# On the Fractional Signals and Systems

(Invited paper)

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**Abstract**

A look into Fractional Calculus and their applications from the Signal Processing point of view is done in this paper. A coherent approach to the fractional derivative is presented leading to notions that are, not only compatible with the classic, but constitute a true generalization. This means that the classic are recovered when the fractional domain is left. This happens in particular with the impulse response and transfer function. An interesting feature of the systems is in the causality that the fractional derivative imposes. The main properties of the derivatives and their representations are presented. A brief and general study of the fractional linear systems is done, by showing how to compute the impulse, step and frequecy responses, how to test the stability and how to insert the initial conditions. The practical realization problem is focussed and it is shown how to perform the input-ouput computations. Some Biomedical applications are described.

## 1 Introduction

Fractional calculus has been attracting the attention of scientists and engineers from long time ago, during this period the main applications involved the using of the so called fractional integral operators to obtain explicit solutions of regular models. However, most of the mentioned development was done by mathematicians [7,18,22,28,30]. Since the nineties of last century fractional calculus has been rediscovered and applied in an increasing number of fields, namely in several areas of Physics [19,20,25,33,81,82,85,86,117-119,133,169,171], control engineering [25,29,36,37,50,107,115,132, 150,152,178-180], and signal processing [38,39,82,83,150,163,173]. A complete theory of the linear systems of fractional differential equation with constants or variables coefficients can be found in the literature [18,20,22,28,30]. On the other hand, we must remark that in the 80% of the papers that appear in the Scientifics literature, in the framework of the fractional calculus and their applications, the corresponding author use different fractional differential operators but at the end they contrast their model using a numerical approach based in a finite number of terms of the series that define the known Grünwald-Letnikov derivative [30]. Then they obtain excellent result. Therefore we can conclude that a generalization of the linear systems of differential equation it is very useful to be used in modeling much process [28,137]. Later he showed that the same results could be obtained by using as starting point the Grünwald-Letnikov derivative [142]. This theory was updated recently [148]. With this approach a linear system theory can be formulated in a fashion very similar to the classic, being effectively a generalization in the sense of obtaining the classic results when the order become integer. This theory will be revised here, taking in account recent developments. We will consider the associated problems: establishment of the initial conditions and the stability of the systems. It is intended to present here a self-contained theory suitable for dealing with problems like: filtering, modeling and realization.

As referred above the number of applications has been increasing. One the areas where such can be verified is the Biomedical [20,45,46,67,88]. Here we describe some of the recent applications in this field. The now classic fractional Brownian motion (fBm) modeling is also considered, as an application of the fractional calculus [21,105,121,146]. We define a fractional noise that is obtained through a fractional derivative of white noise. The fBm is an integral of the fractional noise.

The paper is organized as follows. In section 2, we present the Grünwald-Letnikov fractional derivative and its main properties and relations with other fractional derivatives like the called the Riemann-Liouville derivative and the Caputo derivative. Some examples of derivative computations are shown. The practical implementations and simulation are also considered. The introduction of the fractional linear systems is done in section 3. We define transfer function and impulse response and show how to compute it. The stability and the establishment of the correct initial conditions are also studied. The continuous to discrete conversion is considered in 3.5. In 3.6 In section 4 we describe some applications in Biomedical Engineering. To finish, we study the fractional Brownian motion and present some conclusions.

Remarks: 1 - In this paper we deal with a multivalued expression $z^\alpha$. As is well known, to define a function we have to fix a branch cut line and choose a branch (Riemann surface). It is a common procedure to choose the negative real half-axis as branch cut line. Unless stated the contrary, in what follows we will assume that we adopt the principal branch and assume that the obtained function is continuous above the branch cut line. With this, we will write $(-1)^\alpha = e^{j\alpha\pi}$. 2 - Otherwise stated, we will assume to be in the context of the generalised functions (distributions). We always assume that they are either of exponential order or tempered distributions.

## 2 Fractional derivative

### 2.1 Definitions

Similarly to the classic case, it was introduced the known Grünwald-Letnikov definition of fractional differential equation. We here introduce the following modification of the mentioned fractional derivative by the limit of the fractional incremental ratio [148]

$$D_\theta^\alpha f(z) = e^{-j\theta\alpha} \lim_{|h|\to 0} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z-kh)}{|h|^\alpha} \tag{1}$$

where $\binom{\alpha}{k}$ stands for the binomial coefficients and $h = |h|e^{j\theta}$ is a complex number, with $\theta \in (-\pi,\pi]$. The above definition is valid for any order, real or complex [70]. In order to understand and give an interpretation to the above formula, assume that z is a time and that h is real, $\theta = 0$ or $\theta = \pi$. If $\theta = 0$, only the present and past values are being used, while, if $\theta = \pi$, only the present and future values are used. This means that if we look at (1) as a linear system, the first case is causal, while the second is anti-causal[1] [142].

In general, if $\theta = 0$, we call (1) the forward Grünwald-Letnikov derivative

$$D_f^\alpha f(z) = \lim_{h\to 0+} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z-kh)}{h^\alpha} \tag{2}$$

[1] We will return to this subject later.

If $\theta = \pi$, we put h=-|h| to obtain the backward Grünwald-Letnikov derivative

$$D_b^{\alpha} f(z) = \lim_{h \to 0+} e^{-j\pi\alpha} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z + kh)}{h^{\alpha}} \tag{3}$$

It is important to enhance an interesting fact: when $\alpha$ is a positive integer we obtain the classic expressions for the integer order derivatives.

**2.2 Existence**

It is not a simple task to formulate the weakest conditions that ensure the existence of the fractional derivatives (1), (2) and (3), although we can give some necessary conditions for their existence. To study the existence conditions for the fractional derivatives we must care about the behaviour of the function along the half straight-line $z \pm nh$ with $n \in Z^{+}$. If the function is zero for $Re(z) < a \in R$ (resp. $Re(z) > a$) the forward (backward) derivative exists at every finite point of f(z). In the general case, we must have in mind the behavior of the binomial coefficients. They verify

$$\left| \binom{\alpha}{k} \right| \leq \frac{A}{k^{\alpha+1}}$$

meaning that $f(z).\frac{A}{k^{\alpha+1}}$ must decrease, at least as $\frac{A}{k^{|\alpha|+1}}$ when k goes to infinite. For example considering the forward case, if $\alpha > 0$, it is enough that f(z) be bounded in the left half plane, but if $\alpha < 0$, f(z) must decrease to zero to obtain a convergent series. In particular, this suggests that $Re(h) > 0$ and $Re(h) < 0$ should be adopted for right and left functions ([2]), respectively in agreement with Liouville reasoning [10]. In particular, they should be used for the functions such that f(z)=0 for Re(z)<0 and f(z)=0 for Re(z)>0, respectively [3]. This is very interesting, since we conclude that the existence of the fractional derivative depends only on what happens in one half complex plane, left or right. Consider $f(z) = z^{\beta}$, with $\beta \in R$ with a suitable branch cut line. If $\beta > \alpha$, we conclude immediately that $D^{\alpha}[z^{\beta}]$ defined for every $z \in C$ does not exist, unless $\alpha$ is a positive integer, because the summation in (1) is divergent.

[2] We say that *f*(z) is a right [left] function if $f(-\infty) = 0$ [$f(+\infty) = 0$].

[3] By breach of language we call them causal and anti-causal functions borrowing the system terminology.

### 2.3 Main Properties

We are going to present the main properties of the derivative above presented.

#### 2.3.1 Linearity

The linearity property of the fractional derivative is evident from the above formulae. In fact, we have

$$D_\theta^\alpha[f(z) + g(z)] = D_\theta^\alpha f(z) + D_\theta^\alpha g(z) \tag{4}$$

#### 2.3.2 Causality

The causality property was already referred to above and can also be obtained easily. We only have to use (2), or (3). Assume that $t=z\in R$ and that $f(t) = 0$, for $t < 0$, we conclude immediately from (2) that $D_f^\alpha f(t) = 0$ for $t<0$. For the anti-causal case, the situation is similar.

#### 2.3.3 Scale change

Let $f(z) = g(az)$, where $a=|a|e^{j\varphi}$ is a constant. From (1), we have:

$$D_\theta^\alpha g(az) = |a|^\alpha e^{-j(\theta+\varphi)\alpha} \lim_{|h|\to 0} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} g(az - kah)}{|ah|^\alpha} = |a|^\alpha D_\theta^\alpha g(\tau)\big|_{\tau=az} \tag{5}$$

#### 2.3.4 Time reversal

If $f(z) = g(-z)$, we obtain from the property we just deduced that:

$$D_\theta^\alpha g(-z) = (-1)^\alpha \lim_{h\to 0} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} g(-z + kah)}{(-h)^\alpha} = (-1)^\alpha D_\theta^\alpha g(\tau)\big|_{\tau=-z} \tag{6}$$

in agreement with (2) and (3). This means that the time reversal converts the forward derivative into the backward and vice-versa.

#### 2.3.5 Time shift

The derivative operator is shift invariant:

$$D_\theta^\alpha g(z - a) = D_\theta^\alpha g(\tau)\big|_{\tau=z-a} \tag{7}$$

### 2.3.6 Derivative of a product

We are going to compute the derivative of the product of two functions: $f(t) = \varphi(t).\psi(t)$ assumed to be defined for $t\in R$, by simplicity, although the result we will obtain is valid for $t\in C$, excepting over an eventual branch cut line. Assume that one of them is analytic in a given region. We obtain the derivative of the product [18,22-28,30]:

$$D^{\alpha}[\varphi(t)\psi(t)] = \sum_{n=0}^{\infty} \binom{\alpha}{n} \varphi^{(n)}(t)\psi^{(\alpha-n)}(t) \tag{8}$$

that is the generalized Leibniz rule. This rule gives us a curious result when $\alpha$ is a negative integer and $\psi(t) = 1$. For example, if $\alpha = -1$, we obtain

$$D^{-1}[\varphi(t)] = \sum_{n=0}^{\infty} (-1)^{n} \varphi^{(n)}(t) \frac{t^{n+1}}{(n+1)!}$$

similar to the McLaurin series and can be useful in computing the primitive of some functions.

## 2.4 Group structure of the fractional derivative

### 2.4.1 Additivity and Commutativity of the orders

We are going to apply (1) twice for two orders. We have [140]

$$D_{\theta}^{\alpha}\left[D_{\theta}^{\beta}f(t)\right] = D_{\theta}^{\beta}\left[D_{\theta}^{\alpha}f(t)\right] = D_{\theta}^{\alpha+\beta}f(t) \tag{9}$$

### 2.4.2 Associativity

This property comes easily from the above results. In fact, it is easy to show that

$$D_{\theta}^{\gamma}\left[D_{\theta}^{\alpha+\beta}f(t)\right] = D_{\theta}^{\gamma+\alpha+\beta}f(t) = D_{\theta}^{\alpha+\beta+\gamma}f(t) = D_{\theta}^{\alpha}\left[D_{\theta}^{\beta+\gamma}f(t)\right] \tag{10}$$

### 2.4.3 Neutral element

If we put $\beta = -\alpha$ in (10) we obtain:

$$D_{\theta}^{\alpha}\left[D_{\theta}^{-\alpha}f(t)\right] = D_{\theta}^{0}f(t) = f(t) \tag{11}$$

or again by (10)

$$D_{\theta}^{-\alpha}\left[D_{\theta}^{\alpha}f(t)\right] = D_{\theta}^{0}f(t) = f(t) \tag{12}$$

This is very important because it states the existence of inverse.

### 2.4.4 Inverse element

From the last result we conclude that there is always an inverse element: for every $\alpha$ order derivative, there is always a $-\alpha$ order derivative. This seems to be contradictory with our knowledge from the classic calculus where the $N^{th}$ order derivative has N primitives. To understand the situation we must refer that the inverse is given by (1) and that it does not have any primitivation constant. This forces us to be consistent and careful with the used language. So, when $\alpha$ is positive we will speak of the operator as a derivative. When $\alpha$ is negative, we will use the term anti-derivative or primitive (not integral). This clarifies the situation.

## 2.5 Simple examples

### 2.5.1 The exponential

Let us apply the above definitions to the function $f(z) = e^{sz}$. The convergence of (1) is dependent of s and of h. Let h > 0, the series in (2) becomes

$$e^{sz} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} e^{-ksh}$$

The binomial series

$$\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} e^{-ksh}$$

is convergent to the main branch of

$$F(s) = (1 - e^{-sh})^{\alpha}$$

provided that $|e^{-sh}|<1$, that is if $\mathrm{Re}(s) > 0$. This means that the branch cut line of F(s) must be in the left hand half of the complex plane. Then

$$D_f^{\alpha} f(z) = \lim_{h\to 0+} \frac{(1 - e^{-sh})^{\alpha}}{h^{\alpha}} e^{sz} = \lim_{h\to 0+} \left(\frac{1 - e^{-sh}}{h}\right)^{\alpha} e^{sz} = |s|^{\alpha} e^{j\theta\alpha} e^{sz} \tag{13}$$

iff $\theta \in (-\pi/2, \pi/2)$ which corresponds to be working with the principal branch of $(\ .\ )^{\alpha}$ and assuming a branch cut line in the left hand complex half plane.

Now, consider the series in (3) with $f(z) = e^{sz}$. Proceeding as above, we obtain another binomial series:

$$\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} e^{ksh}$$

that is convergent to the main branch of

$$F(s) = (1 - e^{sh})^{\alpha}$$

provided that Re(s) < 0. This means that the branch cut line of F(s) must be in the right hand half complex plane. We obtain directly for $f(z) = e^{sz}$

$$D_b^{\alpha} f(z) = |s|^{\alpha} e^{j\theta\alpha} e^{sz}$$

with $\theta \in (3\pi/2,\pi/2)$, and

$$D_b^{\alpha} f(z) = |s|^{\alpha} e^{j\theta\alpha} e^{sz}$$

valid iff $\theta \in (\pi/2,3\pi/2)$. These results can be used to generalize a well known property of the Laplace transform. If we return back to equation (2) and apply the bilateral Laplace transform

$$F(s) = \int_{-\infty}^{+\infty} f(t)\, e^{-st}\, dt \tag{14}$$

to both sides. We conclude that:

$$L\,[D_f^{\alpha} f(t)] = s^{\alpha} F(s) \quad \text{for } Re(s) > 0 \tag{15}$$

where in $s^{\alpha}$ we assume the principal branch and a cut line in the left half plane. With equation (3) we obtain:

$$L\,[D_b^{\alpha} f(t)] = s^{\alpha} F(s) \quad \text{for } Re(s) < 0 \tag{16}$$

where now the branch cut line is in the right half plane. These results have a system interpretation: there are two systems (differintegrators) with the same expression for the transfer function $H(s) = s^{\alpha}$, but different regions of convergence. One is causal and the other is anti-causal. Later we will compute the corresponding impulse responses. The $s = j\omega$ case will be considered later also.

### 2.5.2 The constant function

We are going to compute the fractional derivative of the constant function. Let $f(z) = 1$ for every $z \in C$ and $\alpha \in R \backslash Z^{-}$. We have

$$D_f^{\alpha} f(z) = \lim_{h\to 0} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k}}{h^{\alpha}} = \lim_{h\to 0} \frac{(1-1)^{\alpha}}{h^{\alpha}} = \begin{cases} 0 & \alpha>0 \\ \infty & \alpha<0 \end{cases} \tag{17}$$

The $\alpha$ order fractional derivative of f(z) is the null function. If $\alpha < 0$, equation (17) leads to infinite. So there is no fractional “primitive” of a constant.

**The step and impulse functions**

Let u(t) be the Heaviside unit step function. It can be shown, with some work, that [22]

$$D_f^{\alpha} u(t) = \frac{t^{-\alpha}}{\Gamma(-\alpha+1)} u(t) \tag{18}$$

where u(t) is the Heaviside unit step. Relation (18) allows us to obtain an interesting result

$$D_f^{\alpha} \delta(t) = \frac{t^{-\alpha-1}}{\Gamma(-\alpha)} u(t) \tag{19}$$

valid for non positive integer orders. In terms of linear system theory, (15) tells us that the fractional forward differintegrator (a current terminology) is a linear system with impulse response equal to the right hand side in (19). We could use (3) and obtain the impulse response of the anti-causal differintegrator by starting with u(-t). The procedure is similar and the result is [137]

$$D_b^{\alpha} \delta(t) = -\frac{t^{-\alpha-1}}{\Gamma(-\alpha)} u(-t) \tag{20}$$

The impulse responses (19) and (20) of the causal and anti-causal differintegrators have $s^{\alpha}$ as transfer functions with regions of convergence Re(s) > 0 and Re(s) < 0, respectively.

**The power function**

The general power function does not have fractional derivative as it is easy to observe from (1), because it increases without bound as t goes to $\pm\infty$. This does not happen with the causal (or anti-causal power). The results obtained in the above close section, allows us to obtain the derivative of $t^{\beta}u(t)$. In the sequence of computations in the following we shall be assuming that the exponents in the powers are not negative integers. Using (18) again, we obtain:

$$D_f^{\alpha} t^{\beta} u(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha} u(t) \tag{21}$$

which generalizes the usual formula for $\beta\in\mathbf{Z}$ and $\beta\notin\mathbf{N}^-$. Equation (21) can be considered valid for $\beta-\alpha = -1$ provided that we write

$$D_f^{\alpha} \frac{t^{\alpha-1}u(t)}{\Gamma(\alpha)} = \delta(t) \tag{22}$$

To see that this is correct, we use (18) to obtain

$$D_f^{\alpha} \frac{t^{\alpha-1}u(t)}{\Gamma(\alpha)} = D_f^{\alpha}\, D_f^{-\alpha+1}\, u(t) = D_f^{1}\, u(t) = \delta(t)$$

**2.6 Integral representations**

Above we introduced the elemental system base for the fractional system building: the differintegrator. In (19) and (20) we presented the impulse responses corresponding to the forward and backward cases. This means that the output of the differintegrator is given by the convolution of the input with the (15) or (19). This leads to the integral representations of the fractional derivatives (called Liouville derivatives [10]):

$$D_f^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} f(t-\tau)\, \tau^{-\alpha-1}\, d\tau \tag{23}$$

valid for functions with the Laplace Transform converging in a region that includes the right hand side of the complex plane. As the convolution is commutative we can write also:

$$D_f^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^{t} f(\tau).(t-\tau)^{-\alpha-1}\, d\tau \tag{24}$$

Similarly, we have an anti-causal (backward) derivative valid for functions with Laplace Transform converging in a region that includes the left hand side of the complex plane. It is the backward Liouville derivative obtained from (16) and (20)

$$D_b^{\alpha} f(t) = \frac{(-1)^{-\alpha}}{\Gamma(-\alpha)} \int_0^{\infty} f(t+\tau).\tau^{-\alpha-1}\, d\tau \tag{25}$$

These definitions were introduced both exactly with this format by Liouville [6]. Unhappily in the common literature the factor $(-1)^{-\alpha}$ in (25) has been removed and is called Weyl derivative [22,30]. Although the above results were obtained for functions with Laplace transform their validity can be extended to other functions [18,30].

### 2.7 Riemann-Liouville and Caputo derivatives

The Riemann-Liouville and Caputo derivatives are multistep derivatives that use several integer order derivatives and a fractional integration [18,20,22,24-28,30]. To present them, we use (19) and (20) to obtain the following distributions [141]:

$$\delta_{\pm}^{(-\nu)}(t) = \pm \frac{t^{\nu-1}}{\Gamma(\nu)} u(\pm t),\ 0 < \nu < 1 \tag{26}$$

and

$$\delta_{\pm}^{(n)}(t) = \begin{cases} \pm \dfrac{t^{-n-1}}{\Gamma(\nu)} u(\pm t) & \text{for } n < 0 \\ \delta^{(n)}(t) & \text{for } n \geq 0 \end{cases} \tag{27}$$

where $n \in Z$. With them we define two differintegrations usually are classified as left and right sided, respectively:

$$f_{l}^{(\alpha)}(t) = [f(t)\, u(t-a)] * \delta_{+}^{(n)}(t) * \delta_{+}^{(-\nu)}(t) \tag{28}$$

$$f_{r}^{(\alpha)}(t) = [f(t)\, u(b-t)] * \delta_{+}^{(n)}(-t) * \delta_{+}^{(-\nu)}(-t) \tag{29}$$

with $a < b \in \mathbf{R}$. The orders are given by $\alpha = n - \nu$, $n$ being the least integer greater than $\alpha$ and $0 < \nu < 1$. In particular, if $\alpha$ is integer then $\nu = 0$ ([4]). From different orders of commutability and associability in the double convolution we can obtain distinct formulations. For example, from (23) we obtain the left Riemann-Liouville and the Caputo derivatives [141]:

$$f_{RL+}^{(\beta)}(t) = \delta_{+}^{(n)}(t) * \left\{ [f(t)\, u(t-a)] * \delta_{+}^{(-\nu)}(t) \right\} \tag{30}$$

$$f_{C+}^{(\beta)}(t) = \left\{ [f(t)\, u(t-a)] * \delta_{+}^{(n)}(t) \right\} * \delta_{+}^{(-\nu)}(t) \tag{31}$$

For the right the procedure is similar. We are going to study more carefully the characteristics of these derivatives. Consider (23). Let $\varphi^{(-\nu)}(t) = \left\{ [f(t)\, u(t-a)] * \delta_{+}^{(-\nu)}(t) \right\}$. We have:

[4] All the above formulae remain valid in the case of integer integration, provided that we put $\delta^{(0)}(t) = \delta(t)$.

$$\varphi^{(-\nu)}(t) = \begin{cases} \frac{1}{\Gamma(\nu)}\int_a^t f(\tau).(t-\tau)^{\nu-1}\, d\tau & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$$

So, in general when doing the second convolution in (30) we are computing the integer order derivative of a function with a jump. This leads to

$$f_{RL+}^{(\beta)}(t) = \frac{1}{\Gamma(-\alpha)}\int_a^t f(\tau).(t-\tau)^{-\alpha-1}\, d\tau - \sum_{i=0}^{n-1} f^{(\alpha-1-i)}(a)\,\delta^{(i)}(t) \qquad (32)$$

The appearance of the "initial conditions" $f^{(\alpha-1-i)}(a+)$ provoked some confusions because they were used as initial conditions of linear systems. This is not correct in general. They represent what we need to join to the Riemann-Liouville derivative to obtain the Liouville derivative (23) [74]. Now let us do a similar analysis to the Caputo derivative. The expression $\left\{[f(t)\, u(t-a)] * \delta_+^{(n)}(t)\right\}$ states the integer order derivative of the function f(t).u(t-a). The so called jump formula gives [35,139,149]:

$$y^{(n)}(t).u(t-a)=[y(t).u(t-a)]^{(n)} - \sum_{i=0}^{n-1} y^{(n-1-i)}(a)\,\delta^{(i)}(t) \qquad (33)$$

that leads to:

$$f_{C+}^{(\beta)}(t) = \frac{1}{\Gamma(-\alpha)}\int_a^t f(\tau).(t-\tau)^{-\alpha-1}\, d\tau - \sum_{i=0}^{n-1} f^{(n-1-i)}(a)\,\delta^{(i-\nu)}(t) \qquad (34)$$

In this case, we can extract conclusions similar to those we did in the Riemann-Liouville case. Relation (34) explains why sometimes the first n terms of the Taylor series of f(t) are subtracted to it before doing a fractional derivative computation. It is like a regularization.

### 2.8 The Fourier transform of the fractional derivative and the frequency response

Now, we are going to see if the above results can be extended to functions with a Fourier Transform. We note that the multivalued expression $F(s) = s^\alpha$ becomes an analytic function as soon as we fix a branch cut line in all the complex plane excepting the branch cut line. The computation of the derivative of functions with Fourier Transform is dependent on the way used to define $(j\omega)^\alpha$. If we define it doing the limit as $s \to j\omega$ from the right we have

$$(j\omega)^\alpha = |\omega|^\alpha.\begin{cases} e^{j\alpha\pi/2} & \text{if } \omega>0 \\ e^{-j\alpha\pi/2} & \text{if } \omega<0 \end{cases} \quad (35)$$

They mean that the forward derivatives of a cisoid is given by

$$D_f^\alpha\, e^{j\omega t} = e^{j\omega t}\, |\omega|^\alpha.\begin{cases} e^{j\alpha\pi/2} & \text{if } \omega>0 \\ e^{-j\alpha\pi/2} & \text{if } \omega<0 \end{cases} \quad (36)$$

For $x(t) = \cos(\omega_0 t)$ we obtain:

$$D_f^\alpha \cos(\omega_0 t) = |\omega_0|^\alpha \cos(\omega_0 t + \alpha\pi/2) \quad (37)$$

It can be show [148] that these results are not valid in the backward case.

### 2.9 Modeling, Identification, and Implementation

As in the usual systems, modeling, identification, and implementation are very interesting tasks. In the fractional case, they are slightly more difficult due to the fact of having, at least, one extra degree of freedom: the fractional order. However, this difficult increments the possibilities of obtaining more reliable and robust systems. This is challenging and people working in the area have been giving different interesting answers. We can refer the following approaches:

#### 2.9.1 fractional devices

The famous Curie law stating that the current in an insulator increases proportionally to a negative power of the time leads to the known “supercapacitors” that have impedance proportional to $1/s^\alpha$, with $0<\alpha<1$ [34,117]. Electrochemists have used the Constant Phase Elements (CPE) description for over 60 years. The fractors (fractional capacitors) [53,100,106,174] and coils [165] have been presented. he new terminology is “fractance” to indicate an Impedance with fractional order response. As these devices become available commercially, we will be rewriting many of the rules for design of filters and controllers [29,36,37,50].

#### 2.9.2 Trans-finite circuits

The infinite transmission lines are circuits with fractional behavior [65], but there are other interesting circuits with similar characteristics like the tree fractance ( a tree of RC circuits) and chain fractance (a series of parallel RC) circuits [66,102,158].

### 2.9.3 Band-limited approximations

It is an engineering approach. There are several ways of doing the design and implementation we can refer a) the CRONE that uses the Bode diagrams [36-39,107,150,152] and b) the continued fraction approaches [91-93]. Both construct pole-zero systems with interlaced poles and zeros.

Other similar alternative is the approximation by a weighted summation of exponentials, which as the number of elements increases toward infinity describes fractional behaviour. This concept has more recently been used by Anastasio [45] to approximate fractional order operators in his analysis of the vestibulo-ocular system. The basic idea developed by Thorson and Biederman-Thorson [71] is to represent a power law relaxation decay in time (e.g., $t^{-\alpha}$, where $0< \alpha < 1$) by a sum of exponentials weighted in an appropriate manner. Starting with the integral definition of the gamma function,

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}dx \quad \alpha > 0, \tag{38}$$

if we let $x = ta$, where $t > 0$, we can solve for $t^{\alpha}$ yield

$$t^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty v^{\alpha-1} e^{-vt}dv, \tag{39}$$

This integral can be interpreted as the Laplace transform of the function $v^{\alpha-1}/\Gamma(\alpha)$. Hence, we see that (39) provides a representation for the power-law decay as a weighted integral of exponentials. Thus, between the values of v and v+$dv$ there exists an exponential $e^{-vt}u(v)$ with a weight, $v^{\alpha-1}/\Gamma(k)$. Here $v$ has the units of $(sec)^{-1}$, and can be viewed as a rate constant. The overall power law relaxation given by (39) is the summation of all these contributions for the entire range of possible rate constants. In order to convert this time domain representation into a model for fractional operations we take the Laplace transform of both sides of (39). As seen in section 2.4

$$L[t^{-\alpha} u(t)] = \Gamma(1-\alpha)s^{\alpha-1}, \tag{40}$$

and assuming that we can interchange the order of integration for $v$ and $t$ we obtain

$$s^{\alpha-1} = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \frac{v^{\alpha-1}}{s+v} dv = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \frac{v^{\alpha-1}}{s+v} dv \tag{41}$$

which is the Stieltjes transform of $v^{\alpha-1}/\Gamma(\alpha)\,\Gamma(1-\alpha)$ ([5]). Finally, solving for $s^{\alpha}$ and if we let $v = 1/\tau$ where $\tau$ is the relaxation time corresponding to a particular value of $v$ we obtain

$$s^{\alpha-1} = \frac{\sin(\alpha\pi)}{\pi}\int_0^{\infty} \tau^{-\alpha}\frac{\tau s}{\tau s+1}\frac{d\tau}{:\tau} = \frac{\sin(\alpha\pi)}{\pi}\int_0^{\infty} \tau^{-\alpha}\frac{\tau s}{\tau s+1}\,d(\log\tau)\,, \tag{42}$$

Thus, in this interpretation, we see that the fractional derivative operator is represented as an integral or summation of Laplace domain terms that correspond to high-pass filters, and by a similar derivation the fractional integral operator is expressed in terms of an integral of low-pass filters. This is a unifying hypothesis because it extends in a natural way the usual progression of modeling linear systems as a series of exponentials, which typically increases as the degree of the integer order transfer function grows. With the above formulation, we see that the poles are logarithmically distributed.

## 3 Fractional Linear Systems

### 3.1 Transfer function and frequency response

The results of the previous section are very important in applications since they allow us to introduce the useful concept of Transfer Function. In fact, if we define a Linear System through a fractional differential equation with the general format:

$$\sum_{n=0}^{N} a_n D^{\nu_n} y(t) = \sum_{m=0}^{M} b_m D^{\nu_m} x(t) \tag{43}$$

where the differentiation orders, $\nu_n$, are, in the general case, complex numbers. As usual, we apply the LT to the equation (43) and use the results of section 1.4, to obtain the transfer function of the system:

$$H(s)=\frac{\sum_{m=0}^{M} b_m s^{\nu_m}}{\sum_{n=0}^{N} a_n s^{\nu_n}} \tag{44}$$

with region of convergence defined by Re(s) > 0 (causal case) or Re(s) < 0 (anti-causal case).

We may put the question of what happens with the frequency response of a given fractional linear system. From the conclusions we presented in 1.7, we can say that, having a causal fractional linear system with transfer function equal to H(s), the frequency response must be computed from:

[5] To obtain the last expression, we used the reflection property of the gamma function [8]

$$H(j\omega) = \lim_{s \to j\omega} H(s) \tag{45}$$

This is in agreement with other known results. For example, if the input to the system is white noise, with unit power, the output spectrum is given by:

$$S(\omega) = \lim_{s \to j\omega} H(s)\, H(-s) \tag{46}$$

### 3.2 From the Transfer Function to the Impulse Response

The general case represented in (43) is not easy to solve, because it is difficult to find the poles. For this reason, in the following, we shall be restricting our attention to the cases in which

- the $\nu_n$ are irrational numbers but multiples of a given $\nu$,
- the $\nu_n$ are any rational numbers. In this case, write them in the format $p_n/q_n$.

Let $\nu$ be the greater common divider of the $\nu_n$. Then $\nu_n = n\nu$. We will assume that $\nu < 2$, for stability reasons.

With this formulation, the equations (43) and (44) assume the general formats

$$\sum_{n=0}^{N} a_n D^{n\nu}\, y(t) \;=\; \sum_{m=0}^{M} b_m D^{m\nu}\, x(t) \tag{47}$$

and

$$H(s) = \frac{\sum_{m=0}^{M} b_m\, s^{m\nu}}{\sum_{n=0}^{N} a_n\, s^{n\nu}} \tag{48}$$

With a Transfer Function as in (47) we can perform the inversion quite easily, by following the steps:

1) Transform H(s) into H(z), by substitution of $s^{v}$ for z. We are assuming that H(z) is a proper fraction, otherwise, we have to decompose it in a sum of a polynomial (inverted separately) and a proper fraction.

2) The denominator polynomial in H(z) is the indicial polynomial [59,60] or characteristic pseudo-polynomial [22]. Perform the expansion of H(z) in partial fractions.

3) Substitute back $s^{v}$ for z, to obtain the partial fractions in the form

$$F(s)=\frac{1}{(s^{\nu}-a)^{k}} \quad k=1, 2, .. \tag{49}$$

4) Invert each partial fraction.

5) Add the different partial Impulse Responses.

We are going to see how to invert $F(s)=\frac{1}{s^{\nu}-a}$. Using the properties of the geometric series, it is a simple task to obtain:

$$F(s) = s^{-\nu} \sum_{n=0}^{\infty} a^{n} s^{-n\nu} \tag{50}$$

with $Re(s) > |a|^{1/\nu}$ defining the region of convergence. However, all the terms of the series are analytic for $Re(s) > 0$. For this reason, we can invert this series term by term, to obtain:

$$f(t) = t^{\nu-1} \sum_{n=0}^{\infty} \frac{a^{n} t^{n\nu}}{\Gamma(n\nu+\nu)} \; u(t) \tag{51}$$

which is a special case of the two parameter Mittag-Leffler function that is a generalization of the exponential to what it reduces when $\nu=1$. This function is well studied {see [18,20,28,65]}[6]. Equation (51) suggests us to work with the step response instead of the impulse response to avoid derivatives or working with non-regular functions near the origin.

When $\nu=1/q$, q being a positive integer, we obtain a different formulation for the inverse of the partial fraction (49). Using the well known result referring the sum of the first q terms of a geometric sequence we obtain ([7]), [22]:

[6] An interesting implementation was done by Prof. Podlubny and can be found at the site of MatLab. It is an implementation of the two parameter generalized Mittag-Leffler function with precision control – usage: mlf(alfa, beta, z, p).

[7] with reason $r = b/x$, we obtain:

$$\sum_{j=0}^{q-1} r^{j} = \frac{1 - r^{q}}{1 - r} \quad \Rightarrow \quad \sum_{j=0}^{q-1} b^{j}.x^{-j} = \frac{1 - b^{q}.x^{-q}}{1 - b/x} \quad \text{or } x^{q} - b^{q} = (x-b).\sum_{j=1}^{q} b^{j-1}.\,x^{q-j} \quad \text{from where } \frac{1}{x-b} = \frac{\sum_{j=1}^{q} b^{j-1}.\,x^{q-j}}{x^{q} - b^{q}}$$

$$F(s)=\frac{1}{(s^{\nu}-a)}=\frac{\sum_{j=1}^{q} a^{j-1}s^{1-j\nu}}{s-a^{q}} \tag{52}$$

We conclude that the inverse LT of a partial fraction as $F(s)=\frac{1}{s^{1/q}-a}$ is a linear combination of q fractional derivatives of $E_0(t,a^q)=e^{a^q t}u(t)$.

The k >1 case in (45) does not present great difficulties except some additional work. It can be obtained from the k=1 case by repeated convolution or by differentiation [137].

### 3.3 The Stability Problem

The study of the stability of the fractional linear time invariant (FLTI) systems we are going to do is based on the BIBO stability criterion that implies stability when the impulse response is absolutely integrable.

The simplest FLTI system is the system with transfer function $H(s) = s^{\nu}$ with s belonging to the principal Riemann surface. If $\nu>0$, the system is definitely unstable, since the impulse response is not absolutely integrable, even in a finite interval. If $-1<\nu<0$, the impulse response remains a limited function when t increases indefinitely and it is absolutely integrable in every finite interval. Therefore, we will say that the system is wide sense stable. This case is interesting to the study of the fractional stochastic processes. If $\nu=-1$, the normal integrator, the system is wide sense stable. The case $\nu<-1$ corresponds to an unstable system, since the impulse response is not a limited function when t goes to $+\infty$.

Consider the LTI systems with transfer function H(s) a quotient of two polynomials in $s^{\nu}$. The transformation $w=z^q$, transforms the sector $0 \le \theta \le 2\pi/q$ $\{\theta = \arg(z)\}$ into the entire complex plane. So, the sector $\frac{\pi}{2q} \le \theta \le \frac{\pi}{2q}+\frac{\pi}{q}$ is transformed in the left half plane. Consider the first Riemann surface of $z=s^{\nu}$ defined by $\theta = \arg(s) \in(-\pi,\pi]$. This domain is transformed into $\varphi=\arg(z)\in(-\pi\alpha,\pi\alpha]$. However the poles leading to instability must be inside the sector $(-\pi\alpha/2,\pi\alpha/2)$. We have two situations leading to stability:

- There are no poles inside the sector $(-\pi\alpha,\pi\alpha]$.
- There are poles but they are in the sectors: $(-\pi\alpha,-\pi\alpha/2)$ and $(\pi\alpha/2,\pi\alpha)$.

The poles with argument equal to $\pm\pi\alpha/2$ may lead to wide sense stable systems as in the usual systems. These conclusions come from properties of the Mittag-Leffler function [18]. To give a simple example, consider the transfer function $H(s) = \frac{1}{s^{\alpha} + 1}$, with $0 < \alpha < 2$. It is easy to see that there is no pole in the principal Riemann surface. So, it represents a stable system.

### 3.4 The Initial Conditions

When looking for the output, y(t), to a given input, x(t), we must consider the initial conditions. This is a problem that created much confusion and difficulties in the past [49,66,74] motivated by the use of several different derivative definitions and of the one-sided Laplace Transform. In [66,76], we proposed a new way of looking at the problem.

As it is well known, the solution of equation (47) has two terms: the forced (or evoked) and free (or spontaneous). This second term depends only on the state of the system at the reference. This state constitutes or is related to the initial conditions. These are the values at t=0 of variables in the system which are associated with stored energy. <u>It is the structure of the system that imposes the initial conditions, not the eventual way of computing the derivatives</u>. The solution is obtained with the fractional jump formula [149]

$$D^{n\alpha}[y(t)].u(t) = D^{n\alpha}[y(t).u(t)] - \sum_{0}^{n-1} y^{(m\gamma)}(0)\delta^{[(n-m)\gamma-1]}(t)$$

that allows us to transform (47) into

$$\sum_{i=0}^{N} a_i .[y(t).u(t)]^{(\gamma i)} = \sum_{i=0}^{M} b_i .[x(t).u(t)]^{(\gamma i)}$$

$$+ \sum_{i=1}^{N} a_i . \sum_{0}^{i-1} y^{(\gamma m)}(0)\delta^{[(n-m)\gamma-1]}(t) - \sum_{i=1}^{M} b_i \, x^{(\gamma m)}(0)\delta^{[(n-m)\gamma-1]}(t) \qquad (53)$$

We must refer that:

- The initial conditions appear directly in the equation, without using any transform.
- Equation (53) is valid also in the time variant case.

### 3.5 Discrete-time implementations

It is not a simple task to obtain a discrete-time implementation of a fractional differintegrator. There are several algorithms that start from an s to z conversion and design an ARMA model [48,49,63,116,143,145]. However, they are mere approximations and there is no clear statement on the optimality of any approch. It is an open subject needing additional research efforts. The simplest way of doing such approximation consists in starting from the forward GL derivative, remove the limit operation and truncate the series. This is not needed if the we intend to compute the output of the system being approximated for a causal imput. In fact, in this case, the series becomes a finite sum:

$$D_\theta^\alpha f(t) \approx \frac{\sum_{k=0}^{\lfloor t/h \rfloor} (-1)^k \binom{\alpha}{k} f(t-kh)}{h^\alpha} \tag{54}$$

In ths situation we must consider the so called ``short memory principle" [28], also known as ``fixed memory principle", is a useful tool for numerical simulations in large time intervals. Taking into account approximation (2) we can see, that if t >> 0 the final sumation would be too large. From calculation of the binomial coefficients above follows, that the past values of the function f(t) near 0 have only small influence on the new evaluated value of the function. Instead of using "whole memory", only "recent past" of the function is used, e.g. the interval (t-T,t), where T is the "memory length"

$$D_{\theta;T}^\alpha f(z) \approx \frac{\sum_{k=0}^{\lfloor (t-T)/h \rfloor} (-1)^k \binom{\alpha}{k} f(t-kh)}{h^\alpha} \tag{55}$$

It is worth mentioning, that similar approach was introduced in Volterra's work under the name "limited after-effect" assumption [28].

This continuous to discrete conversion is essentially the following. Assume that h is a sampling interval. Then we can also sample f(t) and $D_\theta^\alpha f(t)$ with the same interval. This is equivalent to performing the continuous to discrete (Eular) transformation:

$$s = \frac{1 - z^{-1}}{h} \tag{56}$$

and

$$s^\alpha = \left(\frac{1 - z^{-1}}{h}\right)^\alpha \approx \frac{\sum_{k=0}^{N} (-1)^k \binom{\alpha}{k} z^{-k}}{h^\alpha} \tag{57}$$

where N is a “enough high” integer fixed according to the principle stated above. As it can be seen we are doing an FIR approximation to the differintegrator: the impulse response is $h(n) = \frac{(-\alpha)_n}{n!} u(n)$ for n=0, 1, …, N. Using (48) it is possible to obtain ARMA models for the same operator {see [143,145]}.

With these s to z transformations we arrive into the discrete-time signal processing context and so obtain an easier and more known framework.

Other alternatives to the Euler transformation are the bilinear transformation (Tustin) [62, 175-180]

$$s = \frac{2}{h}\frac{1 - z^{-1}}{1 + z^{-1}} \tag{58}$$

and the mixed operator ( Al-alaoui) [44, 62]

$$s = \frac{8}{h}\frac{1 - z^{-1}}{7 + z^{-1}} \tag{59}$$

As we want to obtain discrete equivalents to the differintegrator, $s^\alpha$, the following considerations have to be mentioned [64]:

1) $s^\alpha$, viewed as a causal operator, has a branch cut line along the negative real axis for arguments of s in $(-\pi,\pi)$ but is free of poles and zeros.

2) A dense interlacing of simple poles and zeros along a line in the s plane is, in some way, equivalent to a branch cut {see the deduction of the Cauchy derivative}.

3) It is well known that, for interpolation or evaluation purposes, rational functions are sometimes superior to polynomials, roughly speaking, because of their ability to model functions with zeros and poles. In other words, for evaluation purposes, ARMA models converge faster than the long MA (FIR).

4) Trapezoidal (bilinear) rule maps adequately the stability regions of the s plane on the z plane, and maps the points s=0 and s=∞ into the points z=1 and z=-1, respectively.

The impulse response of the discrete-time linear system corresponding to the Tustin transformation is given by [145]

$$h_{bil}(n) = \left(\frac{2}{h}\right)^{\alpha} \frac{(-1)^n(\alpha)_n}{n!} \sum_{k=0}^{n} \frac{(-\alpha)_k\,(-n)_k}{(-\alpha-n+1)_k} \frac{(-1)^k}{k!} =$$

$$= \left(\frac{2}{h}\right)^{\alpha} \frac{(-1)^n(\alpha)_n}{n!}\, {}_2F_1(-\alpha,-n,-\alpha-n+1,-1)u(n) \tag{60}$$

where ${}_2F_1(a,b,c,-1)$ is the Gauss hypergeometric function that, for these arguments, does not have a closed form. Similarly, the impulse response corresponding to the Al-Alaoui transformation can be computed following the procedure used in [145] and is given by

$$h_{bil}(n)=\left(\frac{8}{7h}\right)^{\alpha}\frac{(-7)^{-n}(\alpha)_n}{n!}\sum_{k=0}^{n}\frac{(-\alpha)_k(-n)_k}{(-\alpha-n+1)_k}\frac{(-7)^{-k}}{k!}u(n) =$$

$$= (-7)^{-n}\left(\frac{8}{7h}\right)^{\alpha}\frac{(-1)^{-n}(\alpha)_n}{n!}\,{}_2F_1(-\alpha,-n,-\alpha-n+1,-7^{-1})u(n) \tag{61}$$

It is interesting because it decreases quickly. With the above impulse responses, we can obtain ARMA models. There are several methods, like the least-squares method [48,49,143,145], the continued fraction method [62-64]

Another and different way of doing the continuous to discrete conversion is the so called Matrix Approach. The "matrix approach" to discretization of fractional integrals and derivatives has been developed by Podlubny [156,157]. It is based on use of triangular strip matrices. This method significantly simplifies many aspects of numerical computations in the fractional calculus, and especially solving fractional differential equations.

According to what we said above, the fractional derivatives of order $\alpha$ can be approximated at all nodes of the uniform grid $t = nh$, $n\in Z$ at once with the help of the upper triangular strip matrix $B_n^{(\alpha)}$

$$[v_n^{(\alpha)}\ \ v_{n-1}^{(\alpha)}\ \ \ldots v_1^{(\alpha)}\ \ v_0^{(\alpha)}]^T = B_n^{(\alpha)}\,[v_n\ \ v_{n-1}\ \ \ldots v_1\ \ v_0]^T \tag{62}$$

where

$$v_n^{(\alpha)} = D^{\alpha} v(nh) \tag{63}$$

and

$$B_n^{(\alpha)} = \frac{1}{\tau^{\alpha}}\begin{bmatrix} h_0^{(\alpha)} & h_1^{(\alpha)} & \ddots & \ddots & h_{n-1}^{(\alpha)} & h_n^{(\alpha)} \\ 0 & h_0^{(\alpha)} & h_1^{(\alpha)} & \ddots & \ddots & h_{n-1}^{(\alpha)} \\ 0 & 0 & h_0^{(\alpha)} & h_1^{(\alpha)} & \ddots & \ddots \\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & 0 & h_0^{(\alpha)} & h_1^{(\alpha)} \\ 0 & 0 & \cdots & 0 & 0 & h_0^{(\alpha)} \end{bmatrix} \tag{64}$$

where $h_n^{(\alpha)}$ represents the impulse response according to the chozen method (Euler, Tustin or Al-Alaoui). From the properties of the derivative those matrices $B_n^{(\alpha)}$ have also the group structure presented in section 2.4. This method has been presented for the first time in [157] along with several examples of numerical solution of ordinary fractional differential equations with Riemann-Liouvile and Caputo derivatives. From the viewpoint of simplicity of usage, the matrix approach to numerical solution of fractional differential equations can be compared to the Laplace transform method for solving ordinary differential equations. Indeed, in both cases operators of differentiation are simply replaced with other symbols — in the case of the Laplace transform by powers of the Laplace variable, and in the case of the matrix approach by matrices of a known structure. For example, the famous Bagley-Torvik equation {see [28] and references therein}

$$a y''(t) + b\, {}_0D_t^{(3/2)} y(t) + c\, y(t) = f(t), \tag{65}$$

is discretized on a uniform grid with n nodes as

$$\left(a B_n^{(2)} + b B_n^{(3/2)} + c B_n^{(0)}\right) Y_n = F_n, \tag{66}$$

where $Y_n$ is the vector of unknown values of y(t) at the discretization nodes, $F_n$ is the vector of values of the input f(t) and $B_n^{(\alpha)}$ is the triangular strip matrix representing the discrete analogue of the fractional derivative. Clearly, $B_n^{(0)}$ is equal to the identity matrix $E_n$. The matrix approach has been implemented in the form of a publicly available Matlab toolbox.

**4 Input-output numerical computations in general nonlinear systems**

In various applications, e.g. in fluid mechanics, viscoelasticity, biology, physics and engineering [22,24,26-28,50,61,73,98,99,172], considerable attention is given to ordinary and partial differential equations of fractional order, due to their memory and hereditary properties. However, most applications have been directed towards modelling existing situations where no outside interference has place. This means that most studies consider the output of fractional systems under non null initial conditions, but with null input. This will be adopted here. According, for example, to Momani [123-132] most fractional differential equations do not have exact analytic solutions, so approximation and numerical techniques [51, 71-78,27] must be used. There are two main classes of methods for solving fractional differential equations: the frequency-domain methods and the time-domain methods. In this review we deal mainly with time-domain methods. Finding accurate and efficient methods for solving fractional-order differential equations (FODEs) is the goal of many research works. Analytical and numerical methods for solving most of the FODEs must be used, as an exact solutions can not be found easily. Some numerical methods for solving FODEs were presented for instance in [71-78,107]. We will consider DE with the format

$$y^{(\alpha)}(t) = f[t,y(t)] + a\, x(t)\,, \tag{67}$$

where x(t) is the input, y(t) is the output, and t∈R. Unless expressed we will assume that the derivative operation is one step. This is important because we will need only one initial condition, according to the results in section 3.4. We can then modify the above equation to make the initial condition appear explicitly in an equation that is valid for t ≥ 0

$$y^{(\alpha)}(t) = \mathbf{y}(0)\, \delta^{(\alpha-1)}(t) + f[t,y(t)] + a\, x(t), \quad t\in R^{+}$$

or

$$\mathbf{y}^{(\alpha)}(t) = f[t,\mathbf{y}(t)] + a\, x(t) + \mathbf{y}(0)\, \frac{t^{-\alpha}}{\Gamma(-\alpha+1)} u(t), \quad t\in R^{+} \tag{68}$$

It is a current practice to use here integral formulations. We will use the Liouville integral seen at section 2.6.

In the following we will present several approaches for solving equation (69), namely: Diethelm's method based on quadrature and Lubich's difference methods followed with some information about Adams-Bashforth-Moulton method based on the Volterra integral equation, an effective method for fractional order dynamical systems, Adomian's Decomposition method (ADM), Variational iteration method (VIM) and Homotopy-perturbation method (HPM).

### 4.1 Diethelm's method based on quadrature

Let us first start from the Liouville derivative specialized for causal signals and proceed as Weilbeer in [32]. Then apply the linear transformation τ = tu to obtain [72]

$$D^{\alpha} f(t) = \frac{t^{-\alpha}}{\Gamma(-\alpha)} \int_0^1 \tau^{-\alpha-1} g(\tau)\, d\tau, \quad \text{for all} \quad t \in (0,T], \tag{69}$$

where g(τ) = f(t - tτ). In the following D means the forward derivative operator. The algorithm now proceeds as follows: Choose a positive integer N and divide the working interval [0,T] into N subintervals of equal length h = T/N with breakpoints (sampling instants) $t_m$ = mh, m = 0,1,2, …, N that we will use in (71). This yields

$$\frac{t_m^{-\alpha}}{\Gamma(-\alpha)} \int_0^1 \tau^{-\alpha-1} y(t_m - t_m\tau)\, d\tau = f[t_m, \mathbf{y}(t_m)] + \mathbf{y}(0)\, \frac{t_m^{-\alpha}}{\Gamma(1-\alpha)} u(t_m), \tag{70}$$

where we have also taken the input equal to zero. In this relation we replace the integral by the quadrature formula $Q_m$, and additionaly introduce the quadrature error $R_m$. Thus, using the abbreviation $g_m(\tau) = y(t_m - t_{m\tau})$ yields

$$\frac{t_m^{-\alpha}}{\Gamma(-\alpha)}\left(\sum_{k=0}^{m}\omega_{km}g_m(k/m)+R_m[g_m]\right)-\frac{y(0)t_m^{-\alpha}}{\Gamma(1-\alpha)}=f(x_m,y(x_m)), \qquad (71)$$

where for $\omega_{km}$ we can write

$$\omega_{km}=\frac{\tilde{\omega}_{km}\Gamma(2-\alpha)}{-\alpha(1-\alpha)m^{-\alpha}} \qquad (72)$$

We finally solve the left-hand side for $y_m = y(t_m)$ and get

$$y_m = h^{\alpha} f(x_m,y_m)-\sum_{k=1}^{m}\tilde{\omega}_{km}y(x_m-kh)-\mathbf{y}(0)\frac{h^{\alpha}t_m^{-\alpha}}{\Gamma(1-\alpha)}u(t_m) \qquad (73)$$

where the weights $\tilde{\omega}_{km}$ are given by the substitution (74) and so

$$\frac{\tilde{\omega}_{km}}{\Gamma(2-\alpha)}=\begin{cases}1 & \text{for } k=0,\\ -\alpha & \text{for } k=m=1,\\ 2^{1-\alpha}-2 & \text{for } k=1 \text{ and } m\geq 2,\\ (k-1)^{1-\alpha}+(k+1)^{1-\alpha}-2k^{1-\alpha} & \text{for } 2\leq k\leq m-1,\\ (k-1)^{1-\alpha}-(\alpha-1)k^{-\alpha}-k^{1-\alpha} & \text{for } k=m\geq 1.\end{cases} \qquad (74)$$

### 4.2 Lubich's difference methods

Lubich's fractional difference methods form a subset to fractional linear multistep methods, which were first presented by Lubich [112-114] and numerically implemented by Hairer, Lubich and Schlichte in [89] for a special type of Volterra integral equations. Consider again that the input, x(t), is null. It can be prove that the FODE can be rewritten as Abel-Volterra integral equation:

$$\mathbf{y}(t) = \int_0^t (t-\tau)^{\alpha-1} f[\tau,\mathbf{y}(\tau)]\, d\tau + \mathbf{y}(0)u(t), \qquad (75)$$

Lubich [113] showed that, if $\alpha > 0$ and given the method order, $p \in \{1,2,3,4,5,6\}$,

$$y(m)=y(0)+h^{\alpha}\sum_{j=0}^{m}\omega_{m-j}f(t_j,y(t_j))+h^{\alpha}\sum_{j=0}^{s}\omega_{m,j}f(t_j,y(t_j)) \qquad (76)$$

for m = 1, 2, ..., N, where the convolution weights $\omega_m$ are given by the generating function

$$\omega^{\alpha}(\zeta)=\left(\sum_{k=1}^{p}\frac{1}{k}(1-\zeta)^k\right)^{-\alpha}. \qquad (77)$$

and the starting weights $\omega_{m,j}$ can be obtained as:

$$\sum_{j=0}^{s}\omega_{m,j}j^{\gamma}=\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)}m^{\alpha+\gamma}-\sum_{j=1}^{m}\omega_{m-j}j^{\gamma},\quad \gamma\in \mathrm{A} \tag{78}$$

with

$$\mathrm{A}=\{\gamma=k+j\alpha;k,j\in N_0,\gamma\leq p-1\},\quad card\mathrm{A}=s+1. \tag{79}$$

Equation (78) gives an approximation of order $O(h^{p-\varepsilon})$ with a small $\varepsilon \geq 0$ for all fixed mesh points $t_m$.

### 4.3 Adams-Bashforth-Moulton method

Adams-Bashforth-Moulton method [169] is also a numerical method to solve FODE, based on the Abel-Volterra integral equation (77). Even though it seems to be a suitable tool for fractional order dynamical systems, there are some difficulties mentioned in the literature:

- the size of the computational work can be burdensome,
- the rounding-off error can cause loss of accuracy.

This method has been introduced by Diethelm and Freed [8] and discussed as well in [9,76]. The following relationships were used in the work of Weilbeer [32]. The numerical solution of the equation (77) on the interval [0,T], on the above used grid. Let us assume that the approximations $y_j = y(t_j)$ for j=1,2, ...,k have been already evaluated. The task is to find the solution $y_{k+1}$, obtained by replacing the integral in (77) using the product trapezoidal quadrature formula where the nodes $t_j$ for j=0, 1, ..., k+1 are taken with respect to the weight function $(t_{k+1} - .)^{\alpha-1}$. First we will get the approximation:

$$\int_0^{t_{k+1}}(t_{k+1}-z)^{\alpha-1}g(z)dz\approx\int_0^{t_{k+1}}(t_{k+1}-z)^{\alpha-1}\tilde{g}_{k+1}(z)dz, \tag{80}$$

where $\tilde{g}_{k+1}$ is the piecewise linear interpolant for g. The right-hand side can be rewritten as:

$$\int_0^{t_{k+1}}(t_{k+1}-z)^{\alpha-1}\tilde{g}_{k+1}(z)dz=\sum_{j=0}^{k+1}a_{j,k+1}g(t_j), \tag{81}$$

where

$$a_{j,k+1}=\int_0^{t_{k+1}}(t_{k+1}-z)^{\alpha-1}\phi_{j,k+1}(z)dz \tag{82}$$

and

$$\phi_{j,k+1}(z)=\begin{cases}(z-t_{j-1})/(t_j-t_{j-1}) & \text{if } t_{j-1}<z\le t_j,\\ (t_{j+1}-z)/(t_{j+1}-t_j) & \text{if } t_j<z<t_{j+1},\\ 0 & \text{else}.\end{cases} \tag{83}$$

For a uniform grid, we have:

$$a_{j,k+1}=\begin{cases}\dfrac{h^\alpha}{\alpha(\alpha+1)}\left(k^{\alpha+1}-(k-\alpha)(k+1)^\alpha\right) & \text{if } j=0,\\ \dfrac{h^\alpha}{\alpha(\alpha+1)}\left((k-j+2)^{\alpha+1}+(k-j)^{\alpha+1}-2(k-j+1)^{\alpha+1}\right) & \text{if } 1\le j\le k,\\ \dfrac{h^\alpha}{\alpha(\alpha+1)} & \text{if } j=k+1.\end{cases} \tag{84}$$

So we obtain the corrector formula (fractional variant of the one-step Adams-Moulton method):

$$y_{k+1}=\frac{1}{\Gamma(\alpha)}\left(\sum_{j=0}^{k}a_{j,k+1}f(t_j,y_j)+a_{k+1,k+1}f(t_{k+1},y_{k+1}^p)\right)+y(0), \tag{85}$$

where the expression $y_{k+1}^p$ means the predictor formula, which will be calculated using generalized one-step Adams-Bashforth method in the same way, how it was by determining the corrector formula.

We again replace the integral in (77), but now by the product rectangle rule:

$$\int_0^{t_{k+1}}(t_{k+1}-z)^{\alpha-1}g(z)dz\approx\sum_{j=0}^{k}b_{j,k+1}g(t_j), \tag{86}$$

where

$$b_{j,k+1}=\int_{t_j}^{t_{j+1}}(t_{k+1}-z)^{\alpha-1}dz=\frac{(t_{k+1}-x_j)^\alpha-(t_{k+1}-t_{j+1})^\alpha}{\alpha}. \tag{87}$$

Now we are not dealing with a piecewise linear approximation, but with piecewise constant approximation, therefore following holds:

$$\phi_{kj}(t):=\begin{cases}1 & on\ [t_j,t_{j+1}]\\ 0 & everywhere\ else\ on\ the\ inteval\ [0,t_{k+1}]\end{cases} \tag{88}$$

and again in the case of equispaced distribution, we get:

$$b_{j,k+1} = \frac{h^{\alpha}}{\alpha}((k+1-j)^{\alpha} - (k-j)^{\alpha}). \tag{89}$$

Finally, the predictor $y^{p}_{k+1}$ is obtained by the fractional Adams-Bashforth method:

$$y^{p}_{k+1} = y(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} b_{j,k+1} f(x_j, y_j). \tag{90}$$

The equations (95) and (90) with the weights $a_{j,k+1}$ and $b_{j,k+1}$ calculated in (92) and (89) form the fractional Adams-Bashforth-Moulton method.

### 4.4 Adomian's decomposition method

The next to mention is the numerical method based on the Adomian decomposition [1]. This method provides the solution of the fractional order system in the form of a power series with easily computed terms. Adomian's decomposition method was firstly used to obtain approximate solutions of linear or nonlinear differential equations [166]. With the increasing popularity of fractional calculus, the application of the method was recently extended for the case of fractional differential equations [124-132]. This method can be used for finding the solution of the Abel-Volterra integral equation (70) as:

$$y(x) = \sum_{i=1}^{\infty} y_i(t) = g(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \sum_{i=0}^{\infty} {}_f A_i(\tau) d\tau, \tag{91}$$

where the ${}_f A_i(t)$ are the Adomian polynomials and in g(t) we include the input and the initial condition term. The explicit scheme of ADM can be written as:

$$y_0(t) = g(t) \quad y_{i+1}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} {}_f A_i(\tau) d\tau, \quad i = 0,1,2,\ldots, \tag{92}$$

where

$${}_f A_i(t) = \left[ \frac{1}{i!} \frac{d^i}{d\lambda^i} f\left( t, \sum_{j=0}^{i} \lambda^j y_j \right) \right]_{\lambda=0}. \tag{93}$$

Even if one cannot use the infinite scheme, it is possible to obtain a finite expansion corresponding to the differentiability properties of f(t,y(t)) [32].

In the work of Momani and Odibat [87], for the solution of linear fractional differential equations the following algorithm for solving linear fractional differential equation was considered. Let us define a linear FODE in the form:

$$\frac{d^m y}{dt^m} - a\frac{d^\alpha y}{dt^\alpha} - by = f(t), \quad m-1 < \alpha \le m, \tag{94}$$

subject to the initial conditions

$$y^{(j)}(0) = c_j, \quad j = 1,0,\ldots,m-1, \tag{95}$$

where $c_j, j = 0,1,\ldots,m-1$ are arbitrary constants and y(t) is a causal function of time. The system represented by (96) can be interpreted as composite fractional relaxation/oscillation equation for the cases $\{0 < \alpha \le 1, m = 1\}$ and $\{1 < \alpha \le 2, m = 2\}$, respectively.

Let J = $D^{-1}$, the anti-derivative operator.

If we apply the operator $J^m$ to both sides of (96) and we use the initial conditions, we get:

$$u(t) = \phi_1(t) + a\phi_2(t) + J^m f(t) + [aJ^{m-\alpha} + bJ^m]u(t), \tag{96}$$

where

$$\phi_1(t) = \sum_{i=1}^{m-1} c_i \frac{t^i}{i!}, \quad \phi_2(t) = \sum_{i=1}^{m-1} c_i \frac{t^{m-\alpha+i}}{\Gamma(m-\alpha+i+1)}. \tag{97}$$

According to Adomian [6, 7] the solution y(t) be decomposed by the infinite series of components:

$$y(t) = \sum_{n=0}^{\infty} y_n(t), \tag{98}$$

After substitution of the decomposition series (101) into both sides of (99) we obtain:

$$\sum_{n=0}^{\infty} y_n(t) = \phi_1(t) + a\phi_2(t) + J^m f(t) + [aJ^{m-\alpha} + bJ^m]\sum_{n=0}^{\infty} y_n(t). \tag{99}$$

The iterates can be obtained from the previous equation by the following recursive way:

$$y_k = (aJ^{m-\alpha} + bJ^m) y_{k-1} = (aJ^{m-\alpha} + bJ^m)^k [\phi_1(t) + a\phi_2(t) + J^m f(t)].$$

The components of y(t) are then defined as:

$$y(t) = \sum_{k=0}^{\infty} (aJ^{m-\alpha} + bJ^m)^k [\phi_1(t) + a\phi_2(t) + J^m f(t)]. \tag{100}$$

To obtain the solution of (96) in a series form we expand the operator in (103) using the binomial formula. Then the solution is:

$$y(t) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j} J^{km-jx} [\phi_1(t) + a\phi_2(t) + J^m f(t)]. \tag{101}$$

This algorithm can be generalised to solve nonlinear systems of fractional differential equations [86].

Although we obtain an approximate solution, at least because we have to truncate the series, in many cases the exact solution in a closed form may be obtained. Moreover, the decomposition series solutions generally converge very rapidly.

### 4.5 The variational iteration method

The variational iteration method, as proposed by He [13, 591], was successfully applied to autonomous ordinary and partial differential equations [90,110,111,126] and other fields. He was the first to apply the variational iteration method to fractional differential equations. Recently Odibat and Momani implemented the variational iteration method to solve nonlinear differential equations of fractional order [126]. In particular he solved the fractional differential equation (96). The correction functional for (96) can be constructed as:

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda (D^m y_n(\tau) - aD^\alpha \tilde{u}_n(\tau) - b\tilde{u}_n(\tau) - f(\tau)) d\tau, \tag{102}$$

where λ is a general Lagrange multiplier [61], which can be identified optimally via the variational theory [91,111], and $\tilde{u}_n$ and $D^\alpha \tilde{u}_n$ are considered as restricted variations. Let us begin with the initial approximation

$$y_0 = c_0 + c_1 t + c_2 t^2 + \ldots + c_{m-1} t^{m-1}. \tag{103}$$

Making the above functional stationary, noticing that $\delta \tilde{u}_n = 0$,

$$\delta y_{n+1}(t) = \delta y_n(t) + \delta \int_0^t \lambda (D^m y_n(\tau) - f(\tau)) d\tau, \tag{104}$$

yields the Lagrange multipliers:

$\lambda = -1, \quad for\ m = 1,$

$\lambda = \tau - t, \quad for\ m = 2.$

Therefore, for m = 1, we obtain the iteration formula:

$$y_{n+1}(t) = y_n(t) - \int_0^t [D^1 y_n(\tau) - aD^\alpha y_n(\tau) - by_n(\tau) - f(\tau)] d\tau. \tag{105}$$

and for m = 2, the iteration formula:

$$y_{n+1}(t) = y_n(t) + \int_0^t (\tau - t)[D^2 y_n(\tau) - aD^\alpha y_n(\tau) - by_n(\tau) - f(\tau)] d\tau. \tag{106}$$

This method was also generalised for systems of differential equations.

### 4.6 Homotopy-perturbation method

The HPM is a combination of the traditional perturbation method and homotopy in topology. It solves the FODEs by decomposing the complex problem to simple problems, and then the perturbation equation can be easily constructed by a homotopy in topology.

In the works of Momani and Odibat linear and nonlinear partial FODEs were solved [128,129] using this method. The problem and solution proposed in [40-42] can be written in the form:

$$\begin{aligned} D^{\alpha_1} y_1(t) &= f_1(t, y_1, y_2, \ldots, y_n), \\ D^{\alpha_2} y_2(t) &= f_2(t, y_1, y_2, \ldots, y_n), \\ &\vdots \\ D^{\alpha_n} y_n(t) &= f_1(t, y_1, y_2, \ldots, y_n), \end{aligned} \tag{107}$$

subject to the following initial conditions

$$y_k(0) = c_k, \quad k = 1, 2, \ldots, n, \tag{108}$$

where $D^{\alpha_i}$ is the fractional derivative of $y_i$ of order $\alpha_i$,where $0 < \alpha_I \leq 1$ and $f_i$ are arbitrary linear or nonlinear functions. The following homotopy can be constructed in view of the HPM [92,94] as:

$$D^{\alpha_i} y_i = pf_i(t, y_1, y_2, \ldots, y_n), \tag{109}$$

where i = 1,2, …, n and p is an embedding parameter which changes from zero to unity [4]:

- If p = 0, we will obtain the linear equation

$$D^{\alpha_i} y_i = 0.$$

- If p = 1, the homotopy (111) turns out to be the original system given in (109).

The solution of the system (109) can be expanded using the parameter p:

$$y_i(t) = y_{i0} + py_{i1} + p^2 y_{i2} + p^3 y_{i3} + \ldots. \tag{110}$$

Series of linear equations can be obtained after substitution (112) into (111) and collection the terms with the same powers of p, in the form [4]:

$$p^0 : D^{\alpha_i} y_{i0} = 0,$$

$$p^1 : D^{\alpha_i} y_{i1} = f_{i1}(t, y_{10}, y_{20}, \ldots, y_{n0}) \tag{111}$$

$$p^2 : D^{\alpha_i} y_{i2} = f_{i2}(t, y_{10}, y_{20}, \ldots, y_{n0} y_{11}, y_{21}, \ldots, y_{n1}),$$

$$p^3 : D^{\alpha_i} y_{i3} = f_{i3}(t, y_{10}, y_{20}, \ldots, y_{n0}, y_{11}, y_{21}, \ldots, y_{n1}, y_{12}, y_{22}, \ldots, y_{n2}),$$

$$\vdots$$

where the functions $f_{i1}, f_{i2}, \ldots,$ satisfy the following equation:

$$fi(t, y_{10}, py_{11} + p^2 y_{12} + \ldots, \ldots, y_{n0} + py_{n1} + p^2 y_{n2} + \ldots) =$$

$$= f_{i1}(t, y_{10}, y_{20}, \ldots, y_{n0}) + pf_{i2}(t, y_{10}, y_{20}, \ldots, y_{n0}, y_{11}, y_{21}, \ldots, y_{n1}) \tag{112}$$

The following conclusions were made in the work of Abdulaziz, Hashim and Monami [42]:

It is obvious that these linear equations can be easily solved by applying the operator $J^{\alpha_i}$ , i.e., the inverse of the fractional operator $D^{\alpha_i}$. Hence, the components yik, k = 0,1,2, …, of the HPM solution can be determined. That is, by setting p = 1 in (114) we can entirely determine the HPM series solutions,

$$y_i(t) = \sum_{k=0}^{\infty} y_{ik}(t). \qquad (113)$$

The HPM series solution (104) can be approximated by the following N-term truncated series:

$$\phi_{iN}(t) = \sum_{k=0}^{N-1} y_{ik}(t). \qquad (114)$$

According to the authors using this method, HPM yields rapid convergence of the solution series in most cases, usually only a few iterations leading to very accurate solutions.

### 4.7 Some considerations

There are six arguments in the work of Momani and Odibat related to the comparison of VIM and ADM [86]:

1. The VIM and the decomposition method provide the solutions in terms of convergent series with easily computable components.

2. It is clear and remarkable that the approximate solutions in all examples using the two methods are in good agreement.

3. The approximate solutions obtained using the VIM are exactly the same as those obtained by using the decomposition method for linear systems of ordinary differential equations.

4. The VIM is more effective and overcomes the difficulty arising in calculating Adomian polynomials.

5. The two techniques require less computational work than existing approaches while supplying quantitatively reliable results.

6. It is also shown that the solutions of the fractional equations reduces to the solutions of the corresponding integer order equations.

In the paper [131] the VIM and the ADM are used to solve linear differential equations of fractional order. Linear as well as nonlinear systems of FODEs were solved using the ADM in [69] and [99, 127], respectively. The comparison made by Momani and Odibat can be found in [130]. These two methods are concluded as very powerful and efficient in finding analytical as well as numerical solutions for wide classes of fractional differential equations. The series solutions they provide, converge very rapidly in real physical problems. The main advantage of the two methods is that they do not require linearization, discretization, or perturbation [1,93].

## 5 Biomedical Applications

### 5.1 Some considerations concerning fractional order models

The first applications of fractional calculus to biomedical problems were in the areas of membrane biophysics and polymer viscoelasticity [15], where the experimentially observed power law dynamics for current-voltage and stress-strain relationships were concisely captured by fractional order differential equations. Subsequently, the work of Mandelbrot in the field of fractals [21] and of others in the emerging fields of chaos and nonlinear systems attracted much attention to biomedical applications of fractional calculus. For example, there is evidence that biological signals (ECG, EMG, and EEG) have spectra that do not increase or decrease by multiples of 20 dB [2,4,17,20]. Hence, system models with poles and zeros of fractional order are often proposed for both analytical and emprical reasons. Here, we describe examples of biomedical applications of fractional calculus taken from the fields of bioinstrumentation, mechanobiology and biomedical imaging.

Physiological models based on linear differential equations are highly successful in describing a wide range of complex phenomena (e.g., action potential propagation, blood oxygenation and filtration, and feedback control of insulin secretion). Such models, also serve as the basis for understanding normal physiological homeostatis, as well as the changes that arise as a consequence of disease. Physiological models connect events at the molecular level (ion transport, gas diffusion, vesicle formation) to those at the organ level (blood clearance, oxygen uptake/gram tissue, muscle tension). Much current work in biophysics and physiology is directed toward linking molecular processes with whole organ (brain, heart, and muscle) function by developing muliscale models that span the intermediate levels of structure (e.g., from the centimeter dimensions of gross anatomy down to the submicron resolution of histology).

In building multiscale models one can either try to use as much of the available anatomical and histological knowledge as possible - building a highly complex structures with hundreds of components (organelles, membranes, cells, extracellular matrix, etc.) - or try to deal empirically with the complexity by developing whole system descriptions (e.g., linear, non-linear, deterministic, or stochastic models) with embedded chaotic or fractal measures (fractal dimensions, Lyapunov exponents, non-Gaussian probability distributions) that capture important features of the observed behavior [2,31]. A diagram illustrating some of the relationships between these approaches is shown in Figure 1. In this figure the models are characterized on the X-axis by their degree of linearity and on the Y-axis with respect to their deterministic nature. Linear time-invariant causal (LTIC) system models cluster in the first quadrant, while stochastic, probabilistic models fall in the fourth quadrant [4]. In this representation the methods of fractional calculus (linear, deterministic, but non-integer order) bundle together in Figure 1 within the relm of LTIC system models where they interpolate between the conventional integer order differential operators and extend the dynamics to fractional order [33].

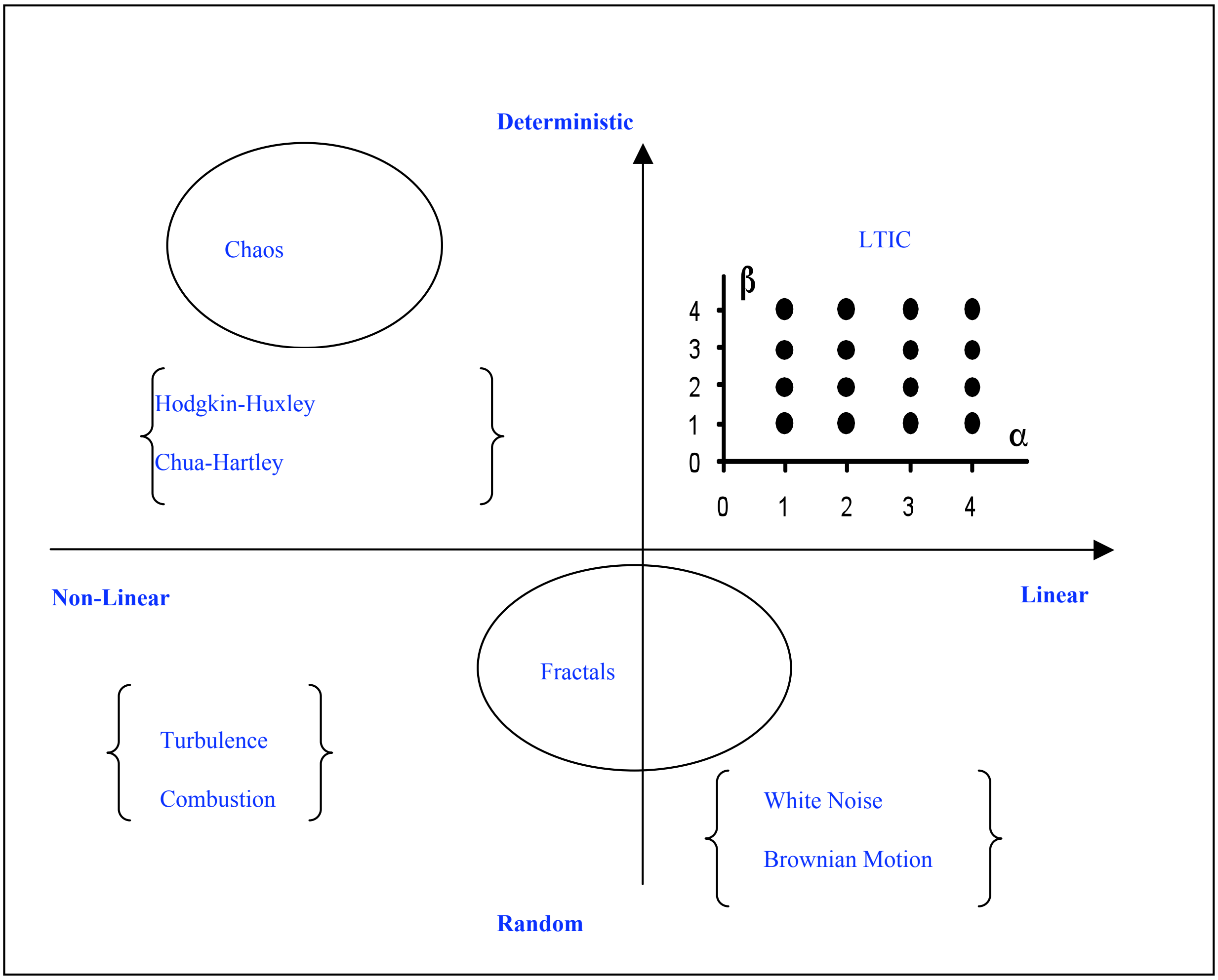


*Fig. 1. Illustration of the relationship between the principle types of models used to describe complex systems. For conventional linear time-invariant causal (LTIC) models the governing differential equations take on only integer order.*

## 5.2 Fractional Dynamics Model

A fractional order model is commonly used to describe the behavior of neural systems (senaory and motor). A simple example is the vestibular-oculomotor system modeled by Anastasio [30, 31] in the Laplace domain as $s^{\alpha}$, where $-1 < \alpha < 1$. The occurrence of $s^{\alpha}$ behavior in the transfer functions for the neural components of vestibulo-oculomotor systems suggests its need to control or monitor the underlying biological, physical, or chemical mechanisms. The $s^{\alpha}$ behavior follows directly from observed power law transient and dynamic behavior unique to the anatomical structure or neurological connections of living systems. Thus, the subthreshold behavior of axons, which mimic at their most basic level lossy (RC) transmission lines with fractional impedance relationships, could play a role in understanding synapse complexity, dendritic convergence and generator potential initiation.

For example, the encoding of head motion by the inner ear arises via convergence of unmyelinated afferent and efferent nerve fibers in the vestibular neuroepithelium. This has been suggested as an anatomical site where summation of excitatory and inhibitory postsynaptic potentials can occur (Figure 2). In a paper on distributed relaxation processes in sensory adaptation, John Thorson and Marguerite Biederman-Thorson [171] reviewed earlier interpretations for fractional dynamics (non-linear spring, transmission line, and Gaussian distribution of exponential rate constants), which they found for the most part, to provide an incomplete explanation for the wide dynamic range of sensory adaptations. These considerations led to the fractional order model presented in section 2.8.

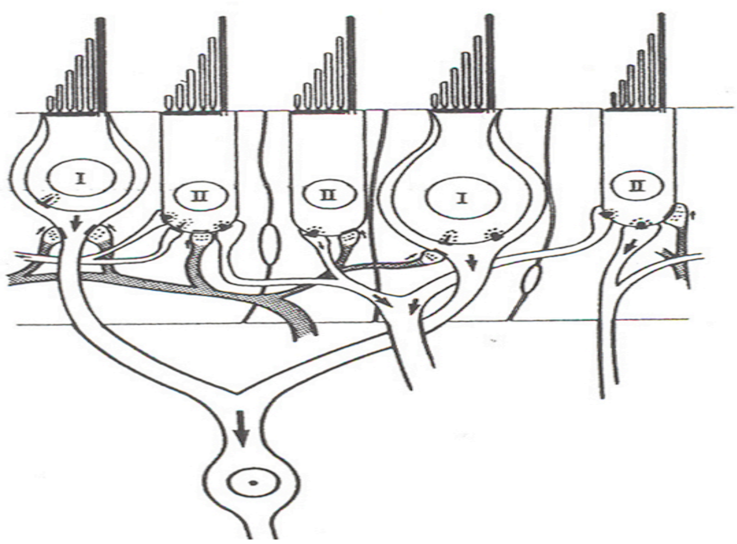


Fig. 2. A drawing of the complex, multiscale neural pathways (hair cells, axons, synapses, neurons) in the vestibular apparatus of the inner ear. (adapted from [20])

### 5.3 Fractional Impedance Model

Distributed relaxation processes appear to be common in cells and tissues. Therefore, it should not be surprising to see that fractional calculus can play an important role in describing the input-output behavior of biological systems. The physical foundations for this behavior may be sought in the fractal or porous structure of the system components or in the physical characteristics of its surfaces and interfaces. Much work [15] is ongoing to develop a direct link between fractal models of molecules, surfaces, and materials and the fractional kinetics or dynamics of the resulting behavior (polymerization electrochemical reactions, viscoelastic relaxation).

A major attribute of fractional dynamic models is that they interpolate between the known integer order behavior by extending the transfer function models, $f(s)$, from rational algebraic functions of integer powers of $s$ to irrational functions involving fractional powers of $s$. This is natural approach that extends the traditional Laplace transform methods of linear systems analysis [20]. Thus, the fractional dynamics hypothesis is accessible to the engineer and scientist through both Laplace and Fourier techniques (for $s = j\omega$, where $\omega$ is the angular frequency in radians/sec).

Fractional order circuit elements, such as the impedance: $Z = Z_0/(s)^{\alpha}$ or $Z = Z_0/(j\omega)^{\alpha}$, where $0< \alpha < 1$, provide a useful model for describing the transient and the sinusoidal steady state frequency response of dielectrics and biological tissues [11,20]. Such circuit elements can also be used to develop an electrical circuit model of the electrode-cardiac tissue interface of a pacemaker electrode (Figure 3). A lumped element circuit model for the cardiac tissue/electrode interface developed by Ovadia and Zavitz, [154] is shown in Figure 4. Accurate impedance models are essential for designing cardiac pacemakers. Fractional calculus appears in the model through the fractional order (or constant phase, $Z = Z_0\omega^{-\alpha}exp(j\tan^{-1}(\pi\alpha/2))$) circuit element $Z_D$ that governs diffusion limited electrochemical reactions at the surface of the electrode.

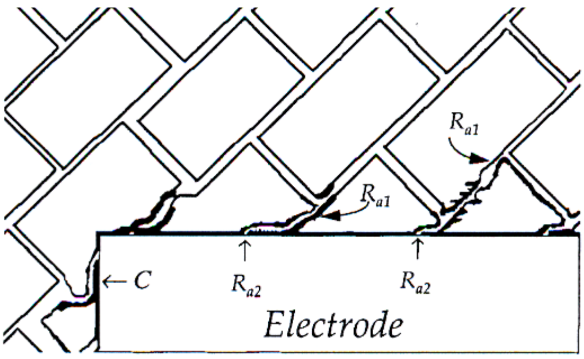


**Fig. 3.** *A drawing of the tissue-electrode interface between cardiac muscle cells and an implanted electrode. (redrawn from Ovadia and Zavitz, 1998)*

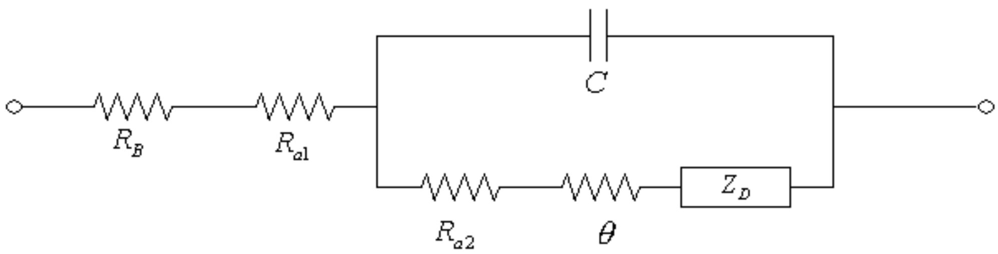


**Fig. 4.** *Tissue-electrode circuit model. $R_B$ is the bulk tissue resistance, $R_{a1}$ and $R_{a2}$ are electrode access resistances, $\theta$ is the charge transfer resistance, C is the dipole layer capacitance and $Z_D$ is the fractional Warburg impedance.*

If we assume that C, the dipole layer capacitance, is small enough so that its reactance can be neglected in comparison with $Z_D$, then the tissue-electrode equivalent circuit reduces to a resistor in series with $Z_D$, which can be approximated by two constant phase elements in series. Thus, in the Laplace domain, the overall impedance can be written as

$$z(s) = \frac{v(s)}{i(s)} = R + \frac{1}{s^{\alpha} C_{\alpha}} + \frac{1}{s^{\beta} C_{\beta}}. \qquad (115)$$

The corresponding impedance plane plot for (115) is shown in Figure 5 for the simple case of $\alpha = ½$ and $\beta = 1$. Such plots match the data measured in experimental studies of Ovadia and Zavitz [154]. The transient voltage response of this circuit to a step in applied current, such as the leading edge of a pacemaker pulse, is described in the time domain by

$$V(t) = I_0 R + \frac{I_0\, t^{\alpha}}{C_{\alpha}\Gamma(\alpha+1)} + \frac{I_0\, t^{\beta}}{C_{\beta}\Gamma(\beta+1)} \qquad (116)$$

which gives a power law response that corresponds to that observed in heart stimulation experiments by Greatbatch and Chardack [88].

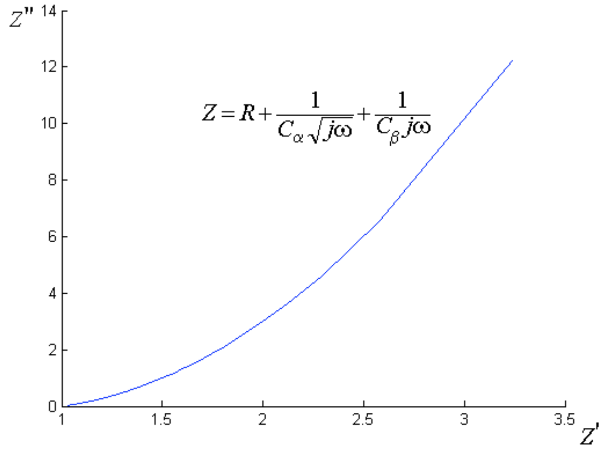


**Fig. 5.** *Impedance plane plot for two constant phase element impedances in series with a resistor. In this example, we set* $R = C_\alpha = C_\beta = 1$, *and* $\alpha = ½$, $\beta = 1$.

Thus, we observe that the basic cardiac tissue electrode impedance can be represented by a series combination of a resistor and two fractional lumped circuit elements. The overall transfer function for this model corresponds to the following fractional order differential equation,

$$C_\alpha \frac{d^\alpha V(t)}{dt^\alpha} = R\, C_\alpha \frac{d^\alpha I(t)}{dt^\alpha} + I(t) + \frac{C_\alpha}{C_\beta} \frac{d^{\alpha-\beta} I(t)}{dt^{\alpha-\beta}} \tag{117}$$

if we assume $\alpha > \beta$.

We can use the correspondence between RC electric circuits and viscoelastic networks of springs and dashpots to construct similar fractional order dynamic models for the biomechanical properties of tissues [19]. For example, Craiem and Armentano [67] have modelled the elastic properties of the aorta, *in vivo* in a Merino sheep, using a fractional order generalization of the relationship between stress $\sigma(t)$ and strain $\varepsilon(t)$. Their generalized Voigt model consists of a spring in parallel with two "springpots" of fractional order $\alpha$ and $\beta$. The governing fractional order differential equation is

$$\sigma(t) = E_0\, \varepsilon(t) + \eta_1 \frac{d^\alpha \varepsilon(t)}{dt^\alpha} + \eta_2 \frac{d^\beta \varepsilon(t)}{dt} \tag{118}$$

where $E_0$ is the elastic constant for a spring, and $\eta_1$ and $\eta_2$ represent the viscosities of two springpots in parallel with the spring. From this equation the complex modulus $E^*(\omega)$ can be defined for sinusoidal signals as the ratio of stress to strain by

$$E^*(\omega) = \frac{\sigma(\omega)}{\varepsilon(\omega)} = E_0 + \eta_1 (j\omega)^\alpha + \eta_2 (j\omega)^\beta . \tag{119}$$

The real part of $E^*(\omega)$ is defined as the storage modulus and the imaginary part of $E^*(\omega)$ is the loss or dissipation modulus. The storage modulus characterizes the elastic property of the arterial wall while the loss modulus describes the tissue's ability to absorb energy. Both properties change with frequency and govern the pulsatile oscillations of the vessel walls in health and disease. This model was found by Craiem and Armentano to give a better fit to *in vivo* data recorded from 2 to 30 Hz than a Voigt model (single spring in parallel with a dashpot) or a fractional Voigt (single spring in parallel with single springpot). A vector plot in the complex plane of the complex modulus for this study is shown in Figure 6.

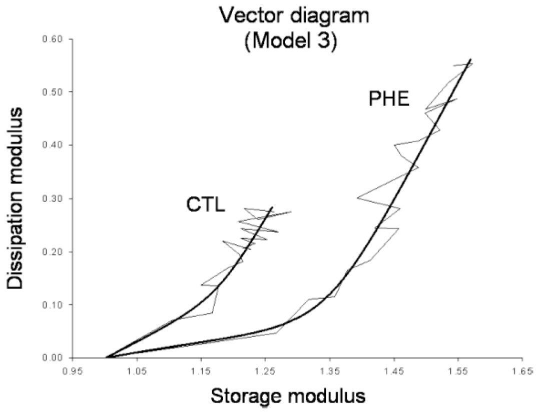


**Fig. 6.** *Vector diagram (complex plane plot) of (10) for in vivo modulus data from an aorta under control (CTL) conditions and following application of a vasoconstrictive agent (PHE). (redrawn from, Craiem and Armentano, 2007)*

In particular, the model (119) captures the changes that arise in vessel wall elasticity when a vascular constriction is induced by the local administration of phenylephrine [67]. The authors conclude that the $\alpha$ springpot appears to describe the stretching of the elastic fibers of the aorta ($\alpha$ is close to zero), while the $\beta$ springpot seems to represent a structural viscous behavior ($\beta$ closer to 1). As expected the elastic contribution increases – $\alpha$ decreases from 0.20 to 0.11 - following administration of phenylephrine while the loss term is relatively unchanged (0.84 to 0.80). Thus, for a complex multi-scale tissue such as the arterial wall, the fractional order model is able to characterize the important features of its dynamic behavior.

Fractional order models have also been used by Sinkus et al [168] to fit magnetic resonance elastography (MRE) data from breast tumors. In this technique, MRI is used to image low frequency (50 – 1,500 Hz) shear wave oscillations in the breast. The wavelength and attenuation of the vibrations directly reflect the elastic shear modulus and the viscosity of the tissue through the complex wave vector: $k(\omega) = \beta(\omega) + j\alpha(\omega)$. In MRE these tissue properties are mapped into an elastogram image through an assumed model of the tissue's mechanical properties – usually a purely elastic spring with zero loss, or a Voigt spring/dashpot model. In his study, Sinkus assumed a power law increase in attenuation with excitation frequency, $\alpha(\omega) = \alpha_0\omega^y$ (where $0 < y < 1$), and invoked causality via the Hilbert transform to obtain the propagation constant as, $\beta(\omega) = \tan(\pi y/2)\ \alpha_0\omega^y$. Thus, for

$$k(\omega) = \alpha_0\ \omega^y\ e^{-j\pi/2}\sqrt{1 + \left(\tan(\pi y/2)^2\right)} \tag{120}$$

$k(\omega)$ is related to the complex shear modulus $G^*(\omega)$ through

$$k(\omega) = \omega\sqrt{\rho / G^*(\omega)}\ , G^*(\omega) = \left|G^*(\omega)\right| e^{j\theta} \tag{121}$$

such that the modulus and phase can be written as,

$$|G^*(\omega)| = \rho\omega^\gamma/\alpha_0^2(1+\chi^2),\ \ \theta = \tan^{-1}(G_l/G_d) = \pi y \tag{122}$$

where $\gamma = 2\text{-}2y$. The advantage of this model is that it does not specify a particular Maxwell, Voigt, or Kelvin rheological model, but simply assumes an underlying fractional order dynamics, $\omega^y$, and then estimates the fractional power law parameters $y$ and $\alpha_0$ from the MRE data. Sinkus first verifies this model for a tissue mimicking breast phantom at a fixed frequency of 65 Hz and then applies the model to human breast tissue by measuring the dynamic modulus at 65, 75, 85, and 100 Hz. A complex plane plot of $G_d$ and $G_l$ gives a straight line with a $y$ value of approximately 0.13 for normal tissue. Analysis of 39 malignant and 29 benign tumors using this method gives a clear separation of the tumors from the normal (and fibrotic) breast tissue, and furthermore separates the malignant from the benign tumors when individual cases are plotted in a graph (Figure 7) of $y$ versus $\alpha_0$ (an increase in specificity of about 20 % at 100 % sensitivity). In earlier studies this group was not able to classify breast tumors on the basis of $G_d$ and $G_l$ alone, so this model provides a significant improvement in cancer detection.

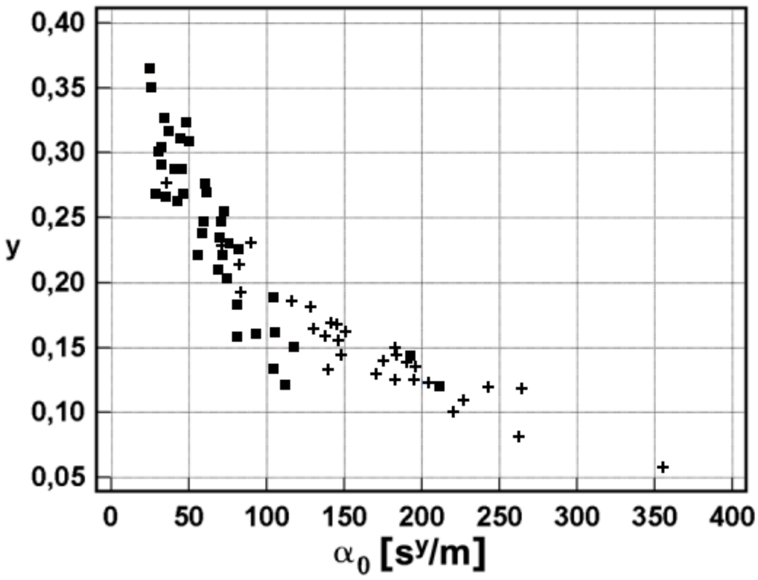


Fig. 7. *Plot of benign (+) and malignant (■) breast tumor MRE data for 39 patients. These data are replotted from Sinkus et al., (2007).*

In the three examples considered here, fractional order models were found to provide better fits to electrical and mechanical measurements made on living tissue. Such studies need replication, but these findings provide useful examples of cases where an extension of the "standard" integer order dynamic models of circuits and mechanical systems is warranted. Fractional order dynamic models of complex, multiscale systems account for anomalous dynamic behavior through a simple extension of the order of the operations from integer to fractional. In the time domain this extension is manifest through incorporation to a variable degree of system memory through convolution with a power law kernel exhibiting fading memory of the past. Perhaps, in the future, the development of integrated space and time domain fractional order models will become a more standard component of linear systems analysis, at least when such models are applied to living systems. Clearly, when the structure in living systems is fractal, or when the measured signals exhibit anomalous properties, one should suspect that such dynamics might best be expressed by fractional order models. Much remains to be done, and we look the philosopher Henri Bergson to provide inspiration, for, as Bergson [3] noted in his 1911 work *Creative Evolution*: "the present contains nothing more than the past, and what is found in the effect was already in the cause".

## 6 The fractional Brownian motion

Fractional Brownian motion was introduced first by Kolmogorov [105]. Later, Mandelbrot and Van Ness [21,121] proposed it as a model for non stationary signals, with stationary increments, that are useful in understanding phenomena with long range dependence and with a frequency dependence of the form $1/f^{\alpha}$, with $\alpha$ non integer [103,181-183]. In [146] an approach based on the fractional derivatives was proposed and will be described next.

Assume now that we are computing the fractional derivative of the white noise, w(t), with power equal to $\sigma^2$. We define a fractional noise by:

$$r_{\alpha}(t) = D^{\alpha}\, w(t) \tag{123}$$

If w(t) is Gaussian, we will call $r_{\alpha}(t)$ fractional Gaussian noise. As known, the autocorrelation function of the white noise is $\sigma^2\delta(t)$. With some work, we obtain for the derivative autocorrelation [65]:

$$R_r^{\alpha}(t) = \lim_{h\to 0} \frac{\Gamma(2\alpha+1)}{h^{2\alpha}} \sum_{-\infty}^{+\infty} \frac{(-1)k}{\Gamma(\alpha-k+1)\,\Gamma(\alpha+k+1)}\,\delta(t - kh) \tag{124}$$

where $R_r^{\alpha}(t) = E[r_{\alpha}(\tau+t)r_{\alpha}(\tau)]$. The right hand side is a sequence of weighted impulses that become close together as h goes to zero. If $\alpha > -1/2$ (124) is a centred derivative [144] of the $\delta(t)$ and can be expressed by

$$R_r^{\alpha}(t) = \frac{1}{2\Gamma(-2\alpha)\cos(\alpha\pi)}\,|t|^{-2\alpha-1} \tag{125}$$

that represents an autocorrelation function, having a maximum at the origin, if

$$\begin{cases} 2\alpha+1>0 \\ \Gamma(-2\alpha)\cos(\alpha\pi)>0 \end{cases} \tag{126}$$

The first condition ($\alpha > -1/2$) was already assumed. As

$$\frac{1}{2\Gamma(-2\alpha)\cos(\alpha\pi)} = -\frac{\Gamma(2\alpha+1).\sin(2\alpha\pi)}{2\pi\cos(\alpha\pi)} = -\frac{\Gamma(2\alpha+1)\,\sin(\alpha\pi)}{\pi} \tag{127}$$

it is not hard to see that for $-1/2 < \alpha < 0$ and $\alpha \in (2n,2n+1)$, $n \in Z^+$ we obtain valid autocorrelation functions. We conclude that, in the interval $-1/2 < \alpha < 1/2$ we obtain a stationary process in the integration case ($\alpha<0$) and nonstationary in the derivative case ($\alpha>0$). This fractional noise will be used next to define the fractional Brownian motion. Let $r_{\alpha}(t)$ be a fractional noise. Define a process $v_{\alpha}(t)$, $t\geq 0$, by:

$$v_{\alpha}(t) = \int_0^t r_{\alpha}(\tau)d\tau \tag{128}$$

We will call this process a fractional Brownian motion (or generalised Wiener-Lévy process). It is not difficult to show that it enjoys all the properties normally required for the fBm [21,121]:

1 - $v_\alpha(0) = 0$ and $E\{v_\alpha(t)\} = 0$ for every $t \geq 0$. If w(t) is Gaussian, so it is $r_\alpha(t)$ and $v_\alpha(t)$. The proposed definitions do not need the Gaussianity.

2 – The covariance is [144]:

$$E[v_\alpha(t)\, v_\alpha(s)] = \frac{\sigma^2}{2\Gamma(-2\alpha+2).\cos\alpha\pi}\left[|t|^{-2\alpha+1}+|s|^{-2\alpha+1} - |t - s|^{-2\alpha+1}\right] \tag{129}$$

valid for $|\alpha| < 1/2$. Putting $H= - \alpha + 1/2$ wth $H\in(0,1)$, we obtain the usual formulation:

$$E[v_\alpha(t)\, v_\alpha(s)] = \frac{V_H}{2}\left[|t|^{2H}+|s|^{2H} - |t - s|^{2H}\right] \tag{130}$$

with

$$V_H= \frac{\sigma^2}{\Gamma(2H+1)\sin H\pi} \tag{131}$$

The variance is readily obtained:

$$E[v_\alpha(t)^2] = V_H\, |t|^{2H} \tag{132}$$

3 – The process has stationary increments.

Letting the increments be defined by

$$\Delta v_\alpha(t,s)= v_\alpha(t) - v_\alpha(s) = \int_s^t r_\alpha(\tau)d\tau \tag{133}$$

its variance is given by [6]:

$$\mathrm{Var}\left\{\Delta v_\alpha\,(t,s)\right\} = \sigma^2 \frac{|t-s|^{-2\alpha+1}}{2\Gamma(-2\alpha+2).\cos\alpha\pi} \tag{134}$$

4 – The process is self similar

From (130), we have:

$$E[v_\alpha(at)\, v_\alpha(as)] = \frac{V_H}{2}\, |a|^{2H} \left[|t|^{2H}+|s|^{2H} - |t - s|^{2H}\right] \tag{135}$$

5 – The incremental process has a $1/f^{\beta}$ spectrum

Defining an incremental process by (133) and choosing s = t-T:

$$d_H(t)= v_H(t) - v_H(t-T) \tag{136}$$

has an autocorrelation function given by

$$R_d(\tau) = \frac{V_H}{2}\left[\,|\tau+T|^{2H} + |\tau-T|^{2H} - 2|\tau|^{2H}\right] \tag{137}$$

and, as [71]

$$FT\left[\frac{1}{2\Gamma(\beta)\cos(\beta\pi/2)}\,|t|^{\beta-1}\right] = \frac{1}{|\omega|^{\beta}} \tag{138}$$

we obtain the spectrum of the incremental process:

$$S_d(\omega) = \sigma^2.\frac{\sin^2(\omega T/2)}{|\omega|^{2H+1}} \tag{139}$$

For | ω | << π/T, the spectrum can be approximated by:

$$S_d(\omega) \approx.\frac{\sigma^2 T^2}{4}\frac{1}{|\omega|^{2H-1}} \tag{140}$$

We conclude that the proposed definition agrees with the Mandelbrot and van Ness results.

The result expressed in (140) is interesting [21,121]:

- If 0<H<1/2, the spectrum is parabolic and corresponds to an antipersistent fBm, because the increments tend to have opposite signs, this case corresponds to the integration of a stationary fractional noise.
- If 1/2<H<1, the spectrum has a hyperbolic character and corresponds to a persistent fBm, because the increments tend to have the same sign, this case corresponds to the integration of a nonstationary fractional noise.

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**8** CONCLUSIONS

Fractional calculus models provide a relatively simple way to describe the physical and electrical properties of complex, heterogeneous, and composite biomaterials. There is a multi-scale generalization inherent in the definition of the fractional derivative that accurately represents interactions occurring over a wide range of space or time. Thus, we can avoid excessive segmentation or compartmentalization of tissues into subsystems or subunits - a system reduction that often creates more computational and compositional complexity than can be experimentally evaluated. Finally, fractional calculus models suggest new experiments and measurements that can shed light on the meaning of biological system structure and dynamics. Thus, by applying fractional calculus to model the behavior of cells and tissues, we can begin to unravel the inherent complexity of individual molecules and membranes in a way that leads to an improved understanding of the overall biological function and behavior of living systems.

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