Multi-phase reliability growth test planning for repairable products sold with a two-dimensional warranty

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Abstract

For products sold with a two-dimensional warranty policy, the warranty servicing cost can be reduced through reliability growth during development. This paper investigates a multi-phase reliability growth test program for repairable products with independent competing failure modes. Considering a *test-find-test* scheme, an accelerated reliability growth model is developed, allowing different failure modes having distinct accelerated relationships. Taking usage heterogeneity and mode-specific failure learning effect into account, the reliability growth is achieved through periodic fixes on a phase-by-phase basis. From the manufacturer's perspective, the main objective is to achieve the optimal trade-off between the warranty cost and the reliability growth test cost by determining the optimal test program that minimizes the expected total cost per product sold, ensuring that the pre-specified reliability growth requirement is met. Assuming the tri-Weibull product failure distribution, we illustrate the proposed optimization model numerically and study the effect of relevant parameters on the optimal reliability growth test program. The results show that the proposed approach yields significant cost reduction and reliability improvement for the examples studied in this paper, especially when the manufacturer possesses high failure learning ability, and the product has expensive repair cost per a warranty failure and extensive warranty coverage.

Keywords: reliability growth test, two-dimensional warranty, cost analysis, failure learning

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1. Introduction

Increasing global competition and rapid changes in technology have resulted in new products appearing on the market at a faster pace. A new generation of products are usually an improvement over earlier ones with changes to design. However, initial prototypes invariably have reliability and performance deficiencies that generally could not be foreseen and eliminated in early design stages. The problem of unexpected failures due to poor product reliability inevitably increases warranty costs resulting from claims servicing. Herein warranty is a contractual agreement that requires the manufacturer to repair or replace the faulty item in the event of failures occurred within specified warranty coverage. Product warranty has been studied by researchers from many different aspects. For detailed information on warranty research, the reader is referred to several review papers [26, 29, 25, 34], and other relevant literature [46, 19, 35, 6, 24, 47, 45, 42, 32].

The warranty cost can be reduced via reliability growth during product developmental process, in which reliability is improved through an iterative *Test-Analyze-And-Fix* (TAAF) procedure. Initial prototypes are exposed to a range of stresses that they are expected to encounter during field use. The observed failures are analyzed to identify the failure modes and determine the root causes. Subsequent fixes are implemented to design, operation, maintenance procedures or the associated manufacturing process for the purpose of improving reliability. This process is repeated until the pre-specified reliability growth target is achieved. Since reliability growth results in both costs and benefits (reduction in warranty cost due to reliability improvement, etc.) [33], it motivates the problem of how to plan a viable reliability growth test program to both prevent prohibitive costs and achieve the reliability growth requirement.

Reliability growth modeling techniques have received significant attention over the years. Fries and Sen [11] and Wong et al. [44] provide a comprehensive survey of significant research work on reliability growth modeling. Generally speaking, these models fall into discrete and continuous

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groups according to the usage of the system. Discrete growth models apply to systems for which usage is measured on an enumerative base such as pass or fail [11, 12]. While continuous growth models apply to systems for which usage is measured on a continuous scale, such as time in hours or distance in miles [4, 5, 7]. Duane [7] introduced the earliest reliability growth model based on the empirical observation that on a log-log scale, there showed an approximately linear relationship between the plots of cumulative mean time between failures (MTBF) and cumulative test time. Crow [4] further derived the stochastic basis of Duane's model, that was wihin a test phase, reliability growth can be modeled by a non-homogeneous Poisson process (NHPP) with a decreasing Weibull intensity function. The resulting model is also known as the Army Material Systems Analysis Activity (AMSAA) model and remains to be the most commonly-used reliability growth method in a variety of applications.

Reliability growth test planning involves addressing test program schedules, resources available, and realism of the program in achieving its requirement [23]. More attention has been paid to the research on reliability growth test planning to perform trade-offs with reliability improvement, incremental cost and test resources, etc [3, 20, 36, 21, 2, 38, 13, 14]. The majority of planning models are Crow/AMSAA-based, which commonly assume large expected number of failures and sufficient opportunities for immediate implementation of fixes, so as to allow the reliability growth to be portrayed as a smooth curve [23]. Such test is conducted under the *test-fix-test* scheme, that is, test stops whenever a failure is observed until a fix is implemented and takes no time. In most situations, test is likely to continue with minimal repair in the event of failures, and the fix will be implemented later. Crow developed an extended reliability growth projection model assuming that all fixes take no time and are delayed until the end of test [5]. Based on AMSAA projection methodology, Ellner and Hall [8] proposed a new reliability growth planning approach, by firstly taking the lag time associated with implementation of fixes into consideration. This is the well-known PM2 approach that can be applied to test programs with limited opportunities for implementation of fixes. Compared to Crow Extended model, the PM2 model is independent of the

NHPP assumption and valid for reliability growth planning over multiple test phases, with fix delays and the associated lag-time incorporated. However, although this model assumes there exists a certain number of independent failure modes, each failure mode has an exponential distributed time to first occurrence, and is assigned with an average fix effectiveness factor (FEF) across the test phases. With limited testing time, reliability growth can be merged with accelerated tests at high levels of one or more accelerating variables (e.g. usage rate, temperature, voltage or pressure) [1, 18]. There has been research that introduces accelerated reliability growth testing by modifying the Crow/AMSAA reliability growth planning model with a system-level acceleration factor included [10]. The approach is based on the assumption of a linear relationship between the reliability growth attained under an accelerated stress and that occurred under only the use-level stress. Intuitively, the time to failure occurrence due to a given failure mode decreases as stress increases. However, as noted by Escobar and Meeker [9], the time compression is not equivalent for each failure mode in the product. When multiple failure modes exist, separate models are required to characterize the effect of the accelerating variable on the mode-specific rate of occurrence of failures (ROCOF).

The existing literature on reliability growth modeling has focused on failures indexed by a single timescale. The previous research conducted by Murthy and Nguyen [28] and Hussain and Murthy [15] is limited to reliability growth modeling in the case of one-dimensional warranties, under which only the time is restricted. In contrast, no work on reliability growth modeling under two-dimensional warranties has been found in the literature. As a natural extension of one-dimensional warranty, a two-dimensional warranty is characterized by a region in two-dimensional plane with one axis representing age and the other usage. Such a warranty policy has been widely used on many complex and repairable commercial products, including printers, automobiles and aircrafts. It is common to observe that different users have different usage rates for the same product. Such customer usage heterogeneity caused by usage rate randomness need to be taken into account when modeling reliability growth. Ignorance of this fact may lead to inaccurate cost estimation and an inferior reliability growth test decision. In addtion, for products sold with two-dimensional warranty, failures are modeled as random points in a two-dimensional plane. Three different approaches have been commonly used to model the two-dimensional product failure process [16], including the one-dimensional approach, the bivariate approach and the composite-scale approach. These approaches differ from the current reliability growth modeling techniques that mainly focus on a single time scale.

In view of the above problematic issues, this paper proposes a modified accelerated reliability growth model for new repairable products sold with a two-dimensional warranty. Considering a multi-phase *test-find-test* scheme, limited product prototypes containing independent competing failure modes, are exposed to accelerated life testing where usage is utilized as a stress to induce failures. Within each test phase, test continues with minimal repair in case of failures. Periodic fixes implemented at the end of each test phase are aimed to reduce the failure intensities of surfaced failure modes. Effects of test usage rate on mode-specific failures are modeled through the Accelerated Failure Time (AFT) approach with distinct accelerated relationships. Based on these settings, the mean number of test failures and warranty failures can be computed by taking usage heterogeneity into account. After that, the expected total cost per product sold, consisting of reliability growth test cost and warranty servicing cost, is estimated from the manufacturer's perspective. The optimal reliability growth test program including the optimal number of fixes and the associated test usage rates under which failure modes are surfaced within each test phase, is determined with the objective of cost minimization and the pre-specified reliability growth requirement being met.

The outline of this paper is organized as follows. In Section 2, we provide model assumptions, notations and model formulation. In Section 3, we derive the expected total cost per product to the manufacturer for the proposed reliability growth test program, and provide an analysis of the cost-based optimization model. A numerical example with sensitivity analysis is conducted to illustrate the proposed model in Section 4. Finally, a summary of our study and future research directions are concluded in Section 5. In the following sections, we will use the terms product and item interchangeably without distinguishing them.

2. Model formulation

In this section, we first present the model assumptions and notations to make the mathematics tractable. Then, we consider a two-dimensional non-renewing free repair warranty (NFRW) policy and model the product failure process when independent competing failure modes are present. Thirdly, an accelerated reliability growth model incorporating mode-specific failure learning effect and usage heterogeneity is developed.

2.1. Model Assumptions and Notations

The following assumptions are taken into account:

- Limited identical product prototypes undergo the reliability growth test concurrently. Each prototype contains a known number of independent latent failure modes competing to the cause of product failure. The time to first occurrence of each failure mode follows a certain probability distribution.

- A multi-phase *test-find-test* program with fixed total testing time L is planned. Product prototypes are exposed to accelerated usage rate θ_j within test phase j ($j = 1, \ldots, n$). Upon a failure, discovery is immediate and the surfaced failure mode is known with certainty.

- In the event of failures, test continues with minimal repairs which make no change in the product failure intensity. Fix is delayed until the conclusion of each test phase, resulting in reliability improvement on a phase-by-phase basis.

- The mean lag time of each fix due to failure root-cause analysis, corrective action review, approval and implementation, etc., is incorporated between test phases. The necessary time for minimal repair is sufficiently small and assumed to be negligible.

- The effect of fixes on the product failure intensity is characterized by a drop in the mode-specific failure intensities considering failure learning effect. For each fix, the mode-specific FEF depends on the number of failures occurred in the preceding test phase due to that failure mode, and the failure learning level.

- After reliability growth, finished products are sold with a two-dimensional NFRW policy. Within the specified warranty coverage, any failed item is repaired minimally by the manufacturer at no cost to the user. Each failure results in an immediate warranty claim that is valid and executed.

- To reduce uncertainty and modeling complexity, the parameters' values in the proposed model are assumed to be known with certainty.

We use the following mathematical notations for the purpose of this paper:

W, U	warranty time and usage limits
x, u	product cumulative age and usage
θ_0	nominal usage rate in product design
ρ	average warranty usage rate
Θ, $g(\theta)$	usage rate (random variable) and density function for Θ
k	number of latent failure modes
$R_X(x\|\theta)$	product reliability function given that the usage rate $\Theta = \theta$
$R_i(x\|\theta)$	mode-specific reliability function conditional on $\Theta = \theta$ $(i = 1, \ldots, k)$
$h_X(x\|\theta)$, $\lambda_X(x\|\theta)$	product hazard function and failure intensity function associated with $R_X(x\|\theta)$
$F_i(x\|\theta_0)$	mode-specific failure distribution function with nominal usage rate θ_0
$F_i(x\|\theta)$	failure distribution function of failure mode i given $\Theta = \theta$
$F_X(x\|\theta)$, $f_X(x\|\theta)$	product failure distribution function and associated density function conditional on $\Theta = \theta$

$\lambda_i(x\|\theta)$, $h_i(x\|\theta)$	initial mode-specific failure intensity function and hazard function given $\Theta = \theta$
L	total testing time
n	number of fixes implemented during test (decision variable)
θ_j	test usage rate in phase j (decision variables, $j = 1, \ldots, n$)
z_i	accelerated coefficient of failure mode i
$\lambda_i^j(x\|\theta)$	conditional failure intensity function of failure mode i after the jth fix
$\lambda_i^{\mathrm{m}}(x\|\theta)$, $\lambda_X^{\mathrm{m}}(x\|\theta)$	minimum achievable conditional failure intensity functions of product and failure mode i, respectively
$\lambda_X^{\mathrm{d}}(x\|\theta)$	product failure intensity function after reliability growth given $\Theta = \theta$
p_i^j	mode-specific fix effectiveness factor (FEF) of the jth fix
t_{f}	mean lag time of fix
ξ_i^j	cumulative *effective* age under failure mode i till the jth test phase starts
b	failure learning level
C_{s}, C_{d}	set-up cost and operational cost per unit time during test for each product to be sold
C_{f}^j	cost of the jth fix per product
C_{r}, C_{m}	minimal repair cost of a test failure and a warranty failure, respectively
ϕ	reliability growth test program with $\phi = \{n, \theta_1, \theta_2, \ldots, \theta_n\}$
$N_{\mathrm{d}}(\phi)$	number of total test failures
$\mathrm{EC}_{\mathrm{d}}(\phi)$, $\mathrm{EC}_{\mathrm{w}}(\phi)$	expected test cost and warranty cost per product, respectively
$\mathrm{TC}(\phi)$	manufacturer's expected total cost per product sold
$E[\cdot]$	expectation of the variable in the bracket

2.2. Warranty policy and product failures modeling

As mentioned earlier, the finished products are repairable and sold with a two-dimensional NFRW policy. The manufacturer is responsible to rectify all item failures that occur within the region specified in the warranty with no charge to the customer. Different shapes for two-dimensional warranty region are available; e.g., the rectangle, triangle, and the L-shape, etc. The commonly used rectangular region $[0, W] \times [0, U]$ is considered here with two parameters-W and U being the time and usage limits. The warranty expires at the first instance when the age of the product reaches W, or its total usage exceeds U.

The products are intended to be sold to a population of customers with heterogeneous usage intensities. It is assumed that the usage rate is a constant for each customer over the use period, but varies across the population. The usage variation is modeled by a random variable Θ with a probability density function $g(\theta)$. Let θ be a realization of Θ. We further assume that the manufacturer knows this distribution either through historical data or detailed customer survey. Given $\Theta = \theta$, the time to first failure occurrence of product has a conditional failure intensity function that is dependent on product age x and field usage rate θ. Such one-dimensional approach treats the usage rate as covariate with the usage being a linear function of age.

Each product is at risk of failures due to k failure modes which are assumed to be mutually independent. Under a given set of conditions, each failure mode competes to be the cause of product failure. If $i = 1, \ldots, k$ is the only failure mode to which the product is exposed, given $\Theta = \theta$, a sequence of latent failure times can be envisioned. Let X_i denote the time to first failure due to failure mode i when none of the other failure modes are present. Therefore the first failure time is denoted by $X = \min(X_1, X_2, \ldots, X_i, \ldots, X_k)$ when all k failure modes are present. With $\Theta = \theta$, the product conditional survivor function and hazard function denoted by $R_X(x|\theta)$ and $h_X(x|\theta)$, are given by

$$R_X(x|\theta) = P\{X_1 > x, X_2 > x, \ldots, X_i > x, \ldots, X_k > x|\theta\} = \prod_{i=1}^{k} R_i(x|\theta), \tag{1}$$

and

$$h_X(x|\theta) = -\frac{d}{dx}\ln R_X(x|\theta) = -\frac{d}{dx}\ln\prod_{i=1}^{k} R_i(x|\theta) = \sum_{i=1}^{k} h_i(x|\theta) \tag{2}$$

where $R_i(x|\theta)$ is the conditional reliability function for failure mode i and $h_i(x|\theta)$ is the associated conditional hazard function with $h_i(x|\theta) = -\frac{d}{dx}\ln R_{X_i}(x|\theta)$.

The subsequently failures depend on the repair strategy performed on the product. We confine our attention to minimal repair with negligible repair time in case of failures occurred within either the test duration or the warranty coverage. Therefore, conditional on $\Theta = \theta$, the number of product failures over time occur according to a NHPP with failure intensity function $\lambda_X(x|\theta)$ having the same form as the hazard function $h_X(x|\theta)$ that is given by Equation (2). Similarly, there is $\lambda_i(x|\theta) = h_i(x|\theta)$ where $\lambda_i(x|\theta)$ is the conditional failure intensity function of failure mode i.

The effect of usage rate on product reliability is modeled through the Accelerated failure time (AFT) approach. More specifically, the product is initially designed with a nominal usage rate θ_0. When the field usage rate θ differs from θ_0, the product reliabiltiy is affected. The stresses on the product increases with θ, and this in turn accelerates the degradation. Using the AFT formulation, let X_i (X_0) be the time to first failure due to failure mode i under usage rate θ (θ_0), then we have

$$\frac{X_i}{X_0} = (\frac{\theta_0}{\theta})^{z_i} \tag{3}$$

where z_i (>0) represents the accelerated coefficient of failure mode i.

Conditional on $\theta = \theta_0$, let $F_i(x|\theta_0)$ be the failure distribution function due to failure mode i. The conditional hazard function associated with $F_i(x|\theta_0)$ is then

$$h_i(x|\theta_0) = \frac{dF_i(x|\theta_0)/dx}{1 - F_i(x|\theta_0)}. \tag{4}$$

Then, under the usage rate θ, the mode-specific failure distribution function is given by

$$F_i(x|\theta) = F[(\frac{\theta}{\theta_0})^{z_i} x|\theta_0] \tag{5}$$

and the associated hazard function is

$$h_i(x|\theta) = \frac{dF_i(x|\theta)/dx}{1 - F_i(x|\theta)} = (\frac{\theta}{\theta_0})^{z_i} h_i[(\frac{\theta}{\theta_0})^{z_i} x|\theta_0]. \tag{6}$$

Finally, the product failure intensity function under random usage rate θ is given by

$$\lambda_X(x|\theta) = \sum_{i=1}^{k} (\frac{\theta}{\theta_0})^{z_i} \lambda_i[(\frac{\theta}{\theta_0})^{z_i} x|\theta_0]. \tag{7}$$

A common standard product reliability metric is the mean time between failures (MTBF), which also refers to the unconditional s-expected time between failures. Suppose the time required to repair the failed item is very short compared to the mean time to failure (MTTF). Upon removing the condition on usage rate θ, the product MTBF is therefore obtained by

$$\begin{aligned} \text{MTBF} = E[X] &= \int_0^\infty \left(\int_0^\infty x f_X(x|\theta) dx \right) dG(\theta) \\ &= -\int_0^\infty \left(\int_0^\infty x R'_X(x|\theta) dx \right) dG(\theta) \\ &= \int_0^\infty \int_0^\infty R_X(x|\theta) dx dG(\theta) \\ &= \int_0^\infty \int_0^\infty e^{-\lambda_X(x|\theta)} dx dG(\theta) \end{aligned} \tag{8}$$

in which $f_X(x|\theta)$ is the probability density function associated with $F_X(x|\theta)$, and $\lambda_X(x|\theta)$ is given by Equation (2).

2.3. Accelerated reliability growth modeling

As illustrated in Figure 1, with an overall duration L, a periodic *test-find-test* program consisting of n test phases with equivalent length τ is considered. Within any test phase j ($j = 1, \ldots, n$), each prototype is tested under usage rate θ_j to activate failure modes. Test still proceeds with minimal repair in case of failures, which makes no change in the product failure intensity. Fixes are scheduled between each two successive test phases. Such *delayed fixes* result in a significant jump in the product reliability by reducing the mode-specific failure intensities on a phase-by-phase basis. Before a fix is carried out, one failure mode may be minimally repaired one or more times. Let t_f be the mean lag time of each fix, as a result, the mean test interval τ is given by $L/n - t_\mathrm{f}$.

The usage rate implemented remains constant throughout a test phase. The proper selection of n and $\theta_1, \ldots, \theta_j, \ldots, \theta_n$ is critical to avoid cost prohibitive as well as achieve required reliability growth. In this study, the reliability growth test program is denoted by ϕ and characterized by the set of parameters $\phi = \{n, \theta_1, \ldots, \theta_j, \ldots, \theta_n; 1 \leq j \leq n\}$. We confine that the reliability growth requirement will be met only if the product MTBF after exceeds the lower limit π, the value of which can be determined by design engineers and failure analysis experts, etc. As can be seen in Figure 1, the idealized reliability growth curve is single and smooth, while the planned growth curve is constructed on a phase-by-phase basis.

Reliability growth is the positive improvement in a reliability parameter over a period of time due to implementation of fixes to product design. In this paper, we characterize the reliability growth by the reduction in the overall product failure intensity which occurs in a series of finite steps corresponding to discrete and periodic fixes. A modified probabilistic approach - (p, q) rule, is used to model the fix effectiveness of each failure mode [30] at the end of each phase. That is, after the jth fix, the failure mode i has a minimum achievable failure intensity - $\lambda_i^\mathrm{m}(x|\theta)$ with probability p_i^j, and the failure intensity in the preceding test phase - $\lambda_i^{j-1}(x|\theta)$ with probability $q_i^j = 1 - p_i^j$. Clearly, if $p_i^j = 1$, the fix reduces the mode-specific failure intensity to the maximum degree. If $p_i^j = 0$, the failure intensity of that discovered failure mode can not be removed by any amount. As a result, after the jth fix, the conditional failure intensity function of failure mode i at time x is given by

$$\lambda_i^j(x|\theta) = p_i^j \lambda_i^{\mathrm{m}}(x|\theta) + (1 - p_i^j)\lambda_i^{j-1}(x|\theta) \tag{9}$$

with $\lambda_i^0(x|\theta) = \lambda_i(x|\theta)$ being the initial failure intensity function of failure mode i conditional on $\Theta = \theta$ before the test initiates.

At the time of jth fix, instead of assuming average FEF for all failure modes across the test phases, the manufacturer can determine the mode-specific FEF through learning from the failures occurred within the jth test phase due to that surfaced failure mode. Learning from failure has been investigated in the operations and maintenance area [22, 40, 41, 39, 37]. Referring to the approach proposed by Tarakci [39] to quantify the effect of failure learning based on the number of failures, we model the mode-specific FEF of the jth fix that is denoted by p_i^j in Equation (9), as the following form:

$$\begin{aligned} p_i^j &= 1 - \left(1 + E[N(\xi_i^j + \tau|\theta_j) - N(\xi_i^j|\theta_j)]\right)^{-b} \\ &= 1 - \left(1 + \int_{\xi_i^j}^{\xi_i^j+\tau} \lambda_i^{j-1}(x|\theta_j)dx\right)^{-b} \end{aligned} \tag{10}$$

in which $E[N(\xi_i^j+\tau|\theta_j)-N(\xi_i^j|\theta_j)]$ is the expected number of mode-specific failures occurred during the jth phase under test usage rate θ_j. The magnitude of learning from test failures is denoted by b, that also refers to the failure learning level. Since $b \geq 0$, the second term on the right-hand side of Equation (10) is always less than or equal to 1. It can be seen that p_i^j increases with b. In addition, if we keep b unchanged, the higher value of mean test interval τ will lead to a larger number of failures triggered within a test phase and hence a higher value of p_i^j.

In Equation (10), the random variable ξ_i^j represents the *cumulative effective age* of the product till the beginning of the jth test phase when exposed to failure mode i. It is necessary to note that the AFT model is able to easily incorporate piecewise constant usage rates into product failure modeling. In this paper, the planned test consisting of n test phases is actually a piecewise usage

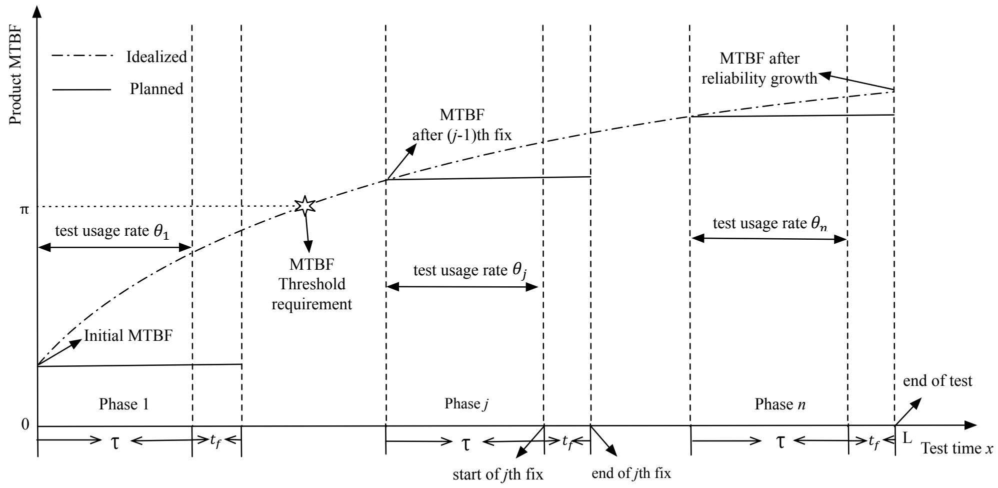

Figure 1: Idealized and planned reliability growth curves with MTBF threshold requirement.

accumulation process. The accelerated usage rate implemented in a test phase is a constant value, but the usage rates differ across test phases. We assume a product is initally tested for τ time units under usage rate θ_1. After fix, it starts a new mission with usage rate θ_2. With AFT formulation, the product's mode-specific *cumulative effective age* ξ_i^2 can be obtained as $\tau(\frac{\theta_1}{\theta_2})^{z_i}$. This means that testing the product under usage rate θ_1 for τ time units is equivalent to testing it under usage rate θ_2 for $\tau(\frac{\theta_1}{\theta_2})^{z_1}$ time units. By that analogy, we can derive the general form of ξ_i^j as

$$\xi_i^j = \begin{cases} 0 & j = 1 \\ \sum_{l=1}^{j-1}(\frac{\theta_l}{\theta_j})^{z_i}\tau & j = 2, \ldots, n. \end{cases} \tag{11}$$

The computation of ξ_i^j doesn't take the mean lag time of fix t_f into account, because the product is not exposed to test during fix hence the usage rate in this period is considered to be zero.

After reliability growth, the conditional product failure intensity denoted by $\lambda_X^\text{d}(x|\theta)$ can be derived as

$$\lambda_X^\text{d}(x|\theta) = \sum_{i=1}^{k} \lambda_i^n(x|\theta) \tag{12}$$

in which $\lambda_i^n(x|\theta)$ is the conditional mode-specific failure intensity function after the nth fix, that is also the end of whole test.

According to Equation (8), the product MTBF after reliability growth test can be given by

$$\mathrm{MTBF}(\phi) = \int_0^\infty \int_0^\infty e^{-\lambda_X^{\mathrm{d}}(x|\theta)} dx dG(\theta) \tag{13}$$

where $\lambda_X^{\mathrm{d}}(x|\theta)$ is given by Equation (12). We confine that the reliability growth requirement will be met only if the product MTBF after test exceeds the lower limit π, that is also referred to as MTBF threshold requirement (see Figure 1). The value of π can be determined by design engineers and failure analysis experts, etc.

3. Model analysis and optimization

The main goal of this section is to obtain the expectation of total cost per product incurred to the manufacturer, including both the reliability growth test cost and the warranty cost. Before deriving the mathematical cost formulas, we evaluate the expected number of test failures and warranty failures. After that, the cost-based optimization model is developed.

3.1. Failures during reliability growth test

Under the test usage rate θ_j and the assumption of minimal repair, the failures occurred due to failure mode i during the jth test phase constitute a NHPP with the failure intensity function $\lambda_i^{j-1}(x|\theta_j)$. Therefore, the associated mean number of mode-specific test failures is Poisson-distributed and equals to the integral of $\lambda_i^{j-1}(x|\theta_j)$ over the mean test interval τ, which is derived as $\int_{\xi_i^j}^{\xi_i^j+\tau} \lambda_i^{j-1}(x|\theta_j)dx$. Considering k failure modes and n test phases, the expected total number of test failures therefore satisfies the following equation:

$$E[N_{\mathrm{d}}(\phi)] = \sum_{j=1}^{n} \left[\sum_{i=1}^{k} \int_{\xi_i^j}^{\xi_i^j+\tau} \lambda_i^{j-1}(x|\theta_j) dx \right] \tag{14}$$

3.2. Failures within warranty coverage

In additional to test failures, we need to obtain the expected number of warranty failures. Define the average warranty usage rate $\rho = U/W$. Let W_θ denote the warranty expiry time when the usage rate is θ, then there is

$$W_\theta = \begin{cases} W & \theta \leq \rho \\ \frac{U}{\theta} & \theta > \rho. \end{cases}$$

Since minimal repair actions are performed within the warranty region, under usage rate θ, the failures under warranty occur according to a NHPP with the product failure intensity function $\lambda_X^{\mathrm{d}}(x|\theta)$. Conditional on $\Theta = \theta$, the number of warranty failures denoted by $N_{\mathrm{w}}(\phi|W_\theta)$ is a Poisson variable with mean number of occurrences that equals to the integration of product failure intensity over the warranty period $[0, W_\theta]$:

$$E[N_{\mathrm{w}}(\phi|W_\theta)] = \int_0^{W_\theta} \lambda_X^{\mathrm{d}}(x|\theta)dx. \tag{15}$$

Combined with Equation (12), the expected number of warranty failures is obtained by taking expectation of Equation (15) with respect to usage rate θ:

$$\begin{aligned} E[N_{\mathrm{w}}(\phi)] &= \int_0^\infty E[N_{\mathrm{w}}(\phi|W_\theta)]dG(\theta) \\ &= \int_0^\infty \left(\int_0^{W_\theta} \sum_{i=1}^{k} \lambda_i^n(x|\theta)dx \right) g(\theta)d\theta. \end{aligned} \tag{16}$$

3.3. Cost analysis

The expected total cost per product sold consists of the reliability growth test cost and the warranty cost, in which the test cost depends on the number of fixes n, and the test usage rates in each phase - $\theta_1, \theta_2, \ldots, \theta_n$. Let C_{s} be the set-up cost of the test per product, C_{d} be the operational cost per unit test time, and C_{r} be the average repair cost to rectify a test failure. The cost of the jth fix is given by

$$C_{\mathrm{f}}^{j} = \sum_{i=1}^{k} C_{\mathrm{f},i}^{j}$$

in which $C_{\mathrm{f},i}^{j}$ represents the model-specific cost of the jth fix. It is reasonable to assume that $C_{\mathrm{f},i}^{j}$ could be divided into a fixed cost and a variable cost modeled by an increasing power function of mode-specific FEF of the jth fix, p_i^j. Thus, we model $C_{\mathrm{f},i}^{j}$ as

$$C_{\mathrm{f},i}^{j} = C_{\mathrm{f},i}^{f} + C_{\mathrm{f},i}^{v}(p_i^j)^{\sigma} \tag{17}$$

with $C_{\mathrm{f},i}^{f}$ and $C_{\mathrm{f},i}^{v}$ being the fixed cost and variable cost, respectively, and $\sigma > 0$.

As a result, the expected test cost per product sold is denoted by $\mathrm{EC_d}(\phi)$ and expressed as

$$\mathrm{EC_d}(\phi) = C_{\mathrm{s}} + C_{\mathrm{d}}L + \sum_{j=1}^{n} C_{\mathrm{f}}^{j} + C_{\mathrm{r}}E[N_{\mathrm{d}}(\phi)] \tag{18}$$

in which $E[N_{\mathrm{d}}(\phi)]$ is given by Equation (14). On the right-hand side of Equation (18), the first two terms represent the set-up cost and variable cost which is proportional to the total test time L. The last two terms refer to the cost of n fixes and the expected minimal repair cost of the test failures.

The warranty cost depends on the product failure intensity after reliability growth, and the maintenance strategy performed within warranty coverage. Let C_{m} be the repair cost per warranty failure, the expected warranty cost per product is denoted by $\mathrm{EC_w}(\phi)$ and then given by

$$\mathrm{EC_w}(\phi) = C_{\mathrm{m}}E[N_{\mathrm{w}}(\phi)] \tag{19}$$

with $E[N_{\mathrm{w}}(\phi)]$ given by Equation (16) and $C_{\mathrm{m}} > C_{\mathrm{r}}$ (since C_{m} includes both shop repair cost and additional repair cost).

By summing up each expected cost obtained in Equations (18) and (19), we derive a formula to estimate the expected total cost per product sold that is denoted by $\mathrm{TC}(\phi)$ as follows:

$$
\begin{aligned}
\mathrm{TC}(\phi) &= \mathrm{EC_d}(\phi) + \mathrm{EC_w}(\phi) \\
&= C_\mathrm{s} + C_\mathrm{d}T + \sum_{j=1}^{n}\sum_{i=1}^{k}[C_{\mathrm{f},i}^{f} + C_{\mathrm{f},i}^{v}(p_i^j)^{\sigma}] + C_\mathrm{r}\sum_{j=1}^{n}\left[\sum_{i=1}^{k}\int_{\xi_i^j}^{\xi_i^j+\tau}\lambda_i^{j-1}(x|\theta_j)dx\right] \\
&\quad + C_\mathrm{m}\int_0^{\infty}\left(\int_0^{W_r}\sum_{i=1}^{k}\lambda_i^{n}(x|\theta)dx\right)dG(\theta).
\end{aligned} \tag{20}
$$

It is necessary to mention that reliability growth test is performed in a few early product prototypes, while the warranty policy applies to all products sold in the marketplace. The cost function $\mathrm{TC}(\phi)$ derived in Equation (20) refers to the expected total cost per product sold in the marketplace instead of the expected total cost per product tested. The values of test cost related parameters including C_s, C_d, C_f^j and C_r are averaged ones for each sale of the product, and they are relatively lower than the value of warranty servicing cost related parameter such as C_m.

3.4. Cost-based optimization model

In this part, given the cost function represented by Equation (20), the cost-based optimization model for reliability growth test program ϕ can be expressed as

$$
\begin{aligned}
&\{n^*, \theta_1^*, \theta_2^*, \ldots, \theta_n^*\} = \operatorname{argmin} \mathrm{TC}(\phi), \\
&s.t. \qquad \mathrm{MTBF}(\phi) \geq \pi, \\
&\qquad\qquad n \in \{1, \ldots, \lfloor T/t_\mathrm{f} \rfloor - 1\}, \\
&\qquad\qquad 0 < \theta_1, \ldots, \theta_j \ldots, \theta_n \leq \overline{\theta},
\end{aligned} \tag{21}
$$

in which $\lfloor T/t_\mathrm{f} \rfloor - 1$ and $\overline{\theta}$ are the upper limits for the feasible number of fixes and the possible test usage rate, respectively. An upper limit for n is necessary since the mean test interval τ must be greater than zero. A ceiling for θ_j is reasonable due to test conditions limitation. An extremely high test usage rate would cause some extraneous failures modes that would not occur at the actual use levels [31].

Considering the special case of no reliability growth test ψ ($n = 0$ and $\theta_1, \theta_2, \ldots, \theta_n = 0$), the expected total cost per product equals to the warranty servicing cost per item. Then there is

$$\mathrm{TC}(\psi) = C_{\mathrm{m}} E[N_{\mathrm{w}}(\psi)] = C_{\mathrm{m}} \int_0^{\infty} \left(\int_0^{W_\theta} \lambda_X(x|\theta) dx \right) dG(\theta), \tag{22}$$

in which $\mathrm{TC}(\psi)$ represents the upper bound of the expected total cost per product, which also refers to the *benchmark cost*. In mathematical terms, the reliability growth test is beneficial if there is $\mathrm{TC}(\phi) < \mathrm{TC}(\psi)$

or

$$\begin{aligned} & C_{\mathrm{s}} + C_{\mathrm{d}} T + \sum_{j=1}^{n} \sum_{i=1}^{k} [C_{\mathrm{f},i}^{f} + C_{\mathrm{f},i}^{v} (p_i^j)^{\sigma}] + C_{\mathrm{r}} \sum_{j=1}^{n} \left[\sum_{i=1}^{k} \int_{\xi_i^j}^{\xi_i^j + \tau} \lambda_i^{j-1}(x|\theta_j) dx \right] \\ & < C_{\mathrm{m}} \int_0^{\infty} \left(\int_0^{W_\theta} \sum_{i=1}^{k} [\lambda_i^0(x|\theta) - \lambda_i^n(x|\theta)] dx \right) dG(\theta). \end{aligned} \tag{23}$$

This implies as long as the test cost incurred under the reliability growth test program ϕ is lower than the benefits derived from the reduction in the number of warranty failures occurred, reliability growth is cost-efficient.

The optimal reliability growth test program is found by minimizing the cost function $\mathrm{TC}(\phi)$ given by Equation (20), ensuring the product MTBF threshold requirement π is achieved as well. The optimal values of n, $\theta_1, \theta_2, \ldots,$ and θ_n are obtained using a two-stage process. In the first stage, we fix n and find $\{\theta_1^*, \theta_2^*, \ldots, \theta_n^*\}$ by minimizing $\mathrm{TC}(n, \theta_1, \theta_2, \ldots, \theta_n)$. In the second stage, n^* is found by minimizing $\mathrm{TC}(n, \theta_1^*(n), \theta_2^*(n), \ldots, \theta_n^*(n))$, and then $n^* = n^*(\theta_1^*(n), \theta_2^*(n), \ldots, \theta_n^*(n))$. Because of the structure of $\mathrm{TC}(n, \theta_1, \theta_2, \ldots, \theta_n)$, it may be impossible to derive general analytical results and so the optimization procedure will be carried out numerically.

4. Numerical example

In this section, we present a numerical example to illustrate the proposed reliability growth test planning model in Section 3 and 4. To generate a prominent bathtub-shaped product hazard function and model multiple failure processes simultaneously, we hereby assume the product lifetime follows a baseline tri-Weibull distribution in this example. As a shape-scale distribution, the tri-Weibull distribution is a typical case of n-fold Weibull competing risk model that describes the minimum of several independent random variables where each is Weibull distributed [17]. The time to failure occurrence due to failure mode i ($i = 1, 2, 3$) follows a two-parameter Weibull distribution. Under nominal design usage rate θ_0, the mode-specific failure intensity function is given by

$$\lambda_i(x|\theta_0) = \frac{\beta_i}{\alpha_i}(\frac{x}{\alpha_i})^{\beta_i - 1}$$

in which α_i and β_i are the scale and shape parameters of failure mode i. Then according to Equations (6) and (7), before reliability growth test is conducted, the initial product failure intensity function given usage rate $\Theta = \theta$ is given by

$$\lambda_X(x|\theta; \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = \sum_{i=1}^{3} \frac{\beta_i}{\alpha_i}(\frac{\theta}{\theta_0})^{z_i\beta_i}(\frac{x}{\alpha_i})^{\beta_i - 1}. \tag{24}$$

Similarly, conditional on $\Theta = \theta$, the minimum achievable product failure intensity function is therefore

$$\lambda_X^{\mathrm{m}}(x|\theta) = \sum_{i=1}^{3} \frac{\beta_i}{\alpha_i^m}(\frac{\theta}{\theta_0})^{z_i\beta_i}(\frac{x}{\alpha_i^m})^{\beta_i - 1}$$

where α_i^m is the minimum value of scale parameter α_i for failure mode i.

Moreover, the product usage rate θ is random and assumed to be Gamma distributed with the density function $g(\theta) = \frac{1}{\delta_2^{\delta_1}\Gamma(\delta_1)}\theta^{\delta_1 - 1}e^{-\frac{r}{\delta_2}}$, where δ_1 and δ_2 are the shape and scale parameters. The parameter values are given in Table 1, of which the time is measured in year and the unit of money is US dollar ($).

Table 1: Model parameter settings.

Notation	Description
Warranty limits	$W = 3$(years), $U = 6(10^4$km), $\rho = \frac{U}{W} = 2$
Product failure distribution	$\theta_0 = 2$, $\alpha_1 = 2.73$, $\alpha_2 = 6$, $\alpha_3 = 10$ ($i = 1, 2, 3$), $\beta_1 = 0.5, \beta_2 = 1, \beta_3 = 5$, $\alpha_1^m = 23$, $\alpha_2^m = 20$, $\alpha_3^m = 12$
Accelerated coefficients	$z_1 = 0.8$, $z_2 = 0.5$, $z_3 = 1.2$
Failure learning level	$b = 6$
Usage rate distribution	$\delta_1 = 5.85$, $\delta_2 = 0.3$
Reliability growth test	$T = 0.5$, $t_\mathrm{f} = 0.03$, $C_\mathrm{s} = 3$, $C_\mathrm{d} = 10$, $C_{\mathrm{f},i}^f = 5$, $C_{\mathrm{f},1}^v = 123$, $C_{\mathrm{f},2}^v = 37$, $C_{\mathrm{f},3}^v = 33$, $\sigma = 3$, $C_\mathrm{r} = 10$
Warranty service	$C_\mathrm{m} = 300$
Product MTBF threshold requirement	$\pi = 3$

4.1. Optimal solution

With no reliability growth, the benchmark cost per product to the manufacturer is calculated through Equation (22). According to Equation (8), the initial product MTBF before test implementation is 2.2348 (years), and the maximum achievable value of product MTBF is 7.2319 (years). Suppose the manufacturer expects the product MTBF after reliability growth to be more than 3 years at least. For (3 years, $6 \times 10,000$km) as the specified warranty contract coverage, we consider n feasible number of fixes with $n \in \{1, \dots, 15\}$ and the possible test usage rates with $\theta_j \in (0, 10]$ ($j = 1, \dots, n$ and $\overline{\theta} = 10$).

Under the above-mentioned two-stage optimization framework, when n is fixed, a penalty function is constructed to transfer the nonlinear optimization problem with inequality constraints to the unconfined extreme problem. Then the gradient descent method is used to search for the optimal values of $\theta_1^*, \theta_2^*, \dots$ and θ_n^*. After that, we determine the optimal set of $\{n^*, \theta_1^*, \theta_2^*, \dots, \theta_n^*\}$ which minimizes TC(ϕ) while assuring MTBF(ϕ^*) $\geq \pi$. In addition, we designed an improved particle swarm optimization (PSO) algorithm to solve the proposed model. Applied with 5 repetitions under an initial population of 30 particles, the typical PSO is terminated at both 20 and 200 iterations to obtain the optimal results. The detailed optimization producedures based on these two algorithms are provided in the Appendix part. At last, the optimal test solutions derived with both

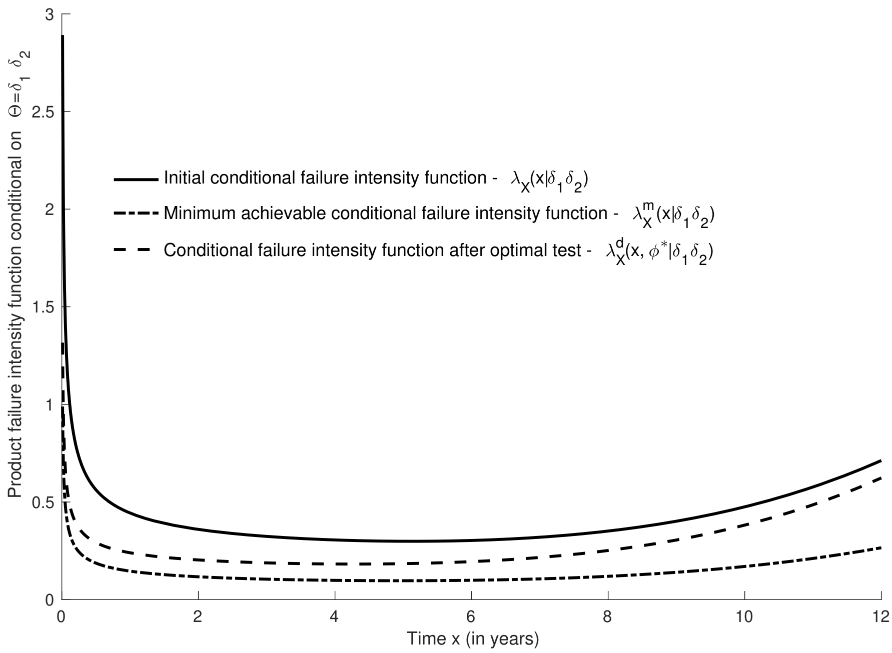

Figure 2: Product failure intensity function conditional on $\Theta = \delta_1\delta_2$.

approaches are presented in Table 2 to compare the computational accuracy.

Table 2: Optimal reliability growth test solutions using two different approaches.

Approach	n^*	$\{\theta_1^*, \ldots, \theta_j^*, \ldots, \theta_n^*\}$	TC(ϕ^*)	MTBF(ϕ^*)	TC(ψ)	MTBF(ψ)
Gradient descent	3	{0.353, 5.443, 10.000}	305.953	3.8966	397.303	2.2348
Improved PSO (20 iterations)	3	{0.262, 5.232, 9.701}	306.488	3.8771	397.303	2.2348
Improved PSO (200 iterations)	3	{0.353, 5.443, 10.000}	305.953	3.8966	397.303	2.2348

As can be seen, using the gradient descent approach, the optimal number of fixes n^* is 3 and the associated optimal test usage rates are obtained with $\theta_1^* = 0.353$, $\theta_2^* = 5.443$ and $\theta_3^* = 10.000$, with the minimum cost being 305.953, which is reduced by 22.99% compared to the benchmark cost being 397.303. The product MTBF after reliability growth increases to be 3.8966 (years). It shows the optimal test program can reduce the expected cost per product to the manufacturer and improve the product reliability as well. Specially, conditional on the mean customer usage level $\delta_1\delta_2 = 1.755$, the product failure intensity functions before and after reliability growth, and the minimum achievable failure intensity function are depicted in Figure 2. By comparison, the optimal test solution results in a little higher minimum cost and lower product MTBF when the improved PSO algorithm is used and terminated with 20 iterations. While with 200 iterations, the optimal results obtained are as same as those derived with the gradient descent method.

4.2. Sensitivity analysis

For simplicity of computation, the model parameter values are given directly. In practice, especially with respect to the product failure distribution parameters, they should be estimated from real data to build a more accurate model. As warranty data contain valuable information on product field reliability and customer behavior, through joint analysis of warranty claim data from products of previous generation and the supplementary tracking data from customers, point estimation for the model parameters could be readily conducted [42, 43]. Point estimators could be determined using maximum likelihood (ML) or Bayesian estimation methods. Before parameter estimation, appropriate mode-specific lifetime models could be selected based on initial tests, past designs and/or prior engineering knowledge of the deterioration mechanisms the items are exposed to. When the mode-specific failure time distributions are unclear to the manufacturer due to lack of knowledge, the nonparametric estimate together with a probability plot is helpful to decide the appropriate lifetime distributions of each failure mode, based on which the subsequent parametric analysis can be conducted. In addition, to account for the uncertainty in the point estimate, confidence interval estimation can be done by using asymptotic normal approximation and bootstrap method.

To reduce parameter uncertainty, sensitivity analysis is performed to investigate how variations in the model inputs cause changes to the model outputs. Several parameter changes are studied as follows: failure learning level b, repair cost per a warranty failure C_{m}, mean lag time of fix t_{f} and average warranty usage rate ρ. For each parameter investigated, the other parameter values are kept the same as in Table 1 and we modify that parameter in five levels. The results are presented in Table 3 and depicted in Figures 3 to 7.

Table 3: Effect of key parameters on the optimal reliability growth test solutions.

Parameter	n^*	$\{\theta_1^*, \ldots, \theta_j^*, \ldots, \theta_n^*\}$	TC(ϕ^*)	MTBF(ϕ^*)	TC(ψ)	MTBF(ψ)
Failure learning level, b						
2	1	{10.000}	324.258	3.5300	397.303	2.2348
4	2	{0.987, 10.000}	309.729	3.7196	397.303	2.2348
6	3	{0.353, 5.443, 10.000}	305.953	3.8966	397.303	2.2348
8	3	{0.175, 2.733, 10.000}	300.336	4.0566	397.303	2.2348
10	3	{0.117, 1.873, 8.707}	299.752	4.1874	397.303	2.2348
Repair cost per a warranty failure, C_{m}						
150	2	{0.142, 1.685}	188.764	3.4172	198.651	2.2348
200	3	{0.202, 2.959, 10.000}	230.067	3.7813	264.869	2.2348
250	3	{0.265, 3.980, 10.000}	268.364	3.8319	331.086	2.2348
300	3	{0.353, 5.443, 10.000}	305.953	3.8966	397.303	2.2348
350	3	{0.426, 6.574, 10.000}	342.872	3.9422	463.520	2.2348
Mean lag time of fix, t_{f}						
0.01	3	{0.295, 4.252, 10.000}	302.970	3.9603	397.303	2.2348
0.03	3	{0.353, 5.443, 10.000}	305.953	3.8966	397.303	2.2348
0.05	2	{0.554, 7.476}	308.323	3.9107	397.303	2.2348
0.07	2	{0.630, 8.935}	308.721	3.9020	397.303	2.2348
0.09	2	{0.680, 10.000}	309.207	3.8582	397.303	2.2348
Average warranty usage rate, ρ (U fixed)						
1 ($W = 6$)	3	{0.440, 6.698, 10.000}	350.284	3.9474	470.911	2.2348
1.5 ($W = 4$)	3	{0.408, 6.266, 10.000}	331.057	3.9303	439.080	2.2348
2 ($W = 3$)	3	{0.353, 5.443, 10.000}	305.953	3.8966	397.303	2.2348
2.5 ($W = 2.4$)	3	{0.293, 4.478, 10.000}	281.331	3.8547	356.394	2.2348
3 ($W = 2$)	3	{0.249, 3.748, 10.000}	260.123	3.8026	321.269	2.2348
Average warranty usage rate, ρ (W fixed)						
1 ($U = 3$)	3	{0.224, 3.365, 10.000}	246.626	3.8018	298.694	2.2348
1.5 ($U = 4.5$)	3	{0.302, 4.626, 10.000}	285.283	3.8613	362.908	2.2348
2 ($U = 6$)	3	{0.353, 5.443, 10.000}	305.953	3.8966	397.303	2.2348
2.5 ($U = 7.5$)	3	{0.377, 5.820, 10.000}	315.511	3.9122	413.055	2.2348
3 ($U = 9$)	3	{0.388, 5.994, 10.000}	319.574	3.9192	419.572	2.2348

Effect of failure learning level, *b* The results are illustrated in Table 3 and Figure 3. It is noted that in Figure 3 (and in the figures for other parameters), the left-hand y-axis represents the minimum expected total cost per product with reliability growth and the benchmark cost per product without reliability growth. Whereas the right-hand y-axis depicts the product MTBF before and after optimal reliability growth. At $b = 2$, the optimal decision is to set only one test phase (one fix) and to make the test condition as harsh as possible (highest test usage rate). It can be seen that the fix frequency increases as the failure learning level goes up. This should make sense since higher levels of failure learning imply more effective fixes which motivates the manufacturer to perform more of them.

An interesting observation is that although we don't confine $\theta_j \leq \theta_{j+1}$ when deriving the optimal test usage rates, it is observed that there is $\theta_j^* < \theta_{j+1}^*$. Such result is reasonable since when the product failure intensity is reduced phase by phase through discrete fixes, the improved product will be less sensitive to the elevated usage rate. Generating additional failures hence requires applying a harsher test condition. When b changes from 6 to 10, the optimal number of fixes n^* keeps to be 3 while the optimal test usage rates in each phase decrease gradually. At $b = 10$, the optimal usage rate implemented in the final test phase is not necessarily the upper limit 10. This implies the manufacturer can afford to keep test usage rates lower when possessing higher failure learning ability. In addition, the higher the failure learning level, the lower the expected total cost per product and the higher product MTBF after optimal reliability growth.

Effect of repair cost of a warranty failure, C_m Table 3 and Figure 4 show that, the optimal number of fixes n^* increases to be 3 when $C_\mathrm{m} \geq 200$ and remains unchanged when C_m is between 200 and 350. Both the minimum cost per product with optimal reliability growth $\mathrm{TC}(\phi^*)$ and the benchmark cost per product $\mathrm{TC}(\psi)$ are higher at higher C_m values. The cost reduction through optimal reliability growth reflected by $\mathrm{TC}(\psi) - \mathrm{TC}(\phi^*)$, is more significant as C_m goes up (see Figure 4). The optimal test usage rate θ_j^* in the jth phase increases with C_m, while when $C_\mathrm{m} \geq 200$, the value of θ_n^* in the final test phase is always the upper limit 10. Presumably the marginal cost of warranty failures becomes higher than that of test failures. Therefore, the manufacturer affords to keep test usage rates higher. As a result, the product MTBF after optimal reliability growth

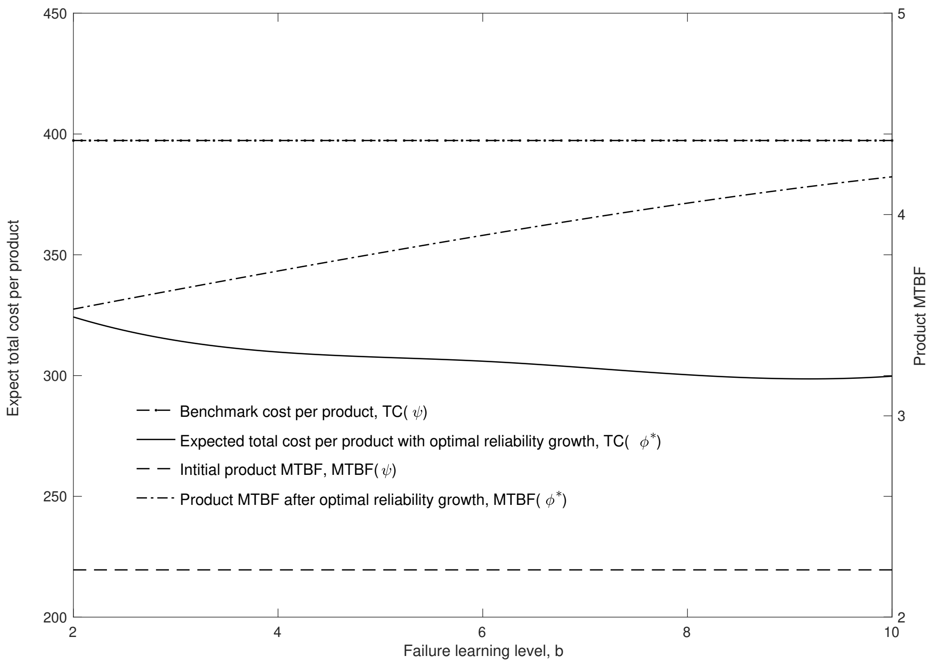

Figure 3: Effect of failure learning level, b.

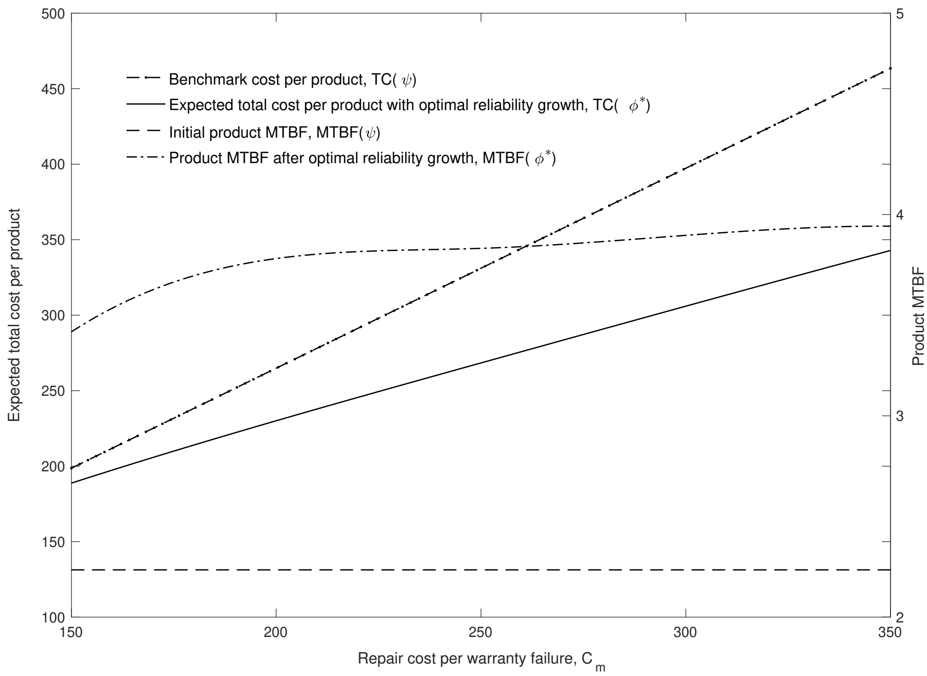

Figure 4: Effect of repair cost per warranty failure, C_{m}.

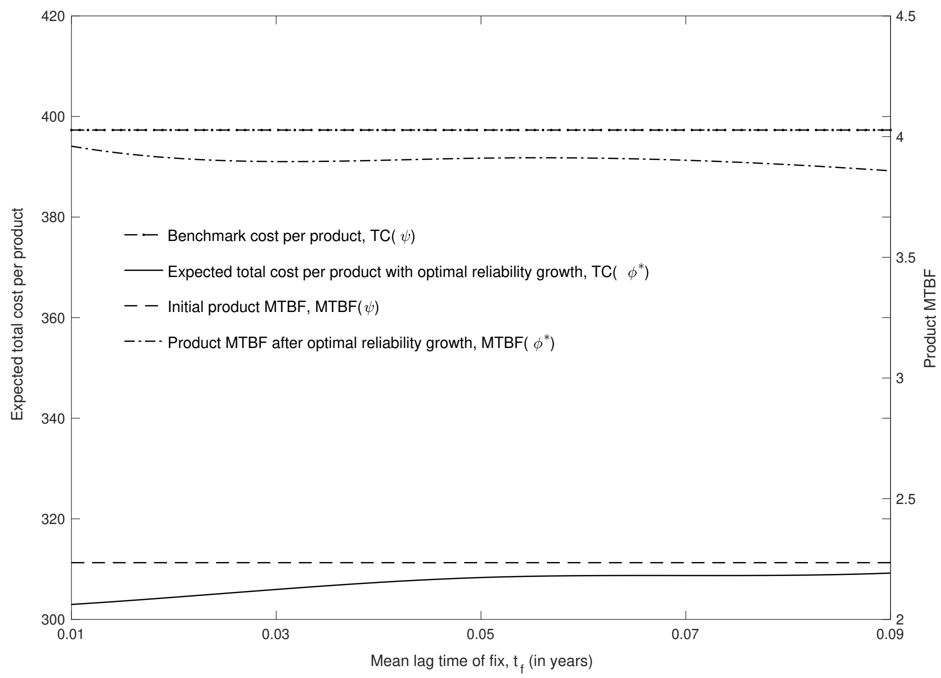

Figure 5: Effect of mean lag time of fix, t_{f} (in years).

increases with C_{m}.

Effect of mean lag time of fix, t_{f} It can be seen from Table 3 and Figure 5 that the optimal fix frequency n^* decreases as the mean lag time of fix t_{f} goes up. With the value of n^* being either 2 or 3, the optimal test usage rates in each phase increase with t_{f}. This should be intuitive since with the total test time L fixed, higher t_{f} will shorten the length of mean test interval τ, thus it makes more sense to implement harsher test conditions (higher test usage rates) in a test phase. Accordingly, as t_{f} increases, the minimum cost per product with reliability growth goes up slightly and the product MTBF after optimal reliability growth decreases little by little.

Effect of average warranty usage rate, ρ This part illustrates the effect of different average warranty usage rates ρ on the optimal test solutions (Table 3, Figures 6 and 7). With $\rho = U/W$, we set five level of ρ $(1, 1.5, 2, 2.5, 3)$ by changing the values of W and U respectively. For fixed value

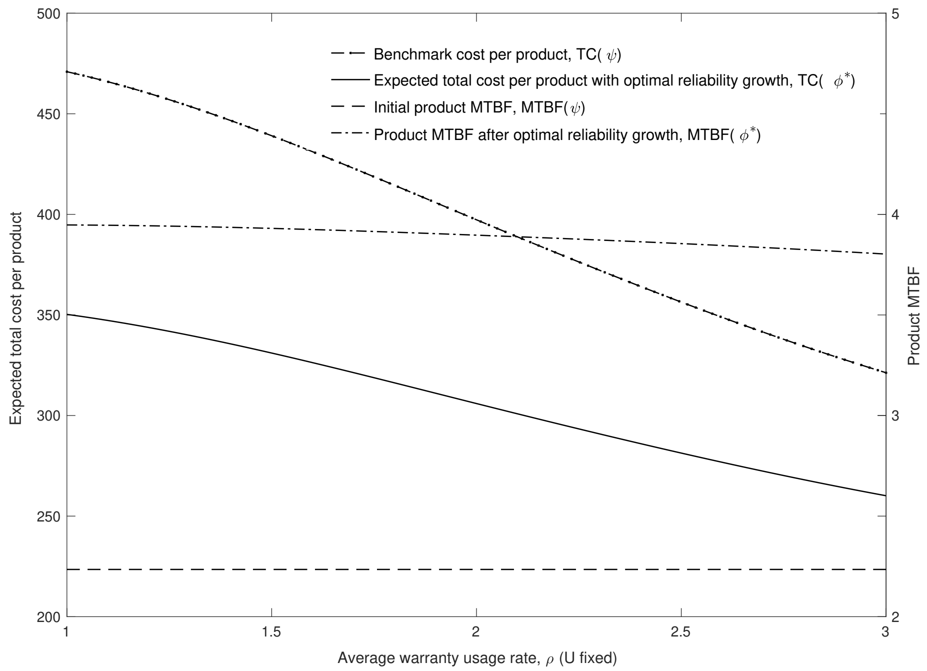

Figure 6: Effect of average warranty usage rate, ρ (U fixed).

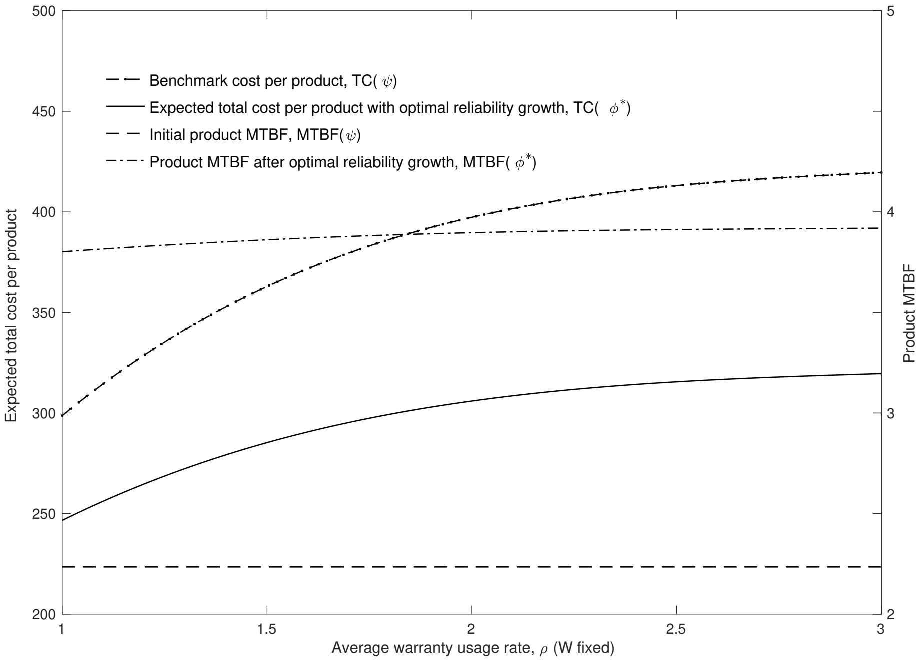

Figure 7: Effect of average warranty usage rate, ρ (W fixed).

of U, the rectangle warranty region becomes narrower when ρ increases, hence the corresponding benchmark cost per product TC(ψ) goes down. The optimal number of fixes n^* remains unchanged equal to 3. The optimal test usage rate θ_j^* ($j = 1, 2$) increases with ρ, while the value of θ_n^* ($n = 3$) is always the upper limit 10. This makes sense to perform less test effort when products are sold with a smaller warranty region. Consequently, the product MTBF after optimal reliability growth drops gradually with ρ.

By contrast, if varying the values of U with fixed W, the warranty coverage enlarges as ρ goes up and results in higher benchmark cost per product without reliability growth. Similarly, the optimal value of n keeps unchanged equal to 3, whereas the corresponding test usage rates in the first two phases show a gradual increase, and the product MTBF after optimal reliability growth rises slowly. The benefit from reliability growth reflected by the reduction in the expected total cost per product, is much more apparent if the manufacturer extends the warranty coverage.

5. Conclusions

This paper has developed a multi-phase reliability growth test planning model for products sold with a two-dimensional non-renewing FRW policy by introducing a modified reliability growth modeling technique that differentiates distinct failure modes, each having a different accelerated coefficient. The aim is to provide decision makers with the insights of effectively allocating limited test time for reliability improvement. Specifically, a *test-find-test* scheme is considered, under which product prototypes are operated under different usage rates in multiple test phases to trigger failures. Taking the difference in accelerated relationships and usage heterogeneity into account, the effect of usage acceleration on the product reliability is modeled through the Accelerated Failure Time (AFT) approach. Fixes are implemented at the end of each test phase to reduce the mode-specific failure intensities with distinct fix effectiveness considering failure learning effect. From the manufacturer's perspective, we have presented a mathematical optimization model to derive the optimal reliability growth test program to minimize the expected total cost per product sold, ensuring the reliability growth requirement is achieved as well.

From the results of numerical example, when the failure learning level increases, the optimal number of fixes and the product MTBF after optimal reliability growth increase, while the optimal test usage rates implemented in each phase as well as the minimum expected total cost per product decreases. When the mean lag time of fix increases, the optimal number of fixes decreases whereas the optimal test usage rates go up. For products sold with high repair cost of a warranty failure and extensive warranty coverage, the optimal reliability growth test program yields significant cost reduction and reliability improvement.

There could be several possible topics for further research. This study has assumed mutual independence among the competing failure modes. It can be modified by introducing a certain degree of dependence between the failure modes, which may have a noticeable effect on the model formulation. In addition, considering that failure learning ability may need to be trained at certain cost instead of happening naturally, the quantification of benefit from failure learning can be incorporated when extending the current model. Other warranty policies other than the FRW, such as the non-renewing free replacement policy and renewing policies, can be considered. Other warranty servicing strategies (e.g., imperfect/perfect repair strategies) and other shapes for the warranty region (e.g. [27] and [46]) also need further investigation. In addition, we suppose that the manufacturer's primary concern in this study is cost minimization given specified reliability growth requirement. Other optimization models with the objectives of reliability maximization or profit maximization including both the cost and the product performance outcome, can be developed. Finally, this study presents the values of entries in the mathematical optimization model directly. In practice, the values should be estimated from real data to construct a more accurate model.

Acknowledgements

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Appendix

Optimization procedure based on gradient descent approach

Step 1 Input model parameters and cost functions. Fix n ($n = 1, \ldots, \lfloor T/t_{\mathrm{f}} \rfloor - 1$).

Step 2 Set $i = 1$. Initialize the penalty factor M_i, accuracy $\varepsilon_M > 0$ and reduction coefficient $\beta \in (0, 1)$.

Step 3 For M_i, construct a new objective function based on Equations (20) and (21) as follow:

$$\min \quad P(\theta; n, M_i) = \mathrm{TC}(\theta; n) + M_i \sum_{l=1}^{2n+1} \frac{1}{g_l(\theta; n)} \tag{25}$$

with $\theta = \{\theta_1, \ldots, \theta_n\}$ and $g_l(\theta; n) \geq 0$ being the lth inequality constraint in Equation (21).

Step 4 Select the initial feasible solution $\theta^{i,j} = \{\theta_1^{i,j}, \ldots, \theta_n^{i,j}\}$ randomly. Set $j = 1$ and select the initial learning rate λ_j, decay rate α and accuracy $\varepsilon_N > 0$.

Step 5 Calculate $\nabla P(\theta^{i,j}; n, M_i)$. If $\| \nabla P(\theta^{i,j}; n, M_i)\| < \varepsilon_N$, stop and accept $\theta^{i,j}$ as the optimal solution, then go to Step 8. Otherwise, go to Step 6.

Step 6 Set $D^{i,j} = - \nabla P(\theta^{i,j}; n, M_i)$. According to the learning rate λ_j, search along $D^{i,j}$ and obtain the next feasible solution $\theta^{i,j+1} = \theta^{i,j} + \lambda_j D^{i,j}$.

Step 7 Calculate $\nabla P(\theta^{i,j+1}; n, M_i)$ and new learning rate $\lambda_{j+1} = \frac{1}{1+\alpha \cdot j}\lambda_j$. Set $j = j + 1$ and go to Step 5.

Step 8 If $\theta^{i,j}$ meets the accuracy requirement that is $M_i \sum_{l=1}^{2n+1} \frac{1}{g_l(\theta^{i,j};n)} < \varepsilon_M$, stop and accept $\theta^{i,j}$ as the optimal solution. Otherwise, set $M_{i+1} = \beta M_i$ and $i = i + 1$, then go back to Step 3.

It is noted that we can select the initial feasible point randomly for multiple times. This helps to escape from the saddle point and reduce the possibility of dropping into local optimum. We find that in this way the approach ensures convergence to the same optimal solution.

Step 9 Output the optimal solution $\theta^{i,j}$ and the corresponding cost $\mathrm{TC}(\theta^{i,j}; n)$.

Step 10 Search $\mathrm{TC}(n, \theta^{i,j}(n))$ to determine n^* which yields the smallest value for $\mathrm{TC}(n, \theta^{i,j}(n))$. Output the optimal test solution $(n^*, \theta^{i,j}(n^*))$ and the associated cost $\mathrm{TC}(n^*, \theta^{i,j}(n^*))$.

Optimization procedure based on improved PSO algorithm

Step 1 Input model parameters and cost functions. Fix n ($n = 1, \ldots, \lfloor T/t_{\mathrm{f}} \rfloor - 1$).

Step 2 Initialize the particle swarm. For each particle k ($k = 1, \ldots, N$), initialize its position $\theta^k = \{\theta_1^k, \ldots, \theta_n^k\}$ and velocity $v^k = \{v_1^k, \ldots, v_n^k\}$ ($j = 1, \ldots, n$) randomly with $\theta_j^k \in (0, \overline{\theta}]$ and $v_j^k \in [-0.5, 0.5]$.

Step 3 Initialize the counting variable $F = 1$. At each iteration time t ($t = 1, \ldots, t_{\max}$), calculate the fitnesss value of particle k denoted by $\mathrm{TC}(\theta^k(t); n)$ based on Equation (20).

Step 4 Compare $\mathrm{TC}(\theta^k(t); n)$ with $\mathrm{TC}(\theta^k_{best}(t); n)$ in which $\theta^k_{best}(t)$ is the individual best position of particle k already found until time t. If $\mathrm{TC}(\theta^k(t); n) < \mathrm{TC}(\theta^k_{best}(t); n)$, update $\theta^k_{best}(t)$ and $\mathrm{TC}(\theta^k_{best}(t); n)$ by $\theta^k_{best}(t) = \theta^k(t)$ and $\mathrm{TC}(\theta^k_{best}(t); n) = \mathrm{TC}(\theta^k(t); n)$ respectively. In addition, update F by $F = F + 1$. Otherwise, set $F = 0$.

Step 5 Compare $\mathrm{TC}(\theta^k(t); n)$ with $\mathrm{TC}(\theta^g_{best}(t); n)$ in which $\theta^g_{best}(t)$ is the globally best position already found in the particle swarm until time t. If $\mathrm{TC}(\theta^k(t); n) < \mathrm{TC}(\theta^g_{best}(t); n)$, update $\theta^g_{best}(t)$ and $\mathrm{TC}(\theta^g_{best}(t); n)$ by $\theta^g_{best}(t) = \theta^k(t)$ and $\mathrm{TC}(\theta^g_{best}(t); n) = \mathrm{TC}(\theta^k(t); n)$ respectively.

Step 6 Update the velocity and position of particle k at time $t + 1$ by

$$v^k(t+1) = \begin{cases} \omega v^k(t) + c_1 r_1(\theta^k_{best}(t) - \theta^k(t)) + c_2 r_2(\theta^g_{best}(t) - \theta^k(t)), & F \leq 5 \\ v^r(t) + c_1 r_1(\theta^k_{best}(t) - \theta^k(t)) + c_2 r_2(\theta^g_{best}(t) - \theta^k(t)), & F > 5 \end{cases} \tag{26}$$

and

$$\theta^k(t+1) = \theta^k(t) + v^k(t+1). \tag{27}$$

In the above two equations, the coefficients c_1 and c_2 are given acceleration constants and $r_1, r_2 \in [0, 1]$ are random values generated. The inertia weight $\omega = 1/t$ aims to provide balance between global and local search. There is $v^r(t) = \{v^r_1(t), \dots, v^r_n(t)\}$ with $v^r_j(t) \in [-0.5, 0.5]$ $(j = 1, \dots, n)$ being random values generated.

Step 7 Check the termination condition of the algorithm. If the maximum iteration times $t_{\max}$ is reached or the convergence criteria is met, go to Step 8. Otherwise, go back to Step 3.

Step 8 Output the globally optimal position θ^g_{best} and the corresponding fitness $\mathrm{TC}(\theta^g_{best}; n)$.

Step 9 Search $\mathrm{TC}(n, \theta^g_{best}(n))$ to determine n^* which yields the smallest value for $\mathrm{TC}(n, \theta^g_{best}(n))$. Output the optimal test solution $(n^*, \theta^g_{best}(n^*))$ and the associated cost $\mathrm{TC}(n^*, \theta^g_{best}(n^*))$.

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