Optimal Replacement Policy under a General Failure and Repair Model: Minimal versus Worse Than Old Repair

F.G. Badía[1], M.D. Berrade[1], Ji Hwan Cha[2] and Hyunju Lee[2]

[1] Departamento de Métodos Estadísticos, Universidad de Zaragoza, Spain.

[2]Department of Statistics, Ewha Womans University, Seoul 120-750, Republic of Korea

Abstract

We analyze the optimal replacement policy for a system subject to a general failure and repair model. Failures can be of one of two types: catastrophic or minor. The former leads to the replacement of the system, whereas minor failures are followed by repairs. The novelty of the proposed model is that, after repair, the system recovers the operational state but its condition is worse than that just prior to failure (worse than old). Undertrained operators or low quality spare parts explain this deficient maintenance. The corresponding failure process is based on the Generalized Pólya Process which presents both the minimal repair and the perfect repair as special cases. The system is replaced by a new one after the first catastrophic failure, and also undergoes two sorts of preventive maintenance based on age and after a predetermined number of minor failures whichever comes first. We derive the long-run average cost rate and study the optimal replacement policy. Some numerical examples illustrate the comparison between the as bad-as-old and the worse than old conditions.

Keywords: Maintenance; Generalized Pólya process; worse-than-minimal-repair; optimum policy

1 Introduction

Both time and use make systems or equipment wear-out and eventually fail. To extend their useful life, systems undergo several maintenance actions (preventive or corrective) before being replaced. Maintenance is crucial when replacement cost is high as in big infrastructures, or when a failure implies risk for human life as in the case of planes or nuclear power plants. After the pioneering work of Barlow and Hunter [2], maintenance modelling has experienced widespread attention and many researchers have analyzed the effects of maintenance in terms of cost or reliability. However less attention has been paid to deficient maintenance when in fact a number of causes are responsible for it: lack of personnel, operators with insufficient qualification or scarce training, low quality spare parts, or underestimation of the particular environmental conditions where the system develops its function. As this is frequently reported in actual cases, the analysis of maintenance policies that fail to improve reliability is by itself a critical issue for research purposes.

Time-based maintenance policies are based on analysis of failure times which determines a programmed calendar for overhauls. Preventive maintenance, carried out before failure, restores the system to the as-good-as-new condition. Corrective maintenance (after failure) can go from minimal repair, that brings the system back to the state just prior to failure (as-bad-as-old), to perfect repair (as-good-as new). The intermediate situations are known as imperfect maintenance. The perfect repair process is described by a renewal, whereas a nonhomogeneous Poisson process (NHPP) is used for the minimal repair process. If the intensity [3] is an increasing (decreasing) function, the corresponding times after each repair constitute an increasing (decreasing) sequence in the hazard rate ordering [4]. Nakagawa [5] provides an overview on time-based maintenance theory. In [6] a review as well as recent advances in minimal repair models is presented. Surveys on various practical maintenance models can be found, e.g., in Sherif and Smith [7], Valdez-Flores and Feldman [8] and Wang [9]. The reviews in Pham and Wang [10] and Tanwar et al. [11] focus on imperfect maintenance.

Alternatively to time-based maintenance policies, condition-based maintenance (CBM) is applied considering the actual condition of the system when some information on the level of damage or the state of the system is available. This procedure reduces the possibility of unnecessary repairs and unwanted side-effects such us defects or deterioration induced by maintenance. The recent work of Alaswad and Xiang [12] presents a review on CBM.

Most maintenance models assume that maintenance leads to a reliability improvement. However, sometimes, maintenance causes an adverse effect and the system results in a worse state than that prior to failure. This situation has been analyzed in Berrade et al. [13] by means of a mixture model where good units are replaced on failure by weak units. This can be the consequence, for example, of instantaneous replacement when the maintainer lacks the required resources or suitable training at maintenance time. The consequences of judgement errors in maintenance actions have also been analyzed in Berrade et al. [13].

Cha [1] presents a new approach, the GPP repair process, which in turn is based on the Generalized Pólya Process. A GPP repair removes failures but the result is not the same as a minimal repair. In fact, it leads to a worse condition than that before the failure. Tanwar et al. [11] refer to this case as worse than old.

As discussed in Lee and Cha [15], when a component in a system fails, the working environment frequently becomes more hostile because of increased pressure, temperature, humidity, etc. This in turn causes instantaneous stress or damage to the adjacent non-failed components. For example, as suggested in [15], "(i) the failure of a still wire cable in a bridge or in an elevator instantaneously increases the stress on the non-failed cables and leads to some damages before repairing the failed one; (ii) when an electric device fails by an external shock (electric or mechanical shock), the non-failed components are also affected by this external shock and their reliability performances become worse than before"(see also [15] for more detailed examples). In this case, the overall state of the system after the repair of the failed component will be worse than the state it had just prior to the failure.

The use of refurbished parts to replace a failed component can also illustrate this type of maintenance. The parts conforming a new system, a car for example, are known as genuine parts. When a genuine part fails, it can be replaced by an identical one built by the original equipment manufacturer (OEM). If so, the new unit presents a similar reliability to that of the failed one and the minimal repair assumption for the whole system is reasonable. Alternatively, a refurbished or recycled part is a less expensive choice. These are used parts where only some characteristics are new but the rest remain unchanged presenting some type of wear. When this type of spares is used, a worse than minimal repair assumption seems to be more realistic. Maintenance models based on an imperfect repair that makes the system return to a condition between perfect renewal and minimal repair (Brown and Proschan [16]), are no longer valid in this context. The situation described in this work requires an assumption of an imperfect repair that leads the system back to the operating state but in a worse condition than that of the systems with the same age of its age at failure.

This paper presents a maintenance policy for a system with two types of failure: catastrophic or minor. A GPP repair follows a minor failure. The maintenance policy is completed with a replacement of the system after a catastrophic failure, or when the system reaches age T, or after the M^{th} minor failure whichever occurs soonest. Previous works have analyzed this maintenance model under the assumption that a minimal repair is carried out after a minor failure. The work of Sheu et al. [17] provides a complete review on this maintenance policy and extends previous models by assuming that the probability of a minor or catastrophic failure depends on the number of failures since the previous replacement. Sheu et al. [17] consider several failure modes and inspections for the non-selfannouncing modes as well as stochastically increasing durations of the repairs. A limited number of spares or impossibility of doing more rework [18] motivate replacement policies with a maximum number of imperfect repairs. Zhao et al. [19] compare replacement polices which are carried out at some periodic times and after a predetermined number of repairs. Our model presents the novelty of carrying out GPP repairs following minor failures. Thus, after each GPP repair, the stochastic intensity increases and this can be interpreted as a higher proneness to failure of the system. In the work of Lee and Cha [15] the system experiences one type of failure and scheduled preventive maintenance at periodic times. In the present paper we consider two failure types and age preventive replacement. Moreover we provide sufficient conditions for the existence of finite optimum policies. These conditions present special interest for maintainers that can deem when replacing the system is a better choice than keeping on a low quality maintenance.

The structure of this paper is as follows. Section 2 contains both concepts, the GPP process and GPP repair whereas the maintenance model is defined in Section 3. Furthermore, the long run average cost rate is obtained and a sufficient condition for the existence of an optimum policy is derived. In Section 4, we analyze the optimal maintenance policy on the basis of some numerical examples, comparing it with the minimal repair policy. We finish in Section 5 with our conclusions about the model and suggestions for further development.

2 The GPP Process and the GPP repair

2.1 Notation

- $\lambda(t)$: baseline failure rate of the unit.
- λ_t: stochastic intensity or failure intensity.
- α, β: parameters defining the stochastic intensity.
- p: probability of catastrophic failure (Type II failure).
- T: preventive replacement age.

- M: maximum number of minor failures in a cycle.
- c_{GPP}: cost of repair of a minor failure.
- c_{PM}: cost of preventive replacement.
- c_R: cost of replacement after a catastrophic failure ($c_R > c_{PM}$).

2.2 Preliminary Results

Let $\{N(t),\ t \geq 0\}$ be an orderly point process and $\mathcal{H}_{t-} \equiv \{N(u),\ 0 \leq u < t\}$ the history (internal filtration) of the process in $[0,t)$. The stochastic intensity (or failure intensity), λ_t, defined below is very useful in describing a counting process (see Aven and Jensen [3], Finkelstein and Cha [20]).

$$\begin{aligned} \lambda_t &\equiv \lim_{\Delta t \to 0} \frac{P\left(N(t, t+\Delta t) = 1 | \mathcal{H}_{t-}\right)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{E\left[N(t, t+\Delta t) | \mathcal{H}_{t-}\right]}{\Delta t}, \end{aligned} \tag{1}$$

where $N(t_1, t_2)$, $t_1 < t_2$, represents the number of events in $[t_1, t_2)$. The foregoing stochastic intensity is heuristically interpreted as: $\lambda_t dt = E\left[dN(t) | \mathcal{H}_{t-}\right]$, which is similar to the ordinary failure rate or hazard rate of a random variable [3].

Doyen and Gaudoin [21] define a new class of imperfect repair models based on reduction of the failure intensity. Other approaches can be found in Guerra de Toledo et al. [22], Wu and Scarf [23], Syamsundar and Achutha Naikan [24] and Guo et al. [25]. As far as we know, there is no attempt to model an increase of the failure intensity due to faulty maintenance. This paper focuses on this issue.

Next, the definition of the the generalized Pólya process is stated.

Definition 1. (Generalized Pólya Process) A counting process $\{N(t),\ t \geq 0\}$ is called the generalized Pólya process (GPP) with the set of parameters $(\lambda(t), \alpha, \beta)$, $\alpha \geq 0$, $\beta > 0$, if
(i) $N(0) = 0$;
(ii) $\lambda_t = (\alpha N(t-) + \beta)\lambda(t)$.

Note that the GPP with $(\lambda(t), \alpha = 0, \beta = 1)$ corresponds to the NHPP with the intensity function $\lambda(t)$. Therefore the GPP constitutes a generalized version of the NHPP.

Observe that the stochastic intensity in Definition 1 can also be formulated as indicated below:

$$\lambda_t = (\alpha N(t-) + \beta)\lambda(t) = \left(\frac{\alpha}{\beta} N(t-) + 1\right) \beta\lambda(t)$$

Thus, the value of the parameter set $(\lambda(t), \alpha, \beta)$ can be expressed as $(\lambda(t), \alpha, 1)$, i.e., $\beta = 1$, without loss of generality.

Lee and Cha [15] define a new type of repair, the "GPP repair", based on the GPP. Repair times are assumed to be negligible. $\{N(t),\ t \geq 0\}$ where $N(t)$ is the total number of failures in $(0, t]$, represents the failure process of the system with baseline failure rate $\lambda(t)$. The GPP repair is formally defined as follows.

Definition 2. GPP Repair [15] For a system with failure rate $\lambda(t)$, a repair is called a "GPP repair" with parameter α if $\{N(t),\ t \geq 0\}$ is the GPP with the parameter set $(\lambda(t), \alpha, 1)$.

Thus, under the GPP repair process, the corresponding stochastic intensity is specified as

$$\lambda_t = (\alpha N(t-) + 1)\lambda(t). \tag{2}$$

It follows from (2) that the state of the system after the GPP repair is worse than the state it had just prior to the failure (worse than old) because its stochastic intensity is larger than that of the minimal repair process (see also [1] for relevant discussions). Note that the parameter α determines the degree of repair. The case $\alpha = 0$ is matched to the minimal repair and $\alpha > 0$ means a repair which is worse than the minimal repair. The larger α the worse the state of the system after the repair. The work of Lee and Cha [15] presents useful practical interpretations of the modelling parameters as well as several practical examples where this type of repair can be applied.

Next we derive some preliminary results for further development. From [1], it follows that

$$P(N(t) = i) = \frac{\Gamma(\frac{\beta}{\alpha} + i)}{\Gamma(\frac{\beta}{\alpha})i!} \left(1 - exp\{-\alpha\Delta(t)\}\right)^i \left(exp\{-\alpha\Delta(t)\}\right)^{\frac{\beta}{\alpha}}), \quad i = 1, 2, \ldots \tag{3}$$

where

$$\Delta(x) = \int_0^x \lambda(t)dt.$$

Let denote by S_i, $i = 1, 2, \cdots$, the arrival time of the ith event in the GPP with the parameter set $(\lambda(t), \alpha, \beta)$. The next lemma states the density and distribution functions of S_i. The proof is given in Appendix.

Lemma 1. Consider the GPP with the parameter set $(\lambda(t), \alpha, \beta)$. The distribution and density functions of S_i are given respectively by

$$F_{S_i}(t) = 1 - \sum_{j=0}^{i-1} P(N(t) = j) = \sum_{j=i}^{\infty} P(N(t) = j), \tag{4}$$

and

$$f_{S_i}(x) = \lambda(x) \prod_{j=0}^{i-1} (\beta + j\alpha) \frac{1}{\alpha^{i-1}(i-1)!} \cdot e^{-\beta\Delta(x)} (1 - e^{-\alpha\Delta(x)})^{i-1}.$$

From Lemma 1 the reliability function corresponding to the ith arrival time is

$$\bar{F}_{S_i}(x) \equiv 1 - F_{S_i}(x) = \sum_{j=0}^{i-1} P(N(x) = j).$$

3 General Failure Model and Replacement Policy

Consider a system with $\lambda(t)$ being its baseline failure rate. Thus, for this system, the survival function of the time until the first failure, S_1, is given by $\bar{F}_{S_1}(t) = \exp\{-\int_0^t \lambda(u)du\}$. The system undergoes any of two types of revealed failures, minor failures (Type I) and catastrophic failures (Type II). A revealed or self-announcing failure means that no inspection or test is required to detect it but, on the contrary, the failure is observed at the very moment it occurs. Failures of this sort usually occur in systems under continuous operation. In the opposite case systems that only work on demand undergo unrevealed failures and should be inspected in the periods while they are not functioning to guarantee they are available when there is a demand of use. Type I and type II failures occur independently of any other events with probabilities $1 - p$ and p, respectively. We assume that each Type I failure can be removed by a GPP repair, whereas a Type II failure can be removed only by a replacement of the system.

We can reformulate the previous mathematical failure model according to Definition 2 of the GPP repair process. Thus, the failure of the system occurs following the GPP with the parameter set $(\lambda(t), \alpha, 1)$. Every time a failure takes place it is a minor failure with probability $1 - p$ independently of the previous failures. If the failure is of the Type II (catastrophic with probability p) then the failure process ends whereas it continues otherwise.

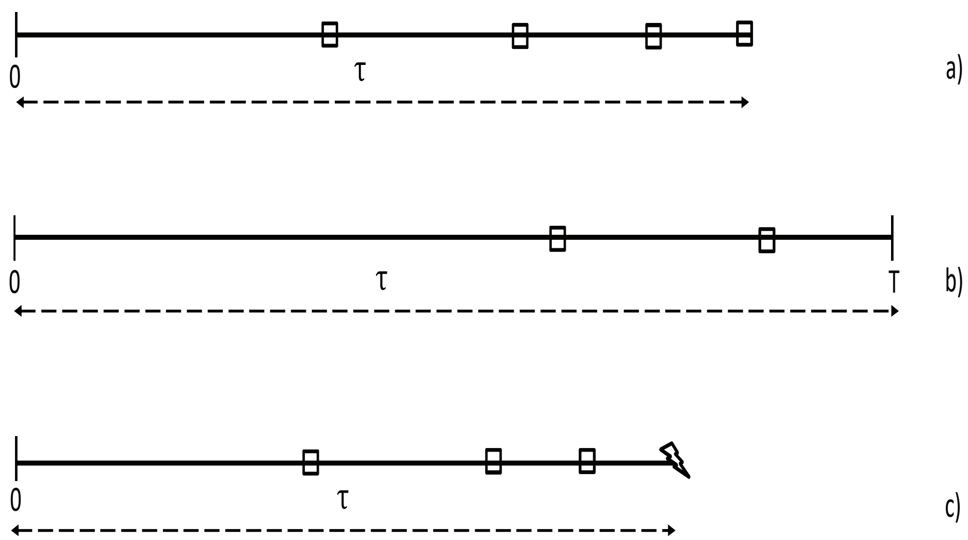

Figure 1: Age for renewal T, minor failure (□), catastrophic failure ( ), renewal period τ.
a) renewal after the M (M=4) minor failure; b) renewal at age T; c) renewal after a catastrophic failure.

Note that, from now on, it is assumed that $\beta = 1$ in the parameter set $(\lambda(t), \alpha, \beta)$ of the GPP process stated in the previous section.

Let X be the number of minor failures previous to the first catastrophic failure. The distribution of X is geometric with parameter p, that is:

$$P(X = k) = (1-p)^k p, \quad k = 0, 1, \ldots$$

The maintenance policy involves the following cost structure:

- Each Type I failure is repaired according a GPP repair with unitary cost c_{GPP}.
- The system is renewed at age T, or after a catastrophic failure or after the M^{th} type I failure, whichever occurs first. The cost of a

preventive replacement is c_{PM} and the cost of a replacement due to a catastrophic failure is c_R, where $c_R > c_{PM}$.

The time until the replacement of the system (i.e., the length of a renewal cycle) is

$$\tau = \min(S_{\min((X+1),M)}, T)$$

where $\min(a, b)$ also represents the minimum of a and b.

The different renewal cycles are described in the sample paths of Figure 1 for $M = 4$. In case a) no catastrophic failure occurs before the system is preventively renewed after M minor failures that happen before T. In case b) the system is renewed at age T since the number of minor failures before T is less that M. In case c) the renewal occurs after a catastrophic failure that takes place before preventive replacement.

The expected length of a renewal cycle is obtained by

$$\begin{aligned} E[\tau] &= \int_0^T \bar{F}_{S_{\min((X+1),M)}}(y)dy \qquad (5) \\ &= \sum_{j=0}^{M-2} (1-p)^j p \sum_{i=0}^{j} \int_0^T P(N(y)=i)dy + \sum_{j=M-1}^{\infty} (1-p)^j p \sum_{i=0}^{M-1} \int_0^T P(N(y)=i)dy \\ &= \sum_{j=0}^{M-2} (1-p)^j p \sum_{i=0}^{j} \int_0^T P(N(y)=i)dy + (1-p)^{M-1} \sum_{i=0}^{M-1} \int_0^T P(N(y)=i)dy. \end{aligned}$$

The first term in the previous equation represents the expected length of a cycle that is completed at age T or after a catastrophic failure whichever comes first. The second term corresponds to a cycle that ends at age T or after the M^{th} minor failure whichever occurs first.

Next proposition states the expected cost incurred in a cycle, $E[C(\tau)]$, which depends on both the age replacement T and the maximum number of minor failures, M, occurring before the replacement of the system.

Proposition 1. $E[C(\tau)]$ is given by

$$\begin{aligned} E[C(\tau)] = \Bigg[& c_{GPP}\Big(\sum_{j=0}^{M-2} (1-p)^j pj (1 - \sum_{i=0}^{j} P(N(T)=i)) \\ & + (M-1)(1-p)^{M-1}(1 - \sum_{i=0}^{M-1} P(N(T)=i)\Big) \\ & + c_{GPP} \left(\sum_{j=0}^{M-2} (1-p)^j p \sum_{i=0}^{j} iP(N(T)=i) + (1-p)^{M-1} \sum_{i=0}^{M-1} iP(N(T)=i) \right) \\ & + (c_R - c_{PM}) \sum_{j=0}^{M-1} (1-p)^j p \cdot (1 - \sum_{i=0}^{j} P(N(T)=j)) + c_{PM} \Bigg] \end{aligned}$$

where $P(N(t) = i)$ is given by (3) with $\beta = 1$.

The cost function considered is the long-run cost per unit time, $Q(T, M)$. According to the renewal reward theorem this function is

$$Q(T, M) = \frac{E[C(\tau)]}{E[\tau]}.$$

with $E[C(\tau)]$ and $E[\tau]$ given in Proposition 1 and equation (5), respectively.

Next we focus on the analysis of conditions for the existence of an optimum policy.

3.1 Optimum policies

The expressions of the density function f_{S_i}, cumulative distribution function, F_{S_i}, and reliability function, $\bar{F}_{S_i}$, of the the arrival time of the ith event, S_i, in the GPP defined by $(\lambda(t);\alpha;\beta)$ were presented in Lemma 1. It follows that

$$P(N(t)=i)=F_{S_i}(t)-F_{S_{i+1}}(t)$$

therefore

$$P(N(t)\geq i)=F_{S_i}(t),$$

$$P(N(t)\leq i)=\bar{F}_{S_{i+1}}(t),$$

and

$$\sum_{i=1}^{j} iP(N(t)=i)=\sum_{i=1}^{j} F_{S_i}(t)-jF_{S_{j+1}}(t)$$

It can be observed that $N(t)$ is a negative binomial random variable with parameters $\frac{\beta}{\alpha}$ and $p=e^{-\alpha\Delta(t)}$ (see (3)). Thus $N(t)$ is log convex (concave) if $\frac{\beta}{\alpha}\leq 1$ ($\frac{\beta}{\alpha}\geq 1$) and $N(t)$ is DFR (IFR) if $\frac{\beta}{\alpha}\leq 1$ ($\frac{\beta}{\alpha}\geq 1$).

Note again that we are assuming $\beta=1$.

The mean length of a cycle given in (5) and the cost function in Proposition 1 can be alternatively expressed as follows

$$\tau(T,M)=E[\tau]=\sum_{j=0}^{M-2}(1-p)^j p\int_0^T \bar{F}_{S_{j+1}}(x)dx+(1-p)^{M-1}\int_0^T \bar{F}_{S_M}(x)dx$$

$$\begin{aligned} C(T,M)=E[C(\tau)] &= c_{GPP}\left(\sum_{j=0}^{M-2}(1-p)^j pjF_{S_{j+1}}(T)+(M-1)(1-p)^{M-1}F_{S_M}(T)\right) \\ &+ \ c_{GPP}\left(\sum_{j=1}^{M-2}(1-p)^j p(\sum_{i=1}^{j}F_{S_i}(T)-jF_{S_{j+1}}(T))\right. \end{aligned}$$

$$+(1-p)^{M-1}(\sum_{i=1}^{M-1} F_{S_i}(T) - (M-1)F_{S_M}(T))\Big)$$

$$+ \quad (c_R - c_{PM}) \sum_{j=0}^{M-1} (1-p)^j p F_{S_{j+1}}(T) + c_{PM}$$

$$= \quad c_{GPP}(\sum_{j=0}^{M-2} (1-p)^j p \sum_{i=1}^{j} F_{S_i}(T) + (1-p)^{M-1} \sum_{i=1}^{M-1} F_{S_i}(T))$$

$$+ \quad (c_R - c_{PM}) \sum_{j=0}^{M-1} (1-p)^j p F_{S_{j+1}}(T) + c_{PM}$$

Next the derivatives of the numerator and denominator of the cost function $Q(T, M)$ are obtained

$$\frac{dC(T,M)}{dT} =$$

$$= \quad c_{GPP}(\sum_{j=1}^{M-2} (1-p)^j p (\sum_{i=1}^{j} f_{S_i}(T)) + (1-p)^{M-1}(\sum_{i=1}^{M-1} f_{S_i}(T)))$$

$$+ \quad (c_R - c_{PM}) \sum_{j=0}^{M-1} (1-p)^j p f_{S_{j+1}}(T)$$

$$\frac{d\tau(T,M)}{dT} = \sum_{j=0}^{M-2} (1-p)^j p \bar{F}_{S_{j+1}}(T) + (1-p)^{M-1} \bar{F}_{S_M}(T)$$

Proposition 2. When the maximum number of GPP repairs, M, is fixed, there exists an optimum T, denoted by T_M^*, minimizing $Q(T, M)$, provided that the following condition holds

$$A > 0$$

where A given by

$$A =$$

$$(c_{GPP}(\sum_{i=1}^{M-2}(1-p)^i p \sum_{j=1}^{i} C_j + (1-p)^{M-1}\sum_{j=1}^{M-1} C_j) + (c_R - c_{PM})\sum_{i=0}^{M-2}(1-p)^i p C_{i+1})$$

$$\times(\sum_{i=0}^{M-2}(1-p)^i p E[S_{i+1}] + (1-p)^{M-1}E[S_M]) - \frac{1}{\beta\lambda(\infty)} \times (\sum_{i=0}^{M-2}(1-p)^i p C_{i+1} + (1-p)^{M-1}C_M)$$

$$\times \quad (c_{GPP}(\sum_{j=0}^{M-2}(1-p)^j p \sum_{i=1}^{j} i + (1-p)^{M-1}(M-1)) + (c_R - c_{PM})\sum_{j=0}^{M-1}(1-p)^j p + c_{PM})$$

being $\lambda(\infty) = \lim_{T\to\infty}\lambda(T)$ and

$$C_i = \frac{\prod_{j=0}^{i-1}(\beta + j\alpha)}{\alpha^{i-1}(i-1)!}, \quad i = 1, 2\ldots$$

Observe that in the foregoing expression of C_i, β can be considered equal to 1.

Moreover, T_M^* is a root of the following equation

$$L(T, M) = \tau(T, M)\frac{dC(T, M)}{dT} - \frac{d\tau(T, M)}{dT}C(T, M) = 0$$

Remark. Regarding the result of Proposition 2, if $\lambda(\infty) = \infty$, then $A > 0$ since $c_R > c_{PM}$ and the condition for the existence of T_M^* is verified.

Next study concerns the analysis of sufficient conditions for the existence of an optimum M when the value of T is fixed. This optimum M is denoted by $M^*(T)$.

Straightforward algebra leads to

$$C_1(T, M) = C(T, M) - C(T, M-1) = (1-p)^{M-1}(c_{GPP}F_{S_{M-1}}(T) + p(c_R - c_{PM})F_{S_M}(T))$$

and

$$\tau_1(T, M) = \tau(T, M) - \tau(T, M-1) = (1-p)^{M-1}\int_0^T (F_{S_{M-1}}(x) - F_{S_M}(x))dx$$

We consider the following auxiliary functions

$$Q_1(T, M) = \frac{C_1(T, M)}{\tau_1(T, M)} = \frac{c_{GPP} + p(c_R - c_{PM})\frac{F_{S_M}(T)}{F_{S_{M-1}}(T)}}{\frac{\int_0^T (F_{S_{M-1}}(x) - F_{S_M}(x))dx}{F_{S_{M-1}}(T)}} \tag{6}$$

$$B(M) = Q_1(T, M)\tau(T, M - 1) - C(T, M - 1) \tag{7}$$

Proposition 3. If the age for replacement T is fixed and the following condition holds

$$\frac{\beta}{\alpha} \leq 1, \tag{8}$$

then there exists an optimum, $M^*(T)$, minimizing $Q(T, M)$ and it can be computed by using $B(M)$ in (7) as follows:

(i)If there exists M_0 such that $M_0 \geq 1$ and $B(M_0) > 0$, then $M^*(T) = \min\{M : B(M) > 0\}$

(ii)If $B(M) \leq 0$ for all M, then $M^*(T) = \infty$.

Observe that β in (8) can be considered equal to 1.

4 Numerical Examples

In the analysis that follows we set the unit of cost equal to the cost of a GPP repair, so that $c_{GPP} = 1$. Considering the example of the refurbished parts which illustrate the worse than old repair, we assume that the cost of a minimal repair, c_{MR}, is greater than c_{GPP}. Following this idea we analyze the consequences of increasing c_{PM} or c_R on both, the optimum age replacement and maximum number of GPP repairs. This study is extended to changes in the probability of catastrophic failures, p, and the degree of reliability of the unit after a GPP repair (quantified by α).

Both, confidentiality reasons and lack of data make it difficult to get the right values of the involved parameters. The ratio of the cost of preventive replacement to the cost of a catastrophic failure is based on the numerical analysis given in Berrade et al. [14]. Obtaining information about the time between consecutive failures results even harder. Si et al. [26] present a review of developments for estimating the useful life left of a system from a given time. This review focuses on statistical methods when observed data are available. These data are usually obtained from condition monitoring. Wang [27] suggests that estimation can be based on expert opinion when data do not exist or are insufficient.

The comparison of the optimum policy and cost in both policies, GPP and minimal repair, is one of the keys in the current study. Table 1 contains the optimum policy $(T^\star, M^\star)$ and the optimum cost $Q(T^\star, M^\star)$ when a GPP repair follows a minor failure whereas the corresponding results when a minor failure is repaired minimally, (T_0^*, M_0^*) and $Q_0(T_0^*, M_0^*)$, are presented in Table 2 for $c_{MR} = 3$. In so doing we see the effect of changes in the parameters under both maintenance policies. Moreover we give some insight into the relations between costs that make the minimal repair a preferable choice to the GPP repair for the maintainer. Following this idea results in Table 3 aim at analyzing the ratio $\frac{c_{MR}}{c_{GPP}}$ to compare both policies.

			c_R							
			30				50			
α	p	c_{PM}	T^*	M^*	$\tau(T^*,M^*)$	$Q(T^*,M^*)$	T^*	M^*	$\tau(T^*,M^*)$	$Q(T^*,M^*)$
0.1	0.1	10	10.1	3	5.35	2.81	8.2	2	4.26	3.34
		15	12.5	5	6.66	3.41	8.1	4	5.77	4.21
		25	18.8	12	8.36	4.10	11.8	6	7.05	5.34
	0.2	10	7.1	3	4.36	3.63	∞	1	2.98	4.19
		15	9.8	4	5.13	4.38	8.0	2	3.78	5.55
		25	17.3	11	6.25	5.14	9.2	5	5.36	7.04
	0.3	10	7.5	2	3.27	4.41	∞	1	2.98	4.79
		15	8.5	4	4.20	5.36	6.8	2	3.21	6.78
		25	16.7	11	4.85	6.32	8.6	4	4.21	8.71
0.2	0.1	10	9.3	3	5.11	2.93	8.0	2	4.16	3.43
		15	10.6	5	6.21	3.64	8.6	3	5.07	4.39
		25	15.9	10	7.37	4.52	11.2	5	6.23	5.73
	0.2	10	8.7	2	3.72	3.73	∞	1	2.98	4.19
		15	8.9	4	4.84	4.61	7.9	2	3.69	5.69
		25	15.3	9	5.70	5.55	9.1	4	4.86	7.42
	0.3	10	7.3	2	3.19	4.52	∞	1	2.98	4.79
		15	8.9	3	3.80	5.59	6.7	2	3.13	6.96
		25	15.0	9	4.53	6.73	9.0	3	3.81	9.11
0.3	0.1	10	8.8	3	4.92	3.04	7.9	2	4.07	3.50
		15	11.3	4	5.53	3.81	8.2	3	4.88	4.55
		25	14.6	9	6.75	4.84	10.1	5	5.89	6.04
	0.2	10	8.5	2	3.64	3.81	∞	1	2.98	4.19
		15	10.0	3	4.32	4.78	7.8	2	3.61	5.82
		25	14.5	8	5.32	5.88	8.6	4	4.64	7.74
	0.3	10	7.2	2	3.12	4.62	∞	1	2.98	4.79
		15	8.6	3	3.68	5.79	6.6	2	3.07	7.11
		25	14.4	8	4.29	7.07	8.7	3	3.68	9.42

Table 1. Optimal policy (T^*,M^*), $\tau(T^*,M^*)$ and $Q(T^*,M^*)$ for different values of α, p, c_{PM} and c_R (with $c_{GPP}=1$ fixed), $\lambda(t)=0.1(t+1)$

It is assumed in all the examples presented in Tables 1, 2, and 3 that the system failure rate is $\lambda(t) = 0.1(t+1),\ t \geq 0$ whereas $\lambda(t) = 0.3t^2$ in Table 4. Tables 1 and 4 also provide the optimum expected lengths of a renewal cycle, $\tau(T^*, M^*)$, and Tables 2 and 3 the corresponding length, $\tau_0(T_0^*, M_0^*)$, when minimal repairs are carried out.

The results in Table 1 reveal that when α increases so does the optimum cost. In addition both, M^* and $\tau(T^*, M^*)$, decrease although T^* is non-monotonic. The higher α, the lower the reliability induced by the GPP repair and these reliability levels do not compensate for the cost incurred. Thus, less GPP repairs before system replacement are recommended. The maintainer gains protection against low quality repairs by an earlier replacement of the system by a new one. The results with $T^\star = \infty$ also match that $M^\star = 1$. They correspond to cases where the cost of preventive replacement is low enough when compared with the cost of GPP repairs and thus it's worth replacing the system the first time a GPP failure happens.

When c_{PM} increases, so do both M^* and T^*. This result indicates that an increasing cost of preventive maintenance makes the maintainer to postpone it extending both, the age for replacement and the maximum number of GPP repairs. A similar behaviour is observed in Table 2 when the unit is minimally repaired. This postponement of the preventive maintenance makes the expected length of a cycle to increase in both cases.

The higher the probability p of a catastrophic failure or its associated cost, c_R, the smaller the time T^* for age replacement. The results derived from the minimal repair in Table 2 also show a decreasing T_0^* when p or c_R increases. This means that in order to prevent the occurrence of a catastrophic failure, an earlier preventive replacement is recommended to reduce this risk.

		c_R							
		30				50			
p	c_{PM}	T_0^*	M_0^*	$\tau_0(T_0^*, M_0^*)$	$Q_0(T_0^*, M_0^*)$	T_0^*	M_0^*	$\tau_0(T_0^*, M_0^*)$	$Q_0(T_0^*, M_0^*)$
0.1	10	14.7	2	4.45	3.26	8.6	2	4.39	3.66
	15	∞	4	6.56	3.96	10.1	3	5.60	4.62
	20	∞	6	7.84	4.43	12.9	4	6.55	5.28
	25	∞	9	9.05	4.74	15.6	6	7.84	5.77
0.2	10	9.3	2	3.93	3.94	∞	1	2.38	4.19
	15	12.8	3	4.90	4.82	8.3	2	3.89	5.80
	20	∞	6	6.30	5.30	9.2	3	4.82	6.70
	25	∞	10	6.94	5.60	10.6	5	5.86	7.32
0.3	10	7.7	2	3.37	4.69	∞	1	2.09	4.79
	15	10.3	3	4.16	5.71	7.0	2	3.30	7.00
	20	14.5	6	5.03	6.28	7.8	3	3.98	8.12
	25	∞	11	5.31	6.63	9.5	4	4.50	8.88

Table 2. Optimal policy (T_0^*, M_0^*), $\tau_0(T_0^*, M_0^*)$ and $Q_0(T_0^*, M_0^*)$ for different values of p, c_{PM} and c_R (with $c_{MR} = 3$ fixed)

The reduction in T^* and T_0^* aims at avoiding the natural wear-out leading to failure. The maintainer should be more concerned with wear-out when the consequences of a catastrophic failure get worse. An increasing value of p or c_R also reduces the number of the maximum number of minor failures before replacement when GPP repairs are performed, that is M^*. However Table 2 shows that M_0^* is not monotonic and therefore a similar result does not hold in the case of minimal repairs. Note that after a minimal repair the reliability is the same as that the system presented just before failure so the probability of a minor failure does not increase as in the case of GPP repairs. Hence M_0^* is not so critical than M^*. The expected length of a cycle decreases with p in Tables 1 and 2 as expected.

Regarding the comparison between GPP and minimal repairs, the results in Table 1 and Table 2 show that in most cases it pays to do GPP repairs instead minimal repairs since $Q(T^*, M^*) < Q_0(T_0^*, M_0^*)$. However a closer look at the results reveals that this economic advantage depends on the parameter values. Thus if α, p, c_{PM} or c_R increase enough, then the inequality is reversed making the minimal repair a more profitable action. The following examples illustrate this idea:

- $p = 0.3$, $c_R = 30$ and $c_{PM} = 25$ then $Q_0(T_0^*, M_0^*) = 6.63$ and $Q(T^*, M^*) = 6.32$ for $\alpha = 0.1$ but $Q(T^*, M^*) = 7.07$ for $\alpha = 0.3$.

- $\alpha = 0.2$, $p = 0.2$, $c_R = 30$ and $c_{PM} = 25$ then $Q_0(T_0^*, M_0^*) = 5.596$ and $Q(T^*, M^*) = 5.55$. For $p = 0.3$, $Q_0(T_0^*, M_0^*) = 6.63$ and $Q(T^*, M^*) = 6.73$.

- $\alpha = 0.3$, $p = 0.3$, $c_R = 30$ and $c_{PM} = 10$ then $Q_0(T_0^*, M_0^*) = 4.69$ and $Q(T^*, M^*) = 4.62$. For $c_{PM} = 25$, $Q_0(T_0^*, M_0^*) = 6.63$ and $Q(T^*, M^*) = 7.07$.

- $\alpha = 0.2$, $p = 0.2$, $c_R = 30$ and $c_{PM} = 25$ then $Q_0(T_0^*, M_0^*) = 5.59$ and $Q(T^*, M^*) = 5.55$. For $c_R = 50$, $Q_0(T_0^*, M_0^*) = 7.32$ and $Q(T^*, M^*) = 7.42$.

When $Q(T^*, M^*) > Q_0(T_0^*, M_0^*)$ for a given c_{PM}, the same inequality holds when c_{PM} increases, therefore the minimal repair remains to be the preferable choice.

In addition $M^* \leq M_0^*$ and $T^* \leq T_0^*$ for those cases where $Q(T^*, M^*) > Q_0(T_0^*, M_0^*)$. The minimal repair produces higher reliability and thus the preventive maintenance can be postponed.

		c_{MR}							
		1.5				2			
p	c_{PM}	T_0^*	M_0^*	$\tau_0(T_0^*,M_0^*)$	$Q_0(T_0^*,M_0^*)$	T_0^*	M_0^*	$\tau_0(T_0^*,M_0^*)$	$Q_0(T_0^*,M_0^*)$
0.1	10	10	4	6.4	2.82	11.5	3	5.64	2.98
	15	15.8	6	7.84	3.32	17.3	5	7.27	3.55
	20	23.6	11	9.58	3.62	26.6	9	9.05	3.91
	25	∞	18	10.57	3.783592	∞	14	10.13	4.13
0.2	10	7.6	3	4.61	3.618986	9.2	2	3.93	3.74
	15	10.7	5	5.87	4.275491	11.4	4	5.51	4.47
	20	16.1	9	6.83	4.624515	16.2	8	6.71	4.86
	25	∞	19	7.24	4.812131	∞	15	7.18	5.08
0.3	10	7.7	2	3.37	4.385567	7.7	2	3.37	4.49
	15	9.2	5	4.65	5.229731	9.5	4	4.50	5.40
	20	13.9	9	5.23	5.68482	14.4	7	5.14	5.89
	25	26.1	20	5.35	5.984506	26.9	16	5.34	6.20

Table 3. Optimal policy (T_0^*, M_0^*), $\tau_0(T_0^*,M_0^*)$ and $Q_0(T_0^*,M_0^*)$ for different values of p, c_{PM} and c_{MR} (with $c_R = 30$ fixed)

The decision between both policies depends also on the ratio $\frac{c_{MR}}{c_{GPP}}$. Table 3 contains the optimum policy and cost under minimal repair for $c_{MR} = 1.5$ and $c_{MR} = 2$. The comparison with Table 1 shows that the minimal repair is now the most economic choice most of the times. This is the case when $c_{MR} = 2$ for $\alpha = 0.3$, $\alpha = 0.2$ and $p = 0.3$, $\alpha = 0.2$ and $p = 0.2$ and $c_{PM} \geq 15$ among others. For $c_{MR} = 1.5$ the advantage of minimal repair is observed in all the examples except one when α, p, and c_{PM} take the smallest values.

			c_R							
			30				50			
α	p	c_{PM}	T^*	M^*	$\tau(T^*,M^*)$	$Q(T^*,M^*)$	T^*	M^*	$\tau(T^*,M^*)$	$Q(T^*,M^*)$
0.2	0.1	10	3.1	2	2.19	5.64	∞	1	1.73	5.78
		15	3.4	3	2.48	7.63	3.0	2	2.18	8.43
		25	4.4	6	2.91	10.49	3.5	3	2.49	12.28
	0.2	10	∞	1	1.54	6.50	∞	1	1.54	6.50
		15	3.3	2	1.97	9.11	∞	1	1.54	9.75
		25	4.2	5	2.41	12.32	3.3	2	1.97	14.92
	0.3	10	∞	1	1.35	7.43	∞	1	1.35	7.43
		15	3.0	2	1.69	10.72	∞	1	1.35	11.14
		25	4.0	5	2.03	14.42	3.0	2	1.69	17.61
0.3	0.1	10	3.0	2	2.16	5.70	∞	1	1.73	5.78
		15	3.3	3	2.42	7.78	2.9	2	2.14	8.53
		25	4.4	5	2.73	10.84	3.4	3	2.44	12.52
	0.2	10	∞	1	1.54	6.50	∞	1	1.54	6.50
		15	3.2	2	1.94	9.22	∞	1	1.54	9.75
		25	4.0	5	2.34	12.66	3.3	2	1.95	15.10
	0.3	10	∞	1	1.35	7.43	∞	1	1.35	7.43
		15	3.0	2	1.68	10.85	∞	1	1.35	11.14
		25	4.1	4	1.95	14.78	3	2	1.68	17.83

Table 4. Optimal policy (T^*,M^*), $\tau(T^*,M^*)$ and $Q(T^*,M^*)$ for different values of α, p, c_{PM} and c_R (with $c_{GPP}=1$ fixed), $\lambda(t)=0.3t^2$

The study of the optimum policy under a different baseline rate is presented in Table 4 where a similar analysis to that in Table 1 is carried out for $\lambda(t) = 0.3t^2$. Thus, the time to failure of the unit is assumed to follow a Weibull distribution with shape and scale parameters equal to 2 and 10^{13} respectively. The dependence of the optimum policy on p, c_{PM} and c_R resembles the previous one. Table 4 contains more cases where $T^\star = \infty$. This is so because when comparing $\lambda(t) = 0.3t^2$ with $\lambda(t) = 0.1(t+1)$, the former describes a system with significantly lower reliability when new. Thus, replacement turns out to be a better option than maintaining the system.

The graphs of the cost function reveal an interesting feature of the optimum policy. Figure 2 contains a high-resolution graph of the cost function $Q(T, M^\star = 2)$ for $\lambda(t) = 0.1(t+1)$, $\alpha = 0.2$, $p = 0.2$, $c_R = 30$ and $c_{PM} = 10$. $Q(T^\star, 2)$ does not differ much from $Q(T, 2)$ for $T > T^\star$. In fact the minimum cannot be told apart from asymptotic values under different values of c_{PM} in Figure 3, where the resolution is lower. This behaviour can also be observed in the corresponding graphs for $c_R = 50$, $\lambda(t) = 0.3t^2$ and even when a minimal repair follows a minor failure. These graphs are omitted to avoid enlarging the paper. As far as costs are concerned, the main conclusion is that replacement at T is unnecessary when the system is going to be replaced after M minor failures. The system is over maintained if both types of preventive replacement are carried out and the age replacement can be ignored. Indeed the consequences on the reliability of not performing the preventive maintenance at T should be analyzed. The maintainer must be aware of the risks incurred before deciding which preventive maintenance to carry out. This analysis is to be assigned for future research.

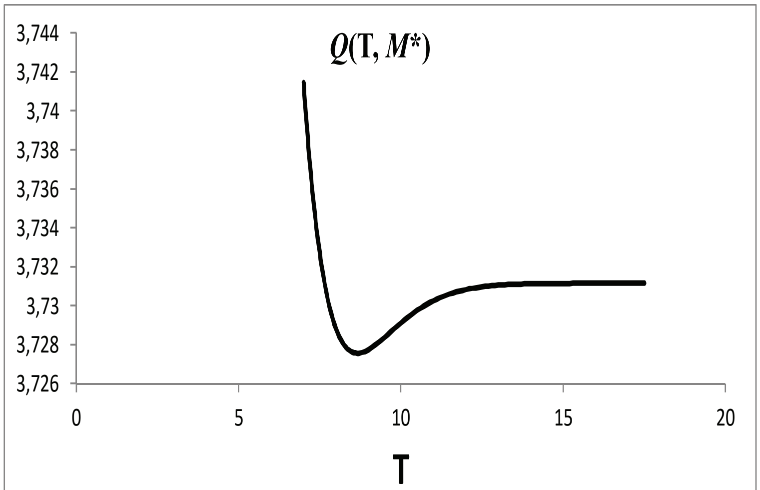

Figure 2: cost function $\alpha = 0.2$, $p=0.2$, $c_R=30$, $c_{PM}=10$, $\lambda(t)=0.1(t+1)$

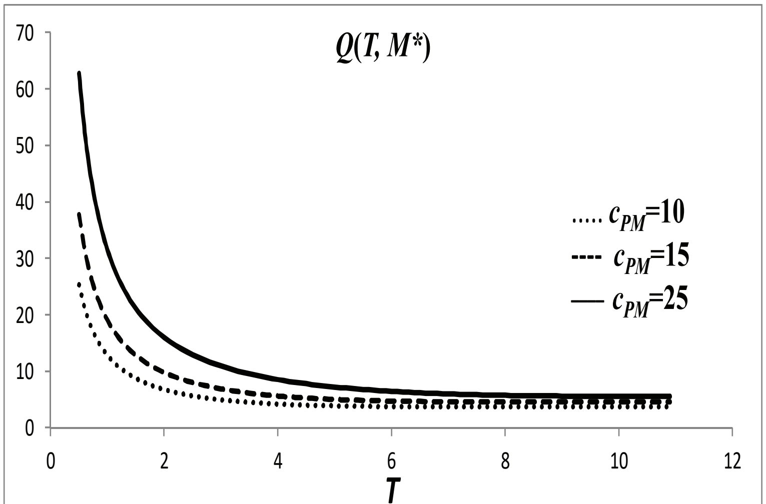

Figure 3: cost function under different values of c_{PM}

5 Concluding Remarks

This paper analyzes a maintenance policy for a system that can undergo failures of two types: minor and catastrophic. Minor failures follow a general failure model which allows a worse than minimal repair after each failure by means of a GPP repair. The motivation for this assumption is the maintenance based on refurbished parts rather than those from the original equipment manufacturing. This practice happens to occur in middle-aged systems or those close to retirement.

Refurbished parts are old components with some degree of wear-out that can be reused once some kind of recycling is performed. In general these components present lower reliability levels and more failures are reported. Thus, the warranty that a user gets from a refurbished component is usually less than that of an original component. The benefit of GPP repairs decreases as the probability of a catastrophic failure or its cost increases.

In addition, we analyze the conditions that make a GPP repair less profitable from an economic point of view than other maintenance procedures such as the minimal repair. This is the case, for example, if the GPP repair leads to such a poor quality (α large) that the reliability after repair is very low. The examples also reveal that the higher the probability of a catastrophic failure, p, or its associated cost, c_R, the less advantageous is the GPP repair when compared to the minimal repair. A reasonable explanation is that the the GPP repair tends to produce more frequent failures than the minimal repair and therefore a catastrophic failure is more likely to occur. When c_R increases the maintainer obtains higher protection against catastrophic failures with the minimal repair. The maintenance model includes a maximum number of GPP repairs before preventive replacement. Availability of spares and warranty restrictions [18] motivate this assumption. When the cost of the scheduled preventive maintenance c_{PM} increases, minimal repairs are preferable to GPP repairs because the time until the Mth failure is delayed. The analysis indicates that the choice between minimal and

GPP repairs cannot be only based on the ratio $\frac{c_{MR}}{c_{GPP}}$ but all the parameters involved. It can be pointed out as a general result that maintainers can consider GPP repairs when these repairs lead to a reliability level not far from that obtained from a better maintenance such as the minimal repair. This type of repairs can also be profitable even when they produce a low reliability if the maintainer is not specially aware with it because the costs derived from preventive replacements and catastrophic failures are low.

The model in this paper focuses on optimum policies based on cost. The analysis of the objective function reveals that the cost incurred when suboptimal solutions, $(T = \infty, M^{\star})$, are applied is near the optimum value. Thus, the interest of age replacement diminishes when the system is replaced after M minor failures. The implication on the reliability of both, suboptimal policies and GPP repairs is to be analyzed.

6 Appendix

Proof of Lemma 1

The event $S_i \leq t$ is equivalent to $N(t) \geq i$, which yields $F_{S_i}(t)$ in (4). The corresponding density function $f_{S_i}(x)$ can be obtained by taking derivative of $F_{S_i}(t)$ as follows:

$$f_{S_i}(x) = -\sum_{j=0}^{i-1} \frac{d}{dt} P(N(t) = j),$$

where

$$
\begin{aligned}
\frac{d}{dt}P(N(t)=0) &= -\beta\lambda(t)\exp\{-\beta\Delta(t)\}, \\
\frac{d}{dt}P(N(t)=1) &= -\beta\lambda(t)\exp\{-\beta\Delta(t)\}\left(-\exp\{-\alpha\Delta(t)\}+\frac{\beta}{\alpha}\left(1-\exp\{-\alpha\Delta(t)\}\right)\right), \\
\frac{d}{dt}P(N(t)=2) &= -\beta\lambda(t)\exp\{-\beta\Delta(t)\}\Bigg[-\left(\frac{\beta}{\alpha}+1\right)\left(1-\exp\{-\alpha\Delta(t)\}\right)\exp\{-\alpha\Delta(t)\} \\
&\qquad +\frac{\beta}{\alpha}\left(\frac{\beta}{\alpha}+1\right)\left(1-\exp\{-\alpha\Delta(t)\}\right)^2\Bigg], \\
\frac{d}{dt}P(N(t)=3) &= -\beta\lambda(t)\exp\{-\beta\Delta(t)\} \\
&\quad \times\Bigg[-\left(\frac{\beta}{2\alpha}+1\right)\left(\frac{\beta}{\alpha}+1\right)\left(1-\exp\{-\alpha\Delta(t)\}\right)^2\exp\{-\alpha\Delta(t)\} \\
&\qquad +\left(\frac{\beta}{2\alpha}+1\right)\left(\frac{\beta}{\alpha}+1\right)\frac{\beta}{3\alpha}\left(1-\exp\{-\alpha\Delta(t)\}\right)^3\Bigg],
\end{aligned}
$$

Thus, it follows that

$$
\begin{aligned}
&\frac{d}{dt}P(N(t)=0)+\frac{d}{dt}P(N(t)=1) \\
&\qquad = -\beta\lambda(t)\exp\{-\beta\Delta(t)\}\left(1+\frac{\beta}{\alpha}\right)\left(1-\exp\{-\alpha\Delta(t)\}\right), \\
&\frac{d}{dt}P(N(t)=0)+\frac{d}{dt}P(N(t)=1)+\frac{d}{dt}P(N(t)=2) \\
&\qquad = -\beta\lambda(t)\exp\{-\beta\Delta(t)\}\left(1+\frac{\beta}{\alpha}\right)\left(1+\frac{\beta}{2\alpha}\right)\left(1-\exp\{-\alpha\Delta(t)\}\right)^2,
\end{aligned}
$$

$\dots$

$$
\begin{aligned}
&\sum_{j=0}^{i-1}\frac{d}{dt}P(N(t)=j) \\
&= -\beta\lambda(t)\exp\{-\beta\Delta(t)\}\left(1+\frac{\beta}{\alpha}\right)\left(1+\frac{\beta}{2\alpha}\right)\cdots\left(1+\frac{\beta}{(i-1)\alpha}\right) \\
&\qquad \times\left(1-\exp\{-\alpha\Delta(t)\}\right)^{i-1} \\
&= -\beta(\beta+\alpha)(\beta+2\alpha)\cdots(\beta+(i-1)\alpha)\frac{1}{\alpha^{i-1}(i-1)!}
\end{aligned}
$$

$$\times\lambda(t)\exp\{-\beta\Delta(t)\}(1-\exp-\alpha\Delta(x))^{i-1}.$$

Finally, we have

$$f_{S_i}(x)=\lambda(x)\prod_{j=0}^{i-1}(\beta+j\alpha)\frac{1}{\alpha^{i-1}(i-1)!}\cdot e^{-\beta\Delta(x)}(1-e^{-\alpha\Delta(x)})^{i-1}.$$

■

Proof of Proposition 1

The cost of a cycle, $C(\tau)$ is given as follows

$$C(\tau)=c_{GPP}((M-1)\wedge X)1_{\{S_{(X+1)\wedge M}\leq T\}}+c_{GPP}N(T)1_{\{S_{(X+1)\wedge M}>T\}}$$
$$+c_R 1_{\{X+1\leq M,S_{X+1}\leq T\}}+c_{PM}1_{\{X+1\leq M,S_{X+1}\leq T\}^c},$$

where $1_{\{\cdot\}}$ is the corresponding indicator function. Then, it can be observed that

$$E[(M-1)\wedge X)1_{\{S_{(X+1)\wedge M\leq T}\}}]=E[(M-1)\wedge X)1_{\{N(T)\geq(X+1)\bigwedge M\}}]$$
$$=\sum_{j=0}^{M-1}j(1-\sum_{i=0}^{j}P(N(T)=j)(1-p)^jp$$
$$+(M-1)(1-\sum_{i=0}^{M-1}P(N(T)=j)\sum_{j=M}^{\infty}(1-p)^jp$$
$$=\sum_{j=0}^{M-1}(1-p)^jpj(1-\sum_{i=0}^{j}P(N(T)=i))$$
$$+(M-1)(1-p)^M(1-\sum_{i=0}^{M-1}P(N(T)=i)$$
$$=\sum_{j=0}^{M-2}(1-p)^jpj(1-\sum_{i=0}^{j}P(N(T)=i))$$
$$+(M-1)(1-p)^{M-1}(1-\sum_{i=0}^{M-1}P(N(T)=i).$$

On the other hand,

$$
\begin{aligned}
&E[N(T)1_{\{S_{(X+1)}\wedge M>T\}}]\\
&=\sum_{j=0}^{M-1}(1-p)^j pE[N(T)1_{\{N(T)\le j\}}]+\sum_{j=M}^{\infty}(1-p)^j pE[N(T)1_{N(T)\le M-1}]\\
&=\sum_{j=0}^{M-1}(1-p)^j p\sum_{i=0}^{j} iP(N(T)=i)+(1-p)^M\sum_{i=0}^{M-1} iP(N(T)=i)\\
&=\sum_{j=0}^{M-2}(1-p)^j p\sum_{i=0}^{j} iP(N(T)=i)+(1-p)^{M-1}\sum_{i=0}^{M-1} iP(N(T)=i).
\end{aligned}
$$

Furthermore,

$$
P(X+1\le M, S_{X+1}\le T)=\sum_{j=0}^{M-1}(1-p)^j p\cdot(1-\sum_{i=0}^{j}P(N(T)=j)).
$$

Therefore

$$
\begin{aligned}
E[C(\tau)]=c_{GPP}\Big(&\sum_{j=0}^{M-2}(1-p)^j pj(1-\sum_{i=0}^{j}P(N(T)=i))\\
&+(M-1)(1-p)^{M-1}(1-\sum_{i=0}^{M-1}P(N(T)=i)\Big)\\
+c_{GPP}\Big(&\sum_{j=0}^{M-2}(1-p)^j p\sum_{i=0}^{j} iP(N(T)=i)+(1-p)^{M-1}\sum_{i=0}^{M-1} iP(N(T)=i)\Big)\\
+c_R&P(X+1\le M, S_{X+1}\le T)+c_{PM}(1-P(X+1\le M, S_{X+1}\le T)).
\end{aligned}
$$

and thus the formula in Proposition 1 is obtained. ∎

Proof of Proposition 2

Notice that

$L(0,M)=-(\sum_{j=0}^{M-2}(1-p)^j p+(1-p)^{M-1})c_{PM}=-c_{PM}<0$

In addition

$$
\lim_{T\to\infty}F_{S_i}(T)=1
$$

and by means of L'Hopital rule we have

$$\lim_{T\to\infty}\frac{\bar{F}_{S_i}(T)}{e^{-\beta\Delta(T)}}=\lim_{T\to\infty}\frac{f_{S_i}(T)}{\beta\lambda(T)e^{-\beta\Delta(T)}}=\frac{C_i}{\beta}$$

Therefore,

$$\lim_{T\to\infty}\frac{L(T,M)}{\lambda(T)e^{-\beta\Delta(T)}}=A$$

and if $A>0$, then there exists an optimum T_M^*. ■

Proof of Proposition 3

First we prove the following property: if $\frac{\beta}{\alpha}\leq 1$, then $Q_1(T,M)$ in (6) is increasing.

If $\frac{\beta}{\alpha}\leq 1$, $N(t)$ is log convex and therefore $N(t)$ is also DFR. Thus, $(P(N(x)\geq M))^2=F_{S_M}^2(x)\leq F_{S_{M-1}}(x)F_{S_{M+1}}(x)=P(N(x)\geq M-1)P(N(x)\geq M+1)$ for all x. Moreover, $\frac{F_{S_M}(T)}{F_{S_{M-1}}(T)}\geq\frac{F_{S_M}(x)}{F_{S_{M-1}}(x)}$, $x\leq T$ as S_M is increasing in the likelihood ratio order ($S_{M-1}\leq_{\mathrm{lr}}S_M$). Therefore

$$\frac{\int_0^T(F_{S_{M-1}}(x)-F_{S_M}(x))dx}{F_{S_{M-1}}(T)}\geq\frac{\int_0^T(F_{S_M}(x)-F_{S_{M+1}}(x))dx}{F_{S_M}(T)}$$

Hence, $Q_1(T,M)$ is increasing in M.

Next we show that Q is decreasing (increasing) in M if $B(M)$ defined in (7) is non positive (non negative)

$$\begin{aligned}&Q(T,M)=\frac{C(T,M-1)+C_1(T,M)}{\tau(T,M-1)+\tau_1(T,M)}\leq Q(T,M-1)=\frac{C(T,M-1)}{\tau(T,M-1)}\\&\Leftrightarrow C_1(T,M)\tau(T,M-1)-\tau_1(T,M)C(T,M-1)\leq 0\\&\Leftrightarrow B(M)=Q_1(T,M)\tau(T,M-1)-C(T,M-1)\leq 0\end{aligned}$$

and $Q(T,M)=\frac{C(T,M-1)+C_1(T,M)}{\tau(T,M-1)+\tau_1(T,M)}>Q(T,M-1)\Leftrightarrow B(M)>0$.

In addition, if Q_1 is increasing in M so does B.

$$B(M+1)-B(M)=(Q_1(T,M+1)-Q_1(T,M))\tau(T,M)$$

and thus $B(M)$ is increasing in M if $\frac{\beta}{\alpha} \leq 1$.

The previous results lead to the sufficient condition stated in Proposition 3 for the existence of an optimum M when T is given. ∎

Acknowledgements

The work of F.G. Badía and M.D. Berrade has been supported by the Spanish Ministry of Economy and Competitiveness under Project MTM2015-63978-P. The work of Ji Hwan Cha was supported by Priority Research Centers Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2009-0093827). The authors acknowledge the helpful comments of the editor and anonymous reviewers which significantly improved the paper.

References

[1] Cha JH. Characterizations of the generalized Polya process and its aplications. Advances in Applied Probability 2014; 46:1148-1171.

[2] Barlow RE, Hunter LC. Optimum preventive maintenance policies. Operations Research 1960; 8:90-100.

[3] Aven T, Jensen U. Stochastic models in reliability, 2nd ed.. Springer, New York; 2013.

[4] Kochar SC. Some results on interarrival times of non homogeneous Poisson processes. Probability in the Engineering and Informational Sciences 1996; 10:75-85.

[5] Nakagawa T. Maintenance theory of reliability. Springer, London; 2005.

[6] Tadj L, Ouali MS, Yacout S, Ait-Kadi D.(eds) Replacement models with minimal repair. Springer, London; 2011.

[7] Sherif YS, Smith ML. Optimal maintenance models for systems subject to failure-A Review. Naval Research Logistics 1981; 28:47-74.

[8] Valdez-Flores C, Feldman RM. A survey of preventive maintenance models for stochastically deteriorating single-unit systems. Naval Research Logistics 1989; 36:419-446.

[9] Wang HZ. A Survey of maintenance policies of deteriorating systems. European Journal of Operational Research 2002; 139:469-489.

[10] Pham H, Wang H. Imperfect maintenance. European Journal of Operational Research 1996; 94:425-438.

[11] Tanwar M, Rai RN, Bolia N. Imperfect repair modeling using Kijima type generalized renewal process. Reliability Engineering and System Safety 2014; 124:34-41.

[12] Alaswad S, Xiang Y. A review on condition-based maintenance optimization models for stocastically deteriorating system. Reliability Engineering and System Safety 2017; 157:54-63.

[13] Berrade MD, Scarf PA, Cavalcante CAV. Some insights into the effect of maintenance quality for a protection system. IEEE Transactions on Reliability 2015; 64:661-672.

[14] Berrade MD, Scarf PA, Cavalcante CAV, Dwight RA. Imperfect inspection and replacement of a system with a defective state: A cost and

reliability analysis. Reliability Engineering and System Safety 2013; 120:80-87.

[15] Lee H, Cha JH. New stochastic models for preventive maintenance and maintenance optimization. European Journal of Operational Research 2016; 255:80-90.

[16] Brown M, Proschan F. Imperfect repair. Journal of Applied Probability 1983; 20: 851-859.

[17] Sheu SH, Tsai HN, Wang FK, Zhang ZG. An extended optimal replacement model for a deteriorating system with inspections. Reliability Engineering and System Safety 2015; 139:33-49.

[18] Kurt M, Kharoufeh JP. Optimally maintaining a markovian deteriorating system with limited imperfect repairs. European Journal of Operational Research 2010; 205:368-380.

[19] Zhao X, Qian C, Nakagawa T. Comparisons of replacement policies with periodic times and repair numbers. Reliability Engineering and System Safety 2017; 168:161-170.

[20] Finkelstein M, Cha JH. Stochastic modelling for reliability. Shocks, burn-in and heterogeneous populations. Springer, London; 2013.

[21] Doyen L, Gaudoin O. Classes of imperfect repair models based on reduction of failure intensity or virtual age. Reliability Engineering and System Safety 2004; 84:45-56.

[22] Guerra de Toledo ML, Freitas MA, Colosimo EA, Gilardoni GL. ARA and ARI imperfect repair models: Estimation, goodness-of-fit and re-

liability prediction. Reliability Engineering and System Safety 2015; 140:107-115.

[23] Wu S, Scarf P. Decline and repair, and covariate effects. European Journal of Operational Research 2015; 244:219-226.

[24] Syamsundar A, Achutha Naikan VN. Imperfect repair proportional intensity models for maintained systems. IEEE Transactions on Reliability 2011; 60:782-787.

[25] Guo C, Liao H, Zhao W, Mettas A. A maintenance optimization model for mission-oriented systems based on Wiener degradation. Reliability Engineering and System Safety 2007; 111:183-194.

[26] Si XS, Wang W, Hu CH, Zhou DH. Remaining useful life estimation - A review on the statistical data driven approaches. European Journal of Operational Research 2011; 213: 1-14.

[27] Wang, W. Subjective estimation of the delay time distribution in maintenance modelling. European Journal of Operational Research 1997; 99: 516-529.