Stability of Three-way Concepts and Its Application to Natural Language Generation

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Abstract

Three-way concept analysis (3WCA) has been an emerging and important methodology for knowledge discovery and data analysis. Particularly, 3WCA can efficiently characterize the information of "jointly possessed" and "jointly not possessed" compared to the classical formal concept only can describe common attributes owned by objects. This property, typical of 3WCA has a huge potential in the field of Natural Language Generation (NLG). However, the construction of a three-way concept lattice is proved as an NP-complete problem and even harder than the construction of conventional concept lattice. This could negatively affect the use of 3WCA for NLG in real contexts. Hence, it is necessary to prune the three-way concept lattice and extract more interesting three-way concepts for knowledge acquisition. To this end, this paper defines the stability of a three-way concept and analyzes the relevant properties. An efficient computational algorithm for calculating the stability of three-way concepts is developed and evaluated by an experiment. In addition, a case study on NLG is conducted for demonstrating the applicability of the proposed technique.

Keywords: Three-way Concept, Stability, Natural Language Generation

1. Introduction

Nowadays, Natural Language Processing (NLP) is finding more and more interest thanks to the new frameworks and techniques developed to solve this task, in particular after the advent of Deep Learning (Otter et al. [2020]). The aim of NLP is to understand the content of documents and text. As can be easily imagined, the more complex

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tasks require finding relations between words, both in terms of syntax and meanings to put in place Natural Language Generation (NLG). The more popular issues are disambiguation (Mennes and van der Waart van Gulik [2020]), sentient capabilities (Tellols et al. [2020]), low resources (Tran and Nguyen [2021]). As in other training-based tasks, also NLG requires a long training time to understand relations between words before generating sentences. For this reason, we propose to apply three-way concept analysis (3WCA) to NLG. 3WCA is a new and powerful technique in the decision field that, with our proposed evaluation of the stability of the three-way concept, has been made fast enough to be applicable to NLG. We notice the relation between NLG and knowledge graph since knowledge graph is demonstrated to be particularly suitable for question generation (Elsahar et al. [2018]) or question answering (Hu et al. [2018]). However, classical knowledge graphs only consider relations between elements and neglect dissimilarity between words.

On the other hand, three-way concept analysis (3WCA) is a promising knowledge discovery theory that integrates the three-way decision model and formal concept analysis (FCA). It embodies the idea of three decision-making theories and has the basic manifestation and concreteness of the formal concept analysis theory. Unlike FCA, a three-way concept can efficiently characterize the information of "jointly possessed" and "jointly not possessed" compared to the classical formal concept only can describe common attributes owned by objects (Hao et al. [2017]). As a fundamental research issue of 3WCA, a three-way concept lattice construction is essential to achieving more efficient decision-making and swarm intelligence in the real world (Ma et al. [2020]). However, the construction of a three-way concept lattice is proved an NP-complete problem. Inspired by concept stability in FCA (Kuznetsov and Makhalova [2018]), we consider that selecting the most interesting three-way concepts is quite important for the sake of saving construction time of three-way concept lattice. Starting by the study of Ibrahim et al. [2018], our previous work (Gao et al. [2020]) proved the equivalence between concept stability and key structures of social networks.

It has been demonstrated that concept lattice can be used to analyze sentences (Serutla and Kourie [1998]). In addition, more recently, concept lattice has been used to learn and connect conceptual relationships from unstructured data (AnoopV. and Asharaf [2019]). This kind of knowledge can also be applied to full documents to perform text understanding and summarizing (Ye et al. [2007]). Hence, the following and more challenging step is to use those lattices to learn how to build a sentence or a text. FCA can be used as an analytical tool to formalize the NLP concept and to visualize them (Ilieva and Ormandjieva [2007]). The 3WCA, as a more inclusive extension of FCA, could be also able to help learn how to build new text starting from its concept lattices, since the relationships between concepts are more deeply analyzed. In this sense, 3WCA can help to conquer some limitations of FCA for NLP, presented by the work by Ilieva and Ormandjieva [2007]. In particular, even if FCA has been demonstrated to be useful for representing types of relation, because of its built-in relational mechanism, cannot offer a methodology for discovering relations, objects and attributes. The ability of making connections between the structures in an horizontal manner to understand the context, is proper of the 3WCA by the construction of two different lattices (i.e., AE-concept lattice and OE-concept lattice) and by analyzing them though the stability. Since the textual analysis and generation has many data and connections between concepts, it is required to search for more efficient way to compute stability, that we propose in this paper.

The major contributions of this paper are summarized as follows.

- (**Stability of Three-way Concepts**): Inspired by concept stability and triadic concept stability, we present a novel stability measure, called *three-way concept stability*, for measuring the importance of three-way concepts and further extracting more interesting patterns by filtering out the unimportant three-way concepts. The three-way concept stability is composed of AE-concept stability and OE-concept stability. And it also can be used for characterizing the partition of the equivalence classes. Besides, the scale of three-way concept stability is investigated and theoretically proved. It is found that the three-way concept stability falls into $[0, \frac{2^{|A|}}{2^{|A|}-1}]$, where $|A|$ is the cardinality of extent of the three-way concepts.

- (**Computation of Three-way Concept Stability**): Computation for the stability of a three-way concept is equivalent to exploring all the elements of the power-set of the extent part of such a three-way concept. Thus, for each element of the power-set, i.e., a subset of the extent part, we have to check whether it can preserve the intent. Therefore, the computation for the stability of three-way concepts is NP-complete. To this end, we first validate that the concept stability depends on the stability of its subconcepts. Therefore, an efficient computational algorithm about three-way concepts stability is presented.

- (**Evaluation**) We evaluate the running time of generating three-way concepts and calculating their stability under baseline and our algorithms. Experimental results demonstrate that our algorithm can significantly improve time efficiency. In addition, a case study on NLG is conducted for illustrating the usefulness of this research. It is found that three-way stability is an alternative efficient solution for addressing NLG problem.

The remainder of this paper is organized as follows. Section 2 provides the methodology of 3WCA. Section 3 defines the stability indices for three-way concepts. The computation for the stability of three-way concepts is elaborated in Section 4. Section 5 shows the experimental results and analysis of computation for the stability of three-way concepts and our potential usage for NLG scenario. Finally, Section 6 concludes this paper.

2. 3WCA Methodology

With the popularity of the three-way decision theory, Qi et al. [2014] extended the conventional FCA methodology and then proposed 3WCA methodology by applying the three-way decision theory to FCA. Suppose (G, M, I) be a formal context, for any $X \subseteq G, B \subseteq M$, 3WCA methodology defines the following two new operators.

$$\begin{aligned} X^{\bar{*}} &= \{m \in M | \forall x \in X : ((x, m) \notin I)\}; \\ B^{\bar{*}} &= \{g \in G | \forall b \in B : ((g, b) \notin I)\}. \end{aligned} \tag{1}$$

Based on the above operators, the three-way operators, i.e., AE-operators and OE-operators are defined.

Definition 1 *Let $K = (G, M, I)$ be a formal context. For any objects subsets $X, Y \subseteq G$ and attributes subset $A, \subseteq M$, a pair of AE-operators $\lessdot$ and $\gtrdot$ are given as follows.*

$$A^{\lessdot} = (A^*, A^{\bar{*}}) \tag{2}$$

$$(X, Y)^{\gtrdot} = \{m \subseteq M | m \in X^* \wedge m \in Y^{\bar{*}}\} = X^* \cap Y^{\bar{*}} \tag{3}$$

Similar to the definition of AE-operators, for any objects subset $X, Y \subseteq G$ and attributes subset $B \subseteq M$, a pair of OE-operators $\lessdot$ and $\gtrdot$ are given as follows.

$$X^{\lessdot} = (X^*, X^{\bar{*}}) \tag{4}$$

$$(A, B)^{\gtrdot} = \{g \subseteq G | g \in A^* \wedge g \in B^{\bar{*}}\} = A^* \cap B^{\bar{*}} \tag{5}$$

Definition 2 *Let $K = (G, M, I)$ be a formal context. For any $X \subseteq G$ and $A, B \subseteq M$, if $X^{\lessdot} = (A, B)$ and $(A, B)^{\gtrdot} = X$, then $(X, (A, B))$ is called an OE concept, where X indicates the extent, and (A, B) denotes the intent.*

We use $OEL(G, M, I)$ to indicate the set of all OE concepts generated from the formal context $K = (G, M, I)$. For any $(X, (A, B)), (Y, (C, D)) \in OEL(G, M, I)$, the order relation between them is as follows.

$$\begin{gathered} (X, (A, B)) \leq (Y, (C, D)) \Leftrightarrow X \subseteq Y \Leftrightarrow \\ (C, D) \subseteq (A, B) \Leftrightarrow C \subseteq A, D \subseteq B \end{gathered} \tag{6}$$

where $(X, (A, B))$ is the son-concept of $(Y, (C, D))$, and $(Y, (C, D))$ is the father-concept of $(X, (A, B))$.

Definition 3 *Let $K = (G, M, I)$ be a formal context. For any $X, Y \subseteq G$ and $A \subseteq M$, if $(X, Y)^{\gtrdot} = A$ and $A^{\lessdot} = (X, Y)$, then $((X, Y), A)$ is called an AE concept, where (X, Y) indicates the extent, and A denotes the intent.*

Similarly, we use $AEL(G, M, I)$ to indicate the set of all AE concepts generated from the formal context $K = (G, M, I)$. For any $((X, Y), A), ((Z, W), B) \in AEL(G, M, I)$, the order relation between them is as follows.

$$((X, Y), A) \leq ((Z, W), B) \Leftrightarrow (X, Y) \subseteq (Z, W) \Leftrightarrow B \subseteq A \tag{7}$$

where $((X, Y), A)$ is the son-concept of $((Z, W), B)$, and $((Z, W), B)$ is the father-concept of$((X, Y), A)$.

Example 1 *Given a formal context $K = (G, M, I)$ with $G = \{1, 2, 3, 4\}$ and $M = \{a, b, c, d, e\}$, the relation I between objects G and attributes M is shown in Table 1.*

Table 1: A Formal Context $K = (G, M, I)$

	a	b	c	d	e
1	1	1		1	1
2	1	1	1		
3				1	
4	1	1	1		1

By using the AE-operator and OE-operator, the following AE-concept lattice and OE-concept lattice can be obtained as shown in Figure 1. Clearly, both AE-concept lattice and OE-concept lattice have 11 concepts. For example, for a node ((124,3),ab) in AE-concept lattice denotes an AE-concept in which (124,3) is the extent of AE-concept, ab is the intent of AE-concept; for a node (24,(abc,d)) in OE-concept lattice indicates an OE-concept in which 24 is the extent of OE-concept, and (abc,d) is the intent of OE-concept.

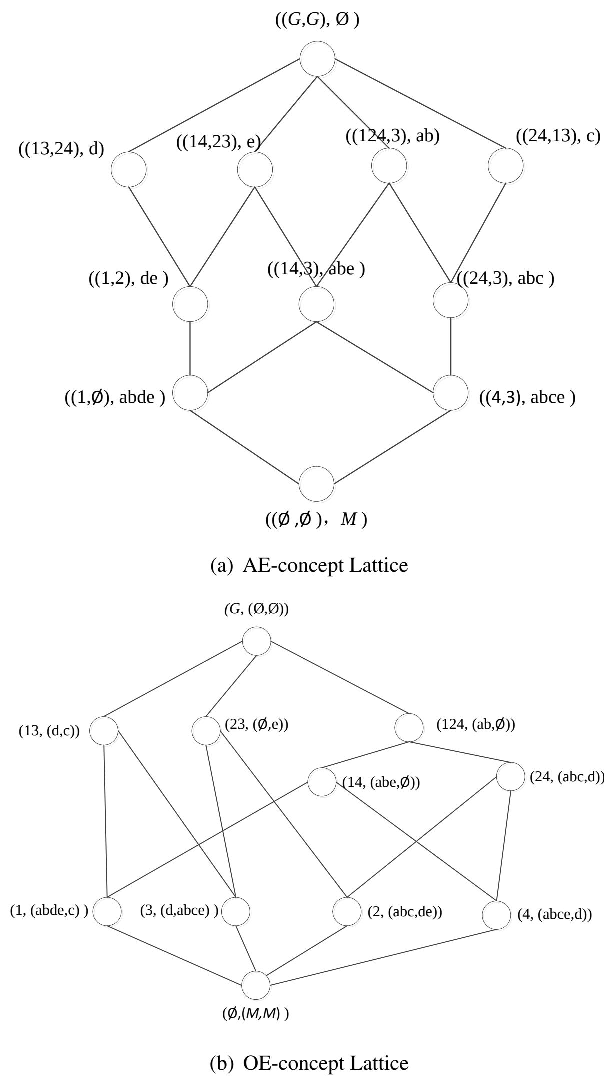


Figure 1: AE-concept Lattice and OE-concept Lattice.

3. Stability Indices for Three-way Concepts

In FCA methodology, Kuznetsov pioneered the stability of formal concepts (Kuzuetsov [1990]), and then modified it and provided the detailed analysis (Kuznetsov [2007]). The stability of formal concepts is used to prune the concept lattice and obtain effective information. Recently, he further extended and defined the stability for triadic concepts (Kuznetsov and Makhalova [2016]). In this section, we define stability indices for the three-way concepts in a similar way. Specifically, the stabilities for AE-concepts and OE-concepts are formalized as follows.

Definition 4 *(**AE-concept Stability**) Given a formal context $K = (G, M, I)$, there exists an AE-concept $((X, Y), A) \in AEL(G, M, I)$. Its stability $\gamma(((X, Y), A))$ is defined as:*

$$\gamma(((X, Y), A)) = \frac{|\{(C, D)|C \subseteq X, D \subseteq Y, (C, D)^{\gtrdot} = A\}|}{2^{(|X|+|Y|)}} \tag{8}$$

Obviously, the above AE-concept stability satisfies $\gamma(((X, Y), A)) \in [0, 1]$.

Similarly, the definition of OE-concept stability is also provided.

Definition 5 *(**OE-concept Stability**) Given a formal context $K = (G, M, I)$, there exists an OE-concept $(X, (A, B)) \in OEL(G, M, I)$. Its stability $\sigma((X, (A, B)))$ is defined as:*

$$\sigma((X, (A, B))) = \frac{|\{C \subseteq X|C^{\lessdot} = (A, B)\}|}{2^{|X|}} \tag{9}$$

Clearly, the above OE-concept stability satisfies $\sigma((X, (A, B))) \in [0, 1]$.

Intuitively, the AE-concept/OE-concept stability measures the degree of dependence of the concept on some specific extents and intents. In particular, it means that when some extents and intents are lost, the probability that the AE-concept/OE-concept can still be preserved, even if the extent with stable intent is noisy. After removing these noisy data, this AE-concept/OE-concept can still be preserved.

From the equivalence class point of view, the AE-concept/OE concept stability indices measure the number of elements of G that are in the same equivalence class of $A/(A, B)$. The equivalence class is defined as follows.

Definition 6 *For $X \subseteq G$ and $Y \subseteq G$, the equivalence class of (X, Y), $<(X, Y)>$ is defined as follows.*

$$<(X, Y)>= \{C \subseteq G; D \subseteq G|(C, D)^{\gtrdot} = (X, Y)^{\gtrdot}\} \tag{10}$$

Hence, the Eq. (8) and Eq. (9) can be rewritten as

$$\gamma(((X, Y), A)) = \frac{|<(X, Y)>|}{2^{(|X|+|Y|)}} \tag{11}$$

$$\sigma((X, (A, B))) = \frac{|<X>|}{2^{|X|}} \tag{12}$$

Example 2 *Continue Example 1, the AE-concept stability and OE-concept stability are easily calculated according to Definition 4 and Definition 5.*

Clearly, the stability for three-way concepts fall into the range (0,1].

From computational results of the above example, we can obtain the following propositions.

Table 2: AE-concept Stability (on the left) and OE-concept Stability (on the right)

No.	AE-concept	AE-S	No.	OE-concept	OE-S
1	$((\emptyset, \emptyset), abcde)$	1	1	$(\emptyset, (abcde, abcde))$	1
2	$((4, 3), abce)$	3/4	2	$(1, (abde, c))$	1/2
3	$((1, \emptyset), abde)$	1/2	3	$(3, (d, abce))$	1/2
4	$((24, 3), abc)$	4/8	4	$(2, (abc, de))$	1/2
5	$((14, 3), abe)$	3/8	5	$(4, (abce, d))$	1/2
6	$((1, 2), de)$	2/4	6	$(14, (abe, \emptyset))$	1/4
7	$((24, 13), c)$	8/16	7	$(24, (abc, d))$	1/4
8	$((124, 3), ab)$	4/16	8	$(124, (ab, \emptyset))$	1/4
9	$((14, 23), e)$	6/16	9	$(13, (d, c))$	1/4
10	$((13, 24), d)$	12/16	10	$(23, (\emptyset, e))$	1/4
11	$((1234, 1234), \emptyset)$	53/64	11	$(1234, (\emptyset, \emptyset))$	5/16

Proposition 1 *Given an OE-concept $(X, (A, B))$ and $|X|$=1, then its stability equals $\frac{1}{2^{|X|}}$.*

Proof *Since the size of X equals 1, i.e. $|X|$=1, the power set of X is $\{X, \emptyset\}$. Obviously, only one subset X satisfies the definition of the stability, thus its stability equals $\frac{1}{2^{|X|}}$.*

Proposition 2 *The bottom concept $(\emptyset, (M, M))$ or $((\emptyset, \emptyset), M)$, its stability equals 1.*

Proof *It can be directly obtained from the definition of the stability of three-way concept.*

Proposition 3 *Given a formal context K=(G, M, I), for any object $g \in G$ and any attribute $m \in M$, $(g, m) \in I$. Its AE-concept $((G, \emptyset), M)$'s stability equals $\frac{2^{|G|}-1}{2^{|G|}}$ and OE-concept $(G, (M, \emptyset))$'s stability equals $\frac{2^{|M|}-1}{2^{|M|}}$.*

Proof *In a formal context K=(G, M, I), for any object $g \in G$ and any attribute $m \in M$, $(g, m) \in I$. The formal context is represented as follows. Obviously, the above formal context is a very special one (there exist binary relationships between objects and attributes), and the generated AE-concept is $((12 \cdots n, \emptyset), ab \cdots m)$ and the generated OE-concept is $(12 \cdots n, (ab \cdots m, \emptyset))$. According to Eq. (10) and Eq. (12), thus the AE-concept stability is $\frac{2^{|G|}-1}{2^{|G|}}$, OE-concept stability is $\frac{2^{|M|}-1}{2^{|M|}}$.*

4. Computation for Stability of Three-way Concepts

Generally speaking, computation for the stability of a three-way concept is equivalent to simply exploring all the elements of the power-set of the extent part of such a three-way concept. Thus, for each element of the power-set, i.e., a subset of the extent part, we have to check whether it can preserve the intent. To be specific, given an AE-concept $((X, Y), A)$/OE-concept $(X, (A, B))$, for each non-empty set from the power-set of X and Y/A and B, we check whether its support is same as the intent A or extent X. Therefore, the computation for the stability of three-way concepts is NP-complete.

Inspired by previous research (Jay et al. [2008]) on stability computation for formal concepts, we know that the computation of concept stability depends on the stability of its subconcepts. Therefore, the computation for the stability of three-way concepts can be calculated with the following propositions.

Proposition 4 *Let $((X,Y),A)$ be an AE-concept of K=(G,M,I).*

$$\gamma((X,Y),A) = 1 - \sum_{C\subset X, D\subset Y, (C,D)=E^{\lessdot\gtrdot}} \gamma((C,D),E)2^{|C|+|D|-|X|-|Y|} \tag{13}$$

Proof *For an AE-concept $((X,Y),A)$, from Eq. (8), we have*

$$\gamma((X,Y),A) = \frac{|<(X,Y)>|}{2^{(|X|+|Y|)}} \tag{14}$$

Let $I_{(X,Y)}$ be the set of subintents of (X,Y): $I_{(X,Y)} = \{C \subseteq X, D \subseteq Y | (C,D) = (C,D)^{\gtrdot\lessdot}\}$. The set of equivalence classes $\{<(C,D)>|(C,D \in I_{(X,Y)})\}$ forms a partition of $2^{(|X|+|Y|)}$. Thus $|2^{(X,Y)}| = \sum_{(C,D)\in I_{(X,Y)}} |<C,D>|$, hence, it has

$$|<(X,Y)>| = |2^{(|X|+|Y|)}| - \sum_{(C,D)\in I_{(X,Y)}, (C,D)\neq(X,Y)} |<C,D>| \tag{15}$$

Dividing by $|2^{(|X|+|Y|)}|$, the above equation can be rewritten as

$$\begin{aligned}\frac{|<(X,Y)>|}{|2^{(|X|+|Y|)}|} = 1 - \sum_{(C,D)\in I_{(X,Y)}, (C,D)\neq(X,Y)} \frac{|<C,D>|}{|2^{(|X|+|Y|)}|}\gamma((X,Y),A) = \\ 1 - \sum_{C\subset X, D\subset Y, (C,D)=(C,D)^{\gtrdot\lessdot}} \gamma((C,D),(C,D)^{\gtrdot\lessdot})2^{(|C|+|D|-|X|-|Y|)}\end{aligned} \tag{16}$$

Proposition 5 *Let $(X,(A,B))$ be an OE-concept of K=(G,M,I).*

$$\gamma(X,(A,B)) = 1 - \sum_{C\subset X, C=C^{\lessdot\gtrdot}} \gamma(C,C^{\lessdot\gtrdot})2^{|C|-|X|} \tag{17}$$

Proof *For an OE-concept $(X,(A,B))$, from Eq. (9), we have*

$$\gamma(X,(A,B)) = \frac{|<X>|}{2^{|X|}} \tag{18}$$

Let I_X be the set of subintents of X: $I_X = \{C \subseteq X | C = C^{\lessdot\gtrdot}\}$. The set of equivalence classes $\{<C>|(C \in I_X)\}$ forms a partition of 2^X. Thus $|2^X| = \sum_{C\in I_X|<C>|}$, hence, it has

$$|<X>| = |2^{|X|}| - \sum_{C\in I_X, C\neq X} |<C>| \tag{19}$$

Dividing by $|2^{|X|}|$, the above equation can be rewritten as

$$\begin{aligned}\frac{|<X>|}{|2^{|X|}|} = 1 - \sum_{C\in I_X, C\neq X} \frac{|<C>|}{|2^{|X|}|}\gamma(X,(A,B)) = \\ 1 - \sum_{C\subset X, C=C^{\lessdot\gtrdot}} \gamma(C,C^{\lessdot\gtrdot})2^{(|C|-|X|)}\end{aligned} \tag{20}$$

Example 3 *Let us take an AE-concept $((14,3),abe)$ in Example 1 as an illustrative example, this example will demonstrate how to calculate the AE-concept stability based on the above proposition. We will calculate the stability for AE-concept $((14,3),abe)$ following the bottom-up approach (as shown in Figure 2).*

Initially, we know that the stability for AE-concept $((\emptyset,\emptyset),M)$ is 1, and it is the sub-concept of AE-concepts $((1,\emptyset),abde)$ and $((4,3),abce)$. First, we execute Step 1 for obtaining the AE-concept stability for $((1,\emptyset),abde)$ and $((4,3),abce)$ according to Proposition 3. The stability for AE-concepts $((1,\emptyset),abde)$ and $((4,3),abce)$ can be calculated as follows.

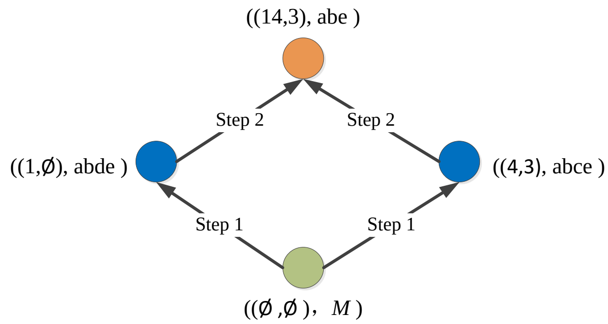


Figure 2: Flow-chart for Computation of AE-concept $((14, 3), abe)$

$$\gamma((1, \emptyset), abde) = 1 - \gamma((\emptyset, \emptyset), M) * 2^{(0+0-1-0)} = 1/2$$
$$\gamma((4, 3), abce) = 1 - \gamma((\emptyset, \emptyset), M) * 2^{(0+0-1-1)} = 3/4$$

Then, we go to Step 2 for further calculating the stability for AE-concept $((14, 3), abe)$.

$$\gamma((14, 3), abe) = 1 - \gamma((\emptyset, \emptyset), M) * 2^{(0+0-2-1)} -$$
$$\gamma((1, \emptyset), abde) * 2^{(1+0-2-1)} - \gamma((4, 3), abce) * 2^{(1+1-2-1)} = 3/8$$

With the above propositions, the corresponding algorithm for the computation of three-way concept stability is developed in Algorithm 1. **Algorithm 1** presents the pseudo-code of our procedure in which the input is the formal context, and the output is the AE-concept stability.

Algorithm 1: 3WCSCal(C)

```
Input: A three-way concept C
Output: Stability S
S ← 0, flag ← true                      // the flag of bottom concept
if flag then
    S = 1 − S
    return S
else if then
    children=GetAllChildren(C)
    for each c in children do
        if cs==0 then
            S = S + 3WCSCal(c) * 2^(|c|−|C|)
        else if then
            S = S + cs * 2^(|c|−|C|)
S = 1 − S
return S
```

A three-way concept C is the input of *3WCSCal* algorithm and the corresponding concept stability S is output. We use $|C|$ to denote the sum of extent's size and intent's size. Let c represents a child concept of C and let cs represents the corresponding concept stability of c.

First, the stability value S and the flag of whether the concept is the bottom concept are initialized with 0 and true, respectively (line 1). Specifically, if C is the bottom concept, its corresponding stability is assigned as 1 (line 2-4). Otherwise, we search all child concepts to compute the stability according to the above proposition 3/4 (line 5-13). The procedure GetAllChildren is invoked to obtain all child concepts of C (line 6). For each child concept c, if its concept stability cs equals 0, which means the child concept c is first searched, we recursively invoke *3WCSCal* to foremost compute its corresponding stability and then update S according to Proposition 3/4 (line 8-9). Otherwise, we directly update S according to Proposition 3/4 (line 10-11). At last, we process the stability according to Proposition 3/4 and then return the concept stability S (line 12-13).

Algorithm 2: GetAllChildren(C)

Input: A concept C, the set of concept L
Output: children
1 *children* $\leftarrow \emptyset$
2 If the partial order relation $\leq$ is not existing then
3 **for** *each concept* $c \in L$ **do**
4   **if** $c \leq C$ **then**
5   *children* = *children* $\cup\ c$
6 return *children*
7 **else if then**
8   **for** *each sonconcept* $sc \in C$ **do**
9   *children* = *children*$\cup$ GetAllChildren(sc)
10 return *children*

Procedure *GetAllChildren* preforms the child concepts search based on the set of concepts L. The children set is fist initialize with empty set (line 1). Then this procedure is divided into two branches according to whether the partial order relationship has been generated (line 2-10). If the partial order relation is not existing, we need to explore all concepts of L and check whether it is a child concept of the input concept C (line 3-6). Noted that we only give the general form, this branch obviously can be implemented using efficient search algorithms. If the partial order relation is existing, it recursively invokes itself to directly update the child set according to the partial order relation in lattice L (line 7-10).

The working process of Algorithm 1 is described as follows. First, it initializes the set of AE-concepts AE, the variables Z and $\sigma(c_i)$ as well as the set of sub-concepts of current AE-concept C (Line 1). Then, this algorithm invokes Yang's algorithm (Yang et al. [2020]), termed **AEConcepts**(K), for obtaining all AE-concepts and stores into the set AE (Line 2). After that, it goes into the AE-concept stability computation module, e.g., if the AE-concept is located at the bottom of the concept lattice, then return 1 (Lines 3-4). Otherwise, it will extract all the sub-concepts of the current AE-concept by sub-algorithm **FindSubConcepts**($(X, Y), A)$) and calculate the stability according to Eq.(15) (Lines 5-10).

Followed by Yang's algorithm, Algorithm 2 is working as follows: Line 1 initializes the complement context $K^c = (G, M, I^c)$, the concept sets $L(K)$ and $L(K^c)$, the set of AE-concepts $AE(K)$, the set of candidate AE-concepts $CAE(K)$, and the set of redundant AE-concepts $RAE(K)$. Then, the complement context is constructed (Lines

Algorithm 3: AE-concept Stability Computational Algorithm **AEStab**$((X, Y), A)$

Input: Formal context K=(G, M, I)
Output: Stability of AE-concept $\gamma(((X, Y), A))$
Initialize AE=0, Z=0, $\sigma(c_i)$=0, C=$\emptyset$
`/* Obtain AE-concepts by invoking Yang's algorithm(Yang et al. [2020]) */`
$AE \leftarrow$ **AEConcepts**(K)
`/* Calculate AE-concept stability` $\gamma(((X, Y), A))$ `*/`
if $X = Y = \emptyset \&\& A = M$ **then**
  return 1
else
  $C \leftarrow$ the sub-concepts of $((X, Y), A)$ by invoking **FindSubConcepts**(((X,Y),A))
  for each AE-concept $c_i = ((X_i, Y_i), A_i) \in C$ **do**
    $\sigma(c_i) \leftarrow AEStab((X_i, Y_i), A_i)) * 2^{|X_i|+|Y_i|-|X|-|Y|}$
  Z=Z+$\sigma(c_i)$
return $1 - Z$

Algorithm 4: AEConcepts(K)

Input: $K = (G, M, I)$
Output: Set of AE-concepts AE(K)
1 Initialize complement context K^c=(G, M, I^c), $L(K) \leftarrow \emptyset$, $L(K^c) \leftarrow \emptyset$, $AE(K) \leftarrow \emptyset$, $CAE(K) \leftarrow \emptyset$, $RAE(K) \leftarrow \emptyset$
2 **while** ($g \in G$ and $m \in M$) **do**
3   **if** $((g, m) \notin I)$ **then**
4     $K^c \leftarrow K^c \cup \{(g, m)\}$
5   **end if**
6 **end while**
7 **while** $((A_1, B_1) \in L(K))$ and $((A_1, B_1) \in L(K^c))$ **do**
8 $CAE(K) = CAE(K) \cup \{((A_1, A_2), B_1 \cap B_2)\}$
9 **if** $((B_1 \cap B_2)^* \supset A_1$ or $(B_1 \cap B_2)^{\bar{*}} \supset A_2)$
10 **then**
11 $RAE(K) = RAE(K) \cup \{((A_1, A_2), B_1 \cap B_2)\}$
12   **end if**
13 **end while**
14 **return** $AE(K) = CAE(K) - RAE(K)$

2-6). Lines 7-8 are in charge of obtaining the candidate AE-concepts by making the intersections between B_1 and B_2 and simply representing it as $\{((A_1, A_2), B_1 \cap B_2)\}$. Especially, if $((B_1 \cap B_2)^* \supset A_1$ or $(B_1 \cap B_2)^{\bar{*}} \supset A_2)$, the redundant AE-concepts are obtained and stored in $RAE(K)$ (Lines 9-12). Finally, Line 14 returns the resulting AE-concepts by filtering out the redundant AE-concepts from candidate AE-concepts.

Complexity Analysis: The first part of the algorithm has $O(t_1)$ time complexity, which is the time required for generating concept lattice. Let n be the number of concepts. The second part of the algorithm has $O(nt_2)$ time complexity, where the time t_2 is required to calculate the stability index. Finally, since traversing all the concepts, the time complexity to calculate the concept stability entropy is $O(n)$.

5. Experiments

This section mainly carries out experiments, including a case study, and evaluates the performance of the proposed approach.

Algorithm 5: FindSubConcepts(((X,Y),A))

Input: $K = (G, M, I)$
Output: Set of SubConcepts
Initialize $SubConcepts \leftarrow \emptyset$
begin
for each AE-concept $(((C, D), (C, D)^{\gtrdot}))$
  if $C \subset X$ && $D \subset Y$&& $(X, Y)^{\gtrdot} \subset (C, D)^{\gtrdot}$ **then**
flag=False
  for$((M, N), (M, N)^{\gtrdot})$
    if $M \subset X$ && $N \subset Y$
    $SubConcepts \leftarrow SubConcepts - ((M, N), (M, N)^{\gtrdot})$
    $SubConcepts \leftarrow SubConcepts \cup ((X, Y), (X, Y)^{\gtrdot})$
    Flag=True
  Flag=True
if not Flag **then**
  $Subconcepts \leftarrow SubConcepts \cup ((X, Y), (X, Y)^{\gtrdot})$
return $SubConcepts$

5.1. Data Sets

We adopt five available benchmark datasets in our experiments. The statics descriptions of these datasets are as follows.

- data1 and data 2: these two datasets are obtained from the benchmark formal context (Lin et al. [2016]) often used in 3WCA research field.
- data3 and data 4: another two relative big datasets, data 3 and data 4 as shown in Figure 3.

$G \times M$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1	1					1		
2	1					1	1	
3	1	1				1	1	
4		1				1	1	1
5	1		1		1			
6	1	1	1		1			
7		1	1	1				
8		1	1		1			

$G \times M$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1	1	1	1		1			
2	1	1				1		1
3		1			1		1	
4	1	1				1		1
5	1	1	1		1			
6		1			1		1	
7		1			1		1	
8							1	
9	1	1	1		1			
1							1	
11	1	1	1		1			
12	1	1				1		1

Figure 3: Formal Context for data 3 (on the left) and data 4 (on the right)

- sushi_1[1]: This dataset includes full rankings given by 10 Japanese users over different types of sushi. Each sushi is associated with 20 attributes, such as style, major group, minor group, heaviness, consumption frequency, normalized price, and selling frequency.
- sushi_2: Different from the above dataset, this dataset includes full rankings given by 10 Japanese users over different types of sushi. Each sushi is associated with 10 attributes, such as style, major group, minor group, heaviness, consumption frequency, normalized price, and selling frequency.

5.2. Comparison System

To better reveal the computational efficiency of the proposed algorithm in terms of time, we also compare the proposed algorithm with the baseline algorithm.

[1] http://www.kamishima.net/sushi/

- **Baseline**: The baseline algorithm is simply enumerating all the elements of the power-set of the extent part of such a three-way concept. Thus, for each element of the power-set, i.e., a subset of the extent part, baseline algorithm has to check whether it can preserve the intent.
- **Ours**: Inspired by Jay's algorithm (Jay et al. [2008]) on stability computation for formal concepts, three-way concepts' stability depends on its subconcepts' stability.

5.3. Experimental Results and Analysis

All algorithms are implemented in JAVA language and were run on an Inter(R) Core (TM) i7-8565U @ 1.80GHz 1.99GHz, 16GB RAM computer. We compare our proposed algorithm with the baseline algorithm in terms of the running time for three-way concepts stability computation (as shown in Figure 4).

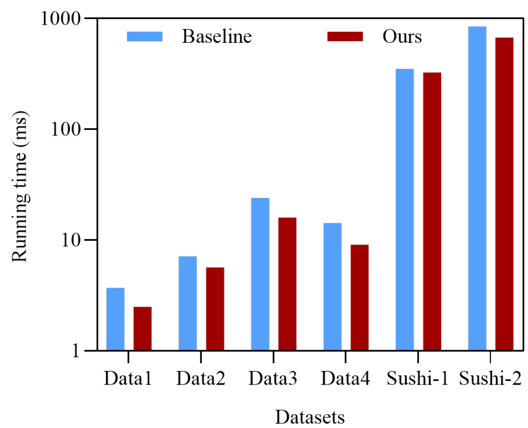


Figure 4: Running time for Three-way Concept Stability Computation under Different Algorithms

Figure 4 shows that our algorithm can quickly calculate their stability compared to the baseline algorithm.

We adopt two more datasets for evaluating the efficiency of the proposed three-way concept stability calculation algorithm.

- **Acute Inflammations Data Set**: This dataset was created by a medical expert as a data set to test the expert system, which will perform the presumptive diagnosis of two diseases of urinary system.
- **Zoo Data Set**: The Zoo dataset contains data items that describe animals[2]. In total 18 attributes are provided,of which one for uniquely identifying each animal (animal name) 16 are Boolean (hair,feathers,eggs,milk and so forth), and one has a predeuned integer range(for number of legs).

[2]http://archive.ics.uci.edu/ml/datasets/Zoo

The running time for calculating the stability of AE-concepts and OE-concepts for the above datasets are 20ms, 1030ms, and 10ms, 990ms, respetively.

In addition, we also conduct a case study on a special formal context that is constructed from a given set of sentences.

5.4. Three-way Concept Stability Analysis for NLG

This section provides a concrete case study on natural language processing, and in particular to improve NLG. In this case study, we will analyse the following paragraph:

"The cats hunt the mice. The mice are frightened by the cats. The cats are frightened by the dogs. Nevertheless, cats and dogs can establish a collaboration. The collaboration may be based on opportunism. However, the collaboration can also ends in a friendship."

To allow the readers to follow working process easily, we will consider only adjectives and names, we will exclude verbs, articles, and conjunctions. To build a sentence that make sense, once we have a set of already built sentences, we have to study the links between names, adjectives as well as the links between adjectives and names. For this reason, both names and adjectives are regarded as objects and attributes in the formal context. Due to the simple sentences considered, we have to analyse: *Dogs*, *Collaboration*, *Friendship*, *Opportunism*, *Cats*, *Frightened* and *Mice*. We can establish that there exists a connection between them iff they are used in the same sentence. The corresponding formal context can be constructed as shown in Table 3.

Table 3: A formal context Constructed by the relations between words in the sentences.

	Dogs	Collaboration	Friendship	Opportunism	Cats	Frightened	Mice
Dogs	1	1			1	1	
Collaboration	1	1	1	1	1		
Friendship		1	1				
Opportunism		1		1			
Cats	1	1			1	1	1
Frightened	1				1	1	1
Mice					1	1	1

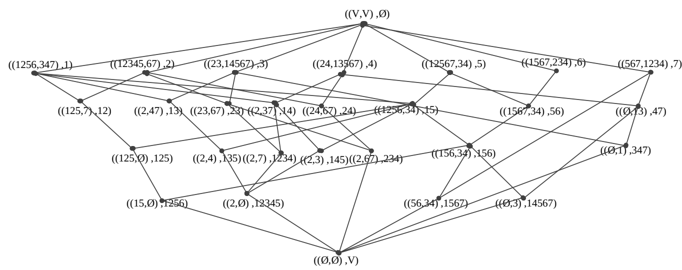
By using the three-way concept construction algorithm, we could extract 33 AE-concepts and 33 OE-concepts as seen in Figure 5. By the formula presented, we can also compute the corresponding AE-concept stability and OE-concept stability.

After calculation it is found that most AE-concepts have stability with 0.5. In fact, these concepts are locating at Level 3 and Level 4 of the lattice, which is coincided with the structure of AE-concept lattice.

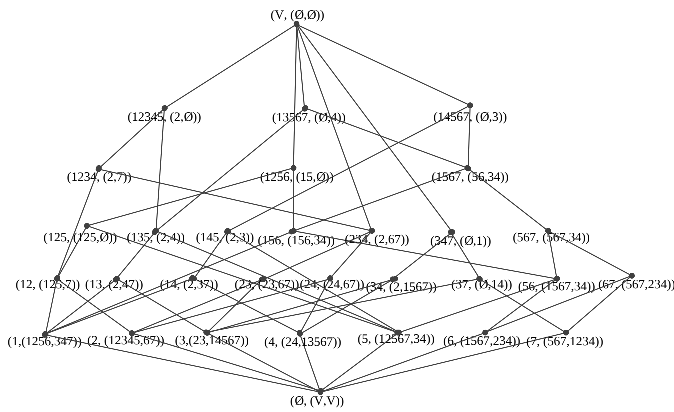
The OE-concept stability increases as the level of OE-concept lattice increases. Its OE-concept stability satisfies the following equation.

$$\sigma(X,(A,B)) = \frac{1}{2^{|X|}} \tag{21}$$

For example, the stability of OE-concept $(1234567,(\emptyset,\emptyset))$ can be calculated as $\sigma(X,(A,B)) = \frac{1}{2^7} = 0.007815$, and also the stability of OE-concept $(12,(125,7))$ can be calculated as $\sigma(X,(A,B)) = \frac{1}{2^2} = 0.25$. Clearly, it is found that most OE-concepts have stability with 0.25. In fact, these concepts are locating at Level 3 and Level 4 of the lattice which is coincided with the structure of OE-concept lattice. From the point of view of NLG, we



(a) AE-concept Lattice



(b) OE-concept Lattice

Figure 5: AE-concept Lattice and OE-concept Lattice. 1: Dogs; 2: Collaboration; 3: Friendship; 4: Opportunism; 5: Cats 6: Frightened; 7: Mice.

can claim that the higher the AE-concept stability, the stronger will be the connection between subjects and adjectives. With this intuition, once we have to build a sentence, we will look for the concepts related, that will have a high AE-stability. At the same time, using 3WCA instead of a classical FCA, we can also avoid the concepts that are not stronger related to be in the same sentence. It is clear that an analysis of a complete book, with training purposes as an example, will require the computation of several stabilities. For this reason, the computational time involved to compute stability needs to be reduced, as we propose in this work.

6. Conclusions

This paper has introduced three-way concept stability that is used for extracting more important three-way concepts. Our aim here is to provide a new benchmark to address the problems posed by NLG. We validated that the AE-concept/OE concept stability indices are able to measure the strength between the objects and attributes. The scale of AE-concept stability and OE-concept stability are numerically studied. This paper developed an efficient computational algorithm for calculating the stability of three-way concepts. Extensive experiments are conducted for validating the efficiency and usefulness of our algorithm. Further, a case study on the NLG, has also been conducted with three-way concept stability analysis and demonstrates that the three-way concept stability can facilitate the generation of sentences both in training than training-free techniques. In the future, we will investigate the dynamic generation of sentences when the sentences are changed in a real-time way. From the practical point of view, it is believed that three-way concept stability can be used for measuring the cohesion of sub-graph, personalized recommendation systems, and team formation in crowd-sourcing systems.

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