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Integrality gap analysis for bin packing games

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ABSTRACT

A cooperative bin packing game is an N-person game, where the player set N consists of k bins of capacity 1 each and n items of sizes $a_1, \ldots, a_n$. The value $v(S)$ of a coalition S of players is defined to be the maximum total size of items in S that can be packed into the bins of S. We analyze the integrality gap of the corresponding 0–1 integer program of the value $v(N)$, thereby presenting an alternative proof for the non-emptiness of the 1/3-core for all bin packing games. Further, we show how to improve this bound $\epsilon \leq 1/3$ (slightly) and point out that the conclusion in Matsui (2000) [9] is wrong (claiming that the bound 1/3 was tight). We conjecture that the true best possible value is $\epsilon = 1/7$. The results are obtained using a new "rounding technique" that we develop to derive good (integral) packings from given fractional ones.

1. Introduction

Nowadays, as online shopping has become so popular, delivering goods by means of transport firms is a steadily growing business. The question therefore arises how transport costs should be compensated in a "fair way". Currently, usually weight and/or volume are used as indicators for transport costs. Motivated by this observation, it seems natural, to study such allocation problems under the framework of cooperative games. Studying allocation rules for bin packing games (as defined below) is a first step towards analyzing allocation problems of this kind.

Formally, a *cooperative game* is defined by a tuple $\langle N, v\rangle$, where N is a set of *players* and $v : 2^N \to \mathbb{R}$ is a *value function* satisfying $v(\emptyset) = 0$. A subset $S \subseteq N$ is called a *coalition* and N itself is the *grand coalition*. The usual goal in cooperative games is to "fairly" allocate the total gain $v(N)$ of the grand coalition N among the individual players. A well known concept is the *core* of a cooperative game, defined by all vectors $x \in \mathbb{R}^N$ satisfying

(i) $x(N) \leq v(N)$,
(ii) $x(S) \geq v(S)$ for all $S \subseteq N$.

As usual, we abbreviate $x(S) = \sum_{i\in S} x_i$.

We say a game is *balanced* if it possesses a nonempty core. Unfortunately, many games are not balanced. Players in a non-balanced game may not cooperate because no matter how the total gain is distributed, there will always be some coalition S with $x(S) < v(S)$, i.e., it gets paid less than what it could earn on its own. For this case, one naturally seeks to relax the condition (ii) above in such a way that the modified core becomes nonempty. Faigle and Kern [3] introduced the multiplicative ϵ-core as follows. Given $\epsilon > 0$, the ϵ-core consists of all vectors $x \in \mathbb{R}^N$ satisfying condition (i) above together with

(ii$'$) $x(S) \geq (1-\epsilon)v(S)$ for all $S \subseteq N$.

We can interpret ϵ as a tax rate in the sense that coalition S is allowed to keep only $(1-\epsilon)v(S)$ on its own. If the value function v is nonnegative, the 1-core is obviously nonempty. In order to approximate the core as close as possible, one would like to have the taxation rate ϵ as small as possible while keeping the ϵ-core nonempty.

As motivated at the beginning of this paper, we study specific games of the following kind: there are two disjoint sets of players, say, A and B. Each player $i \in A$ possesses an item of value/size a_i, for $i = 1, \ldots, n$, and each player $j \in B$ possesses a truck/bin of capacity b_j. The items produce a profit proportional to their size a_i if they are brought to the market place. The value $v(N)$ of the grand coalition thus represents the maximum profit achievable. How should $v(N)$ be allocated to the owners of the items and the owners of the trucks?

We start with some definitions and notations. A *bin packing game* is defined by a set of k bins, of capacity 1 each, and n items $1, 2, \ldots, n$ of sizes $a_1, a_2, \ldots, a_n$, where we assume, w.l.o.g, $0 \leq a_i \leq 1$.

Let A be the set of items and B be the set of bins. A *feasible packing* of an item set $A' \subseteq A$ into a set of bins $B' \subseteq B$ is an assignment of some (or all) elements in A' to the bins in B' such that the total size of items assigned to any bin does not exceed the bin capacity (=1).

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Items that are assigned to a bin are called *packed* and items that are not assigned are called *not packed*. The *value* of a feasible packing is the total size of packed items.

The player set N consists of all items and all bins. The value $v(S)$ of a coalition $S \subseteq N$, where $S = A_S \cup B_S$ with $A_S \subseteq A$ and $B_S \subseteq B$, is the maximum value of all feasible packings of A_S into B_S. A corresponding feasible packing is called an *optimum packing*.

An intriguing problem is to find the 'minimal' taxation rate $\epsilon_{\min}$ such that the $\epsilon_{\min}$-core is nonempty for all bin packing games. It was shown by Faigle and Kern [3] that $1/7 \le \epsilon_{\min} \le 1/2$. Woeginger [12] improved this result to $\epsilon_{\min} \le 1/3$. Kuipers [8] showed that $\epsilon_{\min} = 1/7$ if all item sizes are strictly larger than 1/3.

This is a revised version of our conference paper [7] and the rest part is organized as follows. In Section 2, we introduce an integer linear program for computing $v = v(N)$ and the corresponding fractional packing value $v' = v'(N)$. In Section 3, based on a greedy heuristic, we introduce a new rounding technique to derive integral packings from given fractional ones. As a consequence, we obtain an upper bound on the (relative) integrality gap $(v' - v)/v'$, thereby reproving the known result $\epsilon_{\min} \le 1/3$. In Section 4, we apply the same greedy heuristic w.r.t. modified ("virtual") item sizes to derive a slightly better bound, disproving the result presented in [9]. In Section 5, we conjecture that $\epsilon_{\min} = 1/7$ and mention the connection with the well-known 3-PARTITION problem.

2. Fractional packings

We start with some definitions and notations. A set F of items is called a *feasible set* if its total size does not exceed 1. Denote by $\mathcal{F}$ the set of all feasible sets. Let σ_F be the size of a feasible set and let $\sigma = (\sigma_F) \in \mathbb{R}^{\mathcal{F}}$ for all $F \in \mathcal{F}$, then the total earning $v(N)$ of the grand coalition N equals

$$\begin{aligned}
&\max\ \sigma^T y \\
&\text{s.t.} \sum_{F\in\mathcal{F}} y_F \le k, \\
&\sum_{F\ni i} y_F \le 1 \quad (i = 1, 2, \ldots, n), \\
&y \in \{0, 1\}^{\mathcal{F}}.
\end{aligned} \tag{2.1}$$

The value $v'(N)$ of an *optimum fractional packing* is defined by the relaxation of (2.1), i.e.,

$$\begin{aligned}
&\max\ \sigma^T y \\
&\text{s.t.} \sum_{F\in\mathcal{F}} y_F \le k, \\
&\sum_{F\ni i} y_F \le 1 \quad (i = 1, 2, \ldots, n), \\
&y \in [0, 1]^{\mathcal{F}}.
\end{aligned} \tag{2.2}$$

A *fractional packing* of our bin packing problem is a vector y satisfying the constraints of the linear program (2.2). Accordingly, we refer to the original "feasible packing" as the *integral packing*, which meets the constraints of (2.1). We call item *i fully packed* if $\sum_{F\ni i} y_F = 1$. Observe that for an optimal basic solution y of (2.2) the number of non-zero components $y_F > 0$ can be bounded by $|\text{supp}\ y| \le 1+$ number of fully packed items.

Faigle and Kern [4] have given a sufficient and necessary condition for the non-emptiness of the ϵ-core of a bin packing game (with $v = v(N)$ and $v' = v'(N)$).

Lemma 1 (*See [4] a Short Proof*)**.** *The ϵ-core is nonempty if and only if $\epsilon \ge 1 - v/v'$.*

If all items are packed in a feasible integral packing, we obviously have $v' = v$, thus the core is nonempty. For convenience of description in later sections, we always ignore this trivial case. As a consequence, $v > v'/2$ can always be achieved by filling each bin to at least 1/2 as follows: for a given bin, if there is a yet unpacked item of size $a_i > 1/2$, assign this item to the bin. If all yet unpacked items have size $a_i \le 1/2$, assign as many such items to the bin as possible in a greedy manner. By our assumption that not all items are packed in any feasible packing, this must result in a packing with all bins filled to at least 1/2. So the 1/2-core is nonempty for all bin packing games.

Denote by $\epsilon_N = 1 - v(N)/v'(N)$ the minimal taxation rate of a bin packing game N. We thus seek for good lower bounds on $v(N)/v'(N)$. The first step in [12] is to reduce the analysis to item sizes $a_i > 1/3$. Similarly, if we aim for a bound $\epsilon_N \le \epsilon$ with $\epsilon \in [1/4, 1/3)$, it suffices to investigate instances with item sizes $a_i > 1/4$, as can be seen from the following two lemmas.

Lemma 2. *Let A be a set of items disjoint from N and let $\sigma_A = \sum_{i\in A} a_i$ be the total size of A. Thus, $v(N) + \sigma(A) = v(N \cup A)$ implies $\epsilon_{N\cup A} \le \epsilon_N$.*

Proof. From Lemma 1, we know $\epsilon_N = 1 - v(N)/v'(N)$. Thus,

$$\begin{aligned}
\epsilon_{N\cup A} &= 1 - \frac{v(N\cup A)}{v'(N\cup A)} \le 1 - \frac{v(N) + \sigma(A)}{v'(N) + \sigma(A)} \\
&\le 1 - \frac{v(N)}{v'(N)} = \epsilon_N. \quad \square
\end{aligned}$$

For $\delta \in (0, 1)$, let N_δ denote the restriction of N to items of size $a_i > \delta$.

Lemma 3. *If $\delta, \epsilon_{N_\delta} \le \epsilon$, then $\epsilon_N \le \epsilon$.*

Proof. Assume $\epsilon_{N_\delta} \le \epsilon$, i.e., there exists an integral packing of N_δ with value $v(N_\delta) \ge (1 - \epsilon)v'(N_\delta)$. Let $A = N \setminus N_\delta$ be the set of "small" items. If we can put all of A on top of the already packed N_δ-items, we have $v(N) = v(N_\delta \cup A)$ and $\epsilon_N \le \epsilon$ follows from Lemma 2. Else, i.e. if some of the small items remain not packed, then each bin must be filled to at least $1 - \delta \ge 1 - \epsilon$ and $v(N) \ge (1 - \epsilon)v'(N)$ must hold. $\square$

Thus in what follows, when seeking for an upper bound $\epsilon_N \le \epsilon$ with $\epsilon \in [1/4, 1/3)$, we may assume that all item sizes are at least $a_i > 1/4$. (This is actually a rather interesting class anyway, as it contains all instances of 3-PARTITION, *c.f.* Section 5).

3. Alternative proof of non-emptiness of the 1/3-core

We present an alternative proof for the fact that the 1/3-core of any bin packing game is nonempty. Consider any bin packing game with k bins and item sizes $a_1, \ldots, a_n$ with all $a_i > 1/4$ (although, for the purpose of this section, it would suffice to assume $a_i > \frac{1}{3}$). Let $y = (y_F)$ be an optimal fractional packing. Let $\mathcal{F} = \{F \mid y_F > 0\}$ denote the support of y. First note that if $\sigma_F \le \frac{2}{3}$ for all $F \in \mathcal{F}$ then $v' \le \frac{2}{3}k$ and hence any integral packing filling each bin to at least $\frac{1}{2}$ would achieve a value $v \ge \frac{k}{2} \ge \frac{3}{4}v'$, proving non-emptiness even for the $\frac{1}{4}$-core. More generally, as we will see below, to extract a reasonably good integral packing from the fractional packing y, we may focus on $\bar{\mathcal{F}} := \{F \in \mathcal{F} \mid \sigma_F > \frac{2}{3}\}$, the "interesting part" of the support of y. So assume $\bar{\mathcal{F}} \neq \emptyset$, *i.e.*, it has nonzero *length* $l = \sum_{F\in\bar{\mathcal{F}}} y_F$. Let $\bar{\mathcal{F}} = \{F_1, \ldots, F_m\}$ and

$$\sigma_{F_1} \ge \sigma_{F_2} \ge \cdots \ge \sigma_{F_m} > \frac{2}{3}.$$

Note that the number of fully packed items is at most $3k$ (3 items per bin), so that $m \le |\text{supp}\ y| \le 3k + 1$. The basic idea is to construct an integral solution "greedily" *i.e.*, starting with F_1, we construct a sequence of feasible sets by choosing in each step the largest size F_i in $\bar{\mathcal{F}}$ that is disjoint from all previously chosen ones. Formally: start with $s = 1$ and do the following while $\bar{\mathcal{F}} \neq \emptyset$: let F_{i_s} be the largest size set in $\bar{\mathcal{F}}$ and $\bar{\mathcal{F}}_{i_s} := \{F \in \bar{\mathcal{F}} \mid F \cap F_{i_s} \neq \emptyset\}$. Replace $\bar{\mathcal{F}}$ by $\bar{\mathcal{F}} \setminus \bar{\mathcal{F}}_{i_s}$ and s by $s+1$. Let $F_{i_1}, \ldots, F_{i_r}$ denote the sequence constructed this way. As each F_{i_s} contains at most 3 items, we find that

$$\sum_{F \in \bar{\mathcal{F}}_{i_s}} y_F \leq \sum_{F \cap F_{i_s} \neq \emptyset} y_F \leq 3. \tag{3.1}$$

Define the *length* of $\bar{\mathcal{F}}_{i_s}$ to be $l_{i_s} := \sum_{F \in \bar{\mathcal{F}}_{i_s}} y_F$ and the *value* to be $v_{i_s} := \sum_{F \in \bar{\mathcal{F}}_{i_s}} y_F \sigma_F$. Hence in each step, when removing $\bar{\mathcal{F}}_{i_s}$, we remove at most 3 from the total length $l = \sum_{F \in \bar{\mathcal{F}}} y_F$, so that our construction yields $F_{i_1}, \ldots, F_{i_r}$ with $r \geq l/3$. By the greedy choice of F_{i_s} we have $l_{i_s} \sigma_{F_{i_s}} \geq v_{i_s}$. Hence

$$\sigma_{F_{i_s}} = \frac{l_{i_s}}{3} \sigma_{F_{i_s}} + \left(1 - \frac{l_{i_s}}{3}\right) \sigma_{F_{i_s}} \geq \frac{1}{3} v_{i_s} + \left(1 - \frac{l_{i_s}}{3}\right) \frac{2}{3}.$$

Summation yields

$$\sigma_{F_{i_1}} + \cdots + \sigma_{F_{i_r}} \geq \frac{1}{3} \sum_{F \in \bar{\mathcal{F}}} y_F \sigma_F + \left(r - \frac{l}{3}\right) \frac{2}{3}.$$

Extend this greedy selection by $k-r$ bins, each filled to at least 1/2. As $r \geq \frac{l}{3}$, the resulting packing implies

$$\begin{aligned}
v &\geq \frac{1}{3} \sum_{F \in \bar{\mathcal{F}}} y_F \sigma_F + \left(r - \frac{l}{3}\right) \frac{2}{3} + (k-r) \frac{1}{2} \\
&\geq \frac{1}{3} \sum_{F \in \bar{\mathcal{F}}} y_F \sigma_F + \left(k - \frac{l}{3}\right) \frac{1}{2} \\
&= \frac{1}{3} \sum_{F \in \bar{\mathcal{F}}} y_F \sigma_F + \frac{1}{3} l + \frac{1}{2}(k-l) \\
&\geq \frac{2}{3} \sum_{F \in \bar{\mathcal{F}}} y_F \sigma_F + \frac{1}{2}(k-l),
\end{aligned} \tag{3.2}$$

whereas $v' \leq \sum_{F \in \bar{\mathcal{F}}} y_F \sigma_F + \frac{2}{3}(k-l)$. Hence $v/v' \geq \frac{2}{3}$, as claimed.

4. Modified greedy selection

A previous paper by Matsui [9] claims that the bound 1/3 for $\epsilon_{\min}$ is tight. In his proof, a bin packing game G_α of 3 bins and 5 items of sizes 1/2, 1/2, 1/2, $1/2+\alpha$, $1/2+\alpha$ ($0 \leq \alpha \leq 1/2$) is considered. He "showed" that for any given $\epsilon < 1/3$, by properly choosing α, the ϵ-core of G_α is always empty, based on the fact that items 1–3 (with size 1/2 each) cannot be packed all in an optimum integral packing. Then he claims that an ϵ-core allocation must allocate 0 for each of the 3 players corresponding to items of size 1/2. This implication is only true when one seeks for a core allocation (with $\epsilon = 0$) while obviously incorrect in case of the ϵ-core allocation.

In this section, we aim to improve the bound $\epsilon_{\min} \leq 1/3$ by rounding the fractional packing w.r.t. a modified ordering of its selected feasible sets. First note that actually the inequalities (3.1) must be strict, since all 3 items occur together in $y_{F_{i_s}}$. Indeed, if F_{i_s} contains three items, say, $F_{i_s} = \{j_1, j_2, j_3\}$, then

$$\begin{aligned}
l_{i_s} &= \sum_{F \in \bar{\mathcal{F}}_{i_s}} y_F \leq \sum_{F \ni j_1} y_F + \sum_{F \ni j_2} y_F + \sum_{F \ni j_3} y_F - 2y_{F_{i_s}} \\
&\leq 3 - 2y_{F_{i_s}} < 3.
\end{aligned} \tag{4.1}$$

Note that when F_{i_s} contains less, say, only two items, the same inequality $\sum_{F \in \bar{\mathcal{F}}_{i_s}} y_F \leq 2 - y_{F_{i_s}} \leq 3 - 2y_{F_{i_s}}$ holds.

Summation thus yields

$$l = \sum_{s=1}^{r} l_{i_s} \leq \sum_{s=1}^{r} (3 - 2y_{F_{i_s}}) = 3r - 2 \sum_{s=1}^{r} y_{F_{i_s}}. \tag{4.2}$$

Thus, if $\alpha = \sum_{s=1}^{r} y_{F_{i_s}}$, we find

$$r \geq \frac{1}{3}(l + 2\alpha). \tag{4.3}$$

The estimate in Section 3 can be (slightly) improved by modifying the greedy selection so as to give higher priority to feasible sets $F \in \bar{\mathcal{F}}$ with comparatively large y_F —and thus hopefully increasing α. To this end we modify the size of each $F \in \bar{\mathcal{F}}$ to $\tilde{\sigma}_F := \sigma_F + \frac{1}{9} y_F \geq \sigma_F$. The sizes of $F \in \mathcal{F} \setminus \bar{\mathcal{F}}$ remain unchanged. We then apply greedy selection to $\bar{\mathcal{F}}$ (ordered according to the modified sizes) to obtain certain $F_{i_1}, \ldots, F_{i_r} \in \bar{\mathcal{F}}$ and append $k-r$ bins filled to at least $\frac{1}{2}$ as before.

Now let us analyze the greedy selection w.r.t. the modified ordering. Estimating the value $\tilde{v}$ (*w.r.t.* the modified sizes) of the resulting integral packing as we did in Section 3 (now using $r \geq \frac{l}{3} + \frac{2}{3}\alpha$ instead of $r \geq \frac{l}{3}$), yields

$$\begin{aligned}
\tilde{v} &\geq \tilde{\sigma}_{F_{i_1}} + \cdots + \tilde{\sigma}_{F_{i_r}} + \frac{1}{2}(k-r) \\
&\geq \frac{1}{3} \sum_{\bar{\mathcal{F}}} y_F \tilde{\sigma}_F + \left(r - \frac{l}{3}\right) \frac{2}{3} + (k-r) \frac{1}{2} \\
&\geq \frac{1}{3} \sum_{\bar{\mathcal{F}}} y_F \tilde{\sigma}_F + \left(\frac{2}{3}\alpha\right) \frac{2}{3} + \left(k - \frac{l}{3} - \frac{2}{3}\alpha\right) \frac{1}{2} \\
&= \frac{1}{3} \sum_{\bar{\mathcal{F}}} y_F \tilde{\sigma}_F + \frac{1}{9}\alpha + \left(k - \frac{l}{3}\right) \frac{1}{2} \\
&= \frac{1}{3} \left(\sum_{\bar{\mathcal{F}}} y_F \sigma_F + \frac{1}{9} \sum_{\bar{\mathcal{F}}} y_F^2 \right) + \frac{1}{9}\alpha + \frac{l}{3} + (k-l) \frac{1}{2} \\
&\geq \frac{2}{3} \sum_{\bar{\mathcal{F}}} y_F \sigma_F + \frac{1}{27} \sum_{\bar{\mathcal{F}}} y_F^2 + \frac{1}{9}\alpha + (k-l) \frac{1}{2}.
\end{aligned}$$

By definition of $\tilde{\sigma}$, the true value of our packing equals $\tilde{v} - \frac{1}{9}\alpha$. Subtract $\frac{1}{9}\alpha$ from both sides of the last inequality to conclude

$$v \geq \tilde{v} - \frac{1}{9}\alpha \geq \frac{2}{3} \sum_{\bar{\mathcal{F}}} y_F \sigma_F + \frac{1}{27} \sum_{\bar{\mathcal{F}}} y_F^2 + (k-l) \frac{1}{2}. \tag{4.4}$$

Now write $l = \beta k$ with $\beta \in (0, 1]$. The number of feasible sets $F \in \bar{\mathcal{F}}$ is bounded by $m \leq 3k + 1 \leq 4k$. Hence $\sum_F y_F^2$ can be bounded from below by assuming that all y_F have size $\frac{l}{4k} = \frac{\beta}{4}$ (and their number is $4k$), *i.e.*, $\sum y_F^2 \geq \frac{\beta}{4} l$. Hence (4.4) yields

$$\begin{aligned}
v &\geq \frac{2}{3} \sum_{\bar{\mathcal{F}}} y_F \sigma_F + \frac{\beta}{108} l + (1-\beta) \frac{k}{2} \\
&\geq \left(\frac{2}{3} + \frac{\beta}{108}\right) \sum_{\bar{\mathcal{F}}} y_F \sigma_F + (1-\beta) \frac{k}{2},
\end{aligned} \tag{4.5}$$

whereas

$$v' \leq \sum_{\bar{\mathcal{F}}} y_F \sigma_F + (k-l) \frac{2}{3} = \sum_{\bar{\mathcal{F}}} y_F \sigma_F + (1-\beta) \frac{2}{3} k. \tag{4.6}$$

Let $p(\beta)$ and $q(\beta)$ denote the right hand sides in (4.5) *resp.* (4.6), so that $\frac{v}{v'} \geq \frac{p(\beta)}{q(\beta)}$. The worst case occurs when $\beta = 1$ (as $\frac{p}{q}$ is easily seen to be decreasing), resulting in a slightly improved bound $\epsilon \leq 1/3 - 1/108 = 35/108$ for the minimum taxation rate.

Remark. The factor 1/9 is due to the following: any increase $\Delta\alpha$ in $\alpha = y_{F_{i_1}} + \cdots + y_{F_{i_r}}$ results in an increase of $\Delta r = \frac{2}{3}\Delta\alpha$ in the lower bound for r (*c.f.* (4.3)). This in turn raises the lower bound for

v by $\Delta r(2/3 - 1/2)$ (*c.f.* (3.2)). Thus any increase $\Delta\alpha$ in the total α-value of the selected $F_{i_1}, \ldots, F_{i_r}$ yields a *gain* (i.e., increase in the lower bound for v) of $\Delta\alpha/9$. It can be shown that the factor 1/9 in the definition of modified sizes is optimal in the sense that any alternative choice would lead to a weaker result in our analysis. Yet our analysis is obviously not tight and we expect the true ratio v/v' achieved by the (modified) greedy approach to be significantly better-though very complicated to analyze.

5. Remarks and open problems

Analyzing LP-relaxations and the resulting integrality gap has been a standard issue in combinatorial optimization since long. In recent years, the theoretical analysis of integrality gaps combined with (randomized) rounding techniques has led to interesting results in online algorithms (*c.f.* [1]) as well as approx theory. In particular, the so-called configuration LP, investigated by Verschae/Wiese [11] and Svensson [10] in the context of makespan minimization turns out to be a generalization of our fractional packing formulation, though the objective is different. Other related work aims at approximating the optimum packing value directly without regarding the fractional value. In particular, [2] shows how to compute an integral solution of packing value $\tilde{v} \geq \frac{3}{4}v$ in polynomial time. The relation to our results is rather unclear, as $\tilde{v} \geq \frac{3}{4}v$ does not even imply, say, $\tilde{v} \geq \frac{2}{3}v'$ in general.

Clearly the most straightforward open problem is to determine the smallest ϵ such that all bin packing games have non-empty ϵ-core. We conjecture that 1/7 is best possible (*c.f.* [3] for an example showing that $\epsilon < 1/7$ is impossible and a proof that the ϵ-core is non-empty for any sufficiently large (in terms of k) bin packing game).

A further challenging conjecture due to Woeginger states that $v' - v$ is bounded by a universal constant.

We finally would like to draw the attention of the reader to the well-known 3-PARTITION problem (*c.f.* [5]): given a set of items of sizes $a_1, \ldots, a_{3k}$ with $1/4 < a_i < 1/2$ and k bins, can we pack all items? If the fractional optimum is less than k, the answer is clearly "no". Note that the fractional optimum can be computed efficiently as there are only $O(k^3)$ feasible sets. Thus if $P \neq NP$, then there must be instances with fractional optimum equal to k and integral optimum $< k$. Although we tried hard, we could not exhibit a single such instance. Eventually, Joosten [6] succeeded in computing (probably the smallest) such instances with $k = 6$ bins and 18 items.

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