurations will be denoted by $\mathcal{SM}(\mathbf{w})$ and we use the notation $\mathcal{SM}_{i^a}(\mathbf{w})$ as the set of shift minimal winning configurations in which representative i^a expresses a positive vote.

Definition 8. *Let* $\mathbf{w}$ *be a multi-dimensional rule, the Shift Index for a player* $i \in N$ *is given by:*

$$SF_i(\mathbf{w}) = \sum_{a \in M} \overline{SF}_{i^a}$$

with

$$\overline{SF}_{i^a}(\mathbf{w}) = \frac{|\mathcal{SM}_{i^a}(\mathbf{w})|}{\sum\limits_{i^a \in N^*} |\mathcal{SM}_{i^a}(\mathbf{w})|}.$$

Example 5. *In Example 1, we have the following desirability relation:* $1^1 \succ 1^2 \sim 2^2 \sim 3^2 \succ 2^1 \sim 3^1$. *Therefore,* $\mathcal{SM}(\mathbf{w}) = \mathcal{M}(\mathbf{w})$. *This implies that* $SF_1(\mathbf{w}) = PG_1(\mathbf{w})$ *and* $SF_2(\mathbf{w}) = SF_3(\mathbf{w}) = PG_2(\mathbf{w}) = PG_3(\mathbf{w})$.

We present a second example in which the set of minimal winning configurations is different from the set of shift minimal winning configurations.

Example 6. *Consider that six subjects* ($n = 6$) *are taught in two periods* ($m = 2$). *To pass the year, the following conditions need to be met: i) the test in the first subject is compulsory for the first period; ii) for the second period, at least two tests, including the first subject or at least three tests including the second subject, are needed regardless of subject six.*
We have the following desirability relation: $1^1 \succ 1^2 \succ 2^2 \succ 3^2 \sim 4^2 \sim 5^2 \succ 2^1 \sim 3^1 \sim 4^1 \sim 5^1 \sim 6^1 \sim 6^2$. *Therefore, we have*

$$\boxed{\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \mathcal{SM}(\boldsymbol{w})} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{M}(\boldsymbol{w})$$

The last configuration is not shift minimal since we can replace 2^2 *with* 3^2, *a weaker representative, and the configuration remains a winning one.*
Public Good Index:

For $a = 1$, *we have* $\overline{PG}_{1^1}(\mathbf{w}) = \frac{7}{24}$ *and* $\overline{PG}_{i^1}(\mathbf{w}) = 0$ *for all* $j = 2, \ldots, 6$.

For $a = 2$, we have $\overline{PG}_{1^2}(\mathbf{w}) = \frac{4}{24}$, $\overline{PG}_{2^2}(\mathbf{w}) = \frac{4}{24}$, $\overline{PG}_{i^2}(\mathbf{w}) = \frac{3}{24}$ for $i = 3,4,5$ and $\overline{PG}_{6^2}(\mathbf{w}) = 0$.

Then, $PG_1(\mathbf{w}) = \frac{7}{24}+\frac{4}{24} = \frac{11}{24}$, $PG_2(\mathbf{w}) = \frac{4}{24}$, $PG_i(\mathbf{w}) = \frac{3}{24}$ for $i = 3,4,5$, and $PG_6(\mathbf{w}) = 0$

Shift Index:

For $a = 1$, we have $\overline{SF}_{1^1}(\mathbf{w}) = \frac{6}{21}$ and $\overline{SF}_{i^1}(\mathbf{w}) = 0$ for all $j = 2,...,6$.

For $a = 2$, we have $\overline{SF}_{1^2}(\mathbf{w}) = \frac{3}{21}$, $\overline{SF}_{2^2}(\mathbf{w}) = \frac{3}{21}$, $\overline{PG}_{i^2}(\mathbf{w}) = \frac{3}{21}$ for $i = 3,4,5$ and $\overline{SF}_{6^2}(\mathbf{w}) = 0$.

Then, $SF_1(\mathbf{w}) = \frac{6}{21} + \frac{3}{21} = \frac{9}{21}$, $SF_2(\mathbf{w}) = \frac{3}{21}$, $SF_i(\mathbf{w}) = \frac{3}{21} = \frac{5}{9}$ for $i = 3,4,5$, and $SF_6(\mathbf{w}) = 0$

The axiom of *PG* minimal monotonicity can be replaced by the axiom of *SF* minimal monotonicity .

On the *SF*-minimal monotonicity property

Axiom 10. *(SF-minimal monotonicity)*
Let $\mathbf{w}$, $\bar{\mathbf{w}} \in \mathbb{W}_n^m$, for all representatives $i^a \in N^*$ such that $\mathcal{SM}_{i^a}(\bar{\mathbf{w}}) \subseteq \mathcal{SM}_{i^a}(\mathbf{w})$,

$$\phi_{i^a}(\mathbf{w}) \sum_{i^a \in N^*} |\mathcal{SM}_{i^a}(\mathbf{w})| \geq \phi_{i^a}(\bar{\mathbf{w}}) \sum_{i^a \in N^*} |\mathcal{SM}_{i^a}(\bar{\mathbf{w}})|$$

Theorem 7. *Let ϕ be an index for multi-dimensional rules. Then, ϕ satisfies Axiom 1, Axiom 2, Axiom 3 and Axiom 10, if and only if $\phi = SF$.*

The interpretation of Axiom 10 is similar to the two previous monotonicity axioms.

Proof. The proof is similar to the one used in the proof of Theorem 5. □

4.4 Deegan-Packel index

Similarly to the "Public good" index, the Deegan-Packel index (Deegan and Packel [1978]) results from assuming that only the minimal winning coalitions are likely to form intentionally. But contrary to the "Public good" index where all the players in a minimal winning coalition received one unit of power, the Deegan-Packel index divides this unit of power equally among the players in the minimal coalition. We extend the Deegan-Packel index to multi-dimensional rules as follows.

Definition 9. *Let $\mathbf{w}$ be a multi-dimensional rule, the Deegan-Packel index for a player $i \in N$ is given by:*

$$DP_i(\mathbf{w}) = \sum_{a \in M} \overline{DP}_{i^a}$$

with

$$\overline{DP}_{i^a}(\mathbf{w}) = \frac{1}{|\mathcal{M}(\mathbf{w})|} \sum_{\mathbf{S} \in \mathcal{M}_{i^a}(\mathbf{w})} \frac{1}{\mathbf{s}}.$$

According to this index, only representatives who express a positive vote in a minimal winning configuration receive power. And the representatives with a positive vote who belong to the same minimal winning configuration divide the power equally. Note that each minimal winning configuration has an equal probability of forming.

Example 7. *Once again, we apply our index to the first example.*

For $a = 1$, *we have* $\overline{DP}_{1^1}(\mathbf{w}) = \frac{1}{3} * (\frac{1}{3} + \frac{1}{3} + \frac{1}{3}) = \frac{1}{3}$ *and* $\overline{DP}_{2^1}(\mathbf{w}) = \overline{DP}_{3^1}(\mathbf{w}) = 0$.

For $a = 2$, *we have* $\overline{DP}_{1^2}(\mathbf{w}) = \overline{DP}_{2^2}(\mathbf{w}) = \overline{DP}_{3^2}(\mathbf{w}) = \frac{1}{3} * (\frac{1}{3} + \frac{1}{3}) = \frac{2}{9}$.

Then $DP_1(\mathbf{w}) = \frac{1}{3} + \frac{2}{9} = \frac{5}{9}$ *and* $DP_2(\mathbf{w}) = DP_3(\mathbf{w}) = \frac{2}{9}$

In a simple game context, Lorenzo-Freire et al. [2007] characterized the Deegan-Packel index by introducing a minimal monotonicity property. This property can be extended to multi-dimensional rules as follows:

On the *DP*-minimal monotonicity property

Axiom 11. *(*DP*-minimal monotonicity)*

Let $\mathbf{w}, \bar{\mathbf{w}} \in \mathbb{W}_n^m$, ϕ *satisfies* DP*-minimal monotonicity for multi-dimensional rules if for any rules* $\mathbf{w}$ *and* $\bar{\mathbf{w}}$, *it holds that for all representatives* $i^a \in N^*$ *such that* $\mathcal{M}_{i^a}(\bar{\mathbf{w}}) \subseteq \mathcal{M}_{i^a}(\mathbf{w})$,

$$\phi_{i^a}(\mathbf{w})|\mathcal{M}(\mathbf{w})| \geq \phi_{i^a}(\bar{\mathbf{w}})|\mathcal{M}(\bar{\mathbf{w}})|$$

According to Axiom 11, if the set of minimal winning configurations including a representative i^a in rule $\bar{\mathbf{w}}$ is a subset of the set of minimal winning configurations including the same representative i^a in another multi-dimensional rule $\mathbf{w}$ then, the power of i^a in $\mathbf{w}$ is no lesser than the power of i^a in $\bar{\mathbf{w}}$ (this power must be normalized by the number of minimal winning configuration in $\mathbf{w}$ and $\bar{\mathbf{w}}$).

Theorem 8. *Let* ϕ *be an index for multi-dimensional rules. Then,* ϕ *satisfies Axiom 1, Axiom 2, Axiom 3 and Axiom 11, if and only if* $\phi = DP$.

The interpretation of Axiom 11 is similar to the two previous monotonicity axioms.

Proof. The proof is similar to the one used in the proof of Theorem 5. □

5 Concluding discussion

To conclude, let us summarize all the characterizations in the following tables.

SH	BZ	DP
Transfer	Transfer	***DP* monotonicity**
Anonymity	Anonymity	Anonymity
Efficiency	**Efficiency*** or **Two-Efficiency**	Efficiency
Null player	Null player	Null player

NIF	SF	PG
***NIF* monotonicity**	***SF* monotonicity**	***PG* monotonicity** or **Merger**
Anonymity	Anonymity	Anonymity
Efficiency	Efficiency	Efficiency
Null player	Null player	Null player

The null player and the anonymity properties appear in all the characterizations presented above. The efficiency property characterizes all the indices but not Banzhaf where a modified efficiency property appears. In addition to these properties, which use only one rule in their definitions, another property is necessary to characterize each of the indices presented in this paper. This last property must be one of the groups of properties that establish an inequality between two rules (monotonicity properties) or an equality between the union rule and the component rules (mergeability and transfer properties). Depending on the property used, each one of the six indices is characterized.

A large literature in cooperative game theory is concerned with the study of multiple alternatives of support (see Felsenthal and Machover [1997], Laruelle and Valenciano [2012], Freixas [2005], Courtin et al. [2016, 2017], among others). One way in which this work could be explored in further research, is to study the extension of multi-dimensional rules and the evaluation of power to such games, in which the choice would no longer be binary.

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Appendix: Independence of the axiomatic system of Theorem 1 and Theorem 4

The question of the independence axioms is adressed in the following table. The examples show the independence of any of the axioms we have used with respect to the others in Theorem 1 and Theorem 4.

Let first introduce the following function. If $N^* = \{i^a, j^a\}$, and $\mathcal{M} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then $\hat{\phi}_{i^a}(\mathbf{w}) = \frac{1}{2} + \epsilon$, $\hat{\phi}_{j^a}(\mathbf{w}) = \frac{1}{2} - \epsilon$ and $\hat{\phi}_{i^a}(\mathbf{w}) = SH_{i^a}(\mathbf{w})$ otherwise.

Theorem 1

Examples	Efficiency	Null player	Anonymity	Transfer
$\phi_i(\mathbf{w}) = BZ$	No	Yes	Yes	Yes
$\phi_i(\mathbf{w}) = \frac{1}{n}$	Yes	No	Yes	Yes
$\phi_i(\mathbf{w}) = \sum\limits_{a\in M} \hat{\phi}_{i^a}$	Yes	Yes	No	Yes
$\phi_i(\mathbf{w}) = PG$	Yes	Yes	Yes	No

We introduce the following notation before to present the independance of Theorem 4. Let $(m_{1^1}, m_{2^1}, ..., m_{n^m})$ a system of positive weights such that $\sum\limits_{i^a \in N^*} m_{i^a} = 1$ and $m_{i^a} \neq m_{j^b}$ if $i^a \neq j^b$.

Theorem 4

Examples	Efficiency	Null player	Anonymity	Merger
$\phi_i(\mathbf{w}) = \alpha PG \ (\alpha > 1)$	No	Yes	Yes	Yes
$\phi_i(\mathbf{w}) = \frac{1}{n}$	Yes	No	Yes	Yes
$\phi_i(\mathbf{w}) = \sum\limits_{a\in M} \overline{\phi}_{i^a}$ with $\overline{\phi}_{i^a}(\mathbf{w}) = \frac{m_{i^a}.\lvert\mathcal{M}_{i^a}(\mathbf{w})\rvert}{\sum_{i^a\in N^*} m_{i^a}.\lvert\mathcal{M}_{i^a}(\mathbf{w})\rvert}$	Yes	Yes	No	Yes
$\phi_i(\mathbf{w}) = SH$	Yes	Yes	Yes	No

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