Stochastic comparisons of lifetimes of series and parallel systems with dependent and heterogeneous components

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Abstract

This work considers stochastic comparisons of lifetimes of series and parallel systems with dependent and heterogeneous components having lifetimes following the proportional odds (PO) model. The joint distribution of component lifetimes is modeled by Archimedean survival copula. We discuss some potential applications of our findings in system reliability and actuarial science.

Keywords: Archimedean copula; Majorization; Proportional odds model; Stochastic order.

1 Introduction

Suppose X_1, X_2, \ldots, X_n are the random variables denoting the lifetimes of the components of a system with *n* components. Then the system lifetime is a function of X_1, X_2, \ldots, X_n . Let $X_{k:n}, k = 1, 2, \ldots, n$ denote the *k*th order statistic corresponding to the random variables X_1, X_2, \ldots, X_n . Then the smallest and the largest order statistics $X_{1:n}$ and $X_{n:n}$, respectively, represent the lifetimes of the series and the parallel systems. There have been a number of works on stochastic comparisons of system lifetimes where component lifetimes follow different family of distributions [1, 6, 7, 8, 9, 10, 11, 16, 21]. However, most of the works have considered mutual independence among the concerned random variables. Recently, Fang et al. [9], Li and Fang [16] and Li and Li [17] have considered stochastic comparison of system lifetimes with dependent and heterogeneous component lifetimes following the proportional hazard rate (PHR) model.

The proportional odds (PO) model, as introduced by Bennet [2] and later discussed by Kirmani and Gupta [12] is a very important model in reliability theory and survival analysis. Let

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X and Y be two random variables with distribution functions $F_X(\cdot)$, $F_Y(\cdot)$, survival functions $\bar{F}_X(\cdot)$, $\bar{F}_Y(\cdot)$ and hazard rate functions $r_X(\cdot)$, $r_Y(\cdot)$ respectively. Let the odds functions of X and Y be defined by $\tau_X(t) = \bar{F}_X(t)/F_X(t)$ and $\tau_Y(t) = \bar{F}_Y(t)/F_Y(t)$, respectively. If the random variable X represents the lifetime of a component, then the odds function $\tau_X(t)$ represents the odds of that component functioning beyond time t. The random variables X and Y are said to satisfy PO model if $\tau_Y(t) = \alpha \tau_X(t)$ for all admissible t, where α is a proportionality constant known as proportional odds ratio. Then the survival functions of X and Y are related as

$$\bar{F}_Y(t) = \frac{\alpha \bar{F}_X(t)}{1 - \bar{\alpha} \bar{F}_X(t)},\tag{1}$$

where $\bar{\alpha} = 1-\alpha$. We will say that the random variable Y is following the PO model with baseline survival function $\bar{F}_X(\cdot)$ and parameter (proportionality constant) α . For easy interpretation, we can think of X as the lifetime of a member of control group, and Y as that of a member of treatment group. For two random variables satisfying the PO model, the ratio of hazard rates converges to unity as time tends to infinity, which is in contrast to the PHR model where this ratio remains constant with time. The convergence property of hazard functions makes the PO model reasonable in many practical applications as discussed in [2, 12, 22]. For more applications of PO model one may refer to [4, 25]. Also, the model (1), with $0 < \alpha < \infty$, provides us a method of generating more flexible new family of distributions by introducing the parameter α to an existing family of distributions [18]. The family of distributions so obtained is known as Marshall-Olkin family of distributions [5, 18]. Thus, model (1) has implications both in terms of the PO model and in generating new family of flexible distributions, which makes it worth investigating.

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a random vector with joint distribution function $F(\cdot)$ and joint survival function $\overline{F}(\cdot)$. Also let the distribution function and the survival function of X_i are $F_i(\cdot)$ and $\overline{F}_i(\cdot)$ respectively for $i = 1, 2, \ldots, n$. The joint distribution of X_1, X_2, \ldots, X_n can be represented by a copula model. If there exist $K : [0,1]^n \mapsto [0,1]$ and $\overline{K} : [0,1]^n \mapsto [0,1]$ such that $F(x_1, \ldots, x_n) = K(F_1(x_1), \ldots, F_n(x_n))$ and $\overline{F}(x_1, \ldots, x_n) = \overline{K}(\overline{F}_1(x_1), \ldots, \overline{F}_n(x_n))$ for all $x_i, i \in I_n$, then K and \overline{K} are called the copula and survival copula of X, respectively. If $\varphi : [0, +\infty) \mapsto [0,1]$ with $\varphi(0) = 1$ and $\lim_{t \to +\infty} \varphi(t) = 0$ is (n-2)th differentiable, then $K_{\varphi}(u_1, \ldots, u_n) = \varphi(\varphi^{-1}(u_1) + \ldots + \varphi^{-1}(u_n)) = \varphi(\sum_{i=1}^n \phi(u_i))$ for all $u_i \in (0,1], i \in I_n$ is called an Archimedean survival copula with generator φ provided $(-1)^k \varphi^{(k)}(t) \ge 0, k = 0, 1, \ldots, n-2$ and $(-1)^{n-2} \varphi^{(n-2)}(t)$ is decreasing and convex for all $t \ge 0$. Here $\phi = \varphi^{-1}$ is the right continuous inverse of φ so that $\phi(u) = \varphi^{-1}(u) = \sup\{t \in \mathbb{R} : \varphi(t) > u\}$. Navarro and Spizzichino [21] have derived usual stochastic ordering for lifetimes of series and parallel systems having component lifetimes sharing a common copula, with the idea of mean reliability function associated with the common copula. Li and Fang [16] investigated stochastic order between two samples of dependent random variables following PHR model and having Archimedean survival copula. Fang et al. [9] derived some stochastic ordering results for minimum as well as for maximum of samples equipped with Archimedean survival copulas and following PHR model and proportional reversed hazard rate (PRH) model, respectively. Li and Li [17] investigated hazard rate order on minimums of sample following PHR model, and reversed hazard rate order on maximums of sample following PRH model, where both the samples coupled with Archimedean survival copula.

In case of PO model, some authors, e.g. Kundu and Nanda [14], Kundu et al. [15], Nanda and Das [20] have investigated stochastic comparison of systems with independent components. However, to the best of our knowledge, no research work has been done on stochastic comparison of system lifetimes with dependent and heterogeneous component lifetimes following PO model. In this work, we investigate stochastic comparisons of lifetimes of series and parallel systems with dependent and heterogenous components having lifetimes following the PO model. The joint distribution of component lifetimes is modeled by Archemedian survival copula. It is shown that the usual stochastic ordering and hazard rate ordering hold for series systems under certain conditions whereas for parallel system stochastic ordering and reversed hazard rate ordering hold.

The organization of the paper is as follows. Section 2 recalls some definitions of majorization, stochastic orders, and some lemmas used in the sequel. In Section 3, we investigate stochastic comparisons between series systems of dependent and heterogenous components having lifetimes following the PO model and coupled by Archimedean survival copulas. Section 4 investigates the same in case of parallel systems. Section 5 presents some potential applications of the proposed results.

2 Preliminaries

Given a vector $\boldsymbol{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, denote $x_{(1)} \leq x_{(2)} \leq ... \leq x_{(n)}$ as increasing arrangement of $x_1, x_2, ..., x_n$.

Definition 2.1 Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ in \mathbb{R}^n be any two vectors.

(i) The vector \boldsymbol{x} is said to majorize the vector \boldsymbol{y} , i.e., \boldsymbol{x} is larger than \boldsymbol{y} in majorization order (denoted as $\boldsymbol{x} \succeq \boldsymbol{y}$) if (cf. [19])

$$\sum_{i=1}^{j} x_{(i)} \le \sum_{i=1}^{j} y_{(i)}, \text{ for all } j = 1, 2, \dots, n-1, \text{ and } \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}.$$

(ii) The vector \boldsymbol{x} is said to weakly supermajorize the vector \boldsymbol{y} , denoted as $\boldsymbol{x} \succeq \boldsymbol{y}$ if (cf. [19])

$$\sum_{i=1}^{j} x_{(i)} \le \sum_{i=1}^{j} y_{(i)}, \text{ for all } j = 1, 2, \dots, n.$$

(iii) The vector \boldsymbol{x} is said to be p-larger than the vector \boldsymbol{y} (denoted as $\boldsymbol{x} \succeq \boldsymbol{y}$) if (cf. [3])

$$\prod_{i=1}^{j} x_{(i)} \le \prod_{i=1}^{j} y_{(i)}, \text{ for all } j = 1, 2, \dots, n.$$

It can be seen that

$$\mathbf{x} \stackrel{m}{\succeq} \mathbf{y} \Rightarrow \mathbf{x} \stackrel{w}{\succeq} \mathbf{y} \Rightarrow \mathbf{x} \stackrel{p}{\succeq} \mathbf{y}.\Box$$

Definition 2.2 [23] Let X and Y be nonnegative absolutely continuous random variables with cumulative distribution functions $F_X(\cdot)$, $F_Y(\cdot)$, survival functions $\bar{F}_X(\cdot)$, $\bar{F}_Y(\cdot)$, hazard (failure) rate functions $r_X(\cdot)$, $r_Y(\cdot)$, and the reversed hazard rate functions $\tilde{r}_X(\cdot)$ and $\tilde{r}_Y(\cdot)$, respectively. Then X is said to be smaller than Y in the

- (i) usual stochastic order (denoted as $X \leq_{st} Y$) if $\overline{F}_X(t) \leq \overline{F}_Y(t)$ for all t;
- (ii) hazard rate order (denoted as $X \leq_{hr} Y$) if $\overline{F}_Y(t)/\overline{F}_X(t)$ is increasing in $t \geq 0$, or equivalently if $r_X(t) \geq r_Y(t)$ for all $t \geq 0$;
- (iii) reversed hazard rate order (denoted as $X \leq_{rhr} Y$) if $F_Y(t)/F_X(t)$ is increasing in t > 0, or equivalently if $\tilde{r}_X(t) \leq \tilde{r}_Y(t)$ for all t > 0.

Lemma 2.1 [19] Let $I \subseteq \mathbb{R}$ be an open interval and let $\zeta : I^n \to \mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for ζ to be Schur-convex (resp. Schur-concave) on I^n are that ζ is symmetric on I^n , and for all $i \neq j$,

$$(u_i - u_j)\left(\zeta_{(i)}(\boldsymbol{u}) - \zeta_{(j)}(\boldsymbol{u})\right) \ge (resp. \le) 0 \text{ for all } \boldsymbol{u} = (u_1, u_2, ..., u_n) \in I^n,$$

where $\zeta_{(k)}(\boldsymbol{u}) = \partial \zeta(\boldsymbol{u}) / \partial u_k$.

Lemma 2.2 [19] Let $\mathcal{A} \subseteq \mathbb{R}^n$, and $\zeta : \mathcal{A} \to \mathbb{R}$ be a function. Then, for $x, y \in \mathcal{A}$,

$$\boldsymbol{x} \stackrel{w}{\succeq} \boldsymbol{y} \implies \zeta(\boldsymbol{x}) \ge (resp. \leq) \zeta(\boldsymbol{y})$$

if and only if ζ is both decreasing (resp. increasing) and Schur-convex (resp. Schur-concave) on A. **Lemma 2.3** [13] Let $\zeta : (0, \infty)^n \to \mathbb{R}$ be a function. Then,

$$oldsymbol{x} \stackrel{\mathrm{p}}{\succeq} oldsymbol{y} \implies \zeta(oldsymbol{x}) \geq (resp. \leq) \zeta(oldsymbol{y})$$

if and only if the following two conditions hold:

(i) $\zeta(e^{v_1}, \ldots, e^{v_n})$ is Schur-convex (resp. Schur-concave) in (v_1, \ldots, v_n) , (ii) $\zeta(e^{v_1}, \ldots, e^{v_n})$ is decreasing (resp. increasing) in each v_i , for $i = 1, \ldots, n$, where $v_i = \ln x_i$, for $i = 1, \ldots, n$.

Lemma 2.4 [9] For two n-dimensional Archimedean copulas K_{φ_1} and K_{φ_2} , if $\phi_2 \circ \varphi_1$ is superadditive, then $K_{\varphi_1}(\boldsymbol{u}) \leq K_{\varphi_2}(\boldsymbol{u})$ for all $\boldsymbol{u} \in [0, 1]^n$.

3 Series systems with dependent and heterogeneous component lifetimes following PO Model

Here, we consider the comparison of lifetimes of two series systems with heterogeneous and dependent components. We assume that the lifetime vector $X = (X_1, X_2, ..., X_n)$ is a set of dependent random variables coupled with Archimedean survival copula with generator φ and following the PO model with baseline survival function \overline{F} , denoted as $X \sim PO(\overline{F}, \alpha, \varphi)$, where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}^n_+$ is the proportional odds ratio vector. The survival function and the hazard rate function of X_i are

$$\bar{F}_{\alpha_i}(x) = \frac{\alpha_i F(x)}{1 - \bar{\alpha}_i \bar{F}(x)}$$
 and $r_{\alpha_i}(x) = \frac{r(x)}{1 - \bar{\alpha}_i \bar{F}(x)}$, respectively,

where $\bar{\alpha}_i = 1 - \alpha_i$, i = 1, ..., n and r denotes the baseline hazard rate function. The survival functions of $X_{1:n}$ is given by

$$\bar{F}_{X_{1:n}}(x) = P(X_{1:n} > x) = P(X_i > x, i \in I_n) = \varphi\left(\sum_{i=1}^n \phi\left(\bar{F}_{\alpha_i}(x)\right)\right) = S_1(\bar{F}(x), \boldsymbol{\alpha}, \varphi), \ say,$$
(2)

where $\phi(u) = \varphi^{-1}(u), \ u \in (0, 1].$

The hazard rate function of $X_{1:n}$ is obtained as

$$r_{X_{1:n}}(x) = r(x) \frac{\varphi'\left(\sum_{i=1}^{n} \phi\left(\bar{F}_{\alpha_i}(x)\right)\right)}{\varphi\left(\sum_{i=1}^{n} \phi\left(\bar{F}_{\alpha_i}(x)\right)\right)} \sum_{i=1}^{n} \phi'\left(\bar{F}_{\alpha_i}(x)\right) \frac{\bar{F}_{\alpha_i}(x)}{1 - \bar{\alpha}_i \bar{F}(x)}.$$
(3)

Lemma 3.1 For any $x \in [0,1]$, $S_1(x, \alpha, \varphi)$ is increasing in α_i , $i \in I_n$. Furthermore S_1 is Schur-concave with respect to α .

Proof: For $s \in I_n$,

$$\frac{\partial S_1}{\partial \alpha_s} = \varphi'\left(\sum_{i=1}^n \phi\left(\frac{\alpha_i x}{1 - \bar{\alpha}_i x}\right)\right) \phi'\left(\frac{\alpha_s x}{1 - \bar{\alpha}_s x}\right) \frac{x(1 - x)}{(1 - \bar{\alpha}_s x)^2}.$$

Since both $\varphi(u)$ and $\phi(u)$ are decreasing for all $u \ge 0$, $\frac{\partial S_1}{\partial \alpha_s} \ge 0$. As a result $S_1(x, \alpha, \varphi)$ is increasing in $\alpha_i, i \in I_n$ for any $x \in [0, 1]$. For $s \ne t$,

$$(\alpha_{s} - \alpha_{t}) \left(\frac{\partial S_{1}}{\partial \alpha_{s}} - \frac{\partial S_{1}}{\partial \alpha_{t}} \right)$$

$$= (\alpha_{s} - \alpha_{t})\varphi' \left(\sum_{i=1}^{n} \phi \left(\frac{\alpha_{i}x}{1 - \bar{\alpha}_{i}x} \right) \right) \left[\phi' \left(\frac{\alpha_{s}x}{1 - \bar{\alpha}_{s}x} \right) \frac{x(1 - x)}{(1 - \bar{\alpha}_{s}x)^{2}} - \phi' \left(\frac{\alpha_{t}x}{1 - \bar{\alpha}_{t}x} \right) \frac{x(1 - x)}{(1 - \bar{\alpha}_{t}x)^{2}} \right]$$

$$\stackrel{sign}{=} (\alpha_{s} - \alpha_{t}) \left(-\varphi' \left(\sum_{i=1}^{n} \phi \left(u_{i} \right) \right) \right) \left[\left(-\phi' \left(u_{s} \right) \right) \frac{1}{(1 - \bar{\alpha}_{s}x)^{2}} - \left(-\phi' \left(u_{t} \right) \right) \frac{1}{(1 - \bar{\alpha}_{t}x)^{2}} \right], \quad (4)$$

where $u_i = \frac{\alpha_i x}{1 - \bar{\alpha}_i x}$ and $\stackrel{`sign'}{=}$ means equal in sign. Since both φ and ϕ are decreasing, and ϕ' is increasing, it follows from (4) that $(\alpha_s - \alpha_t) \left(\frac{\partial S_1}{\partial \alpha_s} - \frac{\partial S_1}{\partial \alpha_t}\right) \leq 0$. So from Lemma 2.1, S_1 is Schur-concave in $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$.

Suppose there are two series systems formed out of n statistically dependent and heterogeneous components where the component lifetimes follow the PO model. The joint distribution of lifetimes of components is represented by Archimedean copula. Consider two such series systems with lifetime vectors $X = (X_1, X_2, ..., X_n)$ and $Y = (Y_1, Y_2, ..., Y_n)$ having respective proportionality odds ratio vectors $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, ..., \beta_n)$, where $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n_+$. The following theorem compares the lifetimes of these series systems in the sense of usual stochastic order.

Theorem 3.1 Suppose the lifetime vectors $X \sim PO(\bar{F}, \alpha, \varphi_1)$ and $Y \sim PO(\bar{F}, \beta, \varphi_2)$. If φ_1 or φ_2 is log-convex and $\phi_2 \circ \varphi_1$ is superadditive, then

$$\boldsymbol{\alpha} \succeq \boldsymbol{\beta} \text{ implies } X_{1:n} \leq_{st} Y_{1:n}.$$

Proof: Write $v_i = \ln \alpha_i$, i = 1, 2, ..., n. Then as per (2),

$$\bar{F}_{X_{1:n}}(x) = \varphi_1\left(\sum_{i=1}^n \phi_1\left(\frac{e^{v_i}\bar{F}(x)}{1 - (1 - e^{v_i})\bar{F}(x)}\right)\right) = S_1(\bar{F}(x), (e^{v_1}, e^{v_2}, ..., e^{v_n}), \varphi_1).$$

Here $S_1(\bar{F}(x), (e^{v_1}, e^{v_2}, ..., e^{v_n}), \varphi_1)$ is symmetric with respect to $(v_1, v_2, ..., v_n) \in \mathbb{R}^n$. Now, for

 $s \in I_n$,

$$\frac{\partial S_1}{\partial v_s} = \varphi_1' \left(\sum_{i=1}^n \phi_1 \left(\frac{e^{v_i} x}{1 - (1 - e^{v_i}) x} \right) \right) \phi_1' \left(\frac{e^{v_s} x}{1 - (1 - e^{v_s}) x} \right) \frac{x(1 - x)e^{v_s}}{(1 - (1 - e^{v_s}) x)^2},$$

so that $S_1(x, (e^{v_1}, e^{v_2}, ..., e^{v_n}), \varphi_1)$ is increasing in each $v_i, i = 1, 2, ..., n$ for any $x \in [0, 1]$. Now, for $s \neq t$,

$$(v_{s} - v_{t}) \left(\frac{\partial S_{1}}{\partial v_{s}} - \frac{\partial S_{1}}{\partial v_{t}} \right)$$

$$= (v_{s} - v_{t}) \left(-\varphi_{1}' \left(\sum_{i=1}^{n} \phi_{1} \left(\frac{e^{v_{i}} x}{1 - (1 - e^{v_{i}})x} \right) \right) \right) \left[\left(-\phi_{1}' \left(\frac{e^{v_{s}} x}{1 - (1 - e^{v_{s}})x} \right) \right) \frac{x e^{v_{s}}}{(1 - (1 - e^{v_{s}})x)^{2}} \right]$$

$$- \left(-\phi_{1}' \left(\frac{e^{v_{t}} x}{1 - (1 - e^{v_{t}})x} \right) \right) \frac{x e^{v_{t}}}{(1 - (1 - e^{v_{t}})x)^{2}} \right]$$

$$sign = (v_{s} - v_{t}) \left[\left(-\frac{\varphi_{1}(\phi_{1}(u_{s}))}{\varphi_{1}'(\phi_{1}(u_{s}))} \right) \frac{1}{1 - (1 - e^{v_{s}})x} - \left(-\frac{\varphi_{1}(\phi_{1}(u_{t}))}{\varphi_{1}'(\phi_{1}(u_{t}))} \right) \frac{1}{1 - (1 - e^{v_{t}})x} \right],$$

$$(5)$$

where $u_s = \frac{e^{v_s x}}{1 - (1 - e^{v_s})x}$. If φ_1 is log-convex, from (5) it follows that $(v_s - v_t) \left(\frac{\partial S_1}{\partial v_s} - \frac{\partial S_1}{\partial v_t}\right) \leq 0$. Hence from Lemma 2.1, $S_1(x, (e^{v_1}, e^{v_2}, ..., e^{v_n}), \varphi_1)$ is Schur-concave in $(v_1, v_2, ..., v_n)$ if φ_1 is log-convex. Then from Lemma 2.3, we have

$$\boldsymbol{\alpha} \succeq^{p} \boldsymbol{\beta} \text{ implies } S_{1}(\bar{F}(x), \boldsymbol{\alpha}, \varphi_{1}) \leq S_{1}(\bar{F}(x), \boldsymbol{\beta}, \varphi_{1}).$$
(6)

Since $\phi_2 \circ \varphi_1$ is superadditive, from Lemma 2.4, we have

$$S_1(\bar{F}(x), \boldsymbol{\beta}, \varphi_1) \le S_1(\bar{F}(x), \boldsymbol{\beta}, \varphi_2).$$
(7)

Thus combining (6) and (7) we get $S_1(\bar{F}(x), \boldsymbol{\alpha}, \varphi_1) \leq S_1(\bar{F}(x), \boldsymbol{\beta}, \varphi_2)$, that is $X_{1:n} \leq_{st} Y_{1:n}$. Now suppose φ_2 is log-convex, then $S_1(\bar{F}(x), \boldsymbol{\alpha}, \varphi_2) \leq S_1(\bar{F}(x), \boldsymbol{\beta}, \varphi_2)$. Since $\phi_2 \circ \varphi_1$ is superadditive, we have $S_1(\bar{F}(x), \boldsymbol{\alpha}, \varphi_1) \leq S_1(\bar{F}(x), \boldsymbol{\alpha}, \varphi_2)$. So combining we get $X_{1:n} \leq_{st} Y_{1:n}$. \Box

Corollary 3.1 Suppose the lifetime vectors $X \sim PO(\bar{F}, \alpha, \varphi)$ and $Y \sim PO(\bar{F}, \beta, \varphi)$. If φ is log-convex, then

$$\boldsymbol{\alpha} \stackrel{p}{\succeq} \boldsymbol{\beta} \text{ implies } X_{1:n} \leq_{st} Y_{1:n}. \quad \Box$$

The following counterexample shows that one may not get the the ordering result in Theorem 3.1 if the sufficient conditions on the generator functions are dropped.

Counterexample 3.1 Consider two series systems, each comprising of three dependent and heterogeneous components with respective survival functions $\bar{F}_{X_{1:3}}(x) = S_1(\bar{F}(x), \alpha, \varphi_1)$ and $\bar{F}_{Y_{1:3}}(x) = S_1(\bar{F}(x), \beta, \varphi_2)$ with $\bar{F}(x) = e^{-(x)^{1.5}}$, $x \ge 0$, $\alpha = (2, 3, 5.5)$, $\beta = (2.5, 3.5, 3.8)$ so



Figure 1: Plots of $\overline{F}_{X_{1:3}}(x)$ and $\overline{F}_{Y_{1:3}}(x)$ when $\phi_2 \circ \varphi_1$ is not super additive, and φ_1 and φ_2 are not log-convex.

that $\boldsymbol{\alpha} \succeq \boldsymbol{\beta}$. First we take $\varphi_1(x) = (2/(1+e^x))^{1/\theta}$, $\theta = 0.9$ and $\varphi_2(x) = e^{1-(1+x)^{1/\eta}}$ with $\eta = 0.3$ so that $\phi_2 \circ \varphi_1$ is not super additive, and φ_1 and φ_2 are not log-convex. We depict $\bar{F}_{X_{1:3}}(x)$ and $\bar{F}_{Y_{1:3}}(x)$ in Figure 1 for some finite range of x. From this figure we observe that the stochastic ordering result in Theorem 3.1 is not attained.

Since p-larger order is weaker than weakly supermajorization order, the following theorem shows that we can get the ordering result in Theorem 3.1 under weakly supermajorization order with fewer condition.

Theorem 3.2 Suppose the lifetime vectors $X \sim PO(\bar{F}, \alpha, \varphi_1)$ and $Y \sim PO(\bar{F}, \beta, \varphi_2)$. If $\phi_2 \circ \varphi_1$ is superadditive, then

$$\boldsymbol{\alpha} \succeq \boldsymbol{\beta} \text{ implies } X_{1:n} \leq_{st} Y_{1:n}.$$

Proof: From Lemma 3.1 and Lemma 2.2, we have

$$\boldsymbol{\alpha} \succeq \boldsymbol{\beta}$$
 implies $S_1(\bar{F}(x), \boldsymbol{\alpha}, \varphi_1) \leq S_1(\bar{F}(x), \boldsymbol{\beta}, \varphi_1).$

Since $\phi_2 \circ \varphi_1$ is superadditive, so from Lemma 2.4, we have, $S_1(\bar{F}(x), \beta, \varphi_1) \leq S_1(\bar{F}(x), \beta, \varphi_2)$. Combining the above results we have $S_1(\bar{F}(x), \alpha, \varphi_1) \leq S_1(\bar{F}(x), \beta, \varphi_2)$. That is $X_{1:n} \leq_{st} Y_{1:n}$.

Remark 3.1 It is to be noted that super-additive assumption of $\phi_2 \circ \varphi_1$ is satisfied by many members of Archimedean survival copulas. For example, Archimedean survival copula with generators (i) $\varphi_1(t) = e^{1-(1+t)^{\frac{1}{\theta}}}$ and $\varphi_2(t) = \frac{\theta}{\log(e^{\theta}+t)}$, where $0 < \theta \leq 1$, (ii) $\varphi_1(t) = \frac{\theta}{\log(e^{\theta}+t)}$ and $\varphi_2(t) = \log(e+t)^{-1/\theta}$, where $\theta > 1$ and (iii) $\varphi_1(t) = e^{1-(1+t)^{\frac{1}{\theta_1}}}$ and $\varphi_2(t) = e^{1-(1+t)^{\frac{1}{\theta_2}}}$, where $\theta_2 \geq \theta_1 \geq 1$, satisfy super-additivity. **Corollary 3.2** Suppose the lifetime vectors $X \sim PO(\bar{F}, \alpha, \varphi)$ and $Y \sim PO(\bar{F}, \beta, \varphi)$. Then

$$\boldsymbol{\alpha} \succeq \boldsymbol{\beta} \text{ implies } X_{1:n} \leq_{st} Y_{1:n}.$$

Lemma 3.2 $I_1(\boldsymbol{u}) = \frac{\varphi'(\sum_{i=1}^n u_i)}{\varphi(\sum_{i=1}^n u_i)} \sum_{i=1}^n \frac{\varphi(u_i)}{\varphi'(u_i)} (1 - \varphi(u_i))$ is increasing in $u_s, s \in I_n$ and Schurconvex with respect to $\boldsymbol{u} = (u_1, ..., u_n)$ if φ is log-concave and $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing and concave.

Proof: Here $I_1(\mathbf{u})$ is symmetric in \mathbf{u} . For $s \in I_n$,

$$\frac{\partial I_{1}(\mathbf{u})}{\partial u_{s}} = \frac{\partial}{\partial u_{s}} \left(\frac{\varphi'\left(\sum_{i=1}^{n} u_{i}\right)}{\varphi\left(\sum_{i=1}^{n} u_{i}\right)} \right) \sum_{i=1}^{n} \frac{\varphi\left(u_{i}\right)}{\varphi'\left(u_{i}\right)} \left(1 - \varphi\left(u_{i}\right)\right) + \frac{\varphi'\left(\sum_{i=1}^{n} u_{i}\right)}{\varphi\left(\sum_{i=1}^{n} u_{i}\right)} \frac{\partial}{\partial u_{s}} \left(\frac{\varphi\left(u_{s}\right)}{\varphi'\left(u_{s}\right)} \left(1 - \varphi\left(u_{s}\right)\right)\right).$$

Since φ is log-concave, $\frac{\partial}{\partial u_s} \left(\frac{\varphi'(\sum_{i=1}^n u_i)}{\varphi(\sum_{i=1}^n u_i)} \right) \leq 0$. As $\varphi(1-\varphi)/\varphi'$ is decreasing, $\frac{\partial}{\partial u_s} \left(\frac{\varphi(u_s)}{\varphi'(u_s)} \left(1 - \varphi\left(u_s \right) \right) \right) \leq 0$. Then using the fact that φ is deceasing, we have $\frac{\partial I_1(\mathbf{u})}{\partial u_s} \geq 0$. So $I_1(\mathbf{u})$ is increasing in u_s for any $s \in I_n$. For $s, t \in I_n$ with $s \neq t$,

$$\frac{\partial}{\partial u_s} \left(\frac{\varphi'\left(\sum_{i=1}^n u_i\right)}{\varphi\left(\sum_{i=1}^n u_i\right)} \right) = \frac{\partial}{\partial u_t} \left(\frac{\varphi'\left(\sum_{i=1}^n u_i\right)}{\varphi\left(\sum_{i=1}^n u_i\right)} \right).$$

Then

$$(u_{s} - u_{t}) \left(\frac{\partial I_{1}(\mathbf{u})}{\partial u_{s}} - \frac{\partial I_{1}(\mathbf{u})}{\partial u_{t}} \right)$$

$$= (u_{s} - u_{t}) \frac{\varphi'(\sum_{i=1}^{n} u_{i})}{\varphi(\sum_{i=1}^{n} u_{i})} \left[\frac{\partial}{\partial u_{s}} \left(\frac{\varphi(u_{s})}{\varphi'(u_{s})} \left(1 - \varphi(u_{s}) \right) \right) - \frac{\partial}{\partial u_{t}} \left(\frac{\varphi(u_{t})}{\varphi'(u_{t})} \left(1 - \varphi(u_{t}) \right) \right) \right] \ge 0,$$

where the inequality follows from the fact that $\frac{\varphi(1-\varphi)}{\varphi'}$ is concave. So from lemma 2.1, $I_1(\mathbf{u})$ is Schur-convex with respect to \mathbf{u} .

Next we show hazard rate ordering of two series systems formed out of n statistically dependent and heterogeneous components having lifetimes following PO model.

Theorem 3.3 Suppose the lifetime vectors $X \sim PO(\bar{F}, \alpha, \varphi)$ and $Y \sim PO(\bar{F}, \beta, \varphi)$. If φ is log-concave and $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing and concave (or convex), then

$$\boldsymbol{\alpha} \succeq^{w} \boldsymbol{\beta} \text{ implies } X_{1:n} \leq_{hr} Y_{1:n}.$$

Proof: From (3), we have

$$r_{X_{1:n}}(x) = r(x) \frac{\varphi'\left(\sum_{i=1}^{n} \phi\left(\bar{F}_{\alpha_{i}}(x)\right)\right)}{\varphi\left(\sum_{i=1}^{n} \phi\left(\bar{F}_{\alpha_{i}}(x)\right)\right)} \sum_{i=1}^{n} \phi'\left(\bar{F}_{\alpha_{i}}(x)\right) \frac{\bar{F}_{\alpha_{i}}(x)}{1 - \bar{\alpha}_{i}\bar{F}(x)}$$
$$= \frac{r(x)}{F(x)} \frac{\varphi'\left(\sum_{i=1}^{n} \phi\left(\bar{F}_{\alpha_{i}}(x)\right)\right)}{\varphi\left(\sum_{i=1}^{n} \phi\left(\bar{F}_{\alpha_{i}}(x)\right)\right)} \sum_{i=1}^{n} \frac{\bar{F}_{\alpha_{i}}(x)}{\varphi'\left(\phi\left(\bar{F}_{\alpha_{i}}(x)\right)\right)} \frac{F(x)}{1 - \bar{\alpha}_{i}\bar{F}(x)}$$
$$= \frac{r(x)}{F(x)} I_{1}\left(\phi\left(\bar{F}_{\alpha_{1}}(x)\right), \dots, \phi\left(\bar{F}_{\alpha_{n}}(x)\right)\right)\right),$$

where

$$I_1\left(\phi\left(\bar{F}_{\alpha_1}(x)\right),\ldots,\phi\left(\bar{F}_{\alpha_n}(x)\right)\right) = \frac{\varphi'\left(\sum_{i=1}^n \phi\left(\bar{F}_{\alpha_i}(x)\right)\right)}{\varphi\left(\sum_{i=1}^n \phi\left(\bar{F}_{\alpha_i}(x)\right)\right)} \sum_{i=1}^n \frac{\varphi\left(\phi\left(\bar{F}_{\alpha_i}(x)\right)\right)}{\varphi'\left(\phi\left(\bar{F}_{\alpha_i}(x)\right)\right)} \left(1 - \varphi\left(\phi\left(\bar{F}_{\alpha_i}(x)\right)\right)\right).$$

It is easy to check that $\phi\left(\bar{F}_{\alpha_i}(x)\right)$ is decreasing and convex in α_i . From Theorem A.2 (Chapter 5) of Marshall et al. [19], $\boldsymbol{\alpha} \succeq \boldsymbol{\beta}$ implies $\left(\phi\left(\bar{F}_{\alpha_1}(x)\right), \ldots, \phi\left(\bar{F}_{\alpha_n}(x)\right)\right) \succeq_w \left(\phi\left(\bar{F}_{\beta_1}(x)\right), \ldots, \phi\left(\bar{F}_{\beta_n}(x)\right)\right)$. From Lemma 3.2, $I_1(\mathbf{u})$ is increasing in u_i for $i \in I_n$ and Schur-convex with respect to \mathbf{u} whenever φ is log-concave and $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing and concave. Then from Theorem A.8 (Chapter 3) of Marshall et al. [19], we get

$$I_1\left(\phi\left(\bar{F}_{\alpha_1}(x)\right),\ldots,\phi\left(\bar{F}_{\alpha_n}(x)\right)\right) \ge I_1\left(\phi\left(\bar{F}_{\beta_1}(x)\right),\ldots,\phi\left(\bar{F}_{\beta_n}(x)\right)\right)$$

which implies $r_{X_{1:n}}(x) \ge r_{Y_{1:n}}(x)$, that is $X_{1:n} \le_{hr} Y_{1:n}$. Next we prove the theorem when $\frac{\varphi(1-\varphi)}{\varphi'}$ is convex. Let $z_i = \phi\left(\bar{F}_{\alpha_i}(x)\right)$. Then the hazard rate function is given by

$$r_{X_{1:n}}(x) = \frac{r(x)}{F(x)} \frac{\varphi'\left(\sum_{i=1}^{n} z_i\right)}{\varphi\left(\sum_{i=1}^{n} z_i\right)} \sum_{i=1}^{n} \frac{\varphi\left(z_i\right)}{\varphi'\left(z_i\right)} \left(1 - \varphi\left(z_i\right)\right).$$

Now, for $s \in I_n$,

$$\frac{r_{X_{1:n}}(x)}{\partial \alpha_s} = \frac{r(x)}{F(x)} \left[\frac{\partial}{\partial z_s} \left(\frac{\varphi'\left(\sum_{i=1}^n z_i\right)}{\varphi\left(\sum_{i=1}^n z_i\right)} \right) \frac{\partial z_s}{\partial \alpha_s} \sum_{i=1}^n \frac{\varphi\left(z_i\right)\left(1-\varphi\left(z_i\right)\right)}{\varphi'\left(z_i\right)} + \frac{\varphi'\left(\sum_{i=1}^n z_i\right)}{\varphi\left(\sum_{i=1}^n z_i\right)} \frac{\partial}{\partial z_s} \left(\frac{\varphi\left(z_s\right)\left(1-\varphi\left(z_s\right)\right)}{\varphi'\left(z_s\right)} \right) \frac{\partial z_s}{\partial \alpha_s} \right].$$

Note that z_s is decreasing in α_s and $\frac{\partial z_s}{\partial \alpha_s}$ is increasing in α_s . Since φ is log-concave and $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing, we have $\frac{r_{X_{1:n}}(x)}{\partial \alpha_s} \leq 0$. Again

$$\frac{\partial}{\partial z_s} \left(\frac{\varphi'\left(\sum_{i=1}^n z_i\right)}{\varphi\left(\sum_{i=1}^n z_i\right)} \right) = \frac{\partial}{\partial z_t} \left(\frac{\varphi'\left(\sum_{i=1}^n z_i\right)}{\varphi\left(\sum_{i=1}^n z_i\right)} \right), \text{ for } s \neq t.$$

For $s \neq t$,

$$\begin{aligned} & (\alpha_s - \alpha_t) \left(\frac{r_{X_{1:n}}}{\partial \alpha_s} - \frac{r_{X_{1:n}}}{\partial \alpha_t} \right) \\ \stackrel{sign}{=} & (\alpha_s - \alpha_t) \left(\frac{\partial z_s}{\partial \alpha_s} - \frac{\partial z_t}{\partial \alpha_t} \right) + (\alpha_s - \alpha_t) \frac{\varphi'(\sum_{i=1}^n z_i)}{\varphi(\sum_{i=1}^n z_i)} \times \\ & \left[\left(-\frac{\partial}{\partial z_s} \left(\frac{\varphi(z_s) \left(1 - \varphi(z_s)\right)}{\varphi'(z_s)} \right) \right) \left(-\frac{\partial z_s}{\partial \alpha_s} \right) - \left(-\frac{\partial}{\partial z_t} \left(\frac{\varphi(z_t) \left(1 - \varphi(z_t)\right)}{\varphi'(z_t)} \right) \right) \left(-\frac{\partial z_t}{\partial \alpha_t} \right) \right] \\ \leq & 0, \end{aligned}$$

whenever $\frac{\varphi(1-\varphi)}{\varphi'}$ is convex in addition to the log-concave φ and decreasing $\frac{\varphi(1-\varphi)}{\varphi'}$. Thus we have $r_{X_{1:n}}(x)$ is decreasing in α_i , $i \in I_n$ and Schur-convex in $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$. Then from Lemma 2.2, we have

$$\boldsymbol{\alpha} \succeq \boldsymbol{\beta}$$
 implies $r_{X_{1:n}}(x) \geq r_{X_{1:n}}(x)$.

Hence the theorem follows.

Corollary 3.3 Suppose the lifetime vectors $X \sim PO(\bar{F}, \alpha, \varphi)$ and $Y \sim PO(\bar{F}, \alpha \mathbf{1}, \varphi)$. Then, $X_{1:n} \leq_{hr} Y_{1:n}$ if $\alpha \geq \frac{1}{n} \sum_{i=1}^{n} \alpha_i$, φ is log-concave and $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing and concave (or convex). This follows from the Theorem 3.3 and using the fact that $(\alpha_1, \alpha_2, \ldots, \alpha_n) \succeq (\alpha, \alpha, \ldots, \alpha)$, $\frac{\omega}{n \text{ terms}}$

for
$$\alpha \ge \frac{1}{n} \sum_{i=1}^{n} \alpha_i$$
.

Remark 3.2 It is to be noted that Archimedean copulas with generators $\varphi(t) = 2/(1 + e^t)$ and $\varphi(t) = (-1 + \theta)/(-e^t + \theta)$ for $-1 \le \theta \le 0$ are some examples of survival copula such that φ is log-concave, and $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing and convex.

The following counterexample shows that one may not get the the ordering result in Theorem 3.3 if the sufficient conditions on the generator functions are dropped.

Counterexample 3.2 Consider two series systems, each comprising of three dependent and heterogeneous components with respective hazard rate functions $r_{X_{1:3}}(x)$ and $r_{Y_{1:3}}(x)$, with common baseline survival function $\overline{F}(x) = e^{-(0.5x)^2}$, $x \ge 0$, $\alpha = (0.2, 0.4, 0.6)$, $\beta = (0.35, 0.55, 0.95)$ so that $\alpha \succeq \beta$. First we take the common generator $\varphi(x) = \log(e + x)^{-1/a}$, a = 0.1, which is not log-concave but $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing and convex. Next we take $\varphi(x) = (2/(1 + e^x))^{1/a}$, a = 0.2, which is log-concave but $\frac{\varphi(1-\varphi)}{\varphi'}$ is neither decreasing nor convex. For both the cases $r_{X_{1:3}}(x)$ and $r_{Y_{1:3}}(x)$ are depicted in Figure 2(a) and 2(b) respectively for some finite range of x. From both the figures we observe that the hazard rate ordering result in Theorem 3.3 is not attained.



Figure 2: Plots of $r_{X_{1:3}}(x)$ and $r_{Y_{1:3}}(x)$ for (a) $\varphi(x)$ is not log-concave, (b) $\frac{\varphi(1-\varphi)}{\varphi'}$ is neither decreasing nor convex.

4 Parallel systems with dependent and heterogeneous component lifetimes following PO Model

Here, we compare the lifetimes of two parallel systems consisting of dependent and heterogeneous components having lifetimes following the PO model, with respect to some stochastic orders.

Let the lifetime vector $X = (X_1, X_2, ..., X_n)$ be a set of dependent random variables following the PO model with baseline survival function \overline{F} and having the joint distribution function coupled with Archimedean survival copula with generator φ , denoted as $X \sim PO(\overline{F}, \boldsymbol{\alpha}, \varphi)$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}^n_+$ is the proportional odds ratio vector. The distribution function of X_i is $F_{\alpha_i}(x) = \frac{F(x)}{1-\overline{\alpha_i}\overline{F}(x)}$. The distribution function of $X_{n:n}$ is given by

$$F_{X_{n:n}}(x) = P(X_{n:n} \le x) = P(X_i < x, i \in I_n) = \varphi\left(\sum_{i=1}^n \phi\left(F_{\alpha_i}(x)\right)\right) = S_2(F(x), \boldsymbol{\alpha}, \varphi), \ say,$$

where $\phi(u) = \varphi^{-1}(u), u \in (0, 1]$. The reversed hazard rate function of $X_{n:n}$ is obtained as

$$\tilde{r}_{X_{n:n}}(x) = \tilde{r}(x) \frac{\varphi'\left(\sum_{i=1}^{n} \phi\left(F_{\alpha_i}(x)\right)\right)}{\varphi\left(\sum_{i=1}^{n} \phi\left(F_{\alpha_i}(x)\right)\right)} \sum_{i=1}^{n} \phi'\left(F_{\alpha_i}(x)\right) \frac{\alpha_i F_{\alpha_i}(x)}{1 - \bar{\alpha}_i \bar{F}(x)},\tag{8}$$

where \tilde{r} denotes the baseline reversed hazard rate function.

Lemma 4.1 For any $x \in [0,1]$, $S_2(x, \alpha, \varphi)$ is decreasing in α_i , $i \in I_n$. Furthermore S_2 is Schur-convex with respect to α whenever φ is log-concave.

Proof: For $s \in I_n$,

$$\frac{\partial S_2}{\partial \alpha_s} = -\varphi'\left(\sum_{i=1}^n \phi\left(\frac{x}{1-\bar{\alpha}_i(1-x)}\right)\right)\phi'\left(\frac{x}{1-\bar{\alpha}_s(1-x)}\right)\frac{x(1-x)}{(1-\bar{\alpha}_s(1-x))^2}$$

Since both $\varphi(u)$ and $\phi(u)$ are decreasing for all $u \ge 0$, $\frac{\partial S_2}{\partial \alpha_s} \le 0$. So $S_2(x, \boldsymbol{\alpha}, \varphi)$ is decreasing in α_i for any $x \in [0, 1]$.

For $s \neq t$,

$$\begin{aligned} & (\alpha_s - \alpha_t) \left(\frac{\partial S_2}{\partial \alpha_s} - \frac{\partial S_2}{\partial \alpha_t} \right) \\ &= -(\alpha_s - \alpha_t) \varphi' \left(\sum_{i=1}^n \phi\left(v_i \right) \right) \left[\phi'\left(v_s \right) \frac{x(1-x)}{(1-\bar{\alpha}_s x)^2} - \phi'\left(v_t \right) \frac{x(1-x)}{(1-\bar{\alpha}_t x)^2} \right], \ v_i = \frac{x}{1-\bar{\alpha}_i(1-x)} \\ & \stackrel{sign}{=} (\alpha_s - \alpha_t) \left[- \left(-\frac{\varphi(\phi(v_s))}{\varphi'(\phi(v_s))} \right) \frac{1}{1-\bar{\alpha}_s x} + \left(-\frac{\varphi(\phi(v_t))}{\varphi'(\phi(v_t))} \right) \frac{1}{1-\bar{\alpha}_t x} \right] \\ & \geq 0, \end{aligned}$$

where the last inequality is derived using the fact that φ is log-concave. So from Lemma 2.1, S_2 is Schur-convex in $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$.

Suppose there are two parallel systems with lifetime vectors $X = (X_1, X_2, ..., X_n)$ and $Y = (Y_1, Y_2, ..., Y_n)$, formed out of *n* dependent and heterogeneous components where the component lifetimes follow PO model. The following theorem compares the lifetimes of two such parallel systems in the sense of usual stochastic order.

Theorem 4.1 Suppose the lifetime vectors $X \sim PO(\bar{F}, \alpha, \varphi_1)$ and $Y \sim PO(\bar{F}, \beta, \varphi_2)$. If φ_1 or φ_2 is log-concave and $\phi_1 \circ \varphi_2$ is superadditive, then

$$\boldsymbol{\alpha} \succeq^{w} \boldsymbol{\beta} \text{ implies } X_{n:n} \leq_{st} Y_{n:n}.$$

Proof: If φ_1 is log-concave, then from Lemma 4.1 and Lemma 2.2, we have

$$\boldsymbol{\alpha} \succeq \boldsymbol{\beta} \text{ implies } S_2(F(x), \boldsymbol{\alpha}, \varphi_1) \ge S_2(F(x), \boldsymbol{\beta}, \varphi_1).$$
 (9)

Since $\phi_1 \circ \varphi_2$ is superadditive, so from Lemma 2.4 (by replacing φ_1 by φ_2 and vice versa), we have

$$S_2(F(x), \boldsymbol{\beta}, \varphi_1) \ge S_2(F(x), \boldsymbol{\beta}, \varphi_2).$$
(10)

Combining (9) and (10), we get $S_2(F(x), \boldsymbol{\alpha}, \varphi_1) \geq S_2(F(x), \boldsymbol{\beta}, \varphi_2)$. That is $X_{n:n} \leq_{st} Y_{n:n}$.



Figure 3: Plots of $F_{X_{3:3}}(x)$ and $F_{Y_{3:3}}(x)$ for (a) neither φ_1 nor φ_2 is log-concave, (b) $\phi_1 \circ \varphi_2$ is not super additive.

Now suppose φ_2 is log-concave, then

$$S_2(F(x), \boldsymbol{\alpha}, \varphi_1) \geq S_2(F(x), \boldsymbol{\alpha}, \varphi_2)$$
$$\geq S_2(F(x), \boldsymbol{\beta}, \varphi_2),$$

where the first inequality follows from the fact that $\phi_1 \circ \varphi_2$ is superadditive, whereas the second inequality follows from the fact that $\boldsymbol{\alpha} \succeq \boldsymbol{\beta}$. This proves the result. \Box

Corollary 4.1 Suppose the lifetime vectors $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi)$ and $Y \sim PO(\bar{F}, \boldsymbol{\beta}, \varphi)$. If φ is log-concave, then

$$\boldsymbol{\alpha} \succeq \boldsymbol{\beta} \text{ implies } X_{n:n} \leq_{st} Y_{n:n}.\square$$

The following counterexample shows that one may not get the the ordering result in Theorem 4.1 if the sufficient conditions on the generator functions are dropped.

Counterexample 4.1 Consider two parallel systems, each comprising of three dependent and heterogeneous components with respective distribution functions $F_{X_{3:3}}(x) = S_2(F(x), \alpha, \varphi_1)$ and $F_{Y_{3:3}}(x) = S_2(F(x), \beta, \varphi_2)$, where $F(x) = 1 - e^{-x^{0.5}}$, $x \ge 0$, $\alpha = (0.9, 1.45, 2.15)$, $\beta = (1.2, 1.95, 2.65)$ so that $\alpha \succeq \beta$. First we take $\varphi_1(x) = \theta_1/\log(x + e^{\theta_1})$ and $\varphi_2(x) = e^{1-(1+x)^{1/\theta_2}}$ with $\theta_1 = 0.9$ and $\theta_2 = 8$ so that neither φ_1 nor φ_2 is log-concave but $\phi_1 \circ \varphi_2$ is super additive. Next we take $\varphi_1(x) = e^{(1-e^x)/\theta_1}$ and $\varphi_2(x) = (2/(e^x + 1))^{1/\theta_2}$ with $\theta_1 = 0.9$ and $\theta_2 = 0.2$ so that φ_1 is log-concave but $\phi_1 \circ \varphi_2$ is not super additive. For both the cases $F_{X_{3:3}}(x)$ and $F_{Y_{3:3}}(x)$ are depicted in Figure 3(a) and 3(b) respectively for some finite range of x. From both the figures we observe that the stochastic ordering result in Theorem 4.1 is not attained. \Box

Next we show the reversed hazard rate order of lifetimes of two parallel systems of dependent and heterogeneous components. **Theorem 4.2** Suppose the lifetime vectors $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi)$ and $Y \sim PO(\bar{F}, \boldsymbol{\beta}, \varphi)$. If φ is log-concave and $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing and convex, then

$$\boldsymbol{\alpha} \succeq^{w} \boldsymbol{\beta} \text{ implies } X_{n:n} \leq_{rhr} Y_{n:n}.$$

Proof: From (8), the reversed hazard function of $X_{n:n}$ is given by

$$\begin{split} \tilde{r}_{X_{n:n}}(x) &= \frac{\tilde{r}(x)}{\bar{F}(x)} \frac{\varphi'\left(\sum_{i=1}^{n} \phi\left(F_{\alpha_{i}}(x)\right)\right)}{\varphi\left(\sum_{i=1}^{n} \phi\left(F_{\alpha_{i}}(x)\right)\right)} \sum_{i=1}^{n} \frac{F_{\alpha_{i}}(x)}{\varphi'\left(\phi\left(F_{\alpha_{i}}(x)\right)\right)} \bar{F}_{\alpha_{i}}(x) \\ &= \frac{\tilde{r}(x)}{\bar{F}(x)} \frac{\varphi'\left(\sum_{i=1}^{n} \phi\left(F_{\alpha_{i}}(x)\right)\right)}{\varphi\left(\sum_{i=1}^{n} \phi\left(F_{\alpha_{i}}(x)\right)\right)} \sum_{i=1}^{n} \frac{\varphi\left(\phi\left(F_{\alpha_{i}}(x)\right)\right)}{\varphi'\left(\phi\left(F_{\alpha_{i}}(x)\right)\right)} \left(1 - \varphi\left(\phi\left(F_{\alpha_{i}}(x)\right)\right)\right) \\ &= \frac{\tilde{r}(x)}{\bar{F}(x)} \frac{\varphi'\left(\sum_{i=1}^{n} \xi_{i}\right)}{\varphi\left(\sum_{i=1}^{n} \xi_{i}\right)} \sum_{i=1}^{n} \frac{\varphi\left(\xi_{i}\right)}{\varphi'\left(\xi_{i}\right)} \left(1 - \varphi\left(\xi_{i}\right)\right), \end{split}$$

where $\xi_i = \phi(F_{\alpha_i}(x))$. Now, for $s \in I_n$,

$$\frac{\tilde{r}_{X_{n:n}}(x)}{\partial \alpha_s} = \frac{\tilde{r}(x)}{\bar{F}(x)} \left[\frac{\partial}{\partial \xi_s} \left(\frac{\varphi'\left(\sum_{i=1}^n \xi_i\right)}{\varphi\left(\sum_{i=1}^n \xi_i\right)} \right) \frac{\partial \xi_s}{\partial \alpha_s} \sum_{i=1}^n \frac{\varphi\left(\xi_i\right)\left(1-\varphi\left(\xi_i\right)\right)}{\varphi'\left(\xi_i\right)} + \frac{\varphi'\left(\sum_{i=1}^n \xi_i\right)}{\varphi\left(\sum_{i=1}^n \xi_i\right)} \frac{\partial}{\partial \xi_s} \left(\frac{\varphi\left(\xi_s\right)\left(1-\varphi\left(\xi_s\right)\right)}{\varphi'\left(\xi_s\right)} \right) \frac{\partial \xi_s}{\partial \alpha_s} \right].$$

Note that ξ_s is increasing in α_s and $\frac{\partial \xi_s}{\partial \alpha_s}$ is decreasing in α_s . Since φ is log-concave and $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing, we have $\frac{\tilde{r}_{X_n:n}(x)}{\partial \alpha_s} \ge 0$. Again

$$\frac{\partial}{\partial\xi_s} \left(\frac{\varphi'\left(\sum_{i=1}^n \xi_i\right)}{\varphi\left(\sum_{i=1}^n \xi_i\right)} \right) = \frac{\partial}{\partial\xi_t} \left(\frac{\varphi'\left(\sum_{i=1}^n \xi_i\right)}{\varphi\left(\sum_{i=1}^n \xi_i\right)} \right), \text{ for } s \neq t.$$

For $s \neq t$,

$$\begin{aligned} & (\alpha_s - \alpha_t) \left(\frac{\tilde{r}_{X_{n:n}}}{\partial \alpha_s} - \frac{\tilde{r}_{X_{n:n}}}{\partial \alpha_t} \right) \\ \stackrel{sign}{=} & (\alpha_s - \alpha_t) \left(\frac{\partial \xi_s}{\partial \alpha_s} - \frac{\partial \xi_t}{\partial \alpha_t} \right) + (\alpha_s - \alpha_t) \left(-\frac{\varphi'\left(\sum_{i=1}^n \xi_i\right)}{\varphi\left(\sum_{i=1}^n \xi_i\right)} \right) \times \\ & \left[\left(-\frac{\partial}{\partial \xi_s} \left(\frac{\varphi\left(\xi_s\right)\left(1 - \varphi\left(\xi_s\right)\right)}{\varphi'\left(\xi_s\right)} \right) \right) \frac{\partial \xi_s}{\partial \alpha_s} - \left(-\frac{\partial}{\partial \xi_t} \left(\frac{\varphi\left(\xi_t\right)\left(1 - \varphi\left(\xi_t\right)\right)}{\varphi'\left(\xi_t\right)} \right) \right) \frac{\partial \xi_t}{\partial \alpha_t} \right] \\ \leq & 0, \end{aligned}$$

as $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing and convex. Thus we have $\tilde{r}_{X_{n:n}}(x)$ is increasing in $\alpha_i, i \in I_n$ and Schur-concave in $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$. Then from Lemma 2.2, we have

$$\boldsymbol{\alpha} \succeq^{w} \boldsymbol{\beta}$$
 implies $\tilde{r}_{X_{n:n}}(x) \leq \tilde{r}_{Y_{n:n}}(x)$.



Figure 4: Plots of $\tilde{r}_{X_{4:4}}(x)$ and $\tilde{r}_{Y_{4:4}}(x)$ for (a) $\varphi(x)$ is not log-concave, (b) $\frac{\varphi(1-\varphi)}{\varphi'}$ is neither decreasing nor convex.

Hence the theorem follows. The following counterexample shows that one may not get the the ordering result in Theorem 4.2 if the sufficient conditions on the generator functions are dropped.

Counterexample 4.2 Consider two parallel systems, each comprising of four dependent and heterogeneous components with respective reversed hazard rate functions $\tilde{r}_{X_{4:4}}(x)$ and $\tilde{r}_{Y_{4:4}}(x)$, with common baseline survival function $\bar{F}(x) = e^{-x^3}$, $x \ge 0$, $\alpha = (0.2, 0.6, 1.5, 3.5)$, $\beta = (0.8, 0.9, 4.5, 5.5)$ so that $\alpha \succeq \beta$. First we take the common generator $\varphi(x) = (1/(ax+1))^{1/a}$, a = 0.2, which is not log-concave but $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing and convex. Next we take $\varphi(x) = (2/(1+e^x))^{1/a}$, a = 0.2, which is log-concave but $\frac{\varphi(1-\varphi)}{\varphi'}$ is neither decreasing nor convex. For both the cases $\tilde{r}_{X_{4:4}}(x)$ and $\tilde{r}_{Y_{4:4}}(x)$ are depicted in Figure 4(a) and 4(b) respectively for some finite range of x. From both the figures we observe that the reversed hazard rate ordering result in Theorem 4.2 is not attained.

5 Applications

In this section, we highlight some potential applications of the established results. Consider a series (or a parallel) system of n components having dependent lifetimes. It is quite practical that the odds functions of all the components (i.e. the odds of surviving beyond a specified time t) may not be the same for various possible reasons, like the components are manufactured by different manufacturing units or they are subjected to different levels of stress. So let the odds function of ith component is proportional to a baseline odds function with proportionality constant (odds ratio) α_i , i = 1, 2, ..., n. Now consider another series (or parallel) system of n dependent components having different odds ratios β_i , i = 1, 2, ..., n. Even if a same system operates in two different levels of environments/stress (e.g., voltage, temperature, compression and tension), then reliability characteristics (e.g., odds function) of a component of the system

generally will not be the same in the two different environments. So it is a subject of interest to compare lifetimes of two such systems, i.e. under what conditions one system will be more reliable than other. Theorems 3.1 and 3.2 (resp. Theorem 4.1) give the conditions on the corresponding odds ratio vectors and the generators of the survival copulas under which a series (resp. parallel) system will have stochastically longer lifetime than that of the other. Similarly Theorem 3.3 (resp. Theorem 4.2) gives the conditions under which failure rate of a series (resp. parallel) system will be smaller than that of the other.

Next we show that using our proposed results one can compare the lifetime of two series systems whose components are subjected to random shock instantaneously [8]. Suppose random variable X_i denotes the lifetime of *i*-th component of the series system. Define Bernoulli random variable I_{p_i} associated with X_i , where $I_{p_i} = 1$ if shock does not occur and 0 if shock occurs with $p_i = P(I_{p_i} = 1), i = 1, ..., n$. Assume that $I_{p_1}, ..., I_{p_n}$ are independent random variables, and also they are independent of $X_1, ..., X_n$. Let $X_i^* = X_i I_{p_i}, i = 1, ..., n$, and denote $X_{1:n}^* = \min(X_1^*, ..., X_n^*)$. Similarly assume that $I_{q_1}, ..., I_{q_n}$ are independent Bernoulli random variables, and also they are independent of Y_i 's with $q_i = P(I_{q_i} = 1), i = 1, ..., n$. Denote $Y_{1:n}^* = \min(Y_1^*, ..., Y_n^*)$, where $Y_i^* = Y_i I_{q_i}, i = 1, ..., n$. Here $X_{1:n}^*$ represents the lifetime of a series system whose components are subjected to random shock instantaneously. Similarly $Y_{1:n}^*$ represents the the lifetime of another such series system. Now, if $X \sim PO(\bar{F}, \alpha, \varphi_1)$ and $Y \sim PO(\bar{F}, \beta, \varphi_2)$, then with the help of the Theorems 3.1, 3.2, 3.3, and the associated corollaries 3.1, 3.2, we can establish following stochastic comparisons between such smallest order statistics from the fact that $P(X_{1:n}^* > x) = P(X_1 > x, ..., X_n > x)P(I_{p_i} = 1, i \in I_n) = P(X_{1:n} > x)\prod_i^n p_i$.

Theorem 5.1 Suppose the lifetime vectors $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi_1)$ and $Y \sim PO(\bar{F}, \boldsymbol{\beta}, \varphi_2)$. If φ_1 or φ_2 is log-convex, $\phi_2 \circ \varphi_1$ is superadditive and $\prod_i^n p_i \leq \prod_i^n q_i$, then

$$\boldsymbol{\alpha} \succeq^{p} \boldsymbol{\beta} \text{ implies } X_{1:n}^{*} \leq_{st} Y_{1:n}^{*}.$$

Corollary 5.1 Suppose the lifetime vectors $X \sim PO(\bar{F}, \alpha, \varphi)$ and $Y \sim PO(\bar{F}, \beta, \varphi)$. If φ is log-convex and $\prod_{i=1}^{n} p_i \leq \prod_{i=1}^{n} q_i$, Then

$$\boldsymbol{\alpha} \succeq^{p} \boldsymbol{\beta} \text{ implies } X_{1:n}^{*} \leq_{st} Y_{1:n}^{*}.$$

Theorem 5.2 Suppose the lifetime vectors $X \sim PO(\bar{F}, \alpha, \varphi_1)$ and $Y \sim PO(\bar{F}, \beta, \varphi_2)$. If $\phi_2 \circ \varphi_1$ is superadditive and $\prod_i^n p_i \leq \prod_i^n q_i$, then

$$\boldsymbol{\alpha} \succeq^{w} \boldsymbol{\beta} \text{ implies } X_{1:n}^* \leq_{st} Y_{1:n}^*.$$

Corollary 5.2 Suppose the lifetime vectors $X \sim PO(\bar{F}, \alpha, \varphi)$ and $Y \sim PO(\bar{F}, \beta, \varphi)$. If

 $\prod_{i=1}^{n} p_i \leq \prod_{i=1}^{n} q_i$, then

$$\boldsymbol{\alpha} \succeq \boldsymbol{\beta} \text{ implies } X_{1:n}^* \leq_{st} Y_{1:n}^*.$$

Theorem 5.3 Suppose the lifetime vectors $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi)$ and $Y \sim PO(\bar{F}, \boldsymbol{\beta}, \varphi)$. If φ is log-concave and $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing and concave, then

$$\boldsymbol{\alpha} \succeq^{w} \boldsymbol{\beta} \text{ implies } X_{1:n}^{*} \leq_{hr} Y_{1:n}^{*}$$

We will end this section by mentioning an other potential application. In actuarial science, $X_{1:n}^*$ corresponds to the smallest claim amount in a portfolio of risks [1, 17, 24], where X_i 's represent sizes of random claims of multiple risks covered by a policy that can be made in an insurance period and the corresponding I_{p_i} 's indicate the occurrence of these claims. That means $I_{p_i} = 1$ whenever the *i*th policy makes random claim X_i and $I_{p_i} = 0$ whenever there is no claim. Similarly suppose $Y_{1:n}^*$ represents the smallest claim amount in an another portfolio of risks. The above theorems can be used in stochastic comparisons between the smallest claim amounts of two different portfolio of risks.

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