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Analyzing Process Flexibility: A Distribution-free Approach with Partial Expectations

Hoda Bidkhori[a], David Simchi-Levi[b], Yehua Wei[c,1]

[a] University of Pittsburgh, Pittsburgh, PA 15260
[b] Institute for Data, Systems and Society, Operations Research Center, and Department of Civil and Environmental Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139
[c] Decision Sciences, The Fuqua School of Business, Duke University, Durham, NC 27708

Abstract

We develop a distribution-free model to evaluate the performance of process flexibility structures when only the mean and partial expectation of the demand are known. We characterize the worst-case demand distribution under general concave objective functions, and apply it to derive tight lower bounds for the performance of chaining structures under the balanced systems (systems with the same number of plants and products). We also derive a simple lower bound for chaining-like structures under unbalanced systems with different plant capacities.

Keywords: Process flexibility, distributionally-robust analysis, chaining, production system design

1. Introduction

Product demand has become increasingly volatile, due to global market competition, product proliferation, and the enormous impact social media has on customer behavior. This calls for new production systems that can better cope with an increasingly volatile demand. As a result, process flexibility is quickly becoming an option that manufacturers embrace [1]. Interestingly, firms often do not need to implement a fully flexible system (also known as the full flexibility structure), where each plant has the ability to produce all products in the system [2]. Indeed, the seminal paper of Jordan and Graves [2] shows that in simulation, a sparse flexibility structure known as the long chain (also known as the chaining) often performs almost as well as the full flexibility structure.

Email addresses: `bidkhori@pitt.edu` (Hoda Bidkhori), `dslevi@mit.edu` (David Simchi-Levi), `yehua.wei@duke.edu` (Yehua Wei)

[1] Correspondence Author. 100 Fuqua Drive, Box 90120, The Fuqua School of Business, Durham, NC, 27708

The objective of this paper is to develop a new tool to analyze the performance of various process flexibility structures, and in particular, the popular chaining structure proposed by the seminal work of Jordan and Graves. Instead of taking the traditional approach of computing the expected sales of a flexibility structure under a given demand distribution, our paper takes a different approach and studies the worst expected sales of flexibility structures under a class of stochastic demand distributions with limited information. This is closely related to the distributionally robust literature, where one seeks to identify the optimal solution under the worst-case distribution within a distributional uncertainty set. Unlike most papers in the distributionally robust literature, where the set of distributions is defined by moment constraints, we consider a distributional uncertainty set with given partial expectations. By considering the distributional uncertainty set with given partial expectations, we explicitly characterize the worst-case demand distribution and using this characterization, we derive a simple analytical bound for the expected sales of chaining structures. Because the demand distribution is rarely known to a high degree of accuracy, our method enables us to evaluate the performance of flexibility structures in unbalanced and non-homogenous system where limited demand distributional information is known.

1.1. Literature Review

The findings of [2] led to a series of researches to analytically study the effectiveness of the long chain and other sparse flexibility structures. [3] develops a method to compute the average demand satisfied by the chaining in asymptotically large systems; [4] and [5] analyze the chaining and other sparse flexibility structures under worst-case demand; [6] provides a characterization of the expected sales of the long chain and using the characterization, proves that the long chain always outperforms the shorter chains under i.i.d. demand; [7] uses probabilistic graph expanders to construct asymptotically optimal sparse structures; [8] analyzes the chaining with limited reserved capacities, and finally, [9] studies the problem of finding the optimal sparse flexibility configuration to achieve a given service level.

More closely related to this paper, [10] studies the k-chain (a structure where product i is capable of producing product i, $i+1$, ..., $i+k$) in asymptotically large balanced networks under i.i.d. demand using a distributionally-robust approach. The key difference between [10] and this work is that the former studies the worst-case demand distribution with given first and second moments, while this work studies the worst-case demand distribution with mean and partial expectations.

The advantage of our approach is that we provide the exact characterization of the worst-case demand distribution for any finite flexibility structure, which allows us to develop a tool to study the broader class of non-homogenous unbalanced finite flexibility structures. In contrast, [10] does not fully characterize the worst-case distribution, and their closed-form lower-bound is restricted to symmetric, balanced systems with system size going to infinity. We note that while the characterization of the worst-case distribution with partial expectations was known since the 1970s (see [11]), this paper is the first to apply this idea to analyze process flexibility structures.

2. Model and Assumptions

In this paper, we use $\mathbb{P}[\cdot]$ and $\mathbb{E}[\cdot]$ to denote the probability and the expectation functions of random variables. For two random variables D and D', we use $D \stackrel{d}{=} D'$ to denote that D and D' have the same probability distribution, i.e., $\mathbb{P}[D \leq x] = \mathbb{P}[D' \leq x]$ for all $x \in \mathbb{R}$.

We study a manufacturing system with n plants and m products, with $m \geq n$. For each $1 \leq i \leq n$, $1 \leq j \leq m$, c_i and D_j are used to denote the fixed capacity at plant i and the stochastic demand for product (type) j. A flexibility structure, denoted by $\mathscr{A}$, is a set of arcs connecting plant nodes to product nodes. In a flexibility structure $\mathscr{A}$, an arc $(i,j) \in \mathscr{A}$ implies that plant i is capable of producing product j. Given an instance $\mathbf{d}$ of the demand, the sales achieved by a flexibility structure $\mathscr{A}$, denoted by $S(\mathbf{d}, \mathscr{A})$, is defined as

$$\begin{aligned} S(\mathbf{d}, \mathscr{A}) := \max \quad & \sum_{(i,j)\in\mathscr{A}} f_{ij} \\ \text{s.t.} \quad & \sum_{i=1}^{n} f_{ij} \leq d_j, \; \forall 1 \leq j \leq m \\ & \sum_{j=1}^{n} f_{ij} \leq c_i, \; \forall 1 \leq i \leq n \\ & f_{ij} \geq 0, \forall (i,j) \in \mathscr{A}. \end{aligned}$$

Under stochastic demand $\mathbf{D}$, the expected sales of $\mathscr{A}$ is hence denoted by $\mathbb{E}[S(\mathbf{D}, \mathscr{A})]$. Throughout the paper, we assume that the demand vector $\mathbf{D}$ is consisted of m independent random variables, and use μ_j to denote the expected values of D_j.

In the paper, we are interested in providing a lower bound for $\mathbb{E}[S(\mathbf{D}, \mathscr{A})]$ when the expected demand, e.g., μ_j, and the partial expectations of $D_j - \mu_j$ on interval $[0, \infty)$, e.g., $\mathbb{E}[(D_j - \mu_j)^+]$, are known. Note that

$$\mathbb{E}[(D_j - \mu_j)^+] = \mathbb{E}[(\mu_j - D_j)^+], \text{ and } \mathbb{E}[(D_j - \mu_j)^+] + \mathbb{E}[(\mu_j - D_j)^+] = \mathbb{E}[|D_j - \mu_j|].$$

Therefore, $\mathbb{E}[(D_j - \mu_j)^+]$ is exactly one half of the expected absolute deviation of D_j from its mean. We say D_j is γ-*centralized* if $\mathbb{E}[(D_j - \mu_j)^+] \leq \gamma\mu_j$. Clearly, if γ is small, then D_j has most of its probability measure to be concentrated around its mean. Like variance, the partial expectations under consideration, $\mathbb{E}[(D_j - \mu_j)^+]$, informs us about the degree of centralization of the demand.

3. Characterizing the Worst-case Distribution

In this section, we first characterizes the worst-case distribution which in turn bounds the expected values of general stochastic concave objective functions. Then, we apply this result to provide lower bounds for the expected sales of process flexibility structures.

Proposition 1. *Let $f(\cdot) : \mathbb{R}^m \to \mathbb{R}$ be an arbitrary concave function, and let $\mathbf{E}$ be an independent m-dimensional random vector where for all $1 \leq j \leq m$,*

$$\mathbb{P}[-\Delta_j^- \leq E_j \leq \Delta_j^+] = 1, \mathbb{E}[(-E_j)^+] = \gamma_j^- \Delta_j^-, \mathbb{E}[(E_j)^+] = \gamma_j^+ \Delta_j^+,$$

where Δ_j^- and Δ_j^+ are positive reals. Then, we have that $\mathbb{E}[f(\mathbf{E}^*)] \leq \mathbb{E}[f(\mathbf{E})]$, where $\mathbf{E}^*$ is an independent m-dimensional random vector such that

$$\mathbb{P}[E_j^* = -\Delta_j^-] = \gamma_j^-, \mathbb{P}[E_j^* = \Delta_j^+] = \gamma_j^+, \mathbb{P}[E_j^* = 0] = 1 - \gamma_j^+ - \gamma_j^-, \forall 1 \leq j \leq m.$$

The proof of Proposition 1 is a straightforward application of [12], [11] and is relegated to the appendix. Here, we describe the intuition behind the proof of Proposition 1. For each $1 \leq j \leq m$, we have partial expectations of E_j on intervals $[-\Delta_j^-, 0]$ and $[0, \Delta_j^+]$. Because the objective function f is concave, and E_j is independent with $E_{j'}$ for any $j' \neq j$, we can "transport" the probability of E_j on $[-\Delta_j^-, 0]$ and $[0, \Delta_j^+]$ to the points $\{-\Delta_j^-, 0, \Delta_j^+\}$ and obtain a valid independent distribution with a smaller expected objective value. After we do this for each j from 1 to m, we obtain $\mathbf{E}^*$, a distribution with smaller expected objective value than $\mathbf{E}$.

Recall that D_j is γ_j-centralized if $\mathbb{E}[(D_j - \mu_j)^+] \leq \gamma_j \mu_j$. We next derive the result which allows us to characterize the distribution to lower-bound the expected sales of $\mathscr{A}$, when D_j is γ_j-centralized for each $1 \leq j \leq m$. Our derivation is done in two steps. In the first step, we show that $S(\mathbf{d}, \mathscr{A})$ is concave with respect to $\mathbf{d}$; in the second step, we apply Proposition 1 to obtain the worst-case distribution $\mathbf{D}^*$, for the set of all demand distributions where D_j is γ_j-centralized.

Lemma 1. *For any flexibility structure $\mathscr{A}$, $S(\mathbf{d}, \mathscr{A})$ is concave with respect to $\mathbf{d}$.*

Proof. Recall that $S(\mathbf{d}, \mathscr{A})$ is the objective of a linear program. Moreover, $S(\mathbf{d}, \mathscr{A})$ can be expressed as $S(\mathbf{d}, \mathscr{A}) = F(\mathbf{d}) = \max_{\mathbf{x} \in P(\mathbf{d})} \mathbf{c}^T \mathbf{x}$ for some vector $\mathbf{c}$, and some polyhedral $P(\mathbf{d}) = \{\mathbf{x} | \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$. By Theorem 5.1 on pg. 213 of [13], $-F(\mathbf{d}) = \min_{\mathbf{x} \in P(\mathbf{d})} -\mathbf{c}^T \mathbf{x}$ is convex with respect to $\mathbf{d}$ and therefore, $S(\mathbf{d}, \mathscr{A}) = F(\mathbf{d})$ is concave with respect to $\mathbf{d}$. $\square$

Proposition 2. *Let $\mathbf{D}$ be an m-dimensional independent demand vector where for $1 \leq j \leq m$, $\mathbb{E}[D_j] = \mu_j$, $\mathbb{P}[0 \leq D_j \leq \theta\mu_j] = 1$ with some $\theta > 1$ and D_j is γ_j-centralized, $\gamma_j \leq \frac{\theta-1}{\theta}$. Then, for any flexibility structure $\mathscr{A}$, we have $\mathbb{E}[S(\mathbf{D}^*, \mathscr{A})] \leq \mathbb{E}[S(\mathbf{D}, \mathscr{A})]$, where $\mathbf{D}^*$ is an m-dimensional independent demand vector such that*

$$\mathbb{P}[D_j^* = \theta\mu_j] = \frac{\gamma_j}{\theta - 1}, \mathbb{P}[D_j^* = 0] = \gamma_j, \mathbb{P}[D_j^* = \mu_j] = 1 - \frac{\theta\gamma_j}{\theta - 1}, \forall 1 \leq j \leq m,$$

Proof. For each $1 \leq j \leq m$, let $\gamma_j' := \frac{\mathbb{E}[(D_j - \mu_j)^+]}{\mu_j}$, and by definition, $\gamma_j' \leq \gamma_j$. Also, define function $G(\mathbf{g}) := S(\mathbf{g}+\boldsymbol{\mu}, \mathscr{A})$, where $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_m)$. By Lemma 1, $G(.)$ is concave. Applying Proposition 1, we have $\mathbb{E}[G(\mathbf{E}')] \leq \mathbb{E}[G(\mathbf{D} - \boldsymbol{\mu})]$, where $\mathbf{E}'$ is an independent vector with

$$\mathbb{P}[E_j' = (\theta - 1)\mu_j] = \frac{\gamma_j'}{\theta - 1}, \mathbb{P}[E_j' = -\mu_j] = \gamma_j', \text{ and } \mathbb{P}[E_j' = 0] = 1 - \frac{\theta\gamma_j'}{\theta - 1}, \forall 1 \leq j \leq m.$$

Next, define $\mathbf{D}' = \mathbf{E}' + \boldsymbol{\mu}$. Then we have that $\mathbb{E}[S(\mathbf{D}', \mathscr{A})] \leq \mathbb{E}[S(\mathbf{D}, \mathscr{A})]$. Now, define $\mathbf{D}^{(1)}$ as an independent random vector such that

$$D_j^{(1)} \overset{d}{=} D_j', \forall 2 \leq j \leq m, \mathbb{P}[D_1^{(1)} = \theta\mu_1] = \frac{\gamma_1}{\theta - 1}, \mathbb{P}[D_1^{(1)} = 0] = \gamma_1, \mathbb{P}[D_1^{(1)} = \mu_1] = 1 - \frac{\theta\gamma_1}{\theta - 1};$$

and let $g(x) := \mathbb{E}[S(\mathbf{D'}, \mathscr{A})|D'_1 = x]$. Then

$$\mathbb{E}[S(\mathbf{D'}, \mathscr{A})] = \mathbb{E}[g(D'_1)] = \gamma'_1 g(0) + \frac{\gamma'_1}{\theta - 1} g(\theta\mu_1) + (1 - \frac{\theta\gamma'_1}{\theta - 1}) g(\mu_1),$$
$$\mathbb{E}[S(\mathbf{D}^{(1)}, \mathscr{A})] = \mathbb{E}[g(D_1^{(1)})] = \gamma_1 g(0) + \frac{\gamma_1}{\theta - 1} g(\theta\mu_1) + (1 - \frac{\theta\gamma_1}{\theta - 1}) g(\mu_1),$$
$$\implies \mathbb{E}[S(\mathbf{D'}, \mathscr{A})] - \mathbb{E}[S(\mathbf{D}^{(1)}, \mathscr{A})] = (\gamma'_1 - \gamma_1) \cdot \frac{\theta}{\theta - 1} \cdot (\frac{(\theta - 1)g(0)}{\theta} + \frac{g(\theta\mu_1)}{\theta} - g(\mu_1)).$$

Because $g(.)$ is concave, we have $\frac{(\theta - 1)g(0)}{\theta} + \frac{g(\theta\mu_1)}{\theta} \leq g(\mu_1)$. Combining this with the fact that $\gamma'_1 - \gamma_1 \leq 0$, we have

$$\mathbb{E}[S(\mathbf{D}^{(1)}, \mathscr{A})] \leq \mathbb{E}[S(\mathbf{D'}, \mathscr{A})].$$

Next, for $2 \leq i \leq m$, define $\mathbf{D}^{(i)}$ recursively as an independent random vector such that

$$D_j^{(i)} \overset{d}{=} D_j^{(i-1)}, \forall j \neq i, 1 \leq j \leq m,$$
$$\text{and } \mathbb{P}[D_i^{(i)} = \theta\mu_i] = \frac{\gamma_i}{\theta - 1}, \mathbb{P}[D_i^{(i)} = 0] = \gamma_i, \mathbb{P}[D_i^{(i)} = \mu_i] = 1 - \frac{\theta\gamma_i}{\theta - 1}.$$

Apply the same procedure as we did for establishing $\mathbb{E}[S(\mathbf{D}^{(1)}, \mathscr{A})] \leq \mathbb{E}[S(\mathbf{D'}, \mathscr{A})]$, and we get that $\mathbb{E}[S(\mathbf{D}^{(i)}, \mathscr{A})] \leq \mathbb{E}[S(\mathbf{D}^{(i-1)}, \mathscr{A})]$. Note that $\mathbf{D}^* \overset{d}{=} \mathbf{D}^{(m)}$ and therefore, we have $\mathbb{E}[S(\mathbf{D}^*, \mathscr{A})] \leq \mathbb{E}[S(\mathbf{D'}, \mathscr{A})] \leq \mathbb{E}[S(\mathbf{D}, \mathscr{A})]$. □

Intuitively, Proposition 2 illustrates that for any flexibility structure $\mathscr{A}$, the expected sales of $\mathscr{A}$ under $\mathbf{D}$ is lower-bounded by the expected sales of $\mathscr{A}$ under $\mathbf{D}^*$, where for each j, D_j^* is a discrete random variable with exactly three probability mass points.

4. Worst-Case Distribution Analysis for Chaining Structures

In this section, we study the performance of process flexibility structures by applying Proposition 2. Throughout the section, we assume that the demand of product j, D_j, is always nonnegative and bounded from above almost surely. In particular, we fix an arbitrary θ and assume that $\mathbb{P}[0 \leq D_j \leq \theta\mu_j] = 1$.

4.1. Process Flexibility Structures

First, we formally define several flexibility structures of interest. We assume that a *dedicated* flexibility structure, denoted by $\mathscr{D}$, is the structure each product is produced from exactly one plant. Without loss of generality, we assume that for $1 \le i \le n$, $S_i = \{k_{i-1}+1, \ldots, k_i\}$, for some integers $0 = k_0 \le k_1 \le k_2 \le \cdots \le k_n = m$, and $\mathscr{D} := \{(i,j) | 1 \le i \le n, j \in S_i\}$. One can think of each S_i as a single product family, and in the dedicated structure, the firm only assigns plant i to produce product family i. It is useful to think of $\mathscr{D}$ as the *minimal* structure, because if there is no demand uncertainty, the firm needs at least $\mathscr{D}$ to satisfy the product demand.

Throughout the paper, we assume plant i has just enough capacity to match the total expected demand of all of the products in S_i, i.e., $c_i = \sum_{j=k_{i-1}+1}^{k_i} \mu_j$. This is a standard assumption in the literature (see [2], [3]), and can be interpreted as the firm having built just enough capacity to produce its product families under the dedicated production system. Without loss of generality, we assume that for $1 \le i \le n$, the expected demands of products in S_i are increasing with their labels, i.e., $\mu_{k_i+1} \le \mu_{k_i+2} \le \cdots \le \mu_{k_{i+1}}$; we also assume that the expected demand of the last product in S_i is increasing with i, that is, $\mu_{k_i} \le \mu_{k_{i+1}}$.

One of the flexibility structures that received the most attention in the literature is the long chain, which is defined by $\mathscr{C} := \mathscr{D} \cup \{(i, k_{i+1}) | 1 \le i \le n\}\}$, where we assume $k_{n+1} = k_1$ (see Figure 1 for an example of the long chain with $n = 3$, $m = 6$). In words, the long chain structure has each plant not just producing its own product family, but also one other product in a different product family. We note that the long chain is typically defined in a system with $m = n$ (see [2]). Therefore, our definition can be seen as generalization of the original definition to a more general m plants n products system. Finally, full flexibility structure, denoted as $\mathscr{F} = \{(i,j) | 1 \le i \le n, 1 \le j \le m\}$, is a structure where every plant node is connected to every product node.

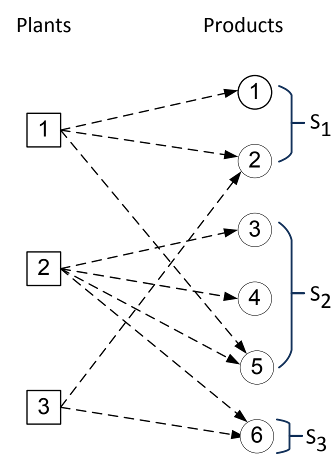

Figure 1: Illustration of a long chain in an unbalanced system

4.2. Balanced and Homogenous Systems

We start the analysis of process flexibility structures in the balanced systems, i.e., $m = n$, and all products have the same expected demand, i.e., $\mu_j = \mu_{j'}$ for any $1 \leq j, j' \leq m$. Without loss of generally, we can assume $\mu_j = 1$ for all $1 \leq j \leq m$. The parsimonious assumption of having $m = n$ and all products having the same expected demand has been popular for understanding the effectiveness of long chain and sparse process flexibility structures (see [2], [3] and [10]). We also assume that $n \geq 2$.

Now, suppose that for all $1 \leq j \leq m$, $\mathbb{P}[0 \leq D_j \leq \theta] = 1$, and D_j is γ-centralized, i.e., $\mathbb{E}[(D_j - 1)^+] = \mathbb{E}[(1 - D_j)^+] \leq \gamma$. The next lemma discusses the range of γ.

Lemma 2. *For any $1 \leq j \leq m$, suppose that $\mathbb{P}[0 \leq D_j \leq \theta] = 1$ and $\mathbb{E}[D_j] = 1$. Then we must have that*

$$\mathbb{E}[(D_j - 1)^+] \leq \frac{\theta - 1}{\theta}.$$

Proof. Note that $f(x) = (x - 1)^+$ is convex in x. Because $\mathbb{E}[D_j] = 1$, by Proposition 1, we have

$$\frac{\theta - 1}{\theta} = \mathbb{E}[(D_j^* - 1)^+] \geq \mathbb{E}[(D_j - 1)^+], \text{ where } \mathbb{P}[D_j^* = \theta] = \tfrac{1}{\theta}, \mathbb{P}[D_j^* = 0] = 1 - \tfrac{1}{\theta}.$$

□

Lemma 2 suggests that under the assumption $\mathbb{P}[0 \leq D_j \leq \theta] = 1$ and $\mathbb{E}[D_j] = 1$, we can always find some $\gamma \leq \frac{\theta-1}{\theta}$ such that D_j is γ-centralized for all $1 \leq j \leq m$. Therefore, in the rest of this section, we always assume that $\gamma \leq \frac{\theta-1}{\theta}$. Applying Proposition 2, we get

$$\mathbb{E}[S(\mathbf{D}^*, \mathscr{A})] \leq \mathbb{E}[S(\mathbf{D}, \mathscr{A})], \tag{1}$$

where $\mathbf{D}^*$ is a vector of m independent random variables such that $\mathbb{P}[D_j^* = \theta] = \gamma/(\theta - 1), \mathbb{P}[D_j^* = 0] = \gamma, \mathbb{P}[D_j^* = 1] = 1 - \theta\gamma/(\theta - 1)$, for each $1 \leq j \leq m$. Next, we show that the expected sales of the long chain (denoted by $\mathscr{C}$) under $\mathbf{D}^*$ has a surprisingly simple analytical form.

Proposition 3. *Suppose $\theta \geq 2$ and let $\mathbf{D}^*$ be an independent demand vector where $\mathbb{P}[D_j^* = \theta] = \frac{\gamma}{\theta-1}, \mathbb{P}[D_j^* = 0] = \gamma$ and $\mathbb{P}[D_j^* = 1] = 1 - \frac{\theta\gamma}{\theta-1}$. Then*

$$\mathbb{E}[S(\mathbf{D}^*, \mathscr{C})] \ = \ m\Big(1 - \frac{\gamma(\theta - 1)}{\theta} - \frac{\gamma}{\theta}(1 - \frac{\theta\gamma}{\theta - 1})^{m-1}\Big). \tag{2}$$

Proof. We first apply Theorem 5 in [6]. Let $W^*_{i+1} = \min(1, (W^*_i + 1 - D^*_i)^+)$, and let $W^*_0 = 0$ with probability 1. By [6], we have

$$\mathbb{E}[S(\mathbf{D}^*, \mathscr{C})] = m \cdot \mathbb{E}[\min(D^*_m, 1 + W^*_{m-1})]. \tag{3}$$

We claim that for each nonnegative integer i,

$$\mathbb{P}[W^*_i = 0] = \frac{1}{\theta} + (1 - \frac{\theta\gamma}{\theta - 1})^i \cdot \frac{\theta - 1}{\theta}, \text{ and } \mathbb{P}[W^*_i = 1] = \frac{\theta - 1}{\theta} - (1 - \frac{\theta\gamma}{\theta - 1})^i \cdot \frac{\theta - 1}{\theta}. \tag{4}$$

The claim is proved by induction. First, by the definition of W^*_0, the claim is true for $i = 0$. Then, note that because $W^*_{i+1} = \min(1, (W^*_i + 1 - D^*_i)^+)$, if the claim is true for W^*_i, then we have

$$\begin{aligned}\mathbb{P}[W^*_{i+1} = 0] &= (1-\gamma) \cdot \Big(\frac{1}{\theta} + (1 - \frac{\theta\gamma}{\theta - 1})^i \cdot \frac{\theta - 1}{\theta}\Big) + \frac{\gamma}{\theta - 1} \cdot \Big(\frac{\theta - 1}{\theta} - (1 - \frac{\theta\gamma}{\theta - 1})^i \cdot \frac{\theta - 1}{\theta}\Big)\\ &= \frac{1}{\theta} + (1 - \frac{\theta\gamma}{\theta - 1})^{i+1} \cdot \frac{\theta - 1}{\theta}.\end{aligned}$$

Finally, note that because $\theta > 2$, $\min(1, (W^*_i + 1 - D^*_i)^+)$ can only take values 0 or 1. This implies that $\mathbb{P}[W^*_{i+1} = 1] = 1 - \mathbb{P}[W^*_{i+1} = 0] = \frac{\theta-1}{\theta} - (1 - \frac{\theta\gamma}{\theta-1})^i \cdot \frac{\theta-1}{\theta}$. This completes the induction.

Finally, applying Equation (4) to Equation (3), we have

$$\mathbb{E}[S(\mathbf{D}^*, \mathscr{C})] = m \cdot \mathbb{E}[\min(D^*_m, 1 + W^*_{m-1})] = m \cdot \Big(1 - \frac{\gamma(\theta - 1)}{\theta} - \frac{\gamma}{\theta}(1 - \frac{\theta\gamma}{\theta - 1})^{m-1}\Big).$$

□

Combining Equation (1) and Proposition 3 immediately provides us with the following lower bound on the expected sales of $\mathscr{C}$ for any $\mathbf{D}$ where D_j is γ-centralized for all $1 \le j \le m$.

Theorem 1. *Fix any $\theta \ge 2$ and any independent demand vector $\mathbf{D}$. Suppose for all $1 \le j \le m$, D_j is γ-centralized, $\mathbb{E}[D_j] = 1$ and $\mathbb{P}[0 \le D_j \le \theta] = 1$, then*

$$m\Big(1 - \frac{\gamma(\theta - 1)}{\theta} - \frac{\gamma}{\theta}(1 - \frac{\theta\gamma}{\theta - 1})^{m-1}\Big) \le \mathbb{E}[S(\mathbf{D}, \mathscr{C})].$$

Proof. Immediate from Equation (1) and Proposition 3. □

We note that the lower bound from Theorem 1 is tight under the three-point distribution described in Proposition 3. We also note that the lower bound is largest when $\theta = 2$, which implies that the demand never exceeds twice of its mean.

Next, we numerically compare the lower bounds provided by Theorem 1 and the paper of Wang and Zhang [10] (abbreviated as WZ). While our bound applies to systems of any finite size, the closed-form bound in WZ only applies to asymptotically large systems. Therefore, in Table 1, we only compare the lower bounds provided by Theorem 1 and WZ for chain structures in asymptotically large systems. We considered five different distributions: continuous uniform distribution over interval [0,2], discrete uniform distribution over set $0, 1, 2$, and three normal distributions all with mean 1 but standard deviations of 0.1, 0.2 and 0.3. Because the probability of the normal distributions being greater than 2 is very small, we compute our bound using $\theta = 2$. The results in Table 1 suggest that Theorem 1 is not as effective as WZ's bound numerically, unless the distribution is close to the worst-case three-point distributions. Nevertheless, the advantage of Theorem 1 is that it provides a simple closed-form expression for any finite system. We note that while [10] describes a method to bound the effectiveness of finite sized systems, the bound can be only obtained numerically via semidefinite programs with a large number of constraints.

	Theorem 1	WZ	Actual
Uniform[0,2]	87.5%	86.8%	89.6%
Uniform{0,1,2}	83.3%	77.7%	83.3%
Normal(1,0.1)	98.0%	99.5%	99.6%
Normal(1,0.2)	96.0%	98.1%	98.4%
Normal(1,0.3)	94.0%	95.9%	96.7%

Table 1: Asymptotic lower bound for long chain

Expected sales is just one way to measure the effectiveness of a flexibility structure. Another way to quantify the effectiveness of a flexibility structure $\mathscr{A}$ is to compute

$$R(\mathscr{A}) := \frac{\mathbb{E}[S(\mathbf{D}, \mathscr{A})] - \mathbb{E}[S(\mathbf{D}, \mathscr{D})]}{\mathbb{E}[S(\mathbf{D}, \mathscr{F})] - \mathbb{E}[S(\mathbf{D}, \mathscr{D})]},$$

which is the increase in expected sales of structure $\mathscr{A}$ over a dedicated (no flexibility) structure, normalized by the maximal possible increase in expected sales achieved by the full flexibility structure. We refer to $R(\mathscr{A})$ as the *effectiveness ratio* of $\mathscr{A}$. Note that given $\mathbb{E}[(1 - D_j)^+] = \gamma$ for all

$j = 1, \ldots m$, the expected sales of $\mathscr{D}$ is then equal to

$$\mathbb{E}[S(\mathbf{D}, \mathscr{D})] = \sum_{j=1}^{m} \mathbb{E}[\min\{1, D_j\}] = \sum_{j=1}^{m} \mathbb{E}[1 - (1 - D_j)^+] = (1 - \gamma)m.$$

Next, we lower bound $R(\mathscr{C})$ in large systems, under general i.i.d. demand distributions.

Corollary 1. *Let the products demand vector* $\mathbf{D}$ *be i.i.d., where* $D_j \overset{d}{=} D$ *for some scalar distribution* D *and any* $1 \leq j \leq m$. *Suppose* $\mathbb{P}[0 \leq D \leq \theta] = 1$ *and* $\mathbb{E}[D] = 1$. *Then as the system size goes to infinity, i.e.* $m \to \infty$, *we have*

$$R(\mathscr{C}) \geq \frac{1}{\theta}.$$

Proof. Fix the system size m, and let $\gamma = \mathbb{E}[(D-1)^+] = \mathbb{E}[(1-D)^+]$. By Proposition 3,

$$R(\mathscr{C}) \geq \frac{m\Big(1 - \frac{\gamma(\theta-1)}{\theta} - \frac{\gamma}{\theta}(1 - \frac{\theta\gamma}{\theta-1})^{m-1}\Big) - m(1-\gamma)}{\mathbb{E}[S(\mathbf{D}, \mathscr{F})] - m(1-\gamma)}.$$

As $m \to \infty$, we know $\frac{\mathbb{E}[S(\mathbf{D}, \mathscr{F})]}{m} \to 1$ by the central limit theorem, while $\frac{\gamma}{\theta}(1 - \frac{\theta\gamma}{\theta-1})^{m-1} \to 0$. Thus, we have that as $m \to \infty$,

$$\frac{m\Big(1 - \frac{\gamma(\theta-1)}{\theta} - \frac{\gamma}{\theta}(1 - \frac{\theta\gamma}{\theta-1})^{m-1}\Big) - m(1-\gamma)}{\mathbb{E}[S(\mathbf{D}, \mathscr{F})] - m(1-\gamma)} \to \frac{\gamma/\theta}{\gamma} = \frac{1}{\theta}.$$

□

Interestingly, the bound in Corollary 1 is completely *independent* of γ. In particular, Corollary 1 implies that the long chain achieves at least 50% of the effectiveness of full flexibility, under any i.i.d. demand where $\mathbb{P}[0 \leq D_1 \leq 2]$. While this is an asymptotic ratio, it has been empirically observed [14], [10] that the long chain is more effective relative to full flexibility in smaller size systems. Therefore, Corollary 1 implies that the long chain should achieve at least 50% of the effectiveness of full flexibility, for any i.i.d. demand that is bounded above by twice of its expected value.

Finally, we briefly discuss the extension of our results to k-chains. In a system with m plants and m products, for any positive integer k, the k-chain is a structure where plant i is capable of producing products $i, i+1, \ldots, i+k$ (modulo m), while the long chain is a special case of the

k-chain when $k = 2$. The k-chains (for $k \geq 3$) are often used when demand uncertainties are large and the long chain is no longer effective (see [10] and [8]). Interestingly, it has been observed that the expected sales of the k-chain for an asymptotically large system can be characterized as a random walk. Next, we apply this characterization to bound the expected sales of the k-chain in asymptotically large systems. The proof of the corollary is relegated to appendix.

Corollary 2. *For any positive integer k, let $\mathscr{C}(k)$ be the k-chain in the system with m plants and m products. Fix any $\theta \geq 2$ and any scalar distribution D, and let the products demand vector $\mathbf{D}$ be i.i.d., where $D_j \overset{d}{=} D$ for all any $1 \leq j \leq m$. Suppose D is γ-centralized, $\mathbb{E}[D] = 1$ and $\mathbb{P}[0 \leq D_j \leq \theta] = 1$. Then we have*

$$\lim_{m\to\infty} \frac{\mathbb{E}[S(\mathbf{D}, \mathscr{C}(k))]}{m} \leq \mathbb{E}[\min(D^*, 1 + W^*)],$$

where $\mathbb{P}[D^* = \theta] = \frac{\gamma}{\theta-1}, \mathbb{P}[D^* = 0] = \gamma$, $\mathbb{P}[D^* = 1] = 1 - \frac{\theta\gamma}{\theta-1}$, and $W^* = \min(k-1, (W^* + 1 - D^*)^+)$.

4.3. Unbalanced Systems with Non-Homogenous Demands and Capacities

In this section, we consider the unbalanced system ($m \geq n$, $m \geq 2$) with non-homogenous independent demands where μ_j is the expected demand of product j. Again, we apply Proposition 2 to provide a lower bound on the expected sales of $\mathscr{C}$. Recall that $\mathscr{C} = \mathscr{D} \cup \{(i, k_{i+1}) | 1 \leq i \leq n\}$. Note that when $m \neq n$, the long chain contains exactly $m + n$ flexibility arcs, which is very sparse comparing given that the full flexibility structure contains mn arcs. Also, for the sake of clarity, we restrict our analysis to the case where D_j always lies in the interval $[0, 2\mu_j]$ for each j.

Theorem 2. *Suppose that for any $1 \leq j \leq m$, and $\mathbb{P}[0 \leq D_j \leq 2\mu_j] = 1$. Let $\gamma = \max_{1\leq j\leq m} \mathbb{E}[(D_j - \mu_j)^+]$, then we have*

$$\mathbb{E}[S(\mathbf{D}, \mathscr{C})] \geq (1 - \gamma + \gamma^2) \sum_{j=1}^{m} \mu_j + \gamma^2(\mu_{k_1} - \mu_{k_n}).$$

In the interest of space, we provide the proof of Theorem 2 in the appendix. A special case of Theorem 2 is when products demand is i.i.d. In that case, let $\mu_1 = \mu_2 = ... = \mu_m = 1$, and we get

$$\mathbb{E}[S(\mathbf{D}, \mathscr{C})] \geq (1 - \gamma + \gamma^2)m. \tag{5}$$

An implication of Theorem 2 is that if demand is i.i.d. and D_j always lies in the interval $[0, 2\mu_j]$ for each $1 \leq j \leq m$, the expected sales of the long chain is at least 75% of the expected sales of the full flexibility for any distribution. To see this, note that the minimal value for $1 - \gamma + \gamma^2$ is minimized at $\gamma = 1/2$, and applying this to Equation 5, we obtain that $\mathbb{E}[S(\mathbf{D}, \mathscr{C})] \geq \frac{3}{4}m$. Because m is an upper bound for the expected sales of full flexibility, we have that long chain the achieves at least 75% of the expected sales of the full flexibility for any independent distribution under the assumption that $\mathbb{P}[0 \leq D_j \leq 2\mu_j] = 1$ for each j from 1 to m.

We also note that while Theorem 2 applies to unbalanced system with non-homogenous plant and products, it is not as tight as the lower bound stated in Theorem 1. To see this, observe that if $k_1 = k_2 = ... = k_n = 1$, we have from Equation (5) that

$$\mathbb{E}[S(\mathbf{D}, \mathscr{C})] \geq (1 - \gamma + \gamma^2)m = (1 - \frac{\gamma}{2} - \frac{\gamma}{2}(1 - 2\gamma))m,$$

while Theorem 1 gives us

$$\mathbb{E}[S(\mathbf{D}, \mathscr{C})] \geq (1 - \frac{\gamma}{2} - \frac{\gamma}{2}(1 - 2\gamma)^{m-1})m.$$

Moreover, for any fixed m, the lower bound from Theorem 1 is tight, but the lower bound from Theorem 2 is only tight when $m = n = 2$. Nevertheless, to the best of our knowledge, Theorem 2 gives us the first distribution-free lower bound for the chaining structure in unbalanced systems with $m > n$.

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Appendix A. Additional Proofs

Available at *http://ssrn.com/abstract=2730345*.