A note on quasi-equilibrium problems

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Abstract

The purpose of this paper is to prove the existence of solutions of quasi-equilibrium problems without any generalized monotonicity assumption. Additionally, we give an application to quasi-optimization problems.

1 Introduction and definitions

Given a real topological vector space X, a subset C of X, a bifunction $f : C \times C \to \mathbb{R}$ and a set-valued map $K : C \rightrightarrows C$, the *quasi-equilibrium problem* (QEP) consists in finding

$$x \in K(x) \text{ such that } f(x, y) \geq 0, \text{ for all } y \in K(x). \tag{1}$$

When $K(x) = C$ for any $x \in C$, the QEP coincides with the classical *equilibrium problem*, which was introduced by Oettli and Blum in [4], and has been extensively studied in recent years (see for instance [3, 5, 10] and the references therein).

The classical example of quasi-equilibrium problem is the quasi-variational inequality problem, which consists in finding $x \in K(x)$, such that there exists $x^* \in T(x)$ with $\langle x^*, y - x \rangle \geq 0$, for all $y \in K(x)$ where $T : X \rightrightarrows X^*$ is a set-valued map, X^* is the dual space of X and $\langle \cdot, \cdot \rangle$ denotes the duality paring between X and X^*. So, if T has compact values, and we define the *representative bifunction* f_T of T by

$$f_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x \rangle,$$

it follows that every solution of the QEP associated to f_T and K is a solution of the quasi-variational inequality problem associated to T and K, and conversely.

Recently in [1] the authors show existence of solution of the QEP using generalized monotonicity for f in a finite dimensional space. Castellani and Giuli ([6]) proved an existence result which does not involve any monotonicity assumption of f in separable Banach spaces.

The aim of this note is to show existence of solution for the QEP without generalized monotonicity assumptions but for Hausdorff locally convex real topological vector spaces which generalizes the spaces in [1, 6].

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2 Existence results

Our existence result will be obtained as a consequence of Kakutani's Fixed Point Theorem which is stated in the next result and it can be found in [9].

Theorem 2.1 (Kakutani's theorem). *Let C be a nonempty compact convex subset of a locally convex space X and let $S : C \rightrightarrows C$ be a set-valued map. If S is upper semicontinuous such that for all $x \in C$, $S(x)$ is nonempty, closed and convex, then S admits a fixed point.*

We denote by $\mathrm{QEP}(f, K)$ the solution set of (1) and we define the following set-valued map $S : C \rightrightarrows C$ by

$$S(x) = \{x_0 \in K(x) : \ f(x_0, y) \geq 0 \text{ for all } y \in K(x)\}.$$

The proposition below plays an important role in our existence result.

Proposition 2.1. *Let $f : C \times C \to \mathbb{R}$ be a bifunction and let $K : C \rightrightarrows C$ be a set-valued map, where C is a nonempty subset of a real topological vector space X. If K is closed and lower semicontinuous and $\{(x, y) \in C \times C : \ f(x, y) \geq 0\}$ is closed; then S is closed.*

Proof. Let $(x_n, z_n)_{n \in \mathbb{N}}$ be a sequence contained in the graph of S converging to (x_0, z_0). Since K is closed, we have $z_0 \in K(x_0)$. The lower semicontinuity of K implies that for any $y \in K(x_0)$, there exists $(y_n)_{\in \mathbb{N}}$ converging to y such that $y_n \in K(x_n)$, for all $n \in \mathbb{N}$. Additionally, as $z_n \in S(x_n)$ we have $f(z_n, y_n) \geq 0$ for every $n \in \mathbb{N}$, which in turn implies by hypothesis that $f(z_0, y) \geq 0$. Therefore, $z_0 \in S(x_0)$. $\qquad\square$

We finish this section with our main existence result.

Theorem 2.2. *Let $f : C \times C \to \mathbb{R}$ be a bifunction and let $K : C \rightrightarrows C$ be a set-valued map, where C is a compact convex and nonempty subset of a Hausdorff locally convex real topological vector space X. If the following hold:*

  i) K is closed and lower semicontinuous with convex values;

  ii) $\{x \in C : \ f(x, y) \geq 0\}$ is convex, for every $y \in C$;

  iii) for any subset $\{x_1, \ldots, x_n\}$ of C, and any $x \in \mathrm{co}(\{x_1, \ldots, x_n\})$ (here co is the convex hull), $\max_{i=1,\ldots,n} f(x, x_i) \geq 0$;

  iv) $\{(x, y) \in C \times C : \ f(x, y) \geq 0\}$ is closed;

then $\mathrm{QEP}(f, K)$ is nonempty.

For the prove of the previous theorem we need the following result.

Theorem 2.3. *[10, Theorem 2.3] Let $f : C \times C \to \mathbb{R}$ be a bifunction, where C is a compact convex and nonempty subset of a Hausdorff real topological vector space X. If for any $\{x_1, \ldots, x_n\} \subset C$ and $x \in \mathrm{co}(\{x_1, \ldots, x_n\})$, $\max_{i=1,\ldots,n} f(x, x_i) \geq 0$; and $\{y \in C : \ f(x, y) \geq 0\}$ is closed, for every $x \in C$; then there exists $x_0 \in C$ such that $f(x_0, y) \geq 0$, for all $y \in C$.*

Proof of Theorem 2.2. By Proposition 2.1, the set-valued map S is closed. For each $x \in C$, $S(x)$ is closed, convex and nonempty due to conditions $ii)$, $iii)$ and $iv)$, and Theorem 2.3. As C is compact, we have that S is upper semicontinuous. Thus, by Kakutani's theorem S has a fixed point. $\qquad\square$

Figure 1: Graphs of h and K.

Remark 1. *Notice that quasiconcavity with respect to the first variable of f implies part ii) of Theorem 2.2. Moreover, if $f(x, \cdot)$ is quasiconvex and $f(x, x) = 0$ for any $x \in C$ then part iii) of Theorem 2.2 holds. However, the converse is not true in general. Consider for instance $f : [0, 1] \times [0, 1] \to \mathbb{R}$ defined by*

$$f(x, y) = \begin{cases} 1, & y \in \mathbb{Q} \cap [0, 1] \\ 0, & y \notin \mathbb{Q} \end{cases} \quad .$$

Clearly f satisfies condition ii) of Theorem 2.2, but it is not quasiconvex with respect to its second argument nor it vanishes on the diagonal of $[0, 1] \times [0, 1]$.

3 Application to quasi-optimization

Given a real-valued function $h : C \to \mathbb{R}$ and a set-valued map $K : C \rightrightarrows C$, where C is a subset of a Hausdorff locally convex real topological vector space X, the *quasi-optimization problem* (QOpt) is described as

$$\text{find } x_0 \in K(x_0) \text{ such that } \quad \min_{z \in K(x_0)} h(z) = h(x_0).$$

The terminology of quasi-optimization problems comes from [8] (see formula (8.3) and Proposition 12) and has been recently used in [2, 7]. It emphasizes the fact that it is not a standard optimization problem since the constraint set depends on the solution and it also highlights the parallelism to quasi-equilibrium problems.

Remark 2. *Under continuity of the constraint set-valued map the continuity of the objective function is not a sufficient condition to guarantee the existence of solution for the QOpt. Consider for instance the function $h : [0, 2] \to \mathbb{R}$ and the set-valued map $K : [0, 2] \rightrightarrows [0, 2]$ both defined by*

$$h(x) = \begin{cases} |x - \frac{1}{2}|, & 0 \leq x \leq 1 \\ |x - \frac{3}{2}|, & 1 < x \leq 2 \end{cases} \qquad K(x) = \begin{cases} [-\frac{3}{2}x + \frac{3}{2}, 2], & 0 \leq x \leq 1 \\ [0, -\frac{3}{2}x + \frac{7}{2}], & 1 < x \leq 2 \end{cases}$$

Figure 1 shows the graphs of h and K. Clearly, h is continuous and K is closed and lower semicontinuous. The set of fixed points of K is the interval $[3/5, 7/5]$. It is not difficult to show that the QOpt does not have a solution.

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Associated to h and K, let us define the bifunction $f^h : C \times C \to \mathbb{R}$ by

$$f^h(x, y) = h(y) - h(x).$$

Now, we can characterize the solutions of the QOpt by solutions of the QEP associated to f^h and K. We denote by $\mathrm{QOpt}(h, K)$ the solution set of the QOpt. The definition of f^h implies the following lemma.

Lemma 3.1. *With the previous notation and assuming that $x_0 \in C$, then $x_0 \in \mathrm{QEP}(f^h, K)$ if and only if $x_0 \in \mathrm{QOpt}(h, K)$.*

Finally, we are ready for our result about the existence of solutions for the QOpt which generalizes [2, Proposition 4.2 and Proposition 4.5].

Theorem 3.1. *Let $h : C \to \mathbb{R}$ be a function and let $K : C \rightrightarrows C$ be a set-valued map, where C is a convex and compact subset of a Hausdorff locally convex real topological vector space X. If K is closed and lower semicontinuous with convex values, and h is quasiconvex and continuous; then* $\mathrm{QOpt}(h,K)$ *is nonempty.*

Proof. We want to verify all assumptions of Theorem 2.2. The first one is trivial. Since h is quasiconvex, the bifunction f^h is quasiconcave with respect to its first argument, which implies $ii)$, and quasiconvex with respect to its second argument. Moreover, f^h vanishes on the diagonal of $C \times C$, hence f^h satisfies condition $iii)$ (see Remark 1). Additionally, as h is continuous then f^h is continuous and hence condition $iv)$ holds. Therefore, there exists $x_0 \in \mathrm{QEP}(f^h, K)$. The result follows from Lemma 3.1. $\square$

References

[1] D. Aussel, J. Cotrina, A. Iusem, *An existence result for quasi-equilibrium problems*, J. Convex Anal. 24 (2017), 55–66.

[2] D. Aussel, J. Cotrina, *Quasimonotone quasivariational inequalities: existence results and applications*, J. Optim. Theory Appl. 158 (2013), 637–652.

[3] M. Bianchi, R. Pini, *A note on equilibrium problems with properly quasimonotone bifunctions*, J. Global Optim. 20 (2001), 67–76.

[4] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Stud. 63 (1993), 1–23.

[5] M. Castellani, M. Giuli, *Refinements of existence results for relaxed quasimonotone equilibrium problems*, J. Global Optim. 57 (2013), 1213–1227.

[6] M. Castellani, M. Giuli, *An existence result for quasiequilibrium problems in separable Banach spaces*, J. Math. Anal. Appl. 425 (2015), 85–95.

[7] F. Facchinei, C. Kanzow, *Generalized Nash equilibrium problems*, Ann. Oper. Res. 175 (2010), 177–211.

[8] F. Giannessi, G. Mastroeni, L. Pellegrini, *On the theory of vector optimization and variational inequalities. Image space analysis and separation.* (English summary) Vector variational inequalities and vector equilibria, 153–215, Nonconvex Optim. Appl., 38, Kluwer Acad. Publ., Dordrecht, 2000.

[9] A. Granas, J. Dugundji, *Fixed point theory* Springer Monographs in Mathematics (2003)

[10] M. Nasri, W. Sosa, *Equilibrium problems and generalized Nash games*, Optimization, 60 (2011), 1161–1170.