# A Polynomial-Time Algorithm for the Tridiagonal and Hessenberg P-Matrix Linear Complementarity Problem 

B. Gärtner and M. Sprecher*

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#### Abstract

We give a polynomial-time dynamic programming algorithm for solving the linear complementarity problem with tridiagonal or, more generally, Hessenberg P-matrices. We briefly review three known tractable matrix classes and show that none of them contains all tridiagonal P-matrices.


## 1 Introduction

Given a matrix $M \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^{n}$, the linear complementarity problem $\operatorname{LCP}(M, q)$ is to find vectors $w, z \in \mathbb{R}^{n}$ such that

$$
\begin{equation*}
w-M z=q, \quad w, z \geq 0, \quad w^{T} z=0 \tag{1}
\end{equation*}
$$

It is NP-complete in general to decide whether such vectors exist [2]. But if $M$ is a $P$-matrix (meaning that all principal minors-determinants of principal submatrices - are positive), then there are unique solution vectors $\tilde{w}, \tilde{z}$ for every right-hand side $q 10$. It is unknown whether these vectors can be found in polynomial time [7].

The matrix $M=\left(m_{i j}\right)_{i, j=1}^{n}$ is tridiagonal if $m_{i j}=$ 0 for $|j-i|>1$. More generally, $M$ is lower Hessenberg if $m_{i j}=0$ for $j-i>1$, and $M$ is upper Hessenberg if $M^{T}$ is lower Hessenberg; see Figure 1

In this note we show that $\operatorname{LCP}(M, q)$ can be solved in polynomial time if $M$ is a lower (or upper) Hessenberg P-matrix. Polynomial-time results already

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Figure 1: A tridiagonal matrix (left) and a lower Hessenberg matrix (right); the nonzero entries are enclosed in bold lines.
exist for other classes of matrices, most notably Zmatrices [1], hidden Z-matrices [6], and transposed hidden K-matrices 9 . Section 6 shows that none of these classes contains all tridiagonal P-matrices.

For the remainder of this note, we fix a P-matrix $M \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^{n}$.

## 2 The optimal basis

For $B \subseteq[n]:=\{1,2, \ldots, n\}$, we let $\bar{M}_{B}$ be the $n \times n$ matrix whose $i$ th column is the $i$ th column of $-M$ if $i \in B$, and the $i$ th column of the $n \times n$ identity matrix $I_{n}$ otherwise. $\bar{M}_{B}$ is invertible for every set $B$, a direct consequence of $M$ having nonzero principal minors. We call $B$ a basis and $\bar{M}_{B}$ the associated basis matrix.

The complementary pair $(w(B), z(B))$ associated with the basis $B$ is defined by

$$
w_{i}(B):= \begin{cases}0 & \text { if } i \in B  \tag{2}\\ \left(\bar{M}_{B}^{-1} q\right)_{i} & \text { if } i \notin B\end{cases}
$$

and

$$
z_{i}(B):= \begin{cases}\left(\bar{M}_{B}^{-1} q\right)_{i} & \text { if } i \in B  \tag{3}\\ 0 & \text { if } i \notin B\end{cases}
$$

for all $i \in[n]$.
Lemma 2.1. For every basis $B \subseteq[n]$, the following two statements are equivalent.
(i) The pair $(w(B), z(B))$ solves $\operatorname{LCP}(M, q)$, meaning that $w=w(B), z=z(B)$ satisfy (1).
(ii) $\bar{M}_{B}^{-1} q \geq 0$.

If both statements hold, $B$ is called an optimal basis for $\operatorname{LCP}(M, q)$.

Proof. As a consequence of (2) and (3), $w=w(B)$ and $z=z(B)$ already satisfy $w-M z=q$ and $w^{T} z=$ 0 , for every $B$. Moreover, $w, z \geq 0$ if and only if $w(B), z(B) \geq 0$; this in turn is equivalent to $\bar{M}_{B}^{-1} q \geq$ 0 .

From now on, we assume w.l.o.g. that $\operatorname{LCP}(M, q)$ is nondegenerate, meaning that $\left(\bar{M}_{B}^{-1} q\right)_{i} \neq 0$ for all $B \subseteq[n]$ and all $i \in[n]$. We can achieve this e.g. through a symbolic perturbation of $q$. In this case, we obtain the following

Lemma 2.2. There is a unique optimal basis $\tilde{B}$ for $\operatorname{LCP}(M, q)$.

Proof. Let $\tilde{w}, \tilde{z}$ be solution vectors of $\operatorname{LCP}(M, q)$, and set $\tilde{B}:=\left\{i \in[n]: \tilde{w}_{i}=0\right\}$. Since $\tilde{w}^{T} \tilde{z}=0$, we have $\tilde{z}_{i}=0$ if $i \in[n] \backslash \tilde{B}$. Hence, the vectors $\tilde{w}, \tilde{z}$ satisfy

$$
q=\left(I_{n} \mid-M\right)\binom{\tilde{w}}{\tilde{z}}=\bar{M}_{\tilde{B}}\binom{\tilde{w}_{[n] \backslash \tilde{B}}}{\tilde{z}_{\tilde{B}}},
$$

so $\tilde{w}=w(\tilde{B}), \tilde{z}=z(\tilde{B})$ follows. Hence, $\tilde{B}$ satisfies statement (i) in Lemma 2.1 and is therefore an optimal basis.

Uniqueness of $\tilde{w}, \tilde{z}[10$ implies via Lemma 2.1 that $(w(B), z(B))=(w(\tilde{B}), z(\tilde{B}))$ for every optimal basis $B$. But then (2) and (3) show that $\left(\bar{M}_{\tilde{B}}^{-1} q\right)_{i}=$ $\left(\bar{M}_{B}^{-1} q\right)_{i}=0$ for all $i \in \tilde{B} \oplus B$. Under nondegeneracy, there can be no such $i$, hence $B=\tilde{B}$.

## 3 Subproblems

For $K \subseteq[n]$, let $M_{K K}$ be the principal submatrix of $M$ consisting of all entries $m_{i j}$ with $i, j \in K$. Furthermore, let $q_{K}$ be the subvector of $q$ consisting of all entries $q_{i}, i \in K$.

By definition, the submatrix $M_{K K}$ is also a Pmatrix, and $\operatorname{LCP}\left(M_{K K}, q_{K}\right)$ is easily seen to inherit nondegeneracy from $\operatorname{LCP}(M, q)$. Hence, Lemma 2.2 allows us to make the following

Definition 3.1. For $k \in[n], B(k) \subseteq[k]$ is the unique optimal basis of $\operatorname{LCP}\left(M_{[k][k]}, q_{[k]}\right)$.

We also set $B(-1)=B(0)=\emptyset$.

## 4 The lower Hessenberg case

Let $M$ be a lower Hessenberg matrix. Then we have the following

Theorem 4.1. For every $k \in[n]$, there exists an index $\ell \in\{-1,0, \ldots, k-1\}$ such that

$$
B(k)=B(\ell) \cup\{\ell+2, \ell+3, \ldots, k\}
$$

Proof. If $B(k)=[k]$, the statement holds with index $\ell=-1$. Otherwise, let $\ell \in\{0,1, \ldots, k-1\}$ be the largest index such that $\ell+1 \notin B(k)$. The matrix $M_{[k][k]}$ is lower Hessenberg as well, which implies that the basis matrix $\bar{M}:=\bar{M}_{[k][k]}{ }_{B(k)}$ associated with $B(k)$ satisfies $\bar{m}_{i j}=0$ if $i \leq \ell<j$; see Figure 2,


$$
\uparrow \text { column } \ell+1 \quad \uparrow \text { column } \ell+1
$$

Figure 2: The basis matrix $\bar{M}={\bar{M} \overline{[k][k]}^{B(k)}}$ in the tridiagonal and lower Hessenberg case if $\ell+1 \notin B_{k}$. We have $\bar{M}_{[\ell][\ell]}={\overline{M_{[\ell][\ell]}}}_{B(k) \cap[\ell]}$.

As a consequence, the system of $k$ equations

$$
\begin{equation*}
{\overline{M_{[k][k]}}}_{B(k)} x=q_{[k]} \tag{4}
\end{equation*}
$$

includes the $\ell$ equations

$$
\begin{equation*}
{\overline{M_{[\ell][\ell]}}}_{B(k) \cap[\ell]} x_{[\ell]}=q_{[\ell]} . \tag{5}
\end{equation*}
$$

Since $B(k)$ is the optimal basis of $\operatorname{LCP}\left(M_{[k][k]}, q_{[k]}\right)$, the unique solution $\tilde{x}$ of (4) satisfies $\tilde{x} \geq 0$; see Lemma 2.1. Vice versa, the unique partial solution $\tilde{x}_{[\ell]} \geq 0$ of subsystem (5) shows that $B(k) \cap[\ell]=B(\ell)$, the unique optimal basis of $\operatorname{LCP}\left(M_{[\ell][\ell]}, q_{[\ell]}\right)$. Together with the choice of $\ell$, the statement of the theorem follows.

We remark that a variant of Theorem 4.1 for upper Hessenberg matrices can be obtained by considering lower right principal submatrices $M_{K K}$.

## 5 Polynomial-time algorithm

A basis test is a procedure to decide whether a given basis $B \subseteq[n]$ is optimal for $\operatorname{LCP}(M, q)$. According to Lemma 2.1, a basis test can be implemented in polynomial time, using Gaussian elimination. In the sequel, we will therefore adopt the number of basis tests as a measure of algorithmic complexity. Here is our main result.

Theorem 5.1. Let $M \in \mathbb{R}^{n \times n}$ be a lower Hessenberg $P$-matrix. The optimal basis $\tilde{B}=B(n)$ of $\operatorname{LCP}(M, q)$ can be found with at most $\binom{n+1}{2}$ basis tests.

Proof. We successively compute the optimal bases $B(-1), B(0), \ldots, B(n)$, where $B(-1)=B(0)=\emptyset$. To determine $B(k), k>0$, we simply test the $k+1$ candidates for $B(k)$ that are given by Theorem 4.1. In fact, we already know $B(k)$ after testing $k$ of the candidates. This algorithm requires a total of $\sum_{k=1}^{n} k=\binom{n+1}{2}$ basis tests.

Using an $O\left(n^{3}\right)$ Gaussian elimination procedure, we obtain an $O\left(n^{5}\right)$ algorithm - this is certainly not best possible. Faster algorithms are available if $M$ is a tridiagonal $Z$-matrix [5] or $K$-matrix [4, 3], but to our knowledge, the above algorithm is the first one to handle tridiagonal (and lower Hessenberg) Pmatrices in polynomial time. The case of upper Hessenberg matrices is analogous, see the remark at the end of Section 4

All upper and lower triangular P-matrices are hidden Z [12, meaning that linear complementarity problems with triangular P-matrices can be solved in polynomial time [6]. We can now also handle the "almost" triangular Hessenberg P-matrices. As we show next, there is a significant combinatorial difference between the two classes.

## 6 A tridiagonal example

Let us consider $\operatorname{LCP}(M, q)$ with

$$
M=\left(\begin{array}{rrrr}
36 & -81 & 0 & 0  \tag{6}\\
147 & 16 & -74 & 0 \\
0 & 114 & 28 & 171 \\
0 & 0 & -33 & 72
\end{array}\right), q=\left(\begin{array}{r}
1 \\
1 \\
-1 \\
1
\end{array}\right)
$$

This linear complementarity problem was found by a computer search, with the goal of establishing Lemma 6.2 below. It can be checked that $M$ is a tridiagonal P-matrix, but not a Z-matrix (a matrix with nonpositive off-diagonal entries). To show that some other known polynomial-time manageable matrix classes fail to contain all tridiagonal P-matrices, we need a new concept.

Definition 6.1. Let $\mathcal{O}(M, q)$ be the digraph with vertex set $2^{[n]}$ and arc set

$$
\left\{(B, B \oplus\{i\}): B \subseteq[n], i \in[n],\left(\bar{M}_{B}^{-1} q\right)_{i}<0\right\}
$$

This digraph was first studied by Stickney \& Watson [11. Under nondegeneracy of $\operatorname{LCP}(M, q)$, it has a unique sink that coincides with the optimal basis.

Lemma 6.2. For $M, q$ as in (6), $\mathcal{O}(M, q)$ contains the directed cycle $\{1,2,3,4\} \rightarrow\{1,2,3\} \rightarrow\{1,2\} \rightarrow$ $\{2\} \rightarrow \emptyset \rightarrow\{3\} \rightarrow\{3,4\} \rightarrow\{2,3,4\} \rightarrow\{1,2,3,4\}$.

We omit the elementary proof. This implies that $M$ cannot be a hidden Z-matrix, since for such matrices, $\mathcal{O}(M, q)$ is the acyclic digraph of some geometric hypercube in $\mathbb{R}^{n}$, with edges directed by a linear function [6]. For the same reason, the tridiagonal P-matrix $M^{T}$ cannot be the transpose of a hidden K-matrix (a hidden Z-matrix that is also a Pmatrix) [9]. We remark that Morris has constructed a family of lower Hessenberg matrices $M \in \mathbb{R}^{n \times n}$ such that $\mathcal{O}(M, q)$ is highly cyclic for suitable $q \in \mathbb{R}^{n}$ [8].

## 7 Beyond Hessenberg matrices

It is natural to ask whether $\operatorname{LCP}(M, q)$ can still be solved in polynomial time if $M$ is a matrix of fixed bandwidth (number of nonzero diagonals), or fixed half-bandwidth (number of nonzero diagonals above or below the main diagonal); see Figure 3


Figure 3: Matrices of bandwidth 5 (left) and right half-bandwidth 2 (right).

Let $M$ be of fixed right half-bandwidth $t$. Generalizing Theorem 4.1 one can prove that there are only polynomially many candidates for $B(k)$, provided that $B(k)$ has a $t$-hole, meaning that $B(k)$ is disjoint from some contiguous $t$-element subset of $[k]$.

The only subset of $[k]$ without a 1 -hole is the set [ $k$ ] itself, and this is why the lower Hessenberg case $t=1$ is easy. But there is already an exponential number of subsets of $[k]$ without a 2-hole. Hence, the above approach fails for $t \geq 2$. It remains open whether there is another polynomial-time algorithm in the case of fixed (right) bandwidth.

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[^0]:    *Institute of Theoretical Computer Science, ETH Zurich, CH-8092 Zurich, Switzerland (gaertner@inf.ethz.ch)

