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The Shapley value for the probability game

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ABSTRACT

The main goal of this paper is to introduce the probability game. On one hand, we analyze the Shapley value by providing an axiomatic characterization. We propose the so-called independent fairness property, meaning that for any two players, the player with larger individual value gets a larger portion of the total benefit. On the other, we use the Shapley value for studying the profitability of merging two agents.

1. Introduction

Game theory has been proved to be a useful tool when analyzing the ballistic missile defense budget allocation and the cooperative *R&D* profit allocation problems. A large quantity of facts has proved that missile interception [6] and cooperative *R&D* issue have always been a significant topic in the field of military tactical ballistic missile and *R&D* problems. Take the ballistic missile defense situation as an example. Once used in the war, it is sure to be anti tactical ballistic missile interceptor weapons [4]. One of the typical cases originates from the Gulf War, in which the interception of Scud missiles by Patriot missile captured worldwide attention [1]. Plenty of research [6,4,1,8] and a number of military exercises illustrate that multiple layered defense system is safer in comparison with the single one. For instance, on 2016, July, 15th, India successfully completed multi-layer ballistic missile defense system test [3]. A positive aspect that is still not yet addressed in the literature is how to distribute the defense project budget among the individual defense layers in cooperative defense situation. The most simple and direct method is to allocate the defense budget according to each layer's defense probability of success. However, it is not taken into account the fact that the defense layers are organized in one system and function as a whole. In the missile example, all the agents operate with the same aim, yielding that for any group of agents it is important that at least one succeeds, as the target is the same for all of them. Therefore, the successful action probability for the group of the agents is an important index which reflects the group action competency. By observing this, we argue that layers certainly enter into alliance and gain through cooperation because once the interceptor system fails, it can be at best costly—at worst, disastrous.

This paper broadens the game theoretic approach to the probability game, as a model of the cooperation ballistic missile defense situation, cooperative *R&D* problems and so on. We introduce the so-called probability game, of which the characteristic function is the successful acting probability of the coalition, i.e., the value of the coalition is described by the probability that at least one event in S is successful. Many concepts of allocations are proposed in the literature. Among them, we study the well known Shapley value, introduced and characterized in [7]. A mass of literature concentrating on this topic can be found to illustrate the fairness of this value, by revealing and emphasizing its properties. Examples include Shapley's efficiency, null player property, linearity and symmetry [7], Young's strong monotonicity [9], and Chun's coalitional strategic equivalence [2]. In the context of probability game, assume that players are mutually independent, we propose the so-called independent fairness property, meaning that, the player with larger successful acting probability will be assigned with more portion of the total profit. We show that for probability game, the independent fairness property can be used to characterize the Shapley value together with linearity, the dummy property and efficiency. Finally, we determine the significant threshold illustrating whether or not merger of any two players produces extra benefits to them.

The paper is organized as follows: in Section 2 we conduct on the probability game and its Shapley value. In Section 3 we deal with the characterization of the Shapley value, by proposing the independent fairness property. In Section 4 we use the Shapley value for studying the profitability of merging two agents.

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2. The Shapley value for probability game

In this section, we determine the Shapley value for the probability game, as a model of the cooperation ballistic missile defense situation, cooperative *R&D* problems, and so on.

Definition 1. The probability game is a triple (N, v, P), where N is the set of mutually independent players, $P = (p_1, p_2, \dots, p_n)$ is an n-dimension vector, with p_i the successful action probability of player i, and its characteristic function $v : 2^N \to R$ satisfying $v(\emptyset) = 0$ and for any $S \subseteq N$,

$$v(S) = \sum_{k\in S} p_k - \sum_{k,l\in S} p_k \cdot p_l + \sum_{k,l,m\in S} p_k \cdot p_l \cdot p_m - \cdots + (-1)^{s-1} \prod_{k\in S} p_k. \quad (1)$$

To interpret Definition 1, assume that each player can have a success or a failure and that successes are probabilistically independent across players. Let p_i be the probability of success of player i, then $v(S)$ is the probability that all players i in S have a success, and all other players have a failure. For the case $p_i = 0$, which means that this player is doomed to fail, then there is no need to take further consideration of this player. We exclude such case by only considering players with positive action probability.

Lots of game theoretic allocations are proposed. The most simple one is the Proportional value [5]. However, this approach ignores the fact that the players function as a congruent whole. Therefore, the players' indices for dividing the total budget should not only take into account the individual successful action probability, but also the coalitional successful action probabilities. This leads to consider the other solutions, such as the Shapley value. If we can work out the index φ_i for dividing the total budget, then we can invest $\frac{\varphi_i}{v(N)} \times W$ to each player, where W is the amount of the total budget. Next, we conduct on the determination of the Shapley value. The Shapley value, as an allocation scheme, is introduced by Shapley in 1953 as follows [7],

$$Sh_i(N, v) = \sum_{S\not\ni i, S\subseteq N} \frac{s!(n-s-1)!}{n!}[v(S\cup i) - v(S)], i \in N. \quad (2)$$

Generally, because of the combinatorial terms of the Shapley value, the computation is rather hulking to deal with. Our purpose is to simplify the solution part of the probability game. Usually, the allocation based on the Shapley value supplies a distribution of the $v(N)$ among all the players according to marginal contributions with the form $v(S \cup \{i\}) - v(S), S \subseteq N \setminus \{i\}$. In the context of probability game, the next lemma denotes the characterization of the marginal contributions, playing a significant role to determine the Shapley value.

Lemma 2.1. *For any probability game (N, v, P), it holds that*

$$v(S \cup i) - v(S) = p_i(1 - v(S)) = p_i \prod_{k\in S}(1 - p_k). \quad (3)$$

The proof of (3) is trivial by the independence of the events and the probability formula.

Theorem 2.2. *(i) For the probability game, the Shapley value of player i, which implies the index for dividing the total budget, is proportional to his individual successful action probability, while inversely proportional to the other players' successful action probability. Namely,*

$$Sh_i(N, v, P) = \frac{p_i}{n} + p_i \sum_{|S|=1,\dots,n-1, S\not\ni i} \alpha_S \prod_{k\in S}(1 - p_k), i \in N, \quad (4)$$

where α_S is the formation probability of coalition S, which is equal to $\frac{s!(n-s-1)!}{n!}$.

(ii) Alternatively, the Shapley allocation $Sh_i(N, v, P)$ can be rewritten as

$$\begin{aligned} &Sh_i(N, v, P) \\ &= p_i - p_i \cdot [\sum_{k\in N\setminus i} \frac{1}{2} p_k - \sum_{k,l\in N\setminus i} \frac{1}{3} p_k p_l - \cdots + (-1)^n \frac{1}{n} \prod_{k\in N\setminus\{i\}} p_k] \quad \text{or} \\ &= p_i - p_i \sum_{|S|=1,\dots,n-1, S\not\ni i} \frac{1}{|S|+1}(-1)^{|S|+1} \prod_{k\in S} p_k, i \in N. \end{aligned} \quad (5)$$

Proof. The validity of the theorem is due to the Shapley value applied to the marginal contribution result (3).

(i) Fix coalition $S \subseteq N$, $S \not\ni i$, by (3), it holds

$$\begin{aligned} Sh_i(N, v, P) &= \sum_{S\not\ni i} \frac{s!(n-s-1)!}{n!}[v(S\cup i) - v(S)] \\ &= \sum_{S=\emptyset} \frac{s!(n-s-1)!}{n!}[v(S\cup i) - v(S)] \\ &\quad + \sum_{S\not\ni i, S\neq\emptyset} \frac{s!(n-s-1)!}{n!}[v(S\cup i) - v(S)] \\ &= \frac{p_i}{n} + p_i \sum_{|S|=1,\dots,n-1, S\not\ni i} \frac{s!(n-s-1)!}{n!} \prod_{k\in S}(1 - p_k). \end{aligned}$$

(ii) Because of (3)

$$\begin{aligned} Sh_i(N, v, P) &= \sum_{S\not\ni i} \frac{s!(n-s-1)!}{n!}[v(S\cup i) - v(S)] \\ &= \sum_{S\not\ni i} \frac{s!(n-s-1)!}{n!} p_i - \sum_{S\not\ni i} \frac{s!(n-s-1)!}{n!} p_i \cdot v(S) \\ &= p_i - p_i \sum_{S\not\ni i} \frac{s!(n-s-1)!}{n!}[\sum_{k\in S} p_k - \sum_{k,l\in S} p_k \cdot p_l - \cdots \\ &\quad + (-1)^{s+1} \prod_{k\in S} p_k] \\ &= p_i - p_i[\sum_{k\in N\setminus\{i\}} \sum_{S\not\ni i, S\ni k} \alpha_S p_k + \cdots + (-1)^n \alpha_{N\setminus\{i\}} \prod_{k\in N\setminus\{i\}} p_k] \\ &= p_i - p_i[\sum_{k\in N\setminus\{i\}} \frac{1}{2} p_k - \sum_{k,l\in N\setminus\{i\}} \frac{1}{3} p_k p_l - \cdots + (-1)^n \frac{1}{n} \prod_{k\in N\setminus\{i\}} p_k] \\ &= p_i - p_i \sum_{|S|=1,\dots,n-1, S\not\ni i} \frac{1}{|S|+1}(-1)^{|S|+1} \prod_{k\in S} p_k. \end{aligned}$$

The last but one equality holds because for any fixed $M = \{i_1, i_2, \dots, i_m\} \subseteq N \setminus \{i\}$,

$$\begin{aligned} &\sum_{S\not\ni i, S\ni i_1, i_2, \dots, i_m} \alpha_S \cdot p_{i_1} \cdot p_{i_2} \cdot \dots \cdot p_{i_m} \\ &= \sum_{|S|=m,\dots,n-1} \frac{s!(n-s-1)!}{n!} \binom{n-m-1}{s-m} p_{i_1} \cdot p_{i_2} \cdot \dots \cdot p_{i_m} \\ &= \frac{1}{m+1} \frac{1}{C_n^{m+1}} p_{i_1} \cdot p_{i_2} \cdot \dots \cdot p_{i_m} \sum_{|S|=m,\dots,n-1} C_s^m \\ &= \frac{1}{m+1} \prod_{k\in M} p_k. \end{aligned}$$

This completes the proof of (ii). □

Although the formula is quite complicated, its economic interpretation is interesting as follows: the Shapley value index of player i is only composed of the portion of $v(N)$ relevant to player i. More specifically, the grand coalition value $v(N)$ can be decomposed into

$$\sum_{k\in N} p_k - \sum_{k,l\in N} p_k p_l + \sum_{k,l,m\in N} p_k p_l p_m - \cdots + (-1)^{n-1}\prod_{k\in N} p_k.$$

Obviously, the components with respect to i are p_i, $-p_i\sum_{k\in N\setminus\{i\}}p_k$, $\ldots$, $\prod_{k\in N}p_k$, while the Shapley value index of i is the combination of these components with each component distributed equally among the existing players appearing in this component, e.g., the coefficient of $p_k p_l p_m$ is $\frac{1}{3}$. By (4), notice that for the probability game, the Shapley value of player i is proportional to the individual successful action probability p_i, and inversely proportional to the other player's successful action probability.

Remark 1. Since the player's index for dividing the total budget is given by the Shapley value by (4) or (5), the profit assigned to player $i \in N$ is $Sh_i(N, v, P)\cdot \frac{W}{v(N)}$.

Proposition 2.3. *For player $i \in N$, the difference between dividing the budget according to the Shapley value and according to the proportional value is*

$$\frac{Sh_i(N,v,P)}{v(N)}\cdot W - \frac{p_i}{\sum_{k\in N}p_k}\cdot W = \frac{W}{v(N)\cdot\sum_{k\in N}p_k}\sum_{k\neq i}(p_i-p_k)\cdot p_k\cdot A_{ik},$$

where $A_{ik} = \frac{1}{2} - \frac{1}{3}\sum_{l\in N\setminus\{k,i\}}p_l + \cdots + \frac{(-1)^{|S|}}{|S|}\sum_{l_1,l_2,\ldots,l_{s-2}\in N\setminus\{k,i\}}p_{l_1}\cdot\ldots p_{l_{s-2}} + \cdots + \frac{(-1)^n}{n}\prod_{l\in N\setminus\{k,i\}}p_l$.

Proof. Because

$$\begin{aligned}
&\frac{Sh_i(N,v,P)}{v(N)}\cdot W - \frac{p_i}{\sum_{k\in N}p_k}\cdot W\\
&= \frac{W}{v(N)\sum_{k\in N}p_k}\Big[\sum_{k\in N}p_k\cdot Sh_i(N,v,P) - v(N)\cdot p_i\Big]\\
&= \frac{W}{v(N)\sum_{k\in N}p_k}\Big[-\sum_{t\in N}p_t\Big(\frac{1}{2}\sum_{k\in N\setminus\{i\}}p_k + \cdots\\
&\quad + \frac{(-1)^{|S|-1}}{|S|}\sum_{l_1,\ldots,l_{s-1}\in N\setminus\{i\}}p_{l_1}\cdots p_{l_{s-1}}\\
&\quad + \cdots + \frac{(-1)^n}{n}\prod_{k\in N\setminus i}p_k\Big) + \Big(\sum_{k,l\in N}p_k p_l - \cdots\\
&\quad - (-1)^{|S|-1}\sum_{l_1,\ldots,l_{s-1}\in N}p_{l_1}\cdots p_{l_{s-1}} - \cdots - (-1)^{n-1}\prod_{k\in N}p_k\Big)\Big], \qquad (6)
\end{aligned}$$

the proof of Proposition 2.3 is immediate if $\sum_{k\neq i}(p_i-p_k)\cdot p_k\cdot\frac{(-1)^{|S|}}{|S|}\sum_{l_1,\ldots,l_{s-2}\in N\setminus\{k,i\}}p_{l_1}\cdot\ldots p_{l_{s-2}}$ coincides with the general term of (6), i.e.,

$$\begin{aligned}
&\sum_{t\in N}p_t\cdot\frac{(-1)^{|S|-1}}{|S|}\sum_{l_1,\ldots,l_{s-1}\in N\setminus\{i\}}p_{l_1}\cdots p_{l_{s-1}}\\
&\quad - (-1)^{|S|-1}\sum_{l_1,\ldots,l_{s-1}\in N}p_{l_1}\cdots p_{l_{s-1}}\\
&= \sum_{k\neq i}(p_i-p_k)\cdot p_k\frac{(-1)^{|S|}}{|S|}\sum_{l_1,\ldots,l_{s-2}\in N\setminus\{k,i\}}p_{l_1}\cdot\ldots p_{l_{s-2}},
\end{aligned}$$

which is left for the reader to check.

3. An axiomatization of the Shapley value for the probability game

In this section, we present one new axiomatization of the Shapley value for the probability game based on the independent fairness property. In the probability game, since the players are mutually independent with each other, the individual player's action is not affected by the other players, yielding the individual value always plays a pivotal role when distributing the profits. Therefore, in the cooperation situation, it is often required that the larger the individual value, the more the profit. This leads to the consideration of the so-called independent fairness property.

Definition 2. For any cooperative game (N, v), an allocation φ satisfies **independent fairness property**, if $v(i) \geq v(j)$, then $\varphi_i(N,v) \geq \varphi_j(N,v)$, for any $i, j \in N$.

This property states that, for any pair of players $i, j \in N$, the player with larger individual value will be assigned with a larger portion of the total profit.

Theorem 3.1. *For the probability game (N, v, P), the Shapley value satisfies efficiency, linearity, dummy property and independent fairness property.*

Proof. It is trivial that the Shapley value satisfies efficiency, linearity, dummy property. It remains to prove the independent fairness property part. To do that, we calculate the gap of the Shapley value of player i and j. Given $v(i) \geq v(j)$, it is equivalent to prove $Sh_i(N,v,P) - Sh_j(N,v,P) \geq 0$. By (3) and (2),

$$\begin{aligned}
&Sh_i(N,v,P) - Sh_j(N,v,P)\\
&= \sum_{S\not\ni i}\frac{s!(n-s-1)!}{n!}[p_i(1-v(S))]\\
&\quad - \sum_{S\not\ni j}\frac{s!(n-s-1)!}{n!}[p_j(1-v(S))]\\
&= \sum_{S\not\ni i,j}\frac{s!(n-s-1)!}{n!}[p_i(1-v(S))]\\
&\quad + \sum_{S\not\ni i,S\ni j}\frac{s!(n-s-1)!}{n!}[p_i(1-v(S))]\\
&\quad + \sum_{S\not\ni i,j}\frac{s!(n-s-1)!}{n!}[p_j(1-v(S))]\\
&\quad + \sum_{S\not\ni j,S\ni i}\frac{s!(n-s-1)!}{n!}[p_j(1-v(S))]\\
&= \sum_{S\not\ni i,j}\frac{s!(n-s-1)!}{n!}[(p_i-p_j)(1-v(S))] + \sum_{S\not\ni i,S\ni j}\frac{s!(n-s-1)!}{n!}\\
&\quad\cdot[p_i(1-v(S\setminus j\cup j))] - \sum_{S\not\ni j,S\ni i}\frac{s!(n-s-1)!}{n!}\\
&\quad\times[p_j(1-v(S\setminus i\cup i))]\\
&= \sum_{S\not\ni i,j}\frac{s!(n-s-1)!}{n!}[(p_i-p_j)(1-v(S))]\\
&\quad + \sum_{T\not\ni i,T\ni j}\frac{(t+1)!(n-t-2)!}{n!}\\
&\quad\cdot[p_i(1-v(T\cup j))] - \sum_{T\not\ni j,T\ni i}\frac{(t+1)!(n-t-2)!}{n!}\\
&\quad\times[p_j(1-v(T\cup i))].
\end{aligned}$$

Denote these three items in the last equality by A, B and C respectively. On one hand, since $p_i \geq p_j$ and $v(S) \leq 1$,

$$A = \sum_{S \not\ni i,j} \frac{s!(n-s-1)!}{n!}[(p_i - p_j)(1 - v(S))] \geq 0.$$

On the other, note that $v(T \cup i) \geq v(T \cup j)$ for all $T \subseteq N \setminus \{i, j\}$ and $p_i \geq p_j$, thus $[p_i(1 - v(T \cup j))] \geq [p_j(1 - v(T \cup i))]$ yielding

$$B - C = \sum_{T \not\ni i, T \ni j} \frac{(t+1)!(n-t-2)!}{n!}[p_i(1 - v(T \cup j))] - \sum_{T \not\ni j, T \ni i} \frac{(t+1)!(n-t-2)!}{n!}[p_j(1 - v(T \cup i))] \geq 0.$$

This completes the proof. □

By the above theorem, the Shapley value for the probability game possesses the independent fairness property and other three properties, which are frequently applied to characterize the values of the cooperative games. As a matter of fact, the Shapley value for probability game can also be completely axiomatized by these three properties together with the proposed independent fairness property. For that purpose, we decompose the probability game into the linear combination of a number of unanimity games U_T, $T \subseteq N$, which is given by

$$U_T(S) = \begin{cases} 1, & \text{if } S \supseteq T; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3.2. *Every probability game (N, v, P) of the form (1) can be decomposed as the following combination of unanimity games:*

$$v = \sum_{T \subseteq N, T \neq \emptyset} \prod_{k \in T} p_k \cdot (-1)^{|T|-1} U_T.$$

Proof. It is sufficient to verify that

$$v(S) = \sum_{T \subseteq N, T \neq \emptyset} \prod_{k \in T} p_k \cdot (-1)^{|T|-1} U_T(S)$$

holds for all $S \subseteq N$, which is valid because

$$\begin{aligned} &\sum_{T \subseteq N, T \neq \emptyset} \prod_{k \in T} p_k \cdot (-1)^{|T|-1} U_T(S) \\ &= \sum_{T \subseteq S, T \neq \emptyset} \prod_{k \in T} p_k \cdot (-1)^{|T|-1} \\ &= \sum_{T \subseteq S, |T|=1} \prod_{k \in T} p_k \cdot (-1)^{|T|-1} + \cdots + \sum_{T \subseteq S, |T|=|S|} \prod_{k \in T} p_k \cdot (-1)^{|T|-1} \\ &= \sum_{k \in S} p_k - \sum_{k,l \in S} p_k \cdot p_l + \sum_{k,l,m \in S} p_k \cdot p_l \cdot p_m - \cdots + (-1)^{s-1} \prod_{k \in S} p_k \\ &= v(S). \end{aligned}$$

The last equality holds by (1). □

Theorem 3.3. *For the Shapley value of probability game, it is the unique value satisfying efficiency, linearity, dummy property and independent fairness property.*

Proof. It remains to prove the uniqueness part, which is equivalent to prove that there is at most one solution concept satisfying the mentioned properties. Let ϕ satisfy the mentioned four properties. For any $T \subseteq N$, note that each player in $N \setminus T$ is a dummy player in U_T, yielding $\phi_i(N, U_T) = 0, i \in N \setminus T$. While for any pair of players $i, j \in T$, it is easy to obtain $U_T(i) \geq U_T(j)$ and $U_T(j) \geq U_T(i)$. Thus, by independent fairness property, it holds $\phi_i(N, U_T) \geq \phi_j(N, U_T)$ and $\phi_j(N, U_T) \geq \phi_i(N, U_T)$, which yield $\phi_i(N, U_T) = \phi_j(N, U_T), i, j \in T$. This, together with efficiency and dummy property, implies $\phi_i(N, U_T) = \frac{1}{|T|}, i \in T$. Hence,

$$\begin{aligned} Sh_i(N, v, P) &= \sum_{T \subseteq N, T \neq \emptyset} \prod_{k \in T} p_k \cdot (-1)^{|T|-1} Sh_i(N, U_T) \\ &= \sum_{T \subseteq N, T \ni i} \prod_{k \in T} p_k \cdot (-1)^{|T|-1} \frac{1}{|T|} \\ &= p_i - p_i \sum_{|S|=1,\ldots,n-1, S \not\ni i} \frac{1}{|S|+1} (-1)^{|S|+1} \prod_{k \in S} p_k, \ i \in N, \end{aligned}$$

which completes the proof. □

In Theorem 3.3, we apply one new property to characterize the Shapley value of probability game, together with efficiency, dummy property and linearity. The independent fairness property is specific for the Shapley value of probability game, and of course, limits the class of games when considering the axiomatization of the Shapley value. Also, the four properties in Theorem 3.3 are independent with each other.

Remark 2. The independence of the four properties in Theorem 3.3 can be illustrated as follows.

(i) The value $\varphi_i^1(v) = v(i), i \in N$, satisfies linearity, dummy property and independent fairness property. But the efficiency is invalid.

(ii) The value $\varphi_i^2(v) = \frac{v(i)}{\sum_{k \in N} v(k)} \cdot v(N)$ for all $i \in N$ satisfies efficiency, dummy property and independent fairness property. But it does not satisfy linearity.

(iii) The value $\varphi_i^3(v) = v(\{1, 2, \ldots, i\}) - v(\{1, 2, \ldots, i-1\})$ for all $i \in N$ satisfies efficiency, linearity and dummy property. But the independent fairness property does not apply. Here $v(\{1, 0\})$ is interpreted as $v(\emptyset) = 0$.

(iv) The value $\varphi_i^4(v) = CIS_i(v) = v(i) + \frac{v(N) - \sum_{i \in N} v(i)}{n}$ for all $i \in N$ satisfies efficiency, linearity and independent fairness property. But the dummy property does not hold for φ^4.

4. The profitability of merging two agents based on Shapley value

In this section, the strength of the Shapley value in the form (5) is embodied in analyzing the behaviors of the players, e.g., the merger situation. One point of concern is the outcome of merger, i.e., whether the merger can benefit the merged players. Note that the Shapley value allocation that we made in Section 2, does not tell us what will happen when players merge. In fact, it only compares the allocations for different players in an existing situation. Mergers are the subject of the next theorem, which predicts the critical value to distinguish if merger can benefit the merged players, when the Shapley value is used to distribute the total profits. In order to facilitate notations, we denote the merged players i, j by $(i, j)^*$, and the profit assigned to the player $(i, j)^*$ in $n-1$ person game $N \setminus \{i, j\} \cup (i, j)^*$, by $Sh_{(i,j)^*}$.

Before we state the theorem, we formally define a merger of players. Let (N, v, P) be a probability game and we assume, without loss of generality, that the player $n-1$ merges into player n. Then, we have a new probability game, $(N \setminus \{n-1, n\} \cup (n-1, n)^*)$, where $(n-1, n)^*$ is regarded as a new player, with the successful action probability $p_{n-1} + p_n - p_{n-1} \cdot p_n$.

Theorem 4.1. *Let (N, v, P) be a probability game and assume that players $n-1, n$ merge into one new player $(n-1, n)^*$. Then the gap of the Shapley value index in front and at the back of amalgamating* *to the merged player* $(n-1,n)^*$ *is as follows,*

$$\begin{aligned} & g_{(n-1,n)^*} \\ = & Sh_{(n-1,n)^*}(N \setminus \{n-1,n\} \cup \{(n-1,n)^*\}) \\ & -(Sh_{n-1}(N)+Sh_n(N)) \\ = & p_{n-1} \cdot p_n \sum_{|S|=1,\ldots,n-2, S\subseteq N\setminus\{n-1,n\}} (-1)^{|S|-2} \\ & \times (\frac{2}{|S|+2} - \frac{1}{|S|+1}) \prod_{k\in S} p_k. \end{aligned} \tag{7}$$

(7) is a direct consequence of (5) applied to the probability games $(N \setminus \{n-1,n\} \cup (n-1,n)^*, v, P)$ and (N, v, P).

Notice that Theorem 4.1 indicates the significant condition if the players can benefit through merging. To be precise, when the change of Shapley value index $g_{(n-1,n)^*}$ is positive, then the amalgamation of two players is always profitable, otherwise might be harmful. Moreover, the theorem only considers the merger by two players simultaneously. For the other complex situations, the merger process continues such that two players merge and then a third player joins, and so on.

Next, we will illustrate with a three-person probability game to show how this theorem works. Consider the probability game (N, v, P), where $N = \{1, 2, 3\}$, and parameters p_1, p_2, p_3 are the successful action probability of different players respectively. By Theorem 2.2, it holds

$$Sh_2(N, v, P) = p_2 - p_2(\frac{1}{2}p_1 + \frac{1}{2}p_3 - \frac{1}{3}p_1 \cdot p_3) \quad \text{and}$$

$$Sh_3(N, v, P) = p_3 - p_3(\frac{1}{2}p_1 + \frac{1}{2}p_2 - \frac{1}{3}p_1 \cdot p_2).$$

When players 2 and 3 merge as one new player $(2, 3)^*$, then by Theorem 2.2,

$$\begin{aligned} Sh_{(2,3)^*}(\{1,(2,3)^*\}, v, P) &= p_{(2,3)^*} - p_{(2,3)^*} \cdot \frac{1}{2}p_1 \\ &= (p_2 + p_3 - p_2 \cdot p_3) - (p_2 \\ &\quad + p_3 - p_2 \cdot p_3) \cdot \frac{1}{2}p_1. \end{aligned}$$

Therefore,

$$\begin{aligned} & Sh_{(2,3)^*}(\{1,(2,3)^*\}, v, P) - Sh_2(N, v, P) \\ & \quad - Sh_3(N, v, P) = -\frac{1}{6}p_1p_2p_3. \end{aligned} \tag{8}$$

On the other, by Theorem 4.1,

$$\begin{aligned} & Sh_{(2,3)^*}(\{1,(2,3)^*\}, v, P) - (Sh_2(N, v, P) + Sh_3(N, v, P)) \\ = & p_2 \cdot p_3 \cdot (-1) \cdot (\frac{2}{3} - \frac{1}{2}) \cdot p_1 = -\frac{1}{6}p_1p_2p_3, \end{aligned}$$

coinciding with the outcome in the form (8). Intuitively, if $p_i > 0, i \in N$, then $-\frac{1}{6}p_1 \cdot p_2 \cdot p_3 < 0$, yielding the merger of the two players decreases the probability index, thus, no players would like to merge in three person situation.

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