A fractional approach for the motion planning of redundant and hyper-redundant manipulators

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ABSTRACT

The trajectory planning of redundant robots through the pseudoinverse control leads to undesirable drift in the joint space. This paper presents a new technique to solve the inverse kinematics problem of redundant manipulators, which uses a fractional differential of order a to control the joint positions. Two performance measures are defined to examine the strength and weakness of the proposed method. The positional error index measures the precision of the manipulator's end-effector at the target position. The repeatability performance index is adopted to evaluate if the joint positions are repetitive when the manipulator execute repetitive trajectories in the operational workspace. Redundant and hyper-redundant planar manipulators reveal that it is possible to choose in a large range of possible values of a in order to get repetitive trajectories in the joint space.

Keywords:
Redundant manipulators, Hyper-redundant manipulators, Robots, Kinematics, Fractional calculus, Trajectory planning

1. Introduction

Kinematic redundancy occurs when a manipulator possesses more degrees of freedom than the required to execute a given task.

Many techniques for solving the kinematics of redundant manipulators that have been suggested control the end-effector indirectly, through the rates at which the joints are driven, using the pseudoinverse of the Jacobian (see, for instance, [1,2]). The pseudoinverse of the Jacobian matrix guarantees an optimal reconstruction of the desired end-effector velocity—in the least-squares sense—with the minimum-norm joint velocity. However, even though the joint velocities are instantaneously minimized, there is no guarantee that the kinematic singularities are avoided [3]. Moreover, this method has the generally undesirable property that repetitive end-effector motions do not necessarily yield repetitive joint motions. Klein and Huang [4] were the first to observe this phenomenon for the case of the pseudoinverse control of a planar three-link manipulator. Baillieul [5] proposed a modified Jacobian matrix called the extended Jacobian matrix. The extended Jacobian is a square matrix that contains the additional information necessary to optimize a certain function. The inverse kinematic solutions are obtained through the inverse of the extended Jacobian. The algorithms, based on the computation of the extended Jacobian matrix, have a major advantage over the pseudoinverse techniques, because they are locally cyclic [6]. A large volume of research has been produced in the last few years in this topic [7–10]. For example, Zhang et al. [11] solve the joint angle drift problem by means of a dual-neural-network based quadratic-programming approach.

Fractional calculus (FC) is a natural extension of the classical mathematics. In fact, since the beginning of theory of differential and integral calculus, several

mathematicians investigated the calculation of noninteger order derivatives and integrals. Nevertheless, the application of FC has been scarce until recently, but the recent scientific advances motivated a renewed interest in this field [12–14].

In this paper, we proposed a modified algorithm to solve the inverse kinematics problem of redundant manipulator, which uses a fractional derivative approach to control the joint positions. Having these ideas in mind, the paper is organized as follows. Sections 2, introduces the fundamentals of the kinematics of redundant manipulators and some basic theory in what concerns the FC. Based in these concepts, Section 3 presents the proposed method for robot trajectory control. Section 4 presents the general conditions of the experiments and the performance measures used to validate the proposed method. In Section 5, the simulation results obtained in various experiments are presented and discussed. Finally, Section 6 draws the main conclusions.

2. Preliminary concepts

In this section are introduced the fundamentals of the kinematics of redundant manipulators and some basic theory in what concerns the FC, used in the following sections.

2.1. Kinematics of redundant manipulators

We consider a manipulator with n degrees of freedom whose joint variables are denoted by $\mathbf{q} = [q_1; q_2; \ldots; q_n]^T$. We assume that the class of tasks we are interested in can be described by m variables, $\mathbf{x} = [x_1; x_2; \ldots; x_m]^T$, $m < n$, and that the relation between $\mathbf{q}$ and $\mathbf{x}$ is given by the direct kinematics:

$$\mathbf{x} = f(\mathbf{q}) \tag{1}$$

Differential kinematics of robot manipulators was introduced by Whitney [15] that proposed the use of differential relationships to solve for the joint motion from the Cartesian trajectory of the end-effector. Differentiating (1) with respect to time yields:

$$\dot{\mathbf{x}} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} \tag{2}$$

where $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{q} \in \mathbb{R}^n$ and $\mathbf{J}(\mathbf{q}) = \partial f(\mathbf{q})/\partial \mathbf{q} \in \mathbb{R}^{m \times n}$. Hence, it is possible to calculate a path $\mathbf{q}(t)$ in terms of a prescribed trajectory $\mathbf{x}(t)$ in the operational space. We assume that the following condition is satisfied:

$$\max \operatorname{rank}\{\mathbf{J}(\mathbf{q})\} = m \tag{3}$$

Failing to satisfy this condition usually means that the selection of manipulation variables is redundant and the number of these variables m can be reduced. When condition (3) is verified, we say that the degree of redundancy of the manipulator is $n-m$. If, for some $\mathbf{q}$ we have:

$$rank\{\mathbf{J}(\mathbf{q})\} < m \tag{4}$$

then the manipulator is in a singular state. This state is not desirable because, in this region of the trajectory, the manipulating ability is very limited.

Eq. (2) can be inverted to provide a solution in terms of the joint velocities:

$$\dot{\mathbf{q}} = \mathbf{J}^{\#}(\mathbf{q})\dot{\mathbf{x}} \tag{5}$$

where $\mathbf{J}^{\#}$ is the Moore–Penrose generalized inverse of the Jacobian $\mathbf{J}$ [2,16].

It can be easily shown that a more general solution to Eq. (2) is given by:

$$\dot{\mathbf{q}} = \mathbf{J}^{\#}(\mathbf{q})\dot{\mathbf{x}} + [\mathbf{I} - \mathbf{J}^{\#}(\mathbf{q})\mathbf{J}(\mathbf{q})]\dot{\mathbf{q}}_0 \tag{6}$$

where $\mathbf{I} \in \mathbb{R}^{n \times n}$ is the identity matrix and $\dot{\mathbf{q}}_0 \in \mathbb{R}^n$ is an arbitrary joint velocity vector. The solution (6) is composed of two terms. The first term is relative to minimum norm joint velocities. The second term, the *homogeneous solution*, attempts to satisfy the additional constraints specified by $\mathbf{q}_0$. Moreover, the matrix $\mathbf{I} - \mathbf{J}^{\#}(\mathbf{q})\mathbf{J}(\mathbf{q})$ allows the projection of $\dot{\mathbf{q}}_0$ in the null space of $\mathbf{J}$. A direct consequence is that it is possible to generate internal motions that reconfigure the manipulator structure without changing the gripper position and orientation [2–17].

Nakamura and Hanafusa [18] proposed a least squares formulation with a damping factor under the name of singularity-robust inverse of Jacobian matrix, to provide continuous and feasible solutions even at or in the neighborhood of singular points. The method compromises between the accuracy with which the desired end-effecter is followed $\|\dot{\mathbf{x}} - \mathbf{J}\dot{\mathbf{q}}\|$ and feasibility of the joint velocities $\|\dot{\mathbf{q}}\|$ of the inverse kinematic solution, by tacking the inverse kinematic problem as:

$$\min\{\|\dot{\mathbf{x}} - \mathbf{J}\dot{\mathbf{q}}\| + k^2\|\dot{\mathbf{q}}\|^2\} \tag{7}$$

where k is known as the damping factor, rather than finding the minimum vector $\dot{\mathbf{q}}$ that gives the best solution to Eq. (2). The solution is given by:

$$\dot{\mathbf{q}}^*_{k^2} = \mathbf{J}^*\dot{\mathbf{x}} = \mathbf{J}^T(\mathbf{J}\mathbf{J}^T + k^2\mathbf{I})^{-1}\dot{\mathbf{x}} \tag{8}$$

where $\mathbf{J}^*$ is the singularity robust inverse and k^2 determines the weighting between the exactness and the feasibility. The limitations of the method are that the damped factor is tuned by trial and error and its optimal value depends on the operating conditions (high damping factors give good behaviour but reduced accuracy in the neighbourhood of singular points).

In the closed-loop pseudoinverse (CLP) method the joint positions can be computed through the time integration of the expression:

$$\Delta\mathbf{q} = \mathbf{J}^{\#}(\mathbf{q})\Delta\mathbf{x} \tag{9}$$

where $\Delta\mathbf{x} = \mathbf{x}_{ref} - \mathbf{x}$ and $\mathbf{x}_{ref}$ is the vector of reference position in the operational space. Nevertheless, in a previous study, addressing the CLP method [19], it was concluded that this method leads to unpredictable, not repeatable, arm configurations and reveals properties resembling those that occur in chaotic systems. As a consequence, the motion in joint space becomes unpredictable for subsequent cycles.

2.2. Introduction to fractional calculus

FC goes back to the beginning of the theory of differential calculus. Nevertheless, the application of FC just emerged in the last two decades, due to the progress in the area of nonlinear and complex systems that revealed subtle relationships with the FC concepts. In the field of dynamical systems theory some work has been carried out, but the proposed models and algorithms are still in a preliminary stage of establishment.

The fundamental aspects of FC theory are addressed in [20–22]. Concerning FC applications research efforts can be mentioned in the area of viscoelasticity, chaos, biology, signal processing, diffusion, wave propagation, percolation, modeling and control [23–27].

FC is a branch of mathematical analysis that extends to real, or even complex, numbers the order of the differential and integral operators. Since its foundation, the generalization of the concept of derivative and integral to a non-integer order a has been the subject of distinct approaches. Due to this reason there are several alternative definitions of fractional derivatives. An approach, based on the concept of fractional differential, is the Grunwald–Letnikov definition given by the equation:

$$ {}_aD_t^{\alpha}[x(t)] = \lim_{\Delta t \to 0}\left[\frac{1}{(\Delta t)^{\alpha}} \sum_{k=0}^{[(t-a)/\Delta t]} x(t-k\Delta t)\gamma(\alpha,k)\right] \quad (10a) $$

$$ \gamma(\alpha,k) = (-1)^k \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \quad (10b) $$

where $\Gamma()$ is the Gamma function.

An important property revealed by expression (10) is that while an integer-order derivative just implies a finite series, the fractional-order derivative requires an infinite number of terms. Therefore, integer derivatives are 'local' operators in opposition with fractional derivatives which have, implicitly, a 'memory' of all past events.

Analyzing (10b) we verify [28] that the series coefficients decay very slowly. Therefore, the samples of the past have a considerable impact upon the calculation of

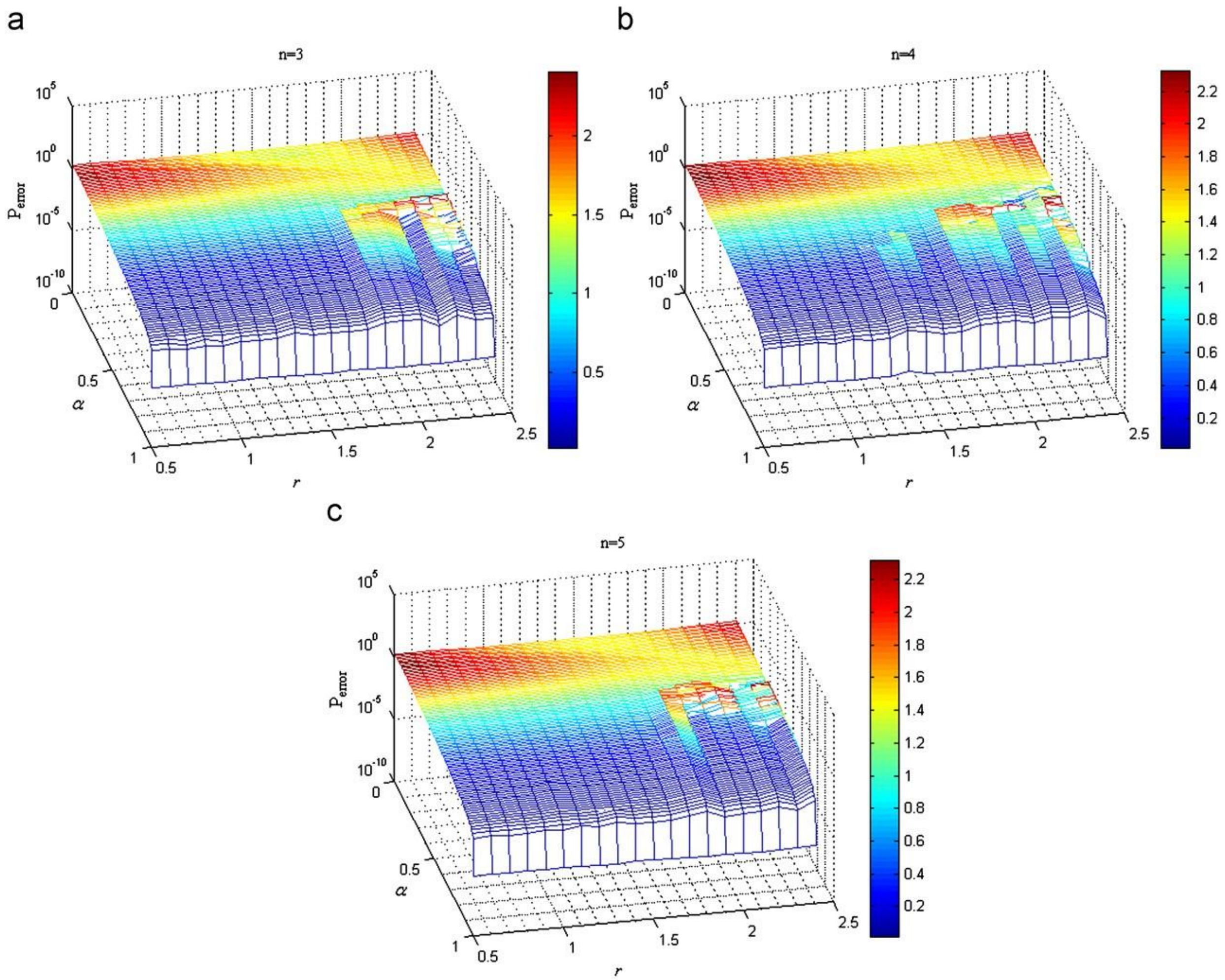

Fig. 1. Charts of $\overline{P}_{error}$ for the nR-robot, n ¼ f3; 4; 5g, vs a and r, with n_C ¼ 300 cycles.

the present value of the fractional derivative. This property leads to a signal variation much more conservative than what we obtain for the case of integer order.

Often, in discrete time implementations expression (10) is approximated by:

$$D^{\alpha}[x(t)] = \frac{1}{(\Delta t)^{\alpha}} \sum_{k=0}^{N} \gamma(\alpha,k)x(t-k\Delta t) \tag{11}$$

where Δt is the sampling period and N is the truncation order.

The characteristics revealed by fractional-order models make this mathematical tool well suited to describe phenomena such as chaos because of its inherent memory property. In this line of thought, the propagation of perturbations and the appearance of long-term dynamic phenomena configure a case where FC tools fit adequately [11].

3. Proposed method for robot trajectory control

In this section we formulate a new method for the trajectory control of a redundant manipulator. The proposed method combines the CLP method with FC, namely the fractional closed-loop pseudoinverse (F-CLP) method.

If $\mathbf{x}_{ref}$ is the vector of reference position in the operational space and $\mathbf{x}(t)$ is a vector representing the current position of the end-effecter in the operational space, then expressions (12)–(13) are the discrete versions of the differential and the integral of order $\alpha = 1$,

$$\Delta\mathbf{x}(t) = \mathbf{x}_{ref} - \mathbf{x}(t-\Delta t) \tag{12}$$

$$\mathbf{q}(t) = \Delta\mathbf{q}(t) + \mathbf{q}(t-\Delta t) \tag{13}$$

These equations yield the standard (integer-order) differential kinematic trajectory planning CLP and inspired the formulation of a fraction-order kinematic scheme. Therefore, in order to take advantage from the longer memory effect provided by (10), in the F-CLP method, expression (12) can be rewritten leading to expression (14):

$$\begin{aligned}\Delta^{\alpha}\mathbf{x}(t) = \mathbf{x}_{ref} &- \gamma(\alpha,1)\mathbf{x}(t-\Delta t) - \gamma(\alpha,2)\mathbf{x}(t-2\Delta t) \\ &- \cdots - \gamma(\alpha,N)\mathbf{x}(t-N\Delta t)\end{aligned} \tag{14}$$

when the first N terms are considered.

Using a fractional perspective, in the F-CLP method, the joint positions are also computed through the time

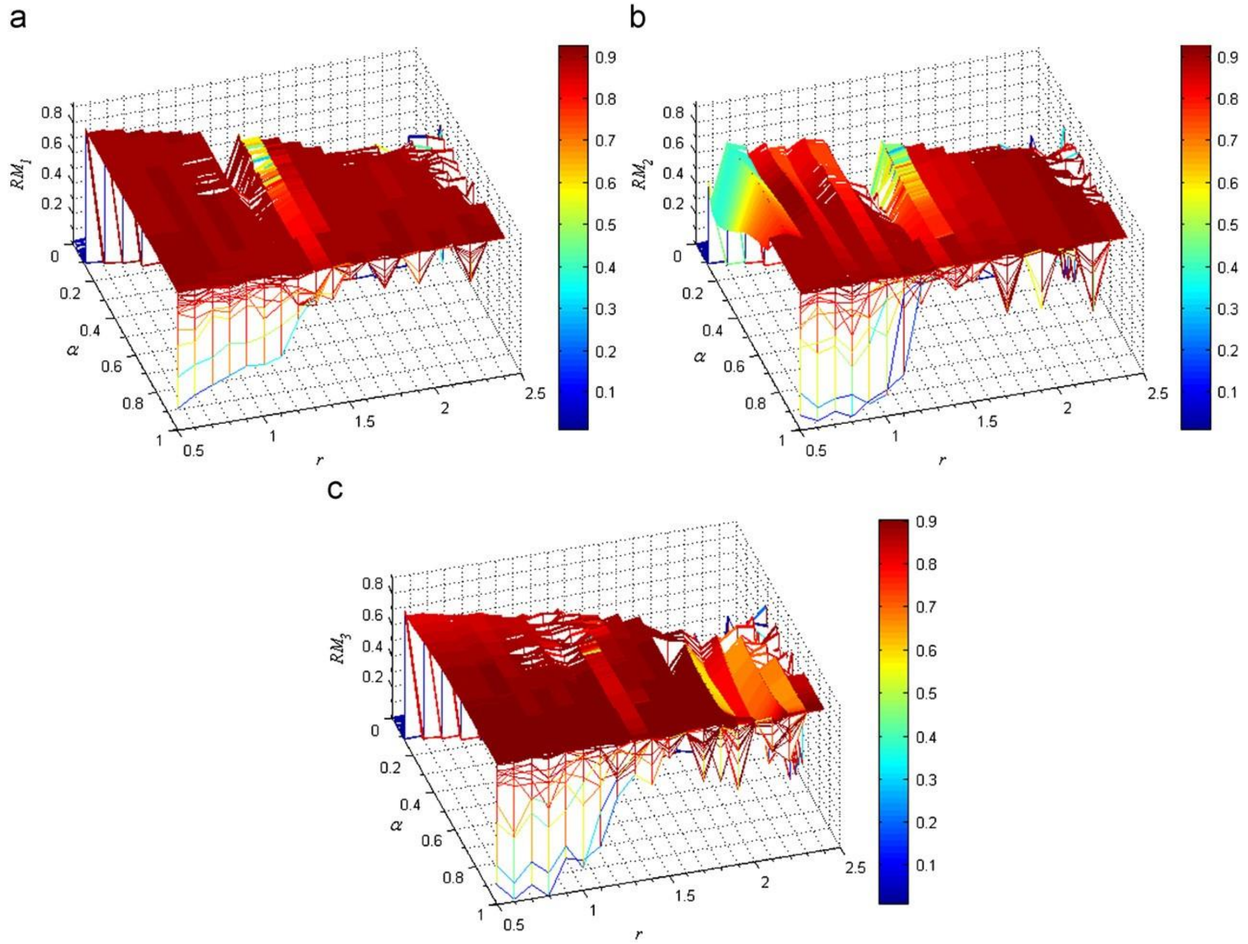

Fig. 2. Charts of RM_j, $j=1,\ldots,3$, for the 3R-robot, vs α and r, respectively, with $n_C = 300$ cycles.

integration of the joint velocities, given by expression (15):

$$\mathbf{q}(t) = \Delta\mathbf{q}(t) + \gamma(\alpha,1)\mathbf{q}(t-\Delta t) + \gamma(\alpha,2)\mathbf{q}(t-2\Delta t) + \cdots + \gamma(\alpha,N)\mathbf{q}(t-N\Delta t) \quad (15)$$

Clearly, when $\alpha = 1$ we get the classical CLP method.

4. General definition of the experiments and performance measures

The Jacobian of a n-link planar manipulator (i.e., $m = 2$) has a simple recursive nature according with the expressions:

$$\mathbf{J} = \begin{bmatrix} -l_1 S_1 - \ldots - l_n S_{1\ldots n} & \ldots & -l_n S_{1\ldots n} \\ l_1 C_1 + \ldots + l_n C_{1\ldots n} & \ldots & l_n C_{1\ldots n} \end{bmatrix} \quad (16)$$

where l_i is the length of link i, $q_{i\ldots k} = q_i + \cdots + q_k$, $S_{i\ldots k} = Sin(q_{i\ldots k})$ and $C_{i\ldots k} = Cos(q_{i\ldots k})$, $i,k \in \mathbb{N}$.

The experiments consist in the analysis of the kinematic performance of a planar manipulator with $n = \{3,4,5\}$ rotational joints, denoted as nR-robot, that is required to repeat a circular motion in the operational space with frequency $\omega_0 = 7.0\,\text{rad}\,\text{s}^{-1}$, center at $r = (x_1^2 + x_2^2)^{1/2}$, radius $\rho = 0.5$ and a step time increment of $\Delta t = 10^{-3}$ s. Without lacking of generality, in the experiments are adopted arms having identical link lengths, $l_1 = l_2 = \cdots = l_n$.

Two performance metrics are defined to examine the strength and the weakness of the proposed method. The positional error measure, P_{error}, is used to measure the precision of the manipulator in the task of positioning the end-effector at the target position. The repeatability performance measure, RM, is used to evaluate if the joint positions are periodic when the manipulator execute repetitive trajectories in the workspace.

4.1. Positional error measure

In order to analyze the precision of the manipulator in the task of positioning the end-effector at the target position, we define a measure based on the positional error at each instant time.

The average of the positional error for n_C cycles is given by the expression:

$$\overline{P}_{error} = \frac{\sum_{i=1}^{k} \sqrt{(x_c - x_f)^2 + (y_c - y_f)^2}}{k} \quad (17)$$

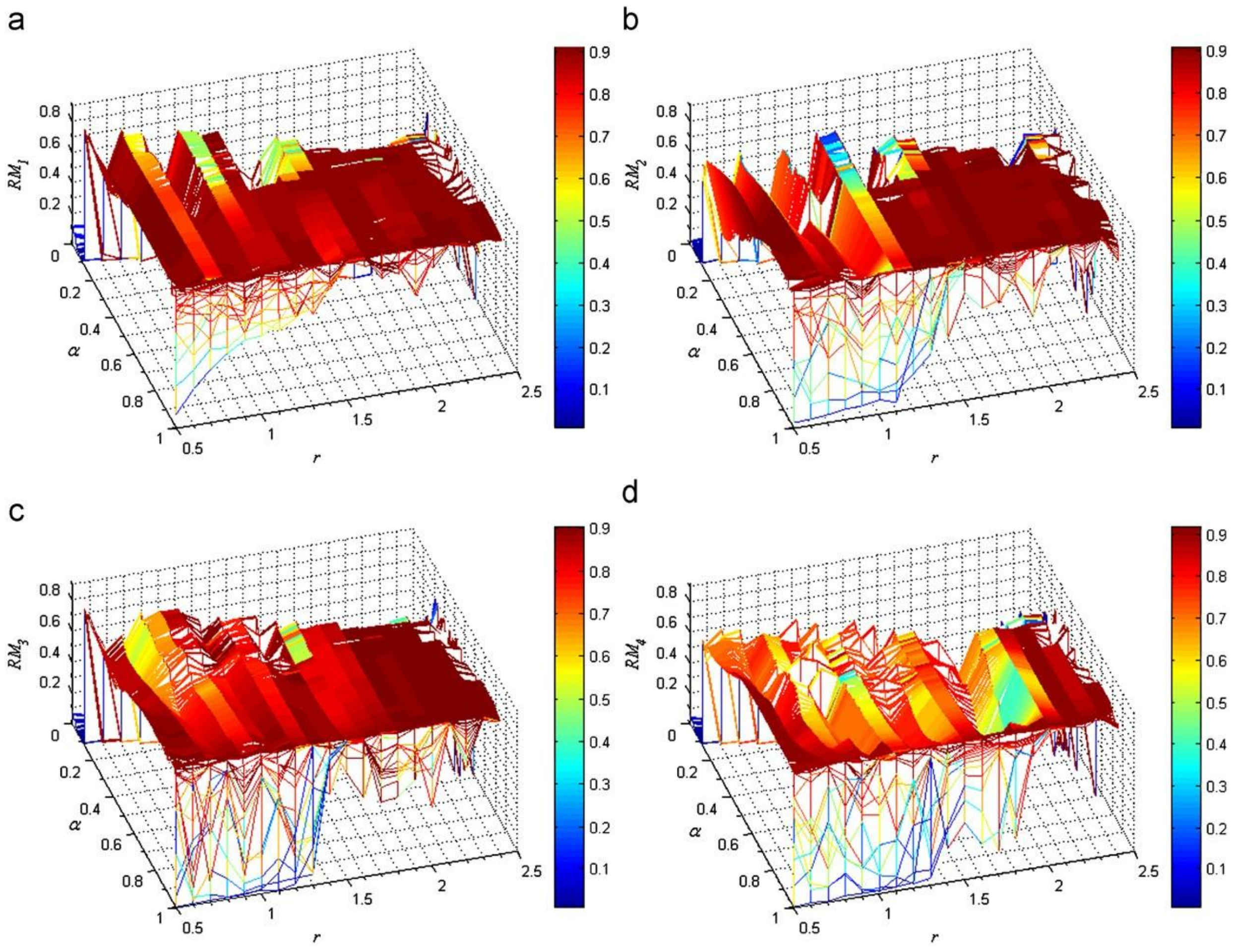

Fig. 3. Charts of RM_j, $j = 1,\ldots,4$, for the 4R-robot, vs α and r, respectively, with $n_C = 300$ cycles.

where $\mathbf{x}_c = (x_c; y_c)$ and $\mathbf{x}_f = (x_f; y_f)$ are vectors representing the end-effector current position and the desired final position, respectively, and k is the number of sampling points and is defined as:

$$k = \frac{2\pi}{\omega_0 \Delta t} n_C \tag{18}$$

4.2. Repeatability performance measure

In order to analyze the repeatability characteristic of the joint positions, we define a measure based on the Fourier analysis of the robot joint velocities. This repeatability measure, *RM*, evaluates the distribution of the energy along the frequencies $\omega \in [\omega_{min}; \omega_{max}]$,

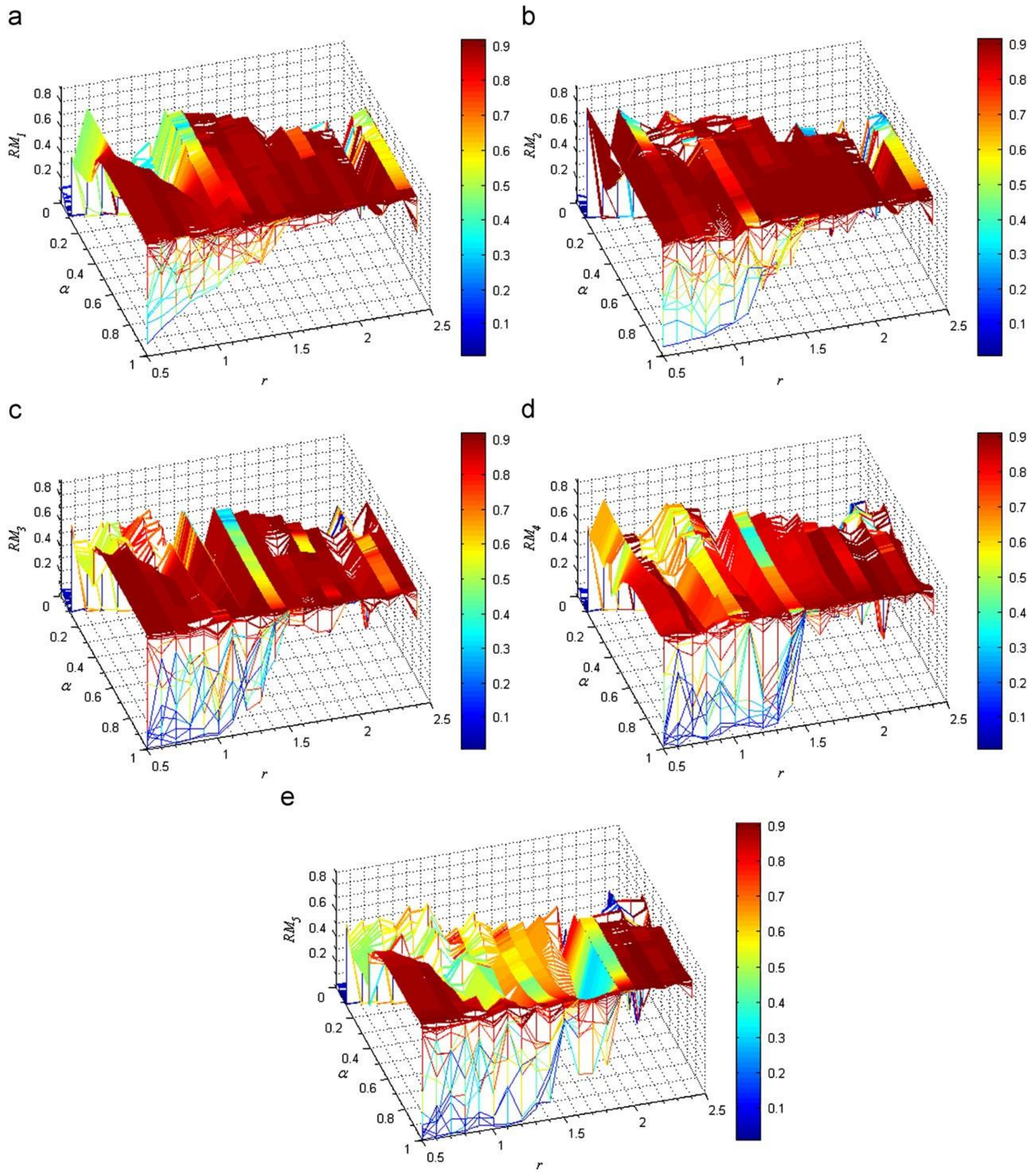

Fig. 4. Charts of RM_j, $j=1,\ldots,5$, of the 5*R*-robot, vs α and *r*, respectively, with $n_C = 300$ cycles.

$\omega_{min} = 0.0\,\mathrm{rad\,s^{-1}}$, $\omega_{max} = n_h\omega_0\,\mathrm{rad\,s^{-1}}$, with a step frequency increment of $\Delta\omega = 0.005\,\mathrm{rad\,s^{-1}}$, where n_h is the number of multiple harmonics and ω_0 is the fundamental frequency. For joint j the index RM, is defined as:

$$RM_j = \frac{E_{j,H}}{E_{j,T}} \tag{19}$$

where $E_{j,H}$ is the energy concentrated in the fundamental and multiple higher harmonics for joint velocity $\dot{q}_j$, and is defined as:

$$E_{j,H} = \sum_{i=0}^{n_h} |F\{\dot{q}_j(t)\}(\omega_0 + i\Delta\omega)| \tag{20}$$

and $E_{j,T}$ is the total energy for the joint velocity $\dot{q}_j$, defined as:

$$E_{j,T} = \sum_{i=0}^{n_\omega} |F\{\dot{q}_j(t)(i\Delta\omega)\}| \tag{21}$$

where n_ω is the number of sampling frequencies:

$$n_\omega = \frac{\omega_{max} - \omega_{min}}{\Delta\omega} \tag{22}$$

5. Simulation results and discussion

This section presents the results of several simulations and discusses the results. In the experiments are considered the first $N = 10$ terms of expression (11). Larger values of N were tested leading to results of the same type. In the motion are considered $n_C = 300$ cycles.

5.1. The performance error

In the following experiments are considered robots with $n = \{3; 4; 5\}$ rotational joints and circular trajectories at radial distances $0.5 \le r \le 2.4$.

The average of the positional error, $\overline{P_{error}}$, for $0.01 \le \alpha \le 1.0$ is depicted in Fig. 1. The case $\alpha = 1.0$ corresponds to the classical CLP method.

We conclude that:

(i) if $\alpha < 0.8$, in general, the precision is better the higher the value of α, but for some radial distances occur slightly variations;
(ii) if $\alpha > 0.8$ the precision is always better the higher the value of α, having a maximum at $\alpha = 1.0$;
(iii) these results are identical for any number of joints.

5.2. The repeatability performance

The repeatability performance measure, RM_j, $j = 1; \ldots; n$, for $n = \{3; 4; 5\}$ rotational joints, $n_h = 5$, radial distances from $0.5 \le r \le 2.4$ and values of α ranging $0.01 \le \alpha \le 1.0$ is depicted in Figs. 2–4. As stated previously for $\alpha = 1.0$ we get the well-known CLP.

We conclude that, for all joints and for the different robot with $n = \{3; 4; 5\}$, that the charts are of the same type. In all cases we have a region of low performance for low values of α, followed by a plateau with almost constant performance, up to a sudden degradation in the close neighborhood of $\alpha = 1.0$. The region of low performance for low α gets larger for higher values of r, but leaves still a considerable region of good behavior. Moreover, we verify also that the performance diminishes slightly for larger values of r and that there are some regions $(\alpha; r)$ with some small ripple in the chart. Nevertheless, these two effects are of minor importance, allowing the designer to choose in a large range of possible values of α.

For example, if the radial distance is $r \ge 1.1$, the classical CLP method leads to unpredictable, non repeatable, arm configurations. If we want to achieve a repetitive joint trajectory we can adopt, for example, $\alpha = 0.99$, for which we get a repeatability performance measure as $RM > \{0.88; 0.71; 0.79\}$, for $n = \{3; 4; 5\}$, respectively.

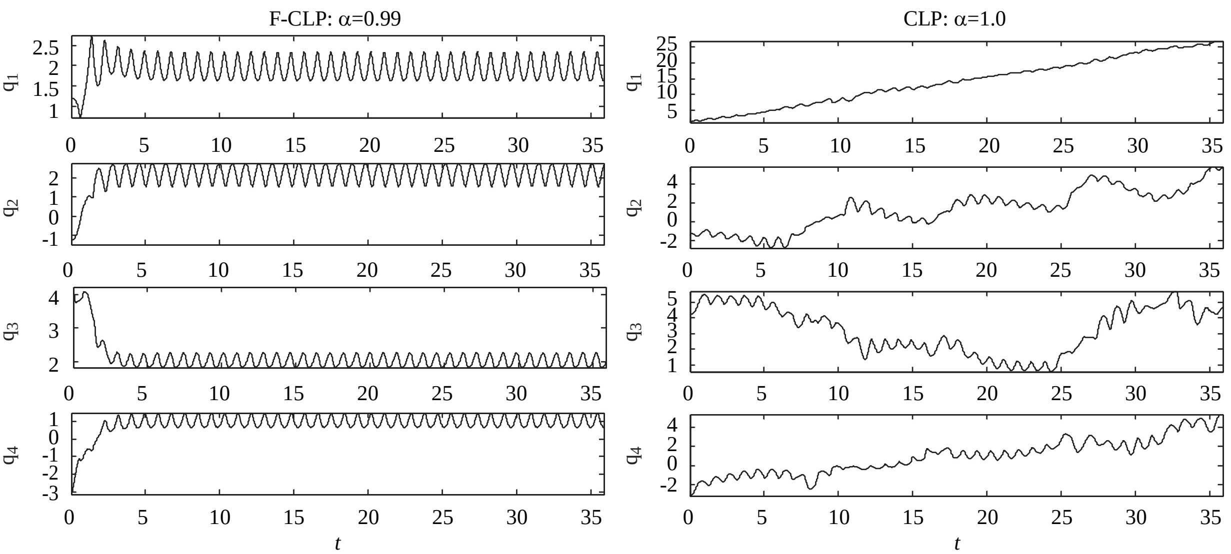

Fig. 5. Charts of the 4R-robot joint positions versus time, for $r = 0.7$, $\alpha = 0.99$ (F-CLP) and $\alpha = 1.0$ (CLP), respectively, with $n_C = 40$ cycles.

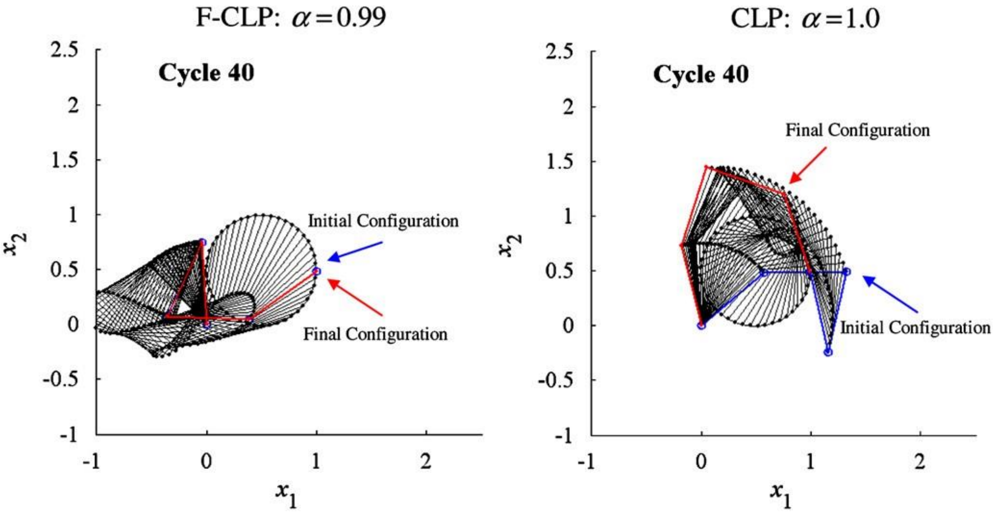

Fig. 6. Charts of the 4R-robot initial and final configurations, for $r = 0.7$, $\alpha = 0.99$ (F-CLP) and $\alpha = 1.0$ (CLP), respectively, during the 40th cycle.

To analyze more deeply this situation, in Fig. 5 are depicted the robot joint positions for $n_C = 40$ cycles and in Fig. 6 are represented the initial and final configurations during the 40th cycle, for $n = 4$, $\alpha = \{0.99; 1.0\}$ and $r = 0.7$. As we can, for $\alpha = 0.99$, after a transient phase, the joints positions start to be repetitive. However, when $\alpha = 1.0$ (i.e., the classical integer-order CLP) the joint positions are not periodic and the motion in joint space becomes unpredictable.

6. Conclusions

A new algorithm to solve the inverse kinematics problem of redundant manipulator, the F-CLP method, that uses a fractional derivative approach to control the joint positions, was presented. Several experiments were developed to study the performance of the F-CLP, when the manipulator is required to repeat a circular motion in the operational space. Two performance measures were defined to examine the strength and weakness of the proposed method. The positional error measure was adopted to measure the precision of the manipulator in the task of positioning the end-effector at the target position. The repeatability performance measure was used to evaluate if the joint positions are repetitive when the manipulator execute repetitive trajectories in the workspace.

The results show that the degradation of the positional error can be negligible while the gain of good repeatability yields significant advantage. It is shown that for all the radial distances, and for the different robot with $n = \{3; 4; 5\}$, it is possible to find a value of α from which the joint positions are repetitive.

The F-CLP follows classical integer-order trajectory planning scheme. Therefore, no dynamical control issues are considered. Nevertheless, a possible direction of future research is clearly the adoption of further concepts borrowed from fractional-order control algorithms.

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