

Proximal Point Method for Vector Optimization on Hadamard Manifolds

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Abstract

In this paper, we extend the proximal point algorithm for vector optimization from the Euclidean space to the Riemannian context. Under suitable assumptions on the objective function the well definition and full convergence of the method to a weak efficient point is proved.

Keywords: Proximal method; Vector optimization; Fejér convergence; Hadamard manifolds.

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1 Introduction

In recent years, extensions to Riemannian manifolds of concepts and techniques which fit in linear spaces are natural. Several algorithms for optimization problem which involve convexity of the objective function have been extended from the linear settings to the Riemannian context; see, for instance, [1, 2, 18, 21, 25, 30, 31] and the references therein. One reason for the success of this extension is the possibility to transform, by introducing a suitable Riemannian metric, nonconvex problems in the linear context into convex problems in the Riemannian context; see [10, 16, 17, 28].

In the last few years, researchers began the study of the vector optimization problems on Riemannian manifolds context; papers dealing with this issues include Bento and Cruz Neto [4], Bento et al. [5], Bento et al. [8] and Bonnel et al. [12]. The present paper deals with the extension of the proximal point method for vector optimization from the Euclidean settings to the Riemannian context, which continues the subject addressed in the following papers [18, 9, 7, 31, 22]. To our best knowledge, this is the first paper extending the proximal point method for vector optimization to the Riemannian settings. Besides our approach is new even in Euclidean context, since we are dealing with general convex cone and the nonlinear scalarization is more flexible than the considered in [6]. Under suitable assumptions on the objective function the well definition and full convergence of the method to a weak efficient point is proved. It is worth to point out that under assumption of null sectional curvature, our algorithm retrieves the proximal point method for multiojective presented in [6] and, somehow, goes further.

This paper is organized as follows. In Section 1.1, we present the notations and terminology used in the paper. In Section 2, we present the vector optimization problem and the proximal point method. Our main results are stated and proved in Section 3, and conclusions are discussed in Section 4.

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1.1 Notation and Terminology

In this section, we introduce some notations about Riemannian geometry, which can be found in any introductory book on Riemannian geometry, such as in Sakai [26], Do Carmo [13]. Let M be a n -dimensional Hadamard manifold. *In this paper, all manifolds M are assumed to be Hadamard finite dimensional.* We denote by T_pM the n -dimensional *tangent space* of M at p , by $TM = \cup_{p \in M} T_pM$ *tangent bundle* of M and by $\mathcal{X}(M)$ the space of smooth vector fields over M . The Riemannian distance between $p, q \in M$, denoted by $d(p, q)$, and given a nonempty set $U \subset M$, the distance function associated to U is given by:

$$M \ni p \mapsto d(p, U) := \inf\{d(p, q) : q \in U\}.$$

Since M is a Hadamard manifold, the Riemannian distance d induces the original topology on M , namely, (M, d) is a complete metric space and bounded and closed subsets are compact. The open metric ball at $p \in M$ is given by $B(p, r) := \{q \in M : d(p, q) < r\}$, where $r > 0$. The Riemannian metric is denoted by $\langle \cdot, \cdot \rangle$ and the corresponding norm by $\|\cdot\|$. The metric induces a map $g \mapsto \text{grad } g \in \mathcal{X}(M)$ which associates to each function differentiable over M its gradient via the rule $\langle \text{grad } g, X \rangle = dg(X)$, $X \in \mathcal{X}(M)$. The geodesic determined by its position p and velocity v at p is denoted by $\gamma = \gamma_v(\cdot, p)$. The restriction of a geodesic to a closed bounded interval is called a *geodesic segment*. Since M is a Hadamard manifold the length of the geodesic segment γ joining p to q is equals $d(p, q)$. Moreover, *exponential map* $\exp_p : T_pM \rightarrow M$ is defined by $\exp_p v = \gamma_v(1, p)$ is a diffeomorphism and, consequently, M is diffeomorphic to the Euclidean space \mathbb{R}^n , $n = \dim M$. Let $q \in M$ and $\exp_q^{-1} : M \rightarrow T_pM$ be the inverse of the exponential map. Note that $d(q, p) = \|\exp_p^{-1} q\|$, the function $d_q^2 : M \rightarrow \mathbb{R}$ defined by $d_q^2(p) = d^2(q, p)$ is C^∞ and $\text{grad } d_q^2(p) := -2\exp_p^{-1} p$. Furthermore, we know that

$$d^2(p_1, p_3) + d^2(p_3, p_2) - \langle \exp_{p_3}^{-1} p_1, \exp_{p_3}^{-1} p_2 \rangle \leq d^2(p_1, p_2). \quad p_1, p_2, p_3 \in M. \quad (1)$$

A set $\Omega \subseteq M$ is said to be *convex* if any geodesic segment with end points in Ω is contained in Ω , that is, if $\gamma : [a, b] \rightarrow M$ is a geodesic such that $x = \gamma(a) \in \Omega$ and $y = \gamma(b) \in \Omega$; then $\gamma((1-t)a + tb) \in \Omega$ for all $t \in [0, 1]$. Let $g : M \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function. The *domain* of g is the set $\text{dom } g := \{p \in M : g(p) < \infty\}$. The function g is said to be *proper* if $\text{dom } g \neq \emptyset$ and *convex* (resp. *strictly convex*, *strongly convex*) on a convex set $\Omega \subset \text{dom } g$ if for any geodesic segment $\gamma : [a, b] \rightarrow \Omega$ the composition $g \circ \gamma : [a, b] \rightarrow \mathbb{R}$ is convex (resp. *strictly convex*, *strongly convex*). It is well known that, for each $q \in M$, d_q^2 is strongly convex. Take $p \in \text{dom } g$. A vector $s \in T_pM$ is said to be a *subgradient* of g at p , if $g(q) \geq g(p) + \langle s, \exp_p^{-1} q \rangle$, for $q \in M$. The set $\partial g(p)$ of all subgradients of g at p is called the *subdifferential* of g at p . We remark that, if g is convex then $\partial g(p) \neq \emptyset$ for all $p \in \text{dom } g$. Moreover, if g is differentiable at p then $\partial g(p) = \{\text{grad } g(p)\}$. The function g is *lower semicontinuous* (lsc) at $\bar{x} \in \text{dom } g$ if for each sequence $\{x^n\}$ converging to \bar{x} we have $\liminf_{n \rightarrow \infty} g(x^n) \geq g(\bar{x})$. Given a closed set $\Omega \subset M$, it is known that indicator function of Ω , $I_\Omega : M \rightarrow \mathbb{R} \cup \{+\infty\}$, is a lower semicontinuous function and, for each $p \in M$, $\partial I_\Omega(p) = N_\Omega(p)$, where

$$N_\Omega(p) := \{v \in T_pM : \langle v, \exp_p^{-1} q \rangle \leq 0, q \in \Omega\}.$$

Definition 1.1. A sequence $\{p^k\} \subset (M, d)$ is said to *Fejér convergence* to a set $W \subset M$ if, for every $q \in W$ we have $d^2(w, p^{k+1}) \leq d^2(w, p^k)$.

Proposition 1.1. Let $\{p^k\}$ be a sequence in (M, d) . If $\{p^k\}$ is Fejér convergent to non-empty set $W \subset M$, then $\{p^k\}$ is bounded. If furthermore, an accumulation point p of $\{p^k\}$ belongs to W , then $\lim_{k \rightarrow \infty} p^k = p$.

2 The Proximal Point Method for Vector Optimization

In this section, we present the vector optimization problem, some concepts and results related to this problem, and introduce the proximal point method for this problem.

Let $C \subset \mathbb{R}^m$ be a closed, pointed and convex cone. We will use the binary relations \preceq_C and \prec_C defined, respectively, by $p \preceq_C q$ means $q - p \in C$ and $p \prec_C q$ means $q - p \in \text{int}C$, for all $p, q \in \mathbb{R}^m$. Given a continuously differentiable vector function $F : M \rightarrow \mathbb{R}^m$, we consider the problem of finding an *efficient point* of F , i.e., a point $p^* \in \mathbb{R}^n$ such that there exists no $p \in M$ with $F(p) \preceq_C F(p^*)$ and $F(p) \neq F(p^*)$. We denote this unconstrained problem as

$$C - \text{Min}_{p \in M} F(p). \quad (2)$$

We say that $p^* \in M$ is a *weakly efficient point* of (2) if there is no $p \in M$ such that $F(p) \prec_C F(p^*)$. The set of the weakly efficient points of (2) is denoted by $C - \text{argmin}_w \{F(p) | p \in M\}$. Throughout this paper we assume that (2) satisfies the following assumption:

(A1) There exists a compact set $Z \subset \mathbb{R}^m \setminus \{0\}$ such that $C = \{y \in \mathbb{R}^m : \langle y, z \rangle \geq 0, z \in Z\}$.

Remark 2.1. In classical optimization $C = \mathbb{R}_+$ and we can take $Z = \{1\}$. For multiobjective optimization, C is the positive orthant of \mathbb{R}^m and we can take Z as the canonical base of \mathbb{R}^m . For a generic cone C we can take $Z = \{z \in C^* : \|z\|_1 = 1\}$, where $C^* := \{y \in \mathbb{R}^m : \langle y, x \rangle \geq 0, x \in \mathbb{R}^m\}$ and $\|z\|_1 := |z_1| + |z_2| + \dots + |z_m|$.

We consider the following nonlinear scalar function $f : \mathbb{R}^m \rightarrow \mathbb{R}$, which will play an important role in our analysis, defined by

$$f(y) := \inf \{t \in \mathbb{R} : te \in y + C\}, \quad (3)$$

where e is any fixed point in $\text{int}C$; see [32]. In [15, Proposition 1.44] it was proved that the nonlinear function above can be rewritten as follow

$$f(y) = \max_{z \in Z} \frac{\langle y, z \rangle}{\langle e, z \rangle}. \quad (4)$$

Remark 2.2. For the multiobjective case, (3) becomes $f(y) = \max_{i \in I} \langle y, e_i \rangle$, where $\{e_i\} \subset \mathbb{R}^m$ is the canonical base of the space in \mathbb{R}^m , which has been used in [6].

Next lemma, which proof is trivial, gives us some properties of the function above that will be useful through the paper.

Lemma 2.1. Let $f : \mathbb{R}^m \rightarrow \mathbb{R}$ be a nonlinear function defined in (4), then following properties hold:

i) Given $y \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$ and $t > 0$, $f(ty + \alpha e) = tf(y) + \alpha$;

ii) If $y \preceq_C z$ then $f(y) \leq f(z)$ for any $y, z \in \mathbb{R}^m$.

Proposition 2.1. Let $F : M \rightarrow \mathbb{R}^m$ be a vectorial function and $C \subset M$ a closed set. Then,

$$\text{argmin}_{p \in C} f(F(p)) \subset \text{argmin}_{p \in C} F(p).$$

Proof. It is similar to the proof of [6, Proposition 3.1]. □

Now we introduce the proximal point method for vector optimization. Let $\{\lambda^k\}$ be a sequence of positive numbers and $\{e^k\} \subset \text{int } C$ such that $\|e^k\| = 1$, for $k = 0, 1, \dots$. Consider the sequence of functions

$$f_k(y) := \max_{z \in Z} \frac{\langle y, z \rangle}{\langle e^k, z \rangle}, \quad k = 0, 1, \dots \quad (5)$$

The *proximal point method* for solving (2), with starting point $p^0 \in M$, is defined by

$$p^{k+1} := \operatorname{argmin}_{p \in \Omega_k} f_k \left(F(p) + \frac{\lambda_k}{2} d^2(p, p^k) e^k \right), \quad k = 0, 1, \dots, \quad (6)$$

where $\Omega_k := \{p \in M : F(p) \preceq_C F(p^k)\}$. From now on $\{p^k\}$ denotes the sequence generated by the proximal point method, with starting point $p^0 \in M$.

3 Convergence Analysis

In this section, we prove the full convergence of the proximal point method to a weak efficient point. For this purpose, we need to define the convexity of a function with respect to the order induced by C . A vectorial function $F : M \rightarrow \mathbb{R}^m$ is called *C-convex* if, for $p, q \in M$ and $\gamma : [0, 1] \rightarrow M$ a geodesic segment joining p to q , there holds $F(\gamma(t)) \preceq_C (1-t)F(p) + tF(q)$, for all $t \in [0, 1]$.

We also need of the following assumption:

$$\mathbf{(A2)} \quad \bar{\Omega} := \bigcap_{k=0}^{\infty} \Omega_k \neq \emptyset.$$

Remark 3.1. In general the set $\bar{\Omega}$ in **(A2)** can be an empty set. One way to guarantee that $\bar{\Omega}$ is nonempty is to assume: for each $p^0 \in M$ the set $(F(p^0) - C) \cap F(M)$ is *C-complete* (see [24, Section 19]), meaning that each sequence $\{q^k\} \subset M$, with $q^0 = p^0$, such that $F(q^{k+1}) \preceq_C F(q^k)$, for $k = 0, 1, \dots$, there exists $q \in M$ such that $F(q) \preceq_C F(q^k)$, for $k = 0, 1, \dots$. This assumption is standard to ensure the convergence of descent methods in vector optimization; see, for instance, [11, 14, 19, 20, 29].

Now we ready to state and prove the main result of this section.

Theorem 3.1. Let $F : M \rightarrow \mathbb{R}^m$ be a *C-convex* function, and assume that **(A1)** and **(A2)** hold and $\{\lambda^k\}$ is bounded. Then, $\{p^k\}$ is well defined and converges to a weakly efficient point.

Proof. Let $\{f_k\}$ be the sequence of functions defined in (5), and define $\varphi_k : M \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$\varphi_k(p) = f_k \left(F(p) + I_{\Omega_k}(p) e^k + \lambda_k d^2(p, p^k) e^k \right), \quad k = 0, 1, \dots,$$

where I_{Ω_k} is the indicator function of Ω_k . From item *i* of Lemma 2.1 we have

$$\varphi_k(p) = f_k(F(p)) + I_{\Omega_k}(p) + \frac{\lambda_k}{2} d^2(p, p^k), \quad k = 0, 1, \dots \quad (7)$$

Since F is *C-convex*, then $f_k \circ F$ is convex and Ω_k is a convex and closed set. Hence φ_k is strongly convex and lower semicontinuous on Ω_k , for $k = 0, 1, \dots$. Thus, there exists a unique $p^{k+1} \in \Omega_k$ such that

$$p^{k+1} = \operatorname{argmin}_{p \in M} \varphi_k(p), \quad k = 0, 1, \dots, \quad (8)$$

which implies that $\{p^k\}$ is well defined and the first part of the proposition is proved.

Using convexity of φ_k and (8) we conclude that $0 \in \partial\varphi_k(p^{k+1})$, which from (7) yields

$$0 \in \partial \left(f_k \circ F(\cdot) + I_{\Omega_k}(\cdot) + \frac{\lambda_k}{2} d^2(\cdot, p^k) \right) (p^{k+1}), \quad k = 0, 1, \dots$$

Last inclusion implies that there exist $w^{k+1} \in \partial(f_k \circ F)(p^{k+1})$ and $v^{k+1} \in N_{\Omega_k}(p^{k+1})$ such that

$$\exp_{p^{k+1}}^{-1} p^k = \frac{1}{\lambda_k} \left(w^{k+1} + v^{k+1} \right), \quad k = 0, 1, \dots \quad (9)$$

On the other hand, using inequality (1) with $p_1 = p \in M$, $p_2 = p^k$ and $p_3 = p^{k+1}$, we have

$$d^2(p, p^{k+1}) + d^2(p^{k+1}, p^k) - \langle \exp_{p^{k+1}}^{-1} p, \exp_{p^{k+1}}^{-1} p^k \rangle \leq d^2(p, p^k), \quad k = 0, 1, \dots$$

Substituting the equality in (9) into the last inequality, we obtain

$$d^2(p, p^{k+1}) \leq d^2(p, p^k) - d^2(p^{k+1}, p^k) + \frac{1}{\lambda_k} \langle \exp_{p^{k+1}}^{-1} p, w^{k+1} + v^{k+1} \rangle, \quad k = 0, 1, \dots$$

Since $f_k \circ F$ is convex and $w^{k+1} \in \partial(f_k \circ F)(p^{k+1})$ then assumption **(A2)** implies

$$\left\langle \exp_{p^{k+1}}^{-1} p, w^{k+1} \right\rangle \leq f_k(F(p)) - f_k(F(p^{k+1})) \leq 0, \quad p \in \bar{\Omega}, \quad k = 0, 1, \dots,$$

where the last inequality follows from item *ii* of Lemma 2.1. Now, taking into account that $\bar{\Omega} \subset \Omega_k$ and $v^{k+1} \in N_{\Omega_k}(p^{k+1})$, for $k = 0, 1, \dots$, we have

$$\langle \exp_{p^{k+1}}^{-1} p, v^{k+1} \rangle \leq 0, \quad p \in \bar{\Omega}, \quad k = 0, 1, \dots$$

Therefore, taking into account **(A2)**, we can combine three last inequalities to conclude that $\{p^k\}$ is Fejér convergence to $\bar{\Omega}$. In particular, Proposition 1.1 implies that $\{p^k\}$ is bounded. Let \bar{p} be a cluster point of $\{p^k\}$ and $\{p^{k_j}\}$ a subsequence of $\{p^k\}$ such that $\lim_{k \rightarrow \infty} p^{k_j} = \bar{p}$. Note that (6) yields

$$F(p^{k+1}) \preceq_C F(p^k), \quad k = 0, 1, \dots$$

Hence, the continuity of F implies that $F(\bar{p}) \preceq_C F(p^k)$, for $k = 0, 1, \dots$, which is equivalent to say $\bar{p} \in \bar{\Omega}$. Using Proposition 1.1 we conclude that $\{p^k\}$ converges to \bar{p} .

It remains to prove that \bar{p} is a weakly efficient point. Let us suppose, by contradiction, that there exists $\hat{p} \in M$ such that $F(\hat{p}) \prec_C F(\bar{p})$ (note that, in particular, $\hat{p} \in \Omega$). Thus, since $\bar{p} \in \bar{\Omega}$, using item *ii*) of Lemma 2.1 we have

$$f_k(F(\hat{p})) - f_k(F(\bar{p})) \geq f_k(F(\hat{p})) - f_k(F(p^{k+1})) \geq \langle w^{k+1}, \exp_{p^{k+1}}^{-1} \hat{p} \rangle, \quad (10)$$

where the last inequality was obtained by using the convexity of the function $f_k \circ F$ and that $w^{k+1} \in \partial(f_k \circ F)(p^{k+1})$. Using (9) we obtain

$$\langle w^{k+1}, \exp_{p^{k+1}}^{-1} \hat{p} \rangle = -\langle v^k, \exp_{p^{k+1}}^{-1} \hat{p} \rangle + \lambda_k \langle \exp_{p^{k+1}}^{-1} p^k, \exp_{p^{k+1}}^{-1} \hat{p} \rangle.$$

Using the definition of f_k we have $f_k(F(\bar{p}) - F(\hat{p})) \geq f_k(F(\bar{p})) - f_k(F(\hat{p}))$, for $k = 0, 1, \dots$. Thus, combining the last equality with (10) and taking into account that $v^{k+1} \in N_{\Omega_k}(p^{k+1})$ we have

$$f_k(F(\bar{p}) - F(\hat{p})) \geq f_k(F(\bar{p})) - f_k(F(\hat{p})) \geq \lambda_k \langle \exp_{p^{k+1}}^{-1} p^k, \exp_{p^{k+1}}^{-1} \bar{p} \rangle.$$

Hence, using again the definition of f_k , the last inequality becomes

$$\max_{z \in Z} \frac{\langle F(\bar{p}) - F(\hat{p}), z \rangle}{\langle e_k, z \rangle} \geq \lambda_k \langle \exp_{p^{k+1}}^{-1} p^k, \exp_{p^{k+1}}^{-1} \bar{p} \rangle.$$

Since Z is a compact set, then there exists $\bar{z} \in Z$ such that

$$\langle F(\bar{p}) - F(\hat{p}), \bar{z} \rangle \geq \lambda_k \langle \exp_{p^{k+1}}^{-1} p^k, \exp_{p^{k+1}}^{-1} \bar{p} \rangle \langle e_k, \bar{z} \rangle.$$

Note that the sequences $\{\langle e_k, \bar{z} \rangle\}$ and $\{\lambda^k\}$ are bounded. Thus, letting k goes to infinity in the last inequality, we have

$$\langle F(\bar{p}) - F(\hat{p}), \bar{z} \rangle \geq 0,$$

which contradicts the fact that $F(\bar{p}) \prec_C F(\hat{p})$ and the desired result follows. \square

4 Final Remarks

It is worth to point out that the nonlinear scalar function, see (5), considered in the iterative step process of the algorithm, see (6), allows a relationship between the weak sharp minima set of the vectorial optimization problem and the weak sharp minima set of the scalarized problem. For state this relationship, we need some definitions and results. Let $G : M \rightarrow \mathbb{R}^m$, $\eta \in \mathbb{R}^m$ and let us define the following level set

$$W_\eta := \{p \in M : G(p) = \eta\}.$$

We denote by $\text{Min}G$ (resp. $\text{WMin}G$) the set of the efficient points (resp. weak efficient points) associated to (2).

Definition 4.1. A point $\hat{p} \in M$ is said to be weak sharp minimum to (2), if there is a constant $\tau > 0$ such that

$$G(p) - G(\hat{p}) \notin B(0, \tau d(p, W_{G(\hat{p})})) - C, \quad p \in M \setminus W_{G(\hat{p})}, \quad (11)$$

The set of all weak sharp minimum to (2) is denoted by WSMin_G .

The above definition has appeared in several contexts, see for example, [3, 6, 27, 32]. Note that the relationship (11) can be expressed in following equivalent form

$$d(G(p) - G(\hat{p}), -C) \geq \tau d(p, W_{G(\hat{p})}), \quad p \in M,$$

and there holds $\text{WSMin}_G \subset \text{Min}G$. In the particular case $m=1$ and $C = \mathbb{R}_+$, the last inequality becomes to the well-known inequality

$$G(p) - G(\hat{p}) \geq \tau d(p, W_{G(\hat{p})}), \quad p \in M,$$

introduced in [23], defining weak sharp minimizer in Riemannian context.

Next result establishes the above mentioned relationship between $\text{WSMin}F$ and the weak sharp minimum associated to the nonlinear scalar function defined in (3), the proof follows by using similar arguments used in the proof of [32, Theorem 3.4].

Theorem 4.1. Let $F : M \rightarrow \mathbb{R}^m$ and $\hat{p} \in M$. Suppose that $W_{F(\hat{p})}$ is closed set and define $\tilde{F} : M \rightarrow \mathbb{R}^n$ by $\tilde{F}(p) = F(p) - F(\hat{p})$. If $\hat{p} \in \text{WSMin}_F$ then $\hat{p} \in \text{WSMin}_{f \circ \tilde{F}}$, where f is given by (4).

We expect that the Theorem 4.1 constitutes a first step towards to establish the following result: “If $\hat{p} \in \text{WSMin}_F$, then $\{p^k\}$ converges, in a finite number of iterations”. We foresee further progress along these line in the nearby future. Similar result has been proven in the Euclidean context; see [6].

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