

# On the Complexity of Instationary Gas Flows

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## Abstract

We study a simplistic model of instationary gas flows consisting of a sequence of  $k$  stationary gas flows. We present efficiently solvable cases and NP-hardness results, establishing complexity gaps between stationary and instationary gas flows (already for  $k = 2$ ) as well as between instationary gas  $s$ - $t$ -flows and instationary gas  $b$ -flows.

*Keywords:* Gas transport network, instationary gas flow, time-dependent flow, complexity, NP-hardness

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## 1. Introduction

This paper studies the algorithmic complexity of time-varying flows in gas transport networks. In the gas transport literature, these flows are called *instationary* in contrast to *stationary* gas flows that describe a steady state situation. This paper presents efficiently solvable problems and identifies complexity gaps between stationary and instationary gas flows, as well as between instationary gas flows with a single source/sink and multi-terminal instationary gas flows. Our ultimate goal is to contribute to a better understanding of the particular difficulty of instationary gas flows. To this end, we introduce a simple model of instationary gas flows in Sect. 3, present an efficiently solvable instationary gas flow problem in Sect. 4, examples of more complicated scenarios in Sect. 5, and finally an NP-hardness result in Sect. 6.

## 2. Stationary Gas Flows

Before turning to the more general case of instationary gas flows, we introduce some basic facts about stationary gas flows. In contrast to classical network flows where, within given capacity bounds, flow may be distributed throughout a network ad libitum, gas flows are governed by the laws of physics.

Essentially, in a gas network the (stationary) flow along an arc (pipeline) is uniquely determined by the pressures at the two endpoints of the arc. For an in-depth treatment of (stationary) flows in gas networks we refer to the recent book [1]. The simplest and most widely adapted model for stationary gas flows is Weymouth's equation [2]: For an arc  $a = (u, v)$ , the flow value  $x_a$  along  $a$  satisfies

$$\beta_a x_a |x_a| = \pi_u - \pi_v, \quad (1)$$

where the node potentials  $\pi_u = p_u^2$  and  $\pi_v = p_v^2$  are the squared pressures at nodes  $u$  and  $v$ , respectively, and  $\beta_a > 0$  is a given constant specifying the resistance of arc  $a$ . Here, a negative flow value  $x_a$  on arc  $a = (u, v)$  represents flow in the opposite direction from node  $v$  to node  $u$ . This stationary gas flow model forms the basis of this paper.

Consider a directed graph  $G$  with node set  $V$  and arc set  $A$ . For given node balances  $b \in \mathbb{R}^V$  with  $\sum_{v \in V} b_v = 0$ , a stationary gas flow satisfying supplies and demands given by  $b$  can be computed by solving the following convex min-cost  $b$ -flow problem [3, 4]

$$\begin{aligned} \min \quad & \sum_{a \in A} \frac{\beta_a}{3} |x_a|^3 \\ \text{s.t.} \quad & \sum_{a \in \delta^{\text{out}}(v)} x_a - \sum_{a \in \delta^{\text{in}}(v)} x_a = b_v \quad \forall v \in V, \end{aligned} \quad (2)$$

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with corresponding dual (strong duality holds)

$$\max_{\pi} \left( \sum_{u \in V} b_u \pi_u - 2 \sum_{(u,v) \in A} \frac{|\pi_u - \pi_v|^{3/2}}{3\sqrt{\beta_{(u,v)}}} \right). \quad (3)$$

The dual variables yield the node potentials in (1). These node potentials are unique up to translation by an arbitrary value. The problems (2) and (3) can be solved efficiently within arbitrary precision.

Throughout this paper, we assume that there are uniform bounds on all node potentials given by an interval  $[\pi_{\min}, \pi_{\max}]$ . A stationary gas flow  $x$  with corresponding node potential  $\pi \in \mathbb{R}^V$  is *feasible* if  $\pi_{\min} \leq \pi_v \leq \pi_{\max}$  for all  $v \in V$ . Before introducing our model of instationary gas flows in the next section, we state an important theorem on stationary gas flows, which essentially follows from the work of Calvert and Keady [5] (see also [6]), and for which we give a short proof for the sake of completeness.

**Theorem 1** ([5]). *In a network with source  $s$ , sink  $t$ , and potential interval  $[\pi_{\min}, \pi_{\max}]$ , the value of a maximal feasible stationary gas  $s$ - $t$ -flow cannot be increased by increasing arc resistances  $\beta_a$ ,  $a \in A$ .*

*Proof.* For a fixed  $s$ - $t$ -flow value  $B = b_s = -b_t$  (and  $b_v = 0$  for  $v \in V \setminus \{s, t\}$ ), consider the primal problem (2) and the dual problem (3). Due to (2), the optimal value  $z^*$  of these two problems is an increasing function of the arc resistances  $\beta_a$ ,  $a \in A$ . Then, by combining (3), (1), and (2), we obtain the following for the optimal solution  $(x^*, \pi^*)$ :

$$z^* = \sum_{v \in V} b_v \pi_v^* - 2 \sum_{a \in A} \frac{\beta_a}{3} |x_a^*|^3 = B(\pi_s^* - \pi_t^*) - 2z^*.$$

For fixed  $s$ - $t$ -flow value  $B$ , the difference of potentials at  $s$  and  $t$  is proportional to  $z^*$ , more precisely  $\pi_s^* - \pi_t^* = 3z^*/B$ , and thus an increasing function of the arc resistances  $\beta_a$ ,  $a \in A$ . Finally, the difference of potentials at  $s$  and  $t$  is also an increasing function of flow value  $B$  and bounded by  $\pi_{\max} - \pi_{\min}$  where the maximum flow value is attained.  $\square$

### 3. A Simple Instationary Gas Flow Model

We introduce a model of instationary gas flows that, while being simple enough to allow for a theoretical analysis, still captures essential characteristics and exhibits interesting properties. In particular, we prove meaningful results that constitute an interesting first step in explaining the increased difficulty of instationary versus stationary gas flows.

For  $k \in \mathbb{Z}_{>0}$ , a  $k$ -stage gas flow  $x$  is a  $k$ -tuple  $(x^1, \dots, x^k)$  of stationary gas flows (where we interpret  $x^1, \dots, x^k$  as a temporal succession). If  $x^i$  satisfies supplies and demands  $b^i \in \mathbb{R}^V$ ,  $i = 1, \dots, k$ , then  $x$  in total satisfies supplies and demands  $b = b^1 + \dots + b^k \in \mathbb{R}^V$  and is called  $k$ -stage gas  $b$ -flow. For two distinguished nodes  $s, t \in V$ ,  $x$  is a  $k$ -stage gas  $s$ - $t$ -flow of value  $q$  if it satisfies supplies and demands  $b \in \mathbb{R}^V$  with  $b_s = -b_t = q$  and  $b_v = 0$  for  $v \in V \setminus \{s, t\}$ . A  $k$ -stage gas flow  $x$  is called *stationary* if  $x^1 = \dots = x^k$ , otherwise  $x$  is called *instationary*. Finally, a  $k$ -stage gas flow  $x$  is *feasible* if  $x^1, \dots, x^k$  are feasible stationary gas flows.

*Remark 2.* Notice that, in marked contrast to actual gas transport, in our model there is no correlation between consecutive flows  $x^i$  and  $x^{i+1}$ . Moreover, the model allows to arbitrarily buffer or borrow flow in each node (i.e., flow may be withdrawn or injected at each node) at each stage as long as the accumulated node balances  $b^1 + \dots + b^k$  add up to the desired  $b$  (cp. examples in Sect. 5 below).

We study the following two algorithmic problems for  $k \in \mathbb{Z}_{>0}$ : first, the *maximum  $k$ -stage gas  $s$ - $t$ -flow problem*, whose input is a network  $G$  with source  $s \in V$ , sink  $t \in V$ , and interval  $[\pi_{\min}, \pi_{\max}]$ , and the task is to find a feasible  $k$ -stage gas  $s$ - $t$ -flow of maximum value; second, the  *$k$ -stage gas  $b$ -flow problem*, whose input is a network  $G$  with supplies and demands  $b \in \mathbb{R}^V$ , as well as interval  $[\pi_{\min}, \pi_{\max}]$ , and the task here is to find a feasible  $k$ -stage gas  $b$ -flow.

### 4. Maximum 2-Stage Gas $s$ - $t$ -Flows

We first show that there exists an efficiently computable stationary solution of the maximum 2-stage gas  $s$ - $t$ -flow problem.

**Theorem 3.** *Taking two copies of the maximum feasible stationary gas  $s$ - $t$ -flow yields an optimal solution to the maximum 2-stage gas  $s$ - $t$ -flow problem.*

In order to prove the theorem, we consider an arbitrary feasible 2-stage gas  $s$ - $t$ -flow  $(x^1, x^2)$  with corresponding node potentials  $\pi^1, \pi^2 \in \mathbb{R}^V$ . By definition, the flow  $\tilde{x} := \frac{1}{2}(x^1 + x^2)$  is an  $s$ - $t$ -flow (not necessarily a stationary gas flow, though), and the value of the feasible 2-stage gas  $s$ - $t$ -flow  $(x^1, x^2)$  is exactly twice the value of  $\tilde{x}$ .

**Lemma 4.** *The node potentials  $\bar{\pi} := \frac{1}{2}(\pi^1 + \pi^2)$  induce a feasible stationary gas flow  $\bar{x}$  with  $\text{sgn}(\bar{x}_a) = \text{sgn}(\tilde{x}_a)$  and  $|\bar{x}_a| \geq |\tilde{x}_a|$  for each  $a \in A$ .*

*Proof.* By definition of  $\bar{x}$  and  $x^i$ ,  $i = 1, 2$ , we have

$$\begin{aligned}\bar{x}_a &= \operatorname{sgn}(\bar{\pi}_u - \bar{\pi}_v) \sqrt{|\bar{\pi}_u - \bar{\pi}_v|} / \sqrt{\beta_a}, \\ x_a^i &= \operatorname{sgn}(\pi_u^i - \pi_v^i) \sqrt{|\pi_u^i - \pi_v^i|} / \sqrt{\beta_a},\end{aligned}$$

for each arc  $a = (u, v) \in A$ . Moreover, by definition of  $\bar{\pi}$ , we get  $\bar{\pi}_u - \bar{\pi}_v = ((\pi_u^1 - \pi_v^1) + (\pi_u^2 - \pi_v^2)) / 2$ . The lemma thus follows from the next observation.  $\square$

**Observation 5.** Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  with  $f(\sigma) = \operatorname{sgn}(\sigma) \sqrt{|\sigma|}$ . Then, for all  $\sigma^1, \sigma^2 \in \mathbb{R}$ ,

$$\begin{aligned}\operatorname{sgn}\left(f\left(\frac{\sigma^1 + \sigma^2}{2}\right)\right) &= \operatorname{sgn}\left(\frac{f(\sigma^1) + f(\sigma^2)}{2}\right), \\ \left|f\left(\frac{\sigma^1 + \sigma^2}{2}\right)\right| &\geq \left|\frac{f(\sigma^1) + f(\sigma^2)}{2}\right|.\end{aligned}$$

*Proof.* Notice that  $f(-\sigma) = -f(\sigma)$  for all  $\sigma \in \mathbb{R}$  (in particular,  $f(0) = 0$ ), and  $f|_{\mathbb{R}_{\geq 0}}$  is non-negative, strictly increasing, and concave. Therefore the statement is clear for the cases that  $\sigma^1$  and  $\sigma^2$  are both non-negative or both non-positive.

It remains to consider the case  $\sigma^1 < 0 < \sigma^2$ . The equality statement on the signs is an immediate consequence of  $f$ 's properties noted above. By symmetry we may assume that  $|\sigma^1| \leq \sigma^2$  such that  $\frac{1}{2}(\sigma^1 + \sigma^2) \geq 0$  and thus  $f(\frac{1}{2}(\sigma^1 + \sigma^2)) \geq 0$ . By concavity of  $f|_{\mathbb{R}_{\geq 0}}$ , we get two inequalities:

$$\begin{aligned}f\left(\frac{\sigma^1 + \sigma^2}{2}\right) &\geq \frac{f(0) + f(\sigma^1 + \sigma^2)}{2} = \frac{f(\sigma^1 + \sigma^2)}{2}, \\ f(\sigma^1 + \sigma^2) - f(\sigma^1) &= f(-|\sigma^1| + \sigma^2) + f(|\sigma^1|) \\ &\geq f(\sigma^2) + f(0) = f(\sigma^2).\end{aligned}$$

The latter inequality implies that  $f(\sigma^1 + \sigma^2) \geq f(\sigma^1) + f(\sigma^2)$ . Together with the former inequality this yields the desired result.  $\square$

It follows from Lemma 4 and (1) that by increasing the  $\beta_a$  values individually for each arc  $a \in A$ , we arrive at a network where the  $s$ - $t$ -flow  $\tilde{x}$  is a feasible stationary gas  $s$ - $t$ -flow induced by the node potentials  $\bar{\pi} := \frac{1}{2}(\pi^1 + \pi^2)$ . More precisely, we need to set  $\tilde{\beta}_a := \beta_a \bar{x}_a^2 / \tilde{x}_a^2 \geq \beta_a$ . Thus, by Theorem 1, the value of the stationary maximal feasible gas  $s$ - $t$ -flow  $x^*$  in the network with original values  $\beta_a$ ,  $a \in A$ , is at least the value of  $\tilde{x}$ , which is half the value of our feasible 2-stage gas  $s$ - $t$ -flow  $(x^1, x^2)$ . Summarizing, the value of the feasible 2-stage gas  $s$ - $t$ -flow  $(x^*, x^*)$  is at least the value of  $(x^1, x^2)$ . This concludes the proof of Theorem 3.

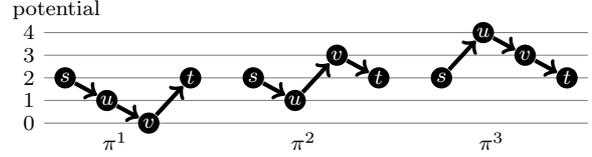


Figure 1: Node potentials of 3-stage gas  $s$ - $t$ -flow with value  $2 - \sqrt{2}$  on the path network described in Example 6

## 5. Examples and Counterexamples

In this section we show that Theorem 3 can neither be generalized to the  $k$ -stage gas  $s$ - $t$ -flow problem for  $k \geq 3$  nor to the 2-stage gas  $b$ -flow problem.

### 5.1. Instationary $k$ -stage gas $s$ - $t$ -flows for $k \geq 3$

We present a network for the maximum 3-stage gas  $s$ - $t$ -flow problem where repeating the maximum feasible stationary gas  $s$ - $t$ -flow three times is not optimal. In order to develop the right intuition for this instance, we first show that fixing the potentials of nodes  $s$  and  $t$  to the same value does not keep us from sending a positive amount of flow in a 3-stage gas  $s$ - $t$ -flow.

**Example 6.** Consider a path network with nodes  $V = \{s, u, v, t\}$ , arcs  $A = \{(s, u), (u, v), (v, t)\}$ , and  $\beta_a = 1$ , for all  $a \in A$ . Moreover,  $[\pi_{\min}, \pi_{\max}] = [0, 4]$ . In Fig. 1 we present the potentials of a 3-stage gas  $s$ - $t$ -flow of value  $2 - \sqrt{2} > 0$ , where the potentials of  $s$  and  $t$  are fixed to 2. Note that, on every arc, there is flow of value 1 in two of the three stages and flow of value  $-\sqrt{2}$  in the remaining stage. In particular, the individual stationary gas flows of the three stages are not  $s$ - $t$ -flows but use the model's freedom to buffer flow at intermediate nodes  $u$  and  $v$  (cf. Remark 2).

In Sect. 4 we have turned a given 2-stage gas flow into a stationary gas flow by considering the average node potentials  $\bar{\pi}$ . Notice that this idea is completely useless with respect to Example 6. The average potential of any node in the given 3-stage gas flow is equal to 2 (cf. Fig. 1). In particular,  $\bar{\pi}$  induces the (stationary) zero flow.

In the next example, we use the intuition behind Example 6 as a gadget to come up with a path network where any maximum  $k$ -stage gas  $s$ - $t$ -flow is instationary. More precisely, we extend the path by adding two additional nodes, one on the left and one on the right, together with arcs of high resistance connecting them to the corresponding ends of the previous path.

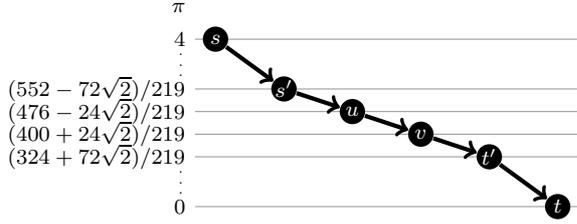


Figure 2: Node potentials of a maximum feasible stationary gas  $s$ - $t$ -flow of value  $\sqrt{(76 - 48\sqrt{2})/219} \approx 0.193$  on the path network described in Example 7

**Example 7.** Consider a path network consisting of node set  $V = \{s, s', u, v, t', t\}$  and arc set  $A = \{(s, s'), (s', u), (u, v), (v, t'), (t', t)\}$  with  $\beta_a = 1$  for all arcs  $a$ , except  $\beta_{(s, s')} = \beta_{(t', t)} = 27 + 18\sqrt{2}$ . Moreover,  $[\pi_{\min}, \pi_{\max}] = [0, 4]$  as in Example 6. In Fig. 2, we give a maximum feasible stationary gas  $s$ - $t$ -flow of value  $\approx 0.193$ . Repeating this flow three times yields a feasible 3-stage gas  $s$ - $t$ -flow of value  $\approx 0.578$ . There is, however, an instationary solution achieving value  $2 - \sqrt{2} \approx 0.586$  which can be achieved as follows. Fix the potentials of node  $s$  to 4 and of node  $t$  to 0; for the remaining ‘inner’ nodes, plug in the solution from Example 6, that is, fix the potentials of  $s'$  and  $t'$  to 2 and let the potentials of  $u$  and  $v$  vary as in Fig. 1.

*Remark 8.* The gap between the instationary solution and the optimal stationary solution in Example 7 is apparently tiny. With a simple trick we can, however, construct path networks where the value of an instationary 3-stage gas  $s$ - $t$ -flow exceeds the value of any stationary 3-stage gas  $s$ - $t$ -flow by an arbitrarily large factor. Such networks can be obtained by replacing the  $s'$ - $t'$ -subnetwork in Example 7 by a serial composition of  $\ell$  copies of this subnetwork. It is not difficult to see that the value of a maximum feasible stationary gas  $s$ - $t$ -flow tends to zero when  $\ell$  tends to infinity. On the other hand, the instationary 3-stage gas  $s$ - $t$ -flow of value  $2 - \sqrt{2}$  described in Example 7 can be extended to the larger network by operating each of the  $\ell$  copies as depicted in Fig. 1.

Finally notice that the examples and results for the maximum 3-stage gas  $s$ - $t$ -flow problem discussed in this section can be generalized in a straightforward way to  $k$ -stages for  $k > 3$ .

### 5.2. Instationary 2-stage gas $b$ -flows

We present a network with supplies and demands  $b \in \mathbb{R}^V$ , where any stationary  $k$ -stage gas

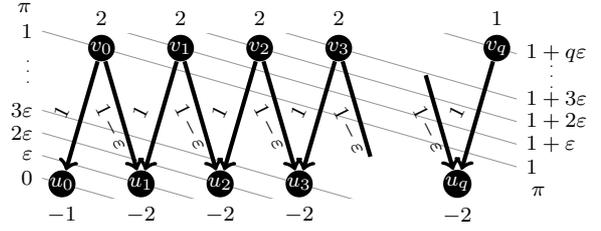


Figure 3: Node potentials  $\pi \in \mathbb{R}^V$  inducing a stationary gas  $b/2$ -flow in the path network of Example 9; the numbers at arcs indicate the  $\beta_a$ -values, the numbers at nodes  $b_u$ -values.

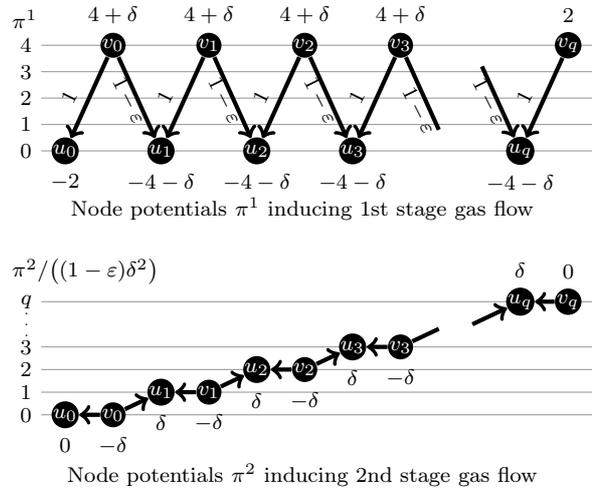


Figure 4: Node potentials  $\pi^1$  and  $\pi^2$  inducing an instationary 2-stage gas  $b$ -flow in the network of Example 9; here  $\delta := 2/\sqrt{1 - \epsilon} - 2 \in \theta(\epsilon)$  for  $\epsilon \rightarrow 0$ .

$b$ -flow requires a considerably larger interval of node potentials than an instationary  $k$ -stage gas  $b$ -flow.

**Example 9.** For some fixed parameter  $0 < \epsilon < 1$  consider the path network with  $2q + 2$  nodes  $V = \{u_0, v_0, u_1, v_1, \dots, u_q, v_q\}$  depicted in Fig. 3. There are arcs  $(v_i, u_i)$  with  $\beta_{(v_i, u_i)} = 1$  for  $i = 0, \dots, q$  and arcs  $(v_i, u_{i+1})$  for  $i = 0, \dots, q - 1$  with  $\beta_{(v_i, u_{i+1})} = 1 - \epsilon$ . The supplies and demands are  $b(u_0) = -2$ ,  $b(u_i) = -4$  for  $i = 1, \dots, q$ ,  $b(v_i) = 4$  for  $i = 0, \dots, q - 1$ , and  $b(v_q) = 2$ . The stationary gas flow induced by the potentials in Fig. 3 sends one unit of flow along each arc and thus fulfills supplies and demands  $b/2$ . It therefore yields the unique stationary 2-stage gas  $b$ -flow, and its range of node potentials is  $1 + q\epsilon$ .

In Fig. 4, we present node potentials inducing an instationary 2-stage gas  $b$ -flow. Notice that, by choice of the node potentials  $\pi^1$ , the first-stage gas flow  $x^1$  overfulfills the supplies at nodes  $v_0, \dots, v_{q-1}$

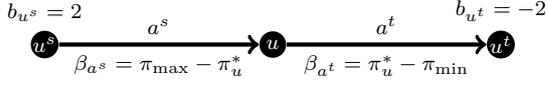


Figure 5: Gadget fixing  $u$ 's potential to given value  $\pi_u^*$

and the demands at nodes  $u_1, \dots, u_q$  slightly by  $\delta := 2/\sqrt{1-\varepsilon} - 2 \in \theta(\varepsilon)$  for  $\varepsilon \rightarrow 0$ . This is compensated for in the second stage (cf. Remark 2). Overall, the range of node potentials has size  $\max\{4, q(1-\varepsilon)\delta^2\}$  (see Fig. 4). If we set  $\varepsilon := 1/\sqrt{q}$  and let  $q$  go to infinity, the node potentials are bounded by a constant. For the stationary 2-stage gas  $b$ -flow (see Fig. 3), however, the size of the range of node potentials is  $1 + \sqrt{q}$  and thus unbounded.

## 6. Complexity Results

We finally prove the following hardness result.

**Theorem 10.** *For a given network with supplies and demands  $b \in \mathbb{R}^V$  and potential interval  $[\pi_{\min}, \pi_{\max}]$ , it is strongly NP-hard to decide whether there exists a feasible 2-stage gas  $b$ -flow.*

In order to prove this result, we first introduce several gadgets, using our insights from Sect. 4.

### 6.1. Nodes with fixed potential

Our first gadget is used to fix the potential of some node  $u$  in any feasible 2-stage gas  $b$ -flow to a given value  $\pi_u^*$  with  $\pi_{\min} < \pi_u^* < \pi_{\max}$ . To this end, we introduce two new nodes  $u^s$  and  $u^t$  whose only incident arcs are  $a^s = (u^s, u)$  and  $a^t = (u, u^t)$ , respectively; see Fig. 5. Moreover, we set  $\beta_{a^s} := \pi_{\max} - \pi_u^*$ ,  $b_{u^s} := 2$ ,  $\beta_{a^t} := \pi_u^* - \pi_{\min}$ , and  $b_{u^t} := -2$ . By construction, the supply and demand at  $u^s$  and  $u^t$ , respectively, can be satisfied by a 2-stage gas flow if the node potentials are set to  $\pi_{u^s}^1 = \pi_{u^s}^2 = \pi_{\max}$ ,  $\pi_u^1 = \pi_u^2 = \pi_u^*$ , and  $\pi_{u^t}^1 = \pi_{u^t}^2 = \pi_{\min}$ .

**Lemma 11.** *In any feasible 2-stage gas flow satisfying the supply and demand at  $u^s$  and  $u^t$ , respectively, node  $u$ 's potential satisfies  $\pi_u^1 = \pi_u^2 = \pi_u^*$ .*

*Proof.* In order to satisfy the supply at  $u^s$ , the total flow on arc  $a^s$  must sum up to  $b_{u^s} = 2$ . Thus, the node potentials  $\pi^1$  and  $\pi^2$  must satisfy

$$\begin{aligned} 2 &= x_{a^s}^1 + x_{a^s}^2 \leq \frac{\sqrt{\pi_{\max} - \pi_u^1} + \sqrt{\pi_{\max} - \pi_u^2}}{\sqrt{\pi_{\max} - \pi_u^*}} \\ &\leq 2 \frac{\sqrt{\pi_{\max} - (\pi_u^1 + \pi_u^2)/2}}{\sqrt{\pi_{\max} - \pi_u^*}}. \end{aligned} \quad (4)$$

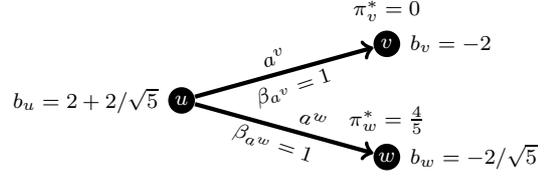


Figure 6: Binary decision gadget with exactly two possibilities for  $u$ 's potential:  $\pi_u^1 = \pi_u^2 = 1$  or  $\{\pi_u^1, \pi_u^2\} = \{0, 4\}$

Here, the first inequality holds since the flow on arc  $a^s$  is maximal if  $\pi_{u^s}^1 = \pi_{u^s}^2 = \pi_{\max}$ . The second inequality follows from the concavity of the square root function. Notice that, in order for the right hand side expression to be at least 2, the average potential  $(\pi_u^1 + \pi_u^2)/2$  must not exceed  $\pi_u^*$ . Using an analogous argument for the total flow on arc  $a^t$ , it can be shown that the average potential  $(\pi_u^1 + \pi_u^2)/2$  must not fall below  $\pi_u^*$ . Thus the average potential must equal  $\pi_u^*$ . As a consequence, equality holds in (4) which, by strict concavity of the square root function, implies  $\pi_u^1 = \pi_u^2 = \pi_u^*$ .  $\square$

### 6.2. Binary decision nodes

Our second gadget is used to create a node  $u$  to model a binary decision. More precisely, there are two possibilities: either  $\pi_u^1 = \pi_u^2 = 1$  or it must attain the two values  $\pi_{\min} = 0$  and  $\pi_{\max} = 4$ , that is,  $\{\pi_u^1, \pi_u^2\} = \{0, 4\}$ . With this end in view, we introduce two additional nodes  $v$  and  $w$  with fixed potentials  $\pi_v^* = 0$ ,  $\pi_w^* = \frac{4}{5}$  and balances  $b_v = -2$  and  $b_w = -2/\sqrt{5}$ . Moreover, nodes  $v$  and  $w$  are connected to  $u$  by the arcs  $a^v = (u, v)$  and  $a^w = (u, w)$  with  $\beta_{a^v} = \beta_{a^w} = 1$ . Finally, we set  $b_u := -(b_v + b_w) = 2 + 2/\sqrt{5}$ ; see Fig. 6.

In order to satisfy the demands at nodes  $v$  and  $w$  in a 2-stage gas flow, the potentials  $\pi_u^1$  and  $\pi_u^2$  need to satisfy the following equations:

$$\begin{aligned} 2 &= \operatorname{sgn}(\pi_u^1) \sqrt{|\pi_u^1|} + \operatorname{sgn}(\pi_u^2) \sqrt{|\pi_u^2|}, \\ \frac{2}{\sqrt{5}} &= \operatorname{sgn}(\pi_u^1 - \frac{4}{5}) \sqrt{|\pi_u^1 - \frac{4}{5}|} \\ &\quad + \operatorname{sgn}(\pi_u^2 - \frac{4}{5}) \sqrt{|\pi_u^2 - \frac{4}{5}|}. \end{aligned} \quad (5)$$

It is straightforward to verify that the only solutions (up to symmetry) to (5) are  $\pi_u^1 = \pi_u^2 = 1$  and  $\{\pi_u^1, \pi_u^2\} = \{0, 4\}$ .

### 6.3. Reduction from Exact Cover By 3-Sets

We prove Theorem 10 via a reduction of the NP-complete problem Exact Cover By 3-Sets (X3C):

the input is a finite set  $X$  of cardinality  $|X| = 3q$ , and a family of subsets  $\mathcal{C} \subseteq 2^X$  with  $|C| = 3$  for all  $C \in \mathcal{C}$ . The question is whether there is a subfamily  $\mathcal{C}' \subseteq \mathcal{C}$  with  $\bigcup_{C \in \mathcal{C}'} C = X$  and  $|\mathcal{C}'| = q$ .

*Proof of Theorem 10.* Given an instance of X3C, we construct an instance of the 2-stage gas  $b$ -flow problem as follows. Set  $\pi_{\min} := 0$  and  $\pi_{\max} := 4$ . For each element  $x \in X$ , we introduce two nodes  $u^x$  and  $v^x$  with fixed potentials  $\pi_{u^x}^* := \frac{16}{25}$  and  $\pi_{v^x}^* = 1$  (see gadget in Fig. 5). For each  $C \in \mathcal{C}$ , we introduce a binary decision node  $u^C$  with  $\pi_{u^C}^1 = \pi_{u^C}^2 = 1$  or  $\{\pi_{u^C}^1, \pi_{u^C}^2\} = \{0, 4\}$  (see gadget in Fig. 6). We say that node  $u^C$  is *off* if  $\pi_{u^C}^1 = \pi_{u^C}^2 = 1$ , otherwise it is *on*. Finally, for each  $C \in \mathcal{C}$  and each of the three  $x \in C$ , we introduce two arcs  $(u^x, u^C)$  and  $(u^C, v^x)$ . By construction, if node  $u^C$  is off,

$$x_{(u^x, u^C)}^1 + x_{(u^x, u^C)}^2 = -2 \frac{\sqrt{1 - \frac{16}{25}}}{\sqrt{\beta_{(u^x, u^C)}}} = \frac{-\frac{6}{5}}{\sqrt{\beta_{(u^x, u^C)}}}$$

and  $x_{(u^C, v^x)}^1 + x_{(u^C, v^x)}^2 = 0$ . If node  $u^C$  is on,

$$x_{(u^x, u^C)}^1 + x_{(u^x, u^C)}^2 = \frac{\frac{4}{5} - \frac{2}{5}\sqrt{21}}{\sqrt{\beta_{(u^x, u^C)}}}$$

and  $x_{(u^C, v^x)}^1 + x_{(u^C, v^x)}^2 = (\sqrt{3} - 1)/\sqrt{\beta_{(u^C, v^x)}}$ . Thus, if we set  $\beta_{(u^x, u^C)} := (2 - \frac{2}{5}\sqrt{21})^2$ ,  $\beta_{(u^C, v^x)} := (\sqrt{3} - 1)^2$ , then changing the state of  $u$  from off to on increases the total flow along arc  $(u^x, u^C)$  and along arc  $(u^C, v^x)$  by 1, leaving the flow balance at node  $u^C$  unchanged.

The idea of the reduction is that the demands of nodes  $u^x$  and  $v^x$  are satisfied if and only if exactly one node  $u^C$  with  $x \in C$  is on. For all  $x \in X$  and  $C \in \mathcal{C}$ , let

$$b_{u^x} := 1 - \frac{\frac{6}{5}|\{C \in \mathcal{C} \mid x \in C\}|}{2 - \frac{2}{5}\sqrt{21}}, \quad b_{v^x} := -1,$$

$$b_{u^C} := 3 \frac{\frac{6}{5}}{2 - \frac{2}{5}\sqrt{21}} + (2 + 2/\sqrt{5}),$$

where the term  $2 + 2/\sqrt{5}$  stems from the decision node gadget (see Fig. 6). By construction of the reduction, there is a one-to-one correspondence between feasible solutions to the X3C instance and feasible solutions to the 2-stage gas  $b$ -flow instance.  $\square$

*Remark 12.* In view of the fact that, due to the irrational numbers involved, stationary and instationary gas flows can only be approximately computed

anyway, the significance of Theorem 10 might seem questionable at first glance. Notice, however, that due to continuity of all functions involved, the gadgets in the proof are robust toward small changes of numbers. In particular, it is even NP-hard to decide whether there exists an almost feasible 2-stage gas flow approximately fulfilling supplies and demands  $b$ .

It is also interesting to compare the hardness result of Theorem 10 to related hardness results for flows over time which constitute a time-dependent variant of classical network flows; see, e.g., the survey article [7]. Most flow over time problems are only weakly NP-hard, if not polynomially solvable. In particular, they can be solved efficiently as long as the number of discrete time steps is polynomially bounded in the input size. In contrast, our instationary gas  $b$ -flows are strongly NP-hard already for only two time steps. This is mainly due to the non-convexity of the square root function describing the relationship between flows and node potentials.

*Remark 13.* We would finally like to point out that the results in this paper are meaningful beyond the area of gas transport. The presented observations can be generalized to potential-based flow models such as those considered in [6], as long as the function  $f$  in  $x_{(u,v)} = \text{sgn}(\pi_u - \pi_v)f(|\pi_u - \pi_v|/\beta_{(u,v)})$  is strictly concave (for gas flows,  $f(z) = \sqrt{z}$ ).

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