

Symmetrical Z-Complementary Code Sets for Optimal Training in Generalized Spatial Modulation

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Abstract

This paper considers the optimal training design for broadband generalized spatial modulation systems over frequency-selective channels using a novel class of code sets introduced, called “symmetrical Z-complementary code sets”, whose aperiodic auto- and cross- correlation sums exhibit zero-correlation zones at both the front-end and tail-end of the entire correlation window. Two constructions of (optimal) symmetrical Z-complementary code sets based on generalized Boolean functions are presented. Numerical evaluations indicate that the proposed training sequences for generalized spatial modulation

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can achieve optimal channel estimation performance and outperform other classes of sequences.

Keywords: Complementary code set, channel estimation, training sequence design, generalized spatial modulation, frequency-selective channels

1. Introduction

Spatial modulation (SM) is a multiple-input multiple-output (MIMO) technique which enjoys zero inter-channel interference over flat fading channels, low hardware complexity and low energy consumption. In SM, information is transmitted over two parts: 1) spatial dimension, coordinated by antenna indices, and 2) the conventional signal constellation of a modulation scheme[1]. That is, information bits are conveyed through not only modulation symbols, but also the index of the active transmit antenna (TA) [2]. Unlike SM system equipped with multiple TA elements but only a single radio-frequency (RF) chain, two or more TAs are activated in generalized spatial modulation (GSM) at each symbol slot and the specific activated pattern itself conveys useful but implicit information [3]. Therefore, GSM can achieve higher spectral efficiency than SM systems while maintaining major advantages of SM [4, 5]. As a result, GSM strikes a flexible trade-off among spectral efficiency, cost of RF chains and energy efficiency by varying the number of RF chains[6, 7]. These advantages make GSM a competitive candidate for the next generation wireless networks [8].

There have been many iterative and non-iterative detector designs for GSM systems [9]. However, most of these detectors, e.g., the tree-search based detector [10] and the near-ML-detector [11], consider narrow-band s-

scenarios where GSM symbols are transmitted over flat-fading channels. This is different from practical wireless channels which may exhibit frequency selective properties. In recent years, various types of GSM detectors have been proposed for broadband GSM to combat the effects of inter-symbol interference imposed by frequency-selective fading channels [12, 13, 14]. For example, a novel soft-decision feedback aided time-domain turbo equalizer based on the minimum mean-square error criterion [14] was proposed for broadband GSM systems. In [15], a frequency-domain based turbo equalizer was developed for GSM under dispersive channels. In these works, perfect channel state information at the receiver was assumed. Recently, there appeared some literature on the channel estimators in GSM systems. For example, a channel estimation scheme based on block pilot pattern and a interpolation method was proposed in [16], which is designed for the GSM-orthogonal frequency-division multiplexing system on high-speed railways. In [17], they proposed pilot-assisted and data-aided super-resolution MIMO channel estimators for GSM-based millimeter-wave systems. Message-passing based blind channel-and-signal estimation and semi-blind channel-and-signal estimation algorithms were developed for massive MIMO systems with GSM [18], which utilize the channel sparsity of the massive MIMO channel and the signal sparsity of GSM. However, to the best of our knowledge, all the existing methods for channel estimation in GSM systems are about the blind and semi-blind estimation algorithms, which usually have high computational complexity, and can not attain optimal estimation in theory.

A major concern of this work is how to attain optimal estimation of channel state information in broadband GSM systems. A common means of ob-

taining channel state information is by sending properly designed preamble sequences at the transmitter, followed by correlating the known preamble sequences at the receiver. It is noted that optimal channel estimation requires preamble sequences with zero nontrivial auto- and cross-correlations. However, the employment of conventional preamble sequences in GSM is not straightforward. Since only a few RF chains whose number is less than that of TAs are activated at each GSM time-slot and hence the transmit signal of GSM is sparse, it is hard to adopt those training sequences which are dense for traditional MIMO systems [19, 20, 21]. On the other hand, when the number of RF chains is greater than one, training sequences designed for SM systems [22] are also inapplicable. To attain optimal training in GSM systems, we propose and construct a novel class of code sets called symmetrical Z-complementary code set (SZCCS). In the sequel, we overview a number of code sets followed by our major contributions of this work.

1.1. Complementary Code Sets

Sequence sets with good correlation properties play an instrumental role in almost every communication system. For example, they can be used for localization [23], synchronization, channel estimation [24, 25], and interference suppression/mitigation in multiuser systems [26, 27, 28]. An interesting sequence family is called “complementary code” which was proposed by Tseng and Liu in [29], where the aperiodic auto-correlation sum of all the constituent sequences equals zero at any nonzero time-shift. A special case of complementary code is Golay complementary pairs (GCPs) which were first found by Golay [30] and each GCP consists of only two sequences.

Extensive research attempts have been made concerning the constructions

and applications of complementary codes and GCPs. A remarkable work was contributed by Davis and Jedwab [31] who constructed polyphase GCPs [32] using the algebraic tool of generalized Boolean functions. Subsequently, representative constructions of complementary codes were proposed in [33, 34, 35, 36]. To obtain more flexible parameters, “complementary code” was extended to Z-complementary code (ZCC) in [37], where the aperiodic auto-correlation sum is zero when the time-shift falls into a region around the in-phase position called zero-correlation zone (ZCZ).

Besides, Tseng and Liu studied in [29] “mutually orthogonal complementary code set (MOCCS)”, which refers to a set of complementary codes with zero aperiodic cross-correlation sums between different complementary codes. Further developments of MOCCSs were reported in [38, 39, 40, 41, 42]. However, MOCCS suffers from limited set size which is upper bounded by the number of constituent sequences (i.e., the flock size) in each code. To overcome this weakness, the concept of MOCCS was extended to Z-complementary code sets (ZCCSs) for higher set sizes [37].

So far, the existing known constructions of ZCCSs only consider the front-end¹ ZCZ of the aperiodic auto-correlation sums and aperiodic cross-correlation sums. Unlike the state-of-the-art works, we consider a specific subclass of ZCCSs named SZCCSs which exhibit both the front-end ZCZ and tail-end ZCZ properties. In practice, a front-end ZCZ and a tail-end ZCZ have particular interest for mitigating interference with small and large delays, respectively. We will show in Section 3 that the properties of SZCCSs

¹with respect to the entire correlation window.

are useful for mitigating multipath interference and multi-antenna interference in GSM systems.

1.2. Novelty and Contributions of This Paper

The main contributions and novelty of this paper are summarized as follows.

- 1) We present a generic training framework for optimal GSM training over frequency selective channels. The most distinctive feature of the proposed generic training framework (compared that for conventional MIMO systems) is that the training matrix should be sparse owing to the sparsity of the GSM transmit signals. Based on this framework, we derive the optimal GSM training criteria under least square channel estimator.

- 2) According to the derived the optimal GSM training criteria, we introduce a novel class of ZCCSs called SZCCSs, each of which displaying zero tail-end auto/cross-correlation sums symmetrical to that of the front-end ones. More restrictive than the conventional ZCCSs, the design challenge of SZCCSs stems from the additional correlation properties associated to the tail-end ZCZ, and show that SZCCS plays a pivotal role in the design of optimal GSM training sequences. Then, we propose two systematic constructions of SZCCS with different set sizes and sequence lengths using generalized Boolean functions. Especially, the SZCCSs generated from the first construction is optimal with power-of-two lengths, and the SZCCSs of non-power-of-two lengths constructed in the second construction have larger zero-correlation ratio than the

former with specific parameters, which means that its sequence length can be smaller than the former when the maximum number of multipaths is same, so as to improve spectral efficiency. To the best of our knowledge, this paper is the first to consider the sequence design for the channel estimation in GSM systems. Numerical evaluations indicate that the proposed GSM training sequences lead to minimum channel estimation mean-square error and significantly outperform other classes of sequences (e.g., Zadoff-Chu sequences and binary random sequences) with different settings of activated transmit antennas and signal-to-noise ratios.

1.3. Organization of This Paper

The remainder of this paper is organized as follows. Section 2 introduces some notations, ZCCSs and the mathematical tools used in the paper followed by a sketch of the basic principle of GSM. In Section 3, first, we present a generic training framework for GSM training over frequency selective channels. Then, we derive the optimal GSM training criteria under LS channel estimator. In Section 4, we first present SZCCSs which play an instrumental role in the design of optimal GSM training sequences, and show an upper bound of the size of SZCCSs. Then, we give two constructions of SZCCSs with different set sizes based on generalized Boolean functions where one is optimal. Section 5 shows some numerical evaluations of the proposed GSM training sequences. Finally, Section 6 concludes this paper with some remarks.

For readability, we summarize in Table 1 all the acronyms which are used in this paper.

Table 1: List of acronyms

Acronyms	Descriptions
CP	cyclic prefix
CZCP	cross Z-complementary pair
GCP	Golay complementary pair
GSM	generalized spatial modulation
LS	least square
MIMO	multiple-input multiple-output
MOCCS	mutually orthogonal complementary code set
MSE	mean-square error
RF	radio-frequency
SM	spatial modulation
SZCCS	symmetrical Z-complementary code set
TA	transmit antenna
ZCC	Z-complementary code
ZCCS	Z-complementary code set
ZCZ	zero-correlation zone
ZP	zero prefix

2. Preliminaries

2.1. Notations

The following notations will be used throughout this paper.

- $T^\tau(\mathbf{X})$ denotes the right-cyclic-shift of matrix \mathbf{X} for τ (non-negative integer) positions over rows;

- $\mathbf{0}_{m \times n}$ denotes an all-0 matrix of order $m \times n$;
- $[j]_J$ denotes the modulo J operation of integer j ;

Let $\mathbf{a} = (a(0), a(1), \dots, a(L-1))$ and $\mathbf{b} = (b(0), b(1), \dots, b(L-1))$ be two complex-valued sequences of length L . The aperiodic cross-correlation function between \mathbf{a} and \mathbf{b} at a time shift u is defined by

$$\rho_{\mathbf{a}, \mathbf{b}}(u) = \begin{cases} \sum_{i=0}^{L-1-u} a(i)b^*(i+u), & 0 \leq u \leq L-1; \\ \sum_{i=0}^{L-1+u} a(i-u)b^*(i), & -(L-1) \leq u \leq -1; \\ 0, & |u| \geq L. \end{cases}$$

It is easy to verify that

$$\rho_{\mathbf{a}, \mathbf{b}}(u) = \rho_{\mathbf{b}, \mathbf{a}}^*(-u). \quad (1)$$

If $\mathbf{a} = \mathbf{b}$, $\rho_{\mathbf{a}, \mathbf{b}}$ is called the aperiodic auto-correlation function, denoted by $\rho_{\mathbf{a}}$ for simplicity.

Also, denote by $\phi_{\mathbf{a}, \mathbf{b}}(u)$ the periodic cross-correlation between \mathbf{a} and \mathbf{b} , i.e.,

$$\phi_{\mathbf{a}, \mathbf{b}}(u) = \sum_{i=0}^{L-1} a(i)b^*([i+u]_L).$$

In particular, let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_M\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_M\}$ be two sequence sets of size M and length L . The aperiodic cross-correlation function between \mathcal{A} and \mathcal{B} at a time shift u is defined by $C_{\mathcal{A}, \mathcal{B}}(u) = \sum_{m=1}^M \rho_{\mathbf{a}_m, \mathbf{b}_m}(u)$. Similarly, if $\mathcal{A} = \mathcal{B}$, $C_{\mathcal{A}, \mathcal{B}}$ is called the aperiodic auto-correlation function of sequence set \mathcal{A} , denoted by $C_{\mathcal{A}}$ for simplicity.

2.2. Generalized Boolean Functions

Let q be a positive integer, for $\mathbf{x} = (x_1, x_2, \dots, x_m) \in \mathbb{Z}_2^m$, a generalized Boolean function $f(\mathbf{x})$ is defined as a mapping f from $\{0, 1\}^m$ to \mathbb{Z}_q . Given

$f(\mathbf{x})$, define

$$\mathbf{f} = (f(0), f(1), \dots, f(2^m - 1)), \quad (2)$$

where $f(i) = f(i_1, i_2, \dots, i_m)$, and (i_1, i_2, \dots, i_m) is the binary representation of $i = \sum_{k=1}^m i_k 2^{k-1}$ with i_m denoting the most significant bit.

In this paper, we consider truncated versions of the sequence \mathbf{f} of (2). Specifically, let $\mathbf{f}^{(L)}$ be a sequence of length L obtained from \mathbf{f} by ignoring the last $2^m - L$ elements of the sequence \mathbf{f} . That is, $\mathbf{f}^{(L)} = (f(0), f(1), \dots, f(L-1))$ is a sequence of length L . Let $\xi_q = \exp(2\pi\sqrt{-1}/q)$ be a primitive q -th complex root of unity. One can naturally associate a complex-valued sequence $\psi(\mathbf{f}^{(L)})$ of length L with $\mathbf{f}^{(L)}$ as

$$\psi(\mathbf{f}^{(L)}) := (\xi_q^{f(0)}, \xi_q^{f(1)}, \dots, \xi_q^{f(L-1)}). \quad (3)$$

From now on, whenever the context is clear, we ignore the superscript of $\mathbf{f}^{(L)}$ unless the sequence length is specified.

2.3. Introduction to ZCCS

Definition 1. Let $\mathcal{A} = \{\mathbf{a}_m\}_{m=1}^M$ be a set of M complex-valued sequences of length L . It is said to be a (aperiodic) (M, L, Z) -ZCC of size M if $C_{\mathcal{A}}(u) = 0$ for any $0 < |u| \leq Z$ where Z is a positive integer with $1 \leq Z \leq L - 1$. In particular, when $Z = L - 1$, the set is called a (aperiodic) complementary code, and when $M = 2$ it is called a (aperiodic) GCP.

The following lemma shows a construction of GCPs, which will be used in the sequel.

Lemma 1 (Corollary 11 of [33], Theorem 3.3 of [43]). *Let q be an even integer and m be a positive integer. Let*

$$a(\mathbf{x}) = \frac{q}{2} \sum_{k=1}^{m-1} x_{\pi(k)} x_{\pi(k+1)} + \sum_{k=1}^m c_k x_k + c,$$

$$b(\mathbf{x}) = a(\mathbf{x}) + \frac{q}{2} x_{\pi(1)}, \quad c(\mathbf{x}) = a(\mathbf{x}) + \frac{q}{2} x_m,$$

where π is a permutation of $\{1, 2, \dots, m\}$ and $\mathbf{x} \in \mathbb{Z}_2^m, c_k, c \in \mathbb{Z}_q$. Then $(\psi(\mathbf{a}), \psi(\mathbf{b}))$ and $(\psi(\mathbf{a}), \psi(\mathbf{c}))$ are GCPs of length 2^m .

Definition 2. Let $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_K\}$, where each $\mathcal{S}_k = \{\mathbf{s}_{k,1}, \mathbf{s}_{k,2}, \dots, \mathbf{s}_{k,M}\}$ ($1 \leq k \leq K$) be a ZCC consisting of M length- L sequences. \mathcal{S} is called a (K, M, L, Z) -ZCCS if $C_{\mathcal{S}_i, \mathcal{S}_k}(u) = 0, \quad \forall \quad 1 \leq i \neq k \leq K$ and $|u| \leq Z$. In particular, when $Z = L$, it is called a (K, M, L) -MOCCS.

The following lemma is about the upper bound of the size of ZCCS, which was proposed in [37].

Lemma 2. ([37], [44]) *Any unimodular (K, M, L, Z) -ZCCS $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_K\}$ satisfies $K \leq \lfloor \frac{ML}{Z+1} \rfloor$, where $\mathcal{S}_k = \{\mathbf{s}_{k,1}, \mathbf{s}_{k,2}, \dots, \mathbf{s}_{k,M}\}$ and $\mathbf{s}_{k,m} = (s_{k,m}(0), s_{k,m}(1), \dots, s_{k,m}(L-1))$ ($1 \leq k \leq K, 1 \leq m \leq M$). In particular, when $K = \lfloor \frac{ML}{Z+1} \rfloor$, \mathcal{S} is called an optimal (K, M, L, Z) -ZCCS.*

2.4. Introduction to GSM

We consider a single-carrier GSM (SC-GSM) system with N_t TA elements, N_r receive antennas and N_{active} transmit RF chains over frequency-selective channels. Moreover, we consider a QAM/PSK modulation with constellation size of \mathcal{M}_{GSM} . An $N_{\text{active}} \times N_t$ switch connects the RF chains to the TAs. In a given channel use, each user selects N_{active} in N_t TAs, and transmit

N_{active} symbols from a QAM/PSK modulation alphabet \mathbb{A} on the selected antennas. The remaining $N_t - N_{\text{active}}$ antennas remain silent. Fig. 1 shows the GSM transmitter at the user terminal. Over each time-slot k , there are

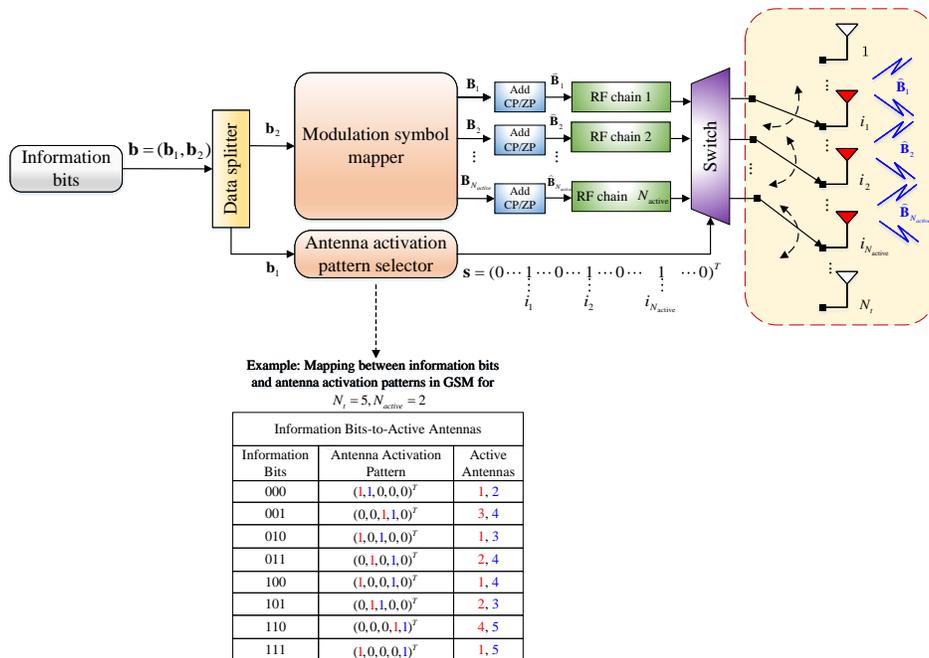


Figure 1: The structure of a GSM transmitter, where $\{i_1, i_2, \dots, i_{N_{\text{active}}}\}$ denotes the index set of activated TAs.

$\left\lceil \log_2 \binom{N_t}{N_{\text{active}}} \right\rceil + \lfloor \log_2 |\mathbb{A}| \rfloor$ bits, denoted by \mathbf{b} , conveyed by a GSM transmitter. Specially, the first $\left\lceil \log_2 \binom{N_t}{N_{\text{active}}} \right\rceil$ bits, denoted by \mathbf{b}_1 , are used to active the i_1 th, i_2 th, ..., $i_{N_{\text{active}}}$ th TAs through a antenna activation pattern selector, which is determined by the mapping between information bits and antenna activation patterns. The table in Fig. 1 gives an example of that mapping for $N_t = 5$ and $N_{\text{active}} = 2$. Suppose the selected antenna activation pattern is denoted by \mathbf{s} , where “1” in \mathbf{s} indicates that the antenna corresponding to that

coordinate is active and silent otherwise. The last $\lfloor \log_2 |\mathbf{A}| \rfloor$ bits, denoted by \mathbf{b}_2 , are used to select N_{active} “constellation symbols” $\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_{N_{\text{active}}}$ through a modulation symbol mapper, conveyed through the N_{active} activated antennas after adding zero prefix (ZP) or cyclic prefix (CP) to combat dispersive GSM channels, respectively. Details of GSM transmit principle can be found in [2].

Example 1. Consider an SC-GSM system with $N_{\text{active}} = 2$ RF chains and $N_t = 5$ TAs using BPSK modulation $\mathcal{M}_{\text{GSM}} = 2$. Specifically, two out of the five transmit antennas are activated at each time-slot using $\lfloor \log_2 \binom{5}{2} \rfloor = 3$ information bits. In total, there are only $2^{\lfloor \log_2 \binom{5}{2} \rfloor} = 8$ activation patterns for signaling. The mapping of information bits to an activation pattern in GSM refers to the table in Fig. 1. For illustration purpose, we consider natural mapping for BPSK modulation over each activated TA. Suppose each SC-GSM block constitutes 4 GSM symbols. Suppose further these symbols correspond to 16 message bits (1001111011000100), with symbols $\mathbf{b}(1) = (1001)$, $\mathbf{b}(2) = (1110)$, $\mathbf{b}(3) = (1100)$ and $\mathbf{b}(4) = (0100)$. Taking the first GSM symbol for example, we have $\mathbf{b}(1) = (\mathbf{b}_1(1), \mathbf{b}_2(1))$ where $\mathbf{b}_1(1) = (100)$ selects the antenna activation pattern $\mathbf{s} = (1, 0, 0, 1, 0)^T$, and $\mathbf{b}_2(1) = (1)$ which implies that the modulated symbols $\mathbf{B}_1 = \mathbf{B}_2 = \xi_2^1 = -1$. This means that during the first time-slot, the first and fourth TAs are activated for the sending of BPSK symbol -1 . Then, the first GSM symbol can be written as $\mathbf{d}_1 = (-1, 0, 0, -1, 0)^T$. The entire SC-GSM block can be expressed by the

sparse matrix:

$$(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, \mathbf{d}_4) = \begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Note that if SM is used instead with the same modulation order, the number of transmit antennas must be increased to eight to maintain the same spectral efficiency.

3. Training Framework for Broadband GSM Systems

In this section, based on the generic training-based SC-MIMO transmission structure as shown in [22], we present a GSM training framework using sparse matrices and derive the correlation properties of the row sequences of such a sparse matrix.

We consider the training setting with a length- λ CP and N_t TAs over quasi-static frequency-selective channel. We denote the channel impulse response (CIR) from the n -th ($1 \leq n \leq N_t$) transmit antenna to the receiver as $\mathbf{h}_n = (h_{n,0}, h_{n,1}, \dots, h_{n,\lambda})^T$ where $h_{n,l}$ ($0 \leq l \leq \lambda$) is the channel coefficient of the l -th path. Note that there is a training sequence followed by data payload in each block at a TA and CP is placed at the front of the training sequence [22]. Let $\mathbf{x}_n = (x_{n,0}, x_{n,1}, \dots, x_{n,L-1})$ be the training sequence transmitted over the n -th TA. We assume that all the training sequences have identical energy of E . Then, the minimum MSE under LS channel estimator

is achieved if and only if

$$\phi_{\mathbf{x}_i, \mathbf{x}_j}(u) = \begin{cases} E, & \text{if } i = j, u = 0, \\ 0, & \text{if } i \neq j, 0 \leq u \leq \lambda, \\ 0, & \text{if } i = j, 1 \leq u \leq \lambda, \end{cases} \quad (4)$$

with

$$\text{minimum MSE} = \frac{\sigma_w^2}{E} \quad (5)$$

where σ_w^2 is the variance of the white complex Gaussian noise [22].

Definition 3. For training sequences $\{\mathbf{x}_n\}_{n=1}^{N_t}$ where the sequences have identical energy of E , they are called optimal training sequences of SC-MIMO systems under LS channel estimator if and only they satisfy (4).

3.1. Proposed Training Framework For Broadband GSM Systems

We define the training matrix Ω as

$$\Omega = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_{N_t} \end{pmatrix} = \begin{pmatrix} x_{1,0} & x_{1,1} & \cdots & x_{1,L-1} \\ x_{2,0} & x_{2,1} & \cdots & x_{2,L-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N_t,0} & x_{N_t,1} & \cdots & x_{N_t,L-1} \end{pmatrix}_{N_t \times L}.$$

Note that in GSM system, there are N_{active} TAs activated over every time-slot. Hence, Ω should be a sparse matrix where each training sequence \mathbf{x}_n ($1 \leq n \leq N_t$) has $Q = L / \lceil N_t / N_{\text{active}} \rceil$ non-zero entries. In this paper, for simplicity, we suppose N_{active} divides N_t . We consider the non-zero entries having identical magnitude of 1, and each training sequence has energy of $E = Q$.

Let $N_t = nN_{\text{active}}$ where n is an integer with $0 \leq n \leq D = N_t/N_{\text{active}}$. Let $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_{N_{\text{active}}}\}$ be a set of sequence sets, where $\mathcal{S}_k = \{\mathbf{s}_{k,1}, \mathbf{s}_{k,2}, \dots, \mathbf{s}_{k,D}\}$ and $\mathbf{s}_{k,m} = (s_{k,m}(0), s_{k,m}(1), \dots, s_{k,m}(Q-1))$ for $1 \leq k \leq N_{\text{active}}$ and $1 \leq m \leq$

$$D. \text{ For } 1 \leq m \leq D, \text{ define } \mathcal{X}_m = \begin{pmatrix} \mathbf{s}_{1,m} \\ \mathbf{s}_{2,m} \\ \vdots \\ \mathbf{s}_{N_{\text{active}},m} \end{pmatrix}_{N_{\text{active}} \times Q} \quad \text{and}$$

$$\Omega = \begin{pmatrix} T^0(\mathcal{X}_1 \mathbf{0}_{N_{\text{active}} \times (D-1)Q}) \\ T^Q(\mathcal{X}_2 \mathbf{0}_{N_{\text{active}} \times (D-1)Q}) \\ \vdots \\ T^{(D-1)Q}(\mathcal{X}_D \mathbf{0}_{N_{\text{active}} \times (D-1)Q}) \end{pmatrix}_{N_t \times L}.$$

An example of training matrix Ω having $N_t = 4$, $N_{\text{active}} = 2$ and $L = 2Q$ is shown below,

$$\Omega = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{s}_{1,1} & \mathbf{0} \\ \mathbf{s}_{2,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{s}_{1,2} \\ \mathbf{0} & \mathbf{s}_{2,2} \end{pmatrix}_{4 \times 2Q},$$

where $\mathbf{0}$ denotes $\mathbf{0}_{1 \times Q}$. Clearly, $\phi_{\mathbf{x}_1, \mathbf{x}_2}(u) = \rho_{\mathbf{s}_{1,1}, \mathbf{s}_{2,1}}(u)$ and $\phi_{\mathbf{x}_1, \mathbf{x}_3}(u) = \rho_{\mathbf{s}_{1,2}, \mathbf{s}_{1,1}}^*(Q-u)$ for $0 \leq u \leq Q-1$, and there are no unimodular sequences having aperiodic zero-correlation zones. This implies that Ω cannot satisfy the optimal condition of GSM training sequences as (4). Note that the good correlation properties of traditional MIMO training sequences may not be maintained due to the sparsity of active antennas, meaning that optimal performance under LS estimator by traditional MIMO training sequences may not be attainable.

To solve this problem, we consider $\mathcal{S}^j = \{\mathcal{S}_1^j, \mathcal{S}_2^j, \dots, \mathcal{S}_{N_{\text{active}}}^j\}$ ($1 \leq j \leq J$) be J sets of sequence sets, where $\mathcal{S}_k^j = \{\mathbf{s}_{k,1}^j, \mathbf{s}_{k,2}^j, \dots, \mathbf{s}_{k,M}^j\}$ and $\mathbf{s}_{k,m}^j = (s_{k,m}^j(0), s_{k,m}^j(1), \dots, s_{k,m}^j(\theta - 1))$ is an unimodular sequence of length θ for $1 \leq k \leq N_{\text{active}}$ and $1 \leq m \leq M$, and the training matrix Ω with the following structure,

$$\Omega = (\Omega_1, \mathbf{0}_{N_t \times \lambda}, \Omega_2, \mathbf{0}_{N_t \times \lambda}, \dots, \Omega_J, \mathbf{0}_{N_t \times \lambda}), \quad (6)$$

$$\Omega_j = \begin{pmatrix} T^0(\mathcal{X}_1^j \mathbf{0}_{N_{\text{active}} \times (D-1)\theta}) \\ T^\theta(\mathcal{X}_2^j \mathbf{0}_{N_{\text{active}} \times (D-1)\theta}) \\ \vdots \\ T^{(D-1)\theta}(\mathcal{X}_D^j \mathbf{0}_{N_{\text{active}} \times (D-1)\theta}) \end{pmatrix}_{N_t \times D\theta},$$

where for $1 \leq n \leq D$,

$$\mathcal{X}_n^j = \begin{pmatrix} \mathbf{s}_{1,n}^j \\ \mathbf{s}_{2,n}^j \\ \vdots \\ \mathbf{s}_{N_{\text{active}},n}^j \end{pmatrix}_{N_{\text{active}} \times \theta}$$

with $J \geq 2$ is a positive integer and $\mathbf{0}_{N_t \times \lambda}$ in (6) is called the λ -length zero-time slot of Ω . Note that $DJ\theta + \lambda J = L$ and $J\theta = Q$. In the sequel, we sometimes write the training matrix Ω as $(N_t, N_{\text{active}}, \lambda, J, \theta) - \Omega$.

Example 2. *Training matrix $(4, 2, \lambda, 2, \theta) - \Omega$ for channel estimation in GSM*

is shown in Fig. 2 with

$$\Omega = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{s}_{1,1}^1 & \mathbf{0}_{1 \times \theta} & \mathbf{0}_{1 \times \lambda} & \mathbf{s}_{1,1}^2 & \mathbf{0}_{1 \times \theta} & \mathbf{0}_{1 \times \lambda} \\ \mathbf{s}_{2,1}^1 & \mathbf{0}_{1 \times \theta} & \mathbf{0}_{1 \times \lambda} & \mathbf{s}_{2,1}^2 & \mathbf{0}_{1 \times \theta} & \mathbf{0}_{1 \times \lambda} \\ \mathbf{0}_{1 \times \theta} & \mathbf{s}_{1,2}^1 & \mathbf{0}_{1 \times \lambda} & \mathbf{0}_{1 \times \theta} & \mathbf{s}_{1,2}^2 & \mathbf{0}_{1 \times \lambda} \\ \mathbf{0}_{1 \times \theta} & \mathbf{s}_{2,2}^1 & \mathbf{0}_{1 \times \lambda} & \mathbf{0}_{1 \times \theta} & \mathbf{s}_{2,2}^2 & \mathbf{0}_{1 \times \lambda} \end{pmatrix} \\ = \begin{pmatrix} \mathcal{X}_1^1 & \mathbf{0}_{2 \times \theta} & \mathbf{0}_{2 \times \lambda} & \mathcal{X}_1^2 & \mathbf{0}_{2 \times \theta} & \mathbf{0}_{2 \times \lambda} \\ \mathbf{0}_{2 \times \theta} & \mathcal{X}_2^1 & \mathbf{0}_{2 \times \lambda} & \mathbf{0}_{2 \times \theta} & \mathcal{X}_2^2 & \mathbf{0}_{2 \times \lambda} \end{pmatrix}.$$

Note that $\phi_{\mathbf{x}_1, \mathbf{x}_2}(u) = \rho_{\mathbf{s}_{1,1}^1, \mathbf{s}_{2,1}^1}(u) + \rho_{\mathbf{s}_{1,1}^2, \mathbf{s}_{2,1}^2}(u)$ and $\phi_{\mathbf{x}_1, \mathbf{x}_3}(u) = \rho_{\mathbf{s}_{1,2}^1, \mathbf{s}_{1,1}^1}^*(\theta - u) + \rho_{\mathbf{s}_{1,2}^2, \mathbf{s}_{1,1}^2}^*(\theta - u)$ for $0 \leq u \leq \theta - 1$, which implies that the periodic zero correlation of the training sequences can be obtained from the complementary aperiodic correlation of the short unimodular sequences in these training sequences.

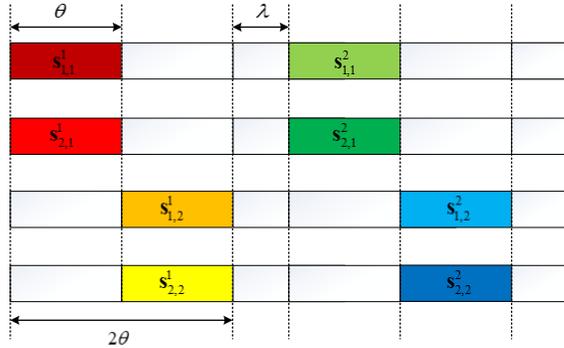


Figure 2: Training matrix with $N_t = 4, N_{\text{active}} = 2, J = 2, L = 4\theta + 2\lambda$

According to (4), the optimal GSM training criteria under the proposed structure of training matrix in (6) can be concluded in the following lemma.

Lemma 3. Ω is called an optimal GSM training sequence matrix in (6) if and only if it satisfies the following three equations:

$$\sum_{j=1}^J \rho_{\mathbf{s}_{k,n}^j}(u) = 0, \quad \text{for } 1 \leq u \leq \lambda, 1 \leq k \leq N_{\text{active}},$$

$$1 \leq n \leq N_t/N_{\text{active}}; \quad (7)$$

$$\sum_{j=1}^J \rho_{\mathbf{s}_{k,n}^j, \mathbf{s}_{k',n}^j}(u) = 0, \quad \text{for } 0 \leq |u| \leq \lambda, 1 \leq k \neq k' \leq N_{\text{active}},$$

$$\text{and } 1 \leq n \leq N_t/N_{\text{active}}; \quad (8)$$

$$\sum_{j=1}^J \rho_{\mathbf{s}_{k,n+1}^j, \mathbf{s}_{k',n}^j}(\theta - u) = 0, \quad \text{for } 1 \leq u \leq \lambda, \text{ and } 1 \leq k \neq k' \leq N_{\text{active}},$$

$$1 \leq n \leq N_t/N_{\text{active}} - 1. \quad (9)$$

It can be easily verified that the following remark proposed in [22] is also valid for our proposed structure.

Remark 1. (Remark 5 of [22]) For any training matrix with q -ary matrix \mathcal{X}_n^j ($1 \leq n \leq D$, $1 \leq j \leq J$) where q is even, J should be even.

Proposed Construction: Consider a $(N_t, N_{\text{active}}, \lambda, J, \theta) - \Omega$ training matrix as follows.

$$\Omega = (\Omega_1, \mathbf{0}_{N_t \times \lambda}, \Omega_2, \mathbf{0}_{N_t \times \lambda}, \dots, \Omega_J, \mathbf{0}_{N_t \times \lambda}), \quad (10)$$

$$\text{where } \Omega_j = \begin{pmatrix} T^0(\mathcal{X}_j \mathbf{0}_{N_{\text{active}} \times (D-1)\theta}) \\ T^\theta(\mathcal{X}_j \mathbf{0}_{N_{\text{active}} \times (D-1)\theta}) \\ \vdots \\ T^{(D-1)\theta}(\mathcal{X}_j \mathbf{0}_{N_{\text{active}} \times (D-1)\theta}) \end{pmatrix}_{N_t \times D\theta} \quad \text{and } \mathcal{X}_j = \begin{pmatrix} \mathbf{a}_{1,j} \\ \mathbf{a}_{2,j} \\ \vdots \\ \mathbf{a}_{N_{\text{active}},j} \end{pmatrix}_{N_{\text{active}} \times \theta}$$

for $1 \leq j \leq J$.

The following lemma is straightforward from (7)-(9) in **Lemma 3**.

Lemma 4. *The training matrix in (10) is optimal if and only if $\{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_{N_{\text{active}}}\}$ satisfies the following two equations.*

$$\sum_{j=1}^J \rho_{\mathbf{a}_{k,j}}(u) = 0, \quad \text{for } u \in \{1, \dots, \lambda\} \cup \{\theta - \lambda, \dots, \theta - 1\},$$

$$1 \leq k \leq N_{\text{active}}; \quad (11)$$

$$\sum_{j=1}^J \rho_{\mathbf{a}_{k,j}, \mathbf{a}_{k',j}}(u) = 0, \quad \text{for } u \in \{0\} \cup \{1, \dots, \lambda\} \cup \{\theta - \lambda, \dots, \theta - 1\},$$

$$1 \leq k \neq k' \leq N_{\text{active}}. \quad (12)$$

The training matrix Ω can be enlarged when J' is a multiple of J , i.e., $J' = tJ$ for some integer t , Ω can be enlarged to $(N_t, N_{\text{active}}, J', \theta) - \Omega'$ as

$$\Omega' = \begin{pmatrix} \underbrace{\Omega \quad \Omega \quad \dots \quad \Omega}_{t \times \Omega} \end{pmatrix}_{N_t \times tJ(D\theta + \lambda)}.$$

3.2. Minimum MSE for the Proposed Training Framework

In this subsection, we show the minimum MSE related to the sequence length of \mathcal{S} , E_b/N_0 and the numbers of transmit antennas and activated antennas and multi-paths which are denoted by N_t , N_{active} and $\lambda + 1$, respectively, under the training framework proposed in *Subsection A* with each training sequence having energy E .

Note that the sequence length of training sequence is $L = N_t/N_{\text{active}} \times (\lambda + \theta)$. Then, the energy per bit is $E_b = \frac{E}{L} = \frac{E}{N_t/N_{\text{active}} \times (\lambda + \theta)}$, where

$$\frac{E_b}{N_0} = \frac{E}{2\sigma_\omega^2 N_t/N_{\text{active}} \times (\lambda + \theta)}, \quad (13)$$

$N_0 = 2\sigma_\omega^2$ is the power spectral density of the white complex Gaussian noise. Combining (5) and (13), we have that

$$\text{minimum MSE} = (2(N_t/N_{\text{active}})(\lambda + \theta)E_b/N_0)^{-1}. \quad (14)$$

Therefore, for given E_b/N_0 and N_t , the minimum MSE decreases with λ and θ , and increases with N_{active} .

4. Symmetrical Z -complementary code set (SZCCS): Properties And Constructions

This section presents the properties and constructions of SZCCSs. The definition and some properties of SZCCSs are as follows.

Definition 4. For a positive integer Z , $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_K\}$ is called a (K, M, L, Z) -SZCCS where $\mathcal{S}_k = \{\mathbf{s}_{k,1}, \mathbf{s}_{k,2}, \dots, \mathbf{s}_{k,M}\}$ and $\mathbf{s}_{k,m} = (s_{k,m}(0), s_{k,m}(1), \dots, s_{k,m}(L-1))$ ($1 \leq k \leq K, 1 \leq m \leq M$), if \mathcal{S} satisfies the following conditions:

C1: $C_{\mathcal{S}_k}(u) = 0$, for $|u| \in \mathcal{T}_1 \cup \mathcal{T}_2$;

C2: $C_{\mathcal{S}_k, \mathcal{S}_j}(u) = 0$, for $k \neq j$ and $|u| \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \{0\}$;

where $\mathcal{T}_1 = \{1, 2, \dots, Z\}$ and $\mathcal{T}_2 = \{L - Z, L + 1 - Z, \dots, L - 1\}$.

According to **Definition 4**, **Lemma 4** can be further expressed as the following theorem.

Theorem 1. The training matrix in (10) is optimal if and only if $\{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_{N_{\text{active}}}\}$ is a $(N_{\text{active}}, J, \theta, \lambda)$ -SZCCS, where $\mathcal{A}_k = \{\mathbf{a}_{k,1}, \mathbf{a}_{k,2}, \dots, \mathbf{a}_{k,J}\}$ ($1 \leq k \leq N_{\text{active}}$).

Remark 2. Note that SZCCS considers the cross-correlation function sum between the different ZCCs, whereas, CZCP proposed in [22] is characterized by the cross-correlation function sum of the sequences in a same ZCP.

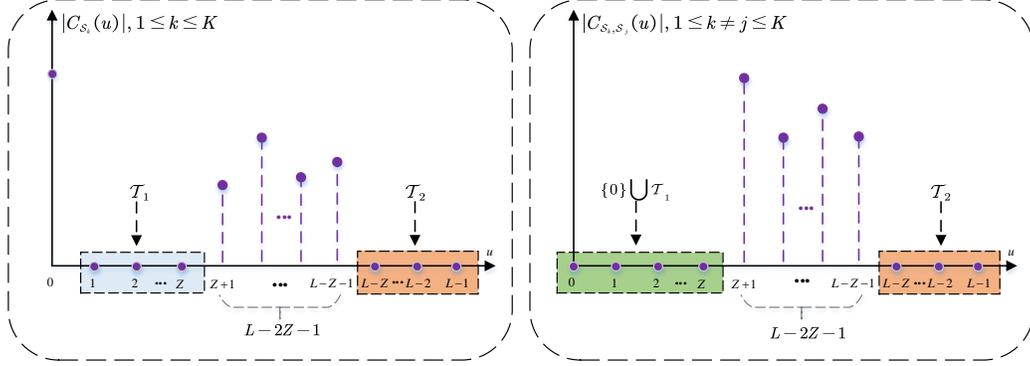


Figure 3: The correlation properties of (K, M, L, Z) -SZCCS

Fig. 3 illustrates the correlation properties of (K, M, L, Z) -SZCCS which shows that there are ZCZs at both the front and tail of auto- or cross-correlation sums. Note that when $Z \geq \frac{L-1}{2}$, a (K, M, L, Z) -SZCCS reduces to a (K, M, L) -MOCCS defined in **Definition 2**. Hence, the following theorem is straightforward from **Lemma 2**.

Theorem 2. Any unimodular (K, M, L, Z) -SZCCS $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_K\}$ satisfies $K \leq \lfloor \frac{ML}{Z+1} \rfloor$, where $\mathcal{S}_k = \{\mathbf{s}_{k,1}, \mathbf{s}_{k,2}, \dots, \mathbf{s}_{k,M}\}$ and $\mathbf{s}_{k,m} = (s_{k,m}(0), s_{k,m}(1), \dots, s_{k,m}(L-1))$ ($l \leq k \leq K, 1 \leq m \leq M$). In particular, when $K = \lfloor \frac{ML}{Z+1} \rfloor$, \mathcal{S} is called an optimal (K, M, L, Z) -SZCCS.

Corollary 1. For any (K, M, L, Z) -SZCCS $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_K\}$ where $\mathcal{S}_k = \{\mathbf{s}_{k,1}, \mathbf{s}_{k,2}, \dots, \mathbf{s}_{k,M}\}$ and $\mathbf{s}_{k,m} = (s_{k,m}(0), s_{k,m}(1), \dots, s_{k,m}(L-1))$ ($1 \leq k \leq K, 1 \leq m \leq M$), it satisfies the following properties:

P1: $\tilde{\mathcal{S}} = \{c_1 \mathcal{S}_1, c_2 \mathcal{S}_2, \dots, c_K \mathcal{S}_K\}$ is also a (K, M, L, Z) -SZCCS, where $c_k \in \mathbb{C}$ ($1 \leq k \leq K$) is nonzero and $c_k \mathcal{S}_k = \{c_k \mathbf{s}_{k,1}, c_k \mathbf{s}_{k,2}, \dots, c_k \mathbf{s}_{k,M}\}$;

P2: $\overleftarrow{\mathcal{S}} = \{\overleftarrow{\mathcal{S}}_1, \overleftarrow{\mathcal{S}}_2, \dots, \overleftarrow{\mathcal{S}}_K\}$ is also a (K, M, L, Z) -SZCCS, where $\overleftarrow{\mathcal{S}}_k = \{\overleftarrow{\mathbf{s}}_{k,1}, \overleftarrow{\mathbf{s}}_{k,2},$

$\dots, \overleftarrow{\mathbf{s}}_{k,M}\}$ and $\overleftarrow{\mathbf{s}}_{k,m} = (s_{k,m}(L-1), s_{k,m}(L-2), \dots, s_{k,m}(0))$ ($1 \leq m \leq M$).

Note that if \mathcal{S} is optimal, $\tilde{\mathcal{S}}$ and $\overleftarrow{\mathcal{S}}$ are also optimal.

Proof. The proof is straightforward according the definitions of aperiodic auto-correlation, aperiodic cross-correlation and SZCCS, so we omit it here. \square

4.1. Proposed construction of optimal SZCCS

Theorem 3. Let q be an even integer, $m \geq 4$ and $L = 2^m$, and let

$$\begin{aligned} f(\mathbf{x}) &= \frac{q}{2} \sum_{s=1}^{m-2} x_{\pi(s)} x_{\pi(s+1)} + \sum_{s=1}^m \mu_s x_s + \mu, \\ a_k(\mathbf{x}) &= f(\mathbf{x}) + d_1^k \left(\frac{q}{2} x_{\pi(m-2)} + \frac{q}{2} x_{\pi(m-2)} x_{\pi(m)} \right) \\ &\quad + d_2^k \left(\frac{q}{2} x_{\pi(m-1)} + \frac{q}{2} x_{\pi(m-2)} x_{\pi(m)} \right) + d_3^k \left(\frac{q}{2} x_{\pi(m)} + \frac{q}{2} x_{\pi(m-1)} x_{\pi(m)} \right) \\ &\quad + d_4^k \left(\frac{q}{2} x_{\pi(m-1)} + \frac{q}{2} x_{\pi(m-2)} x_{\pi(m)} + \frac{q}{2} x_{\pi(m-1)} x_{\pi(m)} \right), \\ b_k(\mathbf{x}) &= a_k(\mathbf{x}) + \frac{q}{2} x_{\pi(1)}, \end{aligned}$$

where $\mathbf{x} \in \mathbb{Z}_2^m$, μ, μ_s are any given elements in \mathbb{Z}_q , π is a permutation of the symbols $\{1, 2, \dots, m\}$ with $\{\pi(m-1), \pi(m)\} = \{m-1, m\}$ and $(\mathbf{D}_k = (d_1^k, d_2^k, d_3^k, d_4^k))_{k=1}^8 = ((0, 0, 0, 0), (1, 0, 1, 0), (1, 1, 0, 0), (0, 1, 1, 0), (0, 0, 0, 1), (1, 0, 1, 1), (1, 1, 0, 1), (0, 1, 1, 1))$. Then, the set

$$\mathcal{S} = \{\mathcal{S}_k = \{\psi(\mathbf{a}_k), \psi(\mathbf{b}_k)\} : k \in \{1, 2, \dots, 8\}\}$$

is an optimal $(8, 2, 2^m, 2^{m-2} - 1)$ -SZCCS.

Proof of Theorem 3: See Appendix A.

Example 3. For $q = 2$ and $m = 4$, let π be an identical permutation of $\{1, 2, 3, 4\}$ with $\pi(i) = i$ for $1 \leq i \leq 4$, and $f(\mathbf{x}) = \sum_{k=1}^2 x_k x_{k+1}$. Then,

$$\mathcal{S} = \{\mathcal{S}_k = \{\psi(\mathbf{a}_k), \psi(\mathbf{b}_k)\} : k \in \{1, 2, \dots, 8\}\}$$

is an optimal $(8, 2, 16, 3)$ -SZCCS, where

$$(|C_{\mathcal{S}_i, \mathcal{S}_j}(u)|)_{u=0}^{15} = (|\rho_{\psi(\mathbf{a}_i), \psi(\mathbf{a}_j)}(u) + \rho_{\psi(\mathbf{b}_i), \psi(\mathbf{b}_j)}(u)|)_{u=0}^{15} \quad (1 \leq i, j \leq 8)$$

can be found in Table 2.

Remark 3. The significance of **Theorem 3** is reflected in the following aspects.

For sequence design:

- It provides an infinite class of optimal SZCCSs, which means that it can generate optimal SZCCSs for infinite groups of parameters.
- This construction is interesting since it is based on two facts we observed.
 1. For the aperiodic correlation sum of sequence pairs in **Lemma 1**, the width of ZCZ Z can be controlled by the permutation π , which means that if $\{\pi(1), \pi(2), \dots, \pi(v)\} = \{1, 2, \dots, v\}$ for a positive v where $1 \leq v < m - 1$, then, we have $Z = 2^v - 1$.
 2. The orthogonality of different sequence pairs in **Lemma 1** can be guaranteed by some offsets composed of some linear functions.
- The $(8, 2, 2^m, 2^{m-2})$ -SZCCS constructed in **Theorem 3** achieves the theoretical upper bound on the size of the SZCCS proposed in **Theorem**

$\begin{array}{c} C \\ \diagdown \\ i \end{array} \quad j$	1	2	3	4	5	6	7	8
1	$(32, 07, 16, 07)$	$(04, 24, 07, 8, 03)$	$(04, -8, 011)$	$(012, 8, 03)$	$(012, 8, 03)$	$(04, 8, 011)$	$(04, 8, 011)$	$(08, 16, 07)$
2	$(04, 24, 07, 8, 03)$	$(32, 07, 16, 07)$	$(012, 8, 03)$	$(04, 8, 011)$	$(04, 8, 011)$	$(012, 8, 03)$	$(08, 16, 07)$	$(012, 8, 03)$
3	$(012, 8, 03)$	$(04, 8, 011)$	$(32, 03, 8, 07, 8, 03)$	$(08, 16, 07)$	$(04, 16, 03, 16, 07)$	$(04, 8, 07, 8, 03)$	$(04, 8, 011)$	$(04, 16, 07, 8, 03)$
4	$(04, 8, 011)$	$(012, 8, 03)$	$(08, 16, 07)$	$(32, 03, 8, 07, 8, 03)$	$(04, 8, 011)$	$(04, 16, 03, 16, 07)$	$(04, 8, 07, 8, 03)$	$(04, -8, 011)$
5	$(04, 8, 011)$	$(012, 8, 03)$	$(04, 16, 03, 16, 07)$	$(04, 8, 011)$	$(32, 03, 8, 07, 8, 03)$	$(08, 16, 07)$	$(04, 16, 07, 8, 03)$	$(04, 8, 011)$
6	$(012, 8, 03)$	$(04, 8, 011)$	$(04, 8, 07, 8, 03)$	$(04, 16, 03, 16, 07)$	$(08, 16, 07)$	$(32, 03, 8, 07, 8, 03)$	$(04, 8, 011)$	$(04, 16, 07, 8, 03)$
7	$(04, 8, 07, 8, 03)$	$(08, 16, 07)$	$(04, 16, 07, 8, 03)$	$(04, 8, 011)$	$(04, 8, 011)$	$(04, 16, 07, 8, 03)$	$(32, 07, 16, 07)$	$(04, 8, 07, 8, 03)$
8	$(08, 16, 07)$	$(04, 8, 07, 8, 03)$	$(04, 8, 011)$	$(04, 16, 07, 8, 03)$	$(04, 16, 07, 8, 03)$	$(04, 8, 011)$	$(04, 8, 07, 8, 03)$	$(32, 07, 16, 07)$

Table 2: $\mathbf{C} = (|C_{S_i, S_j}(u)|)_{u=0}^{15}$ in Example 3

\mathfrak{B} , which implies that for any K , if there exists a $(K, 2, 2^m, 2^{m-2} - 1)$ -SZCCS, then, we have $K \leq 8$. Furthermore, a $(K, 2, 2^m, 2^{m-2} - 1)$ -SZCCS can be obtained by selecting any K codes from an $(8, 2, 2^m, 2^{m-2})$ -SZCCS.

For application in GSM systems:

- For a (K, M, L, Z) -SZCCS, K represents the maximum number of RF chains that can be supported, $Z+1$ represents the number of multi-paths, $M(L+Z)$ represents the length of training sequences under the proposed training framework in the sequel. Hence, for fixed number of multi-path and training sequence length, the bigger the K , the better. Note that $(8, 2, 2^m, 2^{m-2})$ -SZCCS constructed in **Theorem 3** is optimal, so $K = 8$ is the best for the GSM systems under the proposed training framework in the sequel with 2^{m-2} multi-paths and training sequence length $2(2^m + 2^{m-2} - 1)$.

However, the sequence length of the proposed optimal SZCCS is limited to a power-of-two. The following subsection shows a construction of SZCCSs, which are not optimal but with non-power-of-two sequence length.

4.2. Proposed construction of SZCCS with non-power-of-two lengths

Theorem 4. Let q be an even integer, $m \geq 3$, $1 \leq v < m - 1$ and $L = 2^{m-1} + 2^v$, and let

$$g(\mathbf{x}) = \frac{q}{2} \sum_{s=1}^{m-2} x_{\pi(s)} x_{\pi(s+1)} + \sum_{s=1}^{m-1} \lambda_s x_{\pi(s)} x_m + \sum_{s=1}^m \mu_s x_s + \mu,$$

$$a_k(\mathbf{x}) = g(\mathbf{x}) + \frac{q}{2}(k-1)(x_{m-1} + x_m x_{\pi(v)}), \quad b_k(\mathbf{x}) = a_k(\mathbf{x}) + \frac{q}{2} x_{\pi(1)},$$

where $\mathbf{x} \in \mathbb{Z}_2^m$, $k \in \{1, 2\}$, λ_s, μ, μ_s are any given elements in \mathbb{Z}_q , and π is a permutation of the symbols $\{1, 2, \dots, m\}$ with $\{\pi(1), \pi(2), \dots, \pi(v)\} = \{1, 2, \dots, v\}$ and $\pi(m) = m$. Then, $\mathcal{S} = \{\mathcal{S}_k = \{\psi(\mathbf{a}_k), \psi(\mathbf{b}_k)\} : k = 1, 2\}$ is a $(2, 2, 2^{m-1} + 2^v, 2^v - 1)$ -SZCCS.

Proof. The proof of **Theorem 4** is similar with that of **Theorem 3**, so we omit it here. \square

Remark 4. When $v = m - 2$ in **Theorem 4**, one can generate $(2, 2, 2^{m-1} + 2^{m-2}, 2^{m-2} - 1)$ -SZCCSs. However, to obtain SZCCSs with same $Z = 2^{m-2} - 1$ using **Theorem 3**, the sequence length needs to be $L = 2^m$, which is larger than that of SZCCSs generated from **Theorem 4**. That means that when the numbers of activated antennas and multi-paths are 2 and a power-of-two, respectively, the SZCCS generated from **Theorem 4** is a better choice than that from **Theorem 3** because of higher spectral efficiency.

5. Numerical Evaluation

In this section, we analyse the performance of the proposed SZCCSs in **Theorem 3** and **Theorem 4** as the training sequences for GSM systems over frequency-selective channels, based on the framework proposed in the *Subsection 3.1* for different settings of E_b/N_0 , number of multi-paths $\lambda + 1$, number of activated antennas N_{active} and number of zero time-slots θ . Consider the same frequency-selective channel in [22], where the $\lambda + 1$ -path channel (separated by integer symbol durations) has uniform power delay profile as $h[t] = \sum_{n=0}^{\lambda} h_i \delta[t - nT]$ where h_i 's are complex-valued Gaussian random variables with zero mean and $\mathbb{E}(|h_i|^2) = 1$.

5.0.1. MSE comparison with the change of E_b/N_0

We consider a generic training-based single carrier GSM transmission structure with $N_t = 4$ TAs, one receive antenna and $N_{\text{active}} = 2$ RF chains, and a 6-path channel. Let the training matrix of the random binary se-

quences for $J = 2$ be $\Omega_r = \begin{pmatrix} \mathbf{s}_{r,1} & \mathbf{0}_{1 \times 32} & \mathbf{0}_{1 \times 5} & \mathbf{s}_{r,2} & \mathbf{0}_{1 \times 32} & \mathbf{0}_{1 \times 5} \\ \mathbf{s}_{r,3} & \mathbf{0}_{1 \times 32} & \mathbf{0}_{1 \times 5} & \mathbf{s}_{r,4} & \mathbf{0}_{1 \times 32} & \mathbf{0}_{1 \times 5} \\ \mathbf{0}_{1 \times 32} & \mathbf{s}_{r,1} & \mathbf{0}_{1 \times 5} & \mathbf{0}_{1 \times 32} & \mathbf{s}_{r,2} & \mathbf{0}_{1 \times 5} \\ \mathbf{0}_{1 \times 32} & \mathbf{s}_{r,3} & \mathbf{0}_{1 \times 5} & \mathbf{0}_{1 \times 32} & \mathbf{s}_{r,4} & \mathbf{0}_{1 \times 5} \end{pmatrix}$,

where $\mathbf{s}_{r,1}, \mathbf{s}_{r,2}, \mathbf{s}_{r,3}, \mathbf{s}_{r,4}$ are randomly generated binary sequences of length 32. The aperiodic correlation sums of $\{\mathbf{s}_{r,1}, \mathbf{s}_{r,2}, \mathbf{s}_{r,3}, \mathbf{s}_{r,4}\}$ is presented in Fig. 4. It can be observed that there is no ZCZ in the these correlation sums, which implies Ω_r can not satisfy the optimal conditions in (7)-(9). Let Ω_s be the training matrix of the proposed SZCCSs for $J = 2$, which can be generated by replacing $\mathbf{s}_{r,1}, \mathbf{s}_{r,2}, \mathbf{s}_{r,3}, \mathbf{s}_{r,4}$ by $\mathbf{s}_{s,1}, \mathbf{s}_{s,2}, \mathbf{s}_{s,3}, \mathbf{s}_{s,4}$, respectively, where $\mathbf{s}_{s,1}, \mathbf{s}_{s,2}, \mathbf{s}_{s,3}, \mathbf{s}_{s,4}$ are the sequences in the first two sequence sets of the proposed optimal (8, 2, 32, 7)-SZCCS generated in **Example 3**, where

$$\begin{aligned} \mathbf{s}_{s,1} &= (+, +, +, -, +, +, -, +, +, +, +, -, -, -, +, -, +, +, +, -, +, +, -, \\ &\quad +, +, +, +, -, -, -, +, -), \\ \mathbf{s}_{s,2} &= (+, -, +, +, +, -, -, -, +, -, +, +, -, +, +, +, +, -, +, +, +, -, -, \\ &\quad -, +, -, +, +, -, +, +, +), \\ \mathbf{s}_{s,3} &= (+, +, +, -, -, -, +, -, -, -, -, +, -, -, +, -, +, +, +, -, -, -, +, \\ &\quad -, -, -, -, +, -, -, +, -), \\ \mathbf{s}_{s,4} &= (+, -, +, +, -, +, +, +, -, +, -, -, -, +, +, +, +, -, +, +, -, +, +, \\ &\quad +, -, +, -, -, -, +, +, +). \end{aligned}$$

The aperiodic correlation sums of $\{\mathbf{s}_{s,1}, \mathbf{s}_{s,2}, \mathbf{s}_{s,3}, \mathbf{s}_{s,4}\}$ is presented in Fig. 5, where shows that there are symmetric ZCZs of width greater or equal to 7 in these aperiodic correlation sums. Hence, $\{\mathbf{s}_{s,1}, \mathbf{s}_{s,2}, \mathbf{s}_{s,3}, \mathbf{s}_{s,4}\}$ satisfies the optimal conditions in (7)-(9) when $\lambda \leq 7$. The argument for the cases that $J = 6, 18$ are the same. In Fig. 6 we evaluate the channel estimation MSE performances of Ω_r and Ω_s when $J = 2, 6, 18$ and the E_b/N_0 runs over $\{0, 2, \dots, 20\}$. It can be observed that when the number of multi-paths is 6, the proposed (8, 2, 32, 7)-SZCCS can be used to design optimal GSM training matrix which attains the MSE lower bound, which outperforms that of random binary sequences. Those performances are consistent with the discussions of Figs. 4 and 5 above. Moreover, the MSE decreases when J increases, which can be seen as the length of training sequences increases.

5.0.2. MSE comparison with the change of the number of multi-paths

We evaluate the channel estimation MSE performances for different values of multi-paths for $E_b/N_0 = 16$ dB with 4 transmit antennas and 2 RF chains in Fig. 9. We employ the first two sequence sets of the optimal (8,2,32,7)-SZCCS constructed through **Theorem 3** and the (2,2,24,7)-SZCCS in **Theorem 4** to generate our GSM training matrices. We compare their channel estimation performances with those of the (16,8)-CZCP given in [22], the Zadoff-Chu sequences of length 32 and binary random sequences of length 32. Let the training matrix of (16,8)-CZCP $\{\mathbf{a}, \mathbf{b}\}$ be

$$\Omega_c = \begin{pmatrix} \mathbf{a} & \mathbf{0}_{1 \times 16} & \mathbf{0}_{1 \times \lambda} & \mathbf{b} & \mathbf{0}_{1 \times 16} & \mathbf{0}_{1 \times \lambda} \\ \mathbf{b} & \mathbf{0}_{1 \times 16} & \mathbf{0}_{1 \times \lambda} & \mathbf{a} & \mathbf{0}_{1 \times 16} & \mathbf{0}_{1 \times \lambda} \\ \mathbf{0}_{1 \times 16} & \mathbf{a} & \mathbf{0}_{1 \times \lambda} & \mathbf{0}_{1 \times 16} & \mathbf{b} & \mathbf{0}_{1 \times \lambda} \\ \mathbf{0}_{1 \times 16} & \mathbf{b} & \mathbf{0}_{1 \times \lambda} & \mathbf{0}_{1 \times 16} & \mathbf{a} & \mathbf{0}_{1 \times \lambda} \end{pmatrix},$$

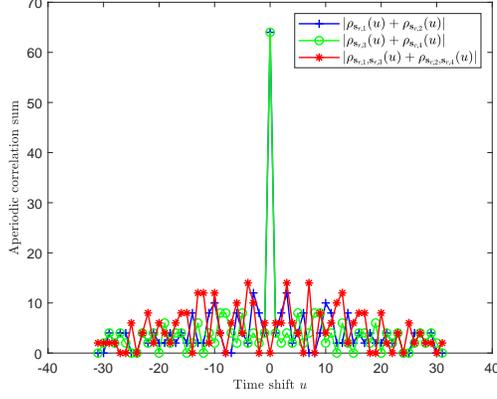


Figure 4: Aperiodic correlation sums of the random training matrix

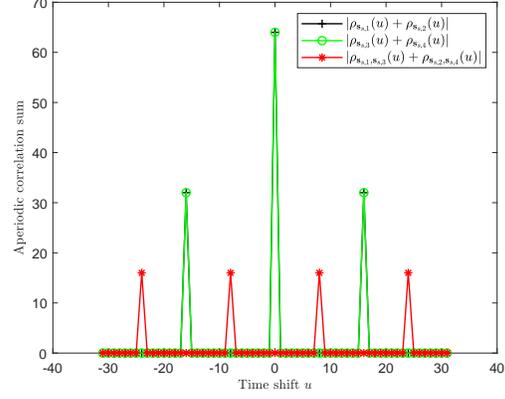


Figure 5: Aperiodic correlation sums of the training matrix of the proposed SZCCS

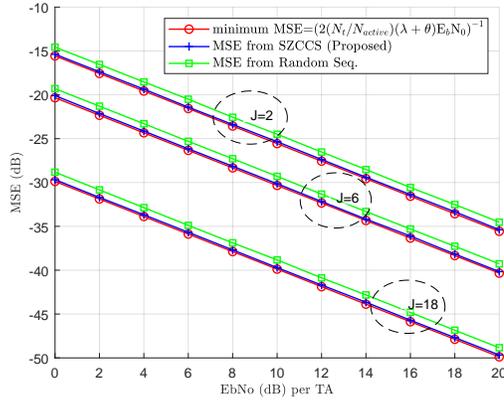


Figure 6: MSE comparison, for (8, 2, 16, 7)-SZCCS, where the number of multi-paths is 6, $N_t = 4$, $N_{\text{active}} = 2$ and $J \in \{2, 6, 18\}$

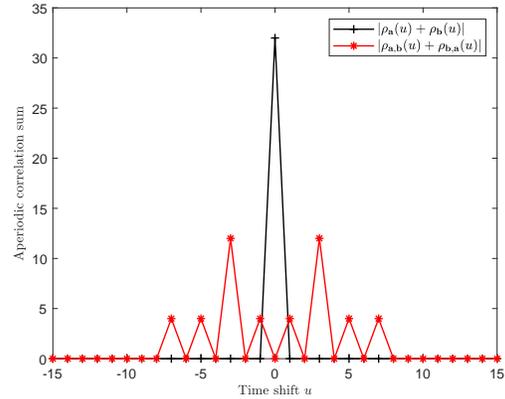


Figure 7: Aperiodic correlation sums of the training matrix of (16,8)-CZCP

where $\mathbf{a} = (1, 1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1, 1, -1, -1, -1)$, $\mathbf{b} = (1, 1, 1, -1, 1, 1, -1, 1, -1, 1, -1, 1, -1, -1, -1, 1, 1, 1)$. The aperiodic correlation sums of $\{\mathbf{a}, \mathbf{b}\}$ is shown in Fig. 7, it can be observed that $\{\mathbf{a}, \mathbf{b}\}$ satisfies optimal conditions in (7) and (9) without satisfying (8) when $\lambda \leq 8$. Let the training matrix of

four Zadoff-Chu sequences $\mathbf{s}_{zc,1}, \mathbf{s}_{zc,2}, \mathbf{s}_{zc,3}, \mathbf{s}_{zc,4}$ be

$$\Omega_{zc} = \begin{pmatrix} \mathbf{s}_{zc,1} & \mathbf{0}_{1 \times 32} & \mathbf{0}_{1 \times \lambda} & \mathbf{s}_{zc,2} & \mathbf{0}_{1 \times 32} & \mathbf{0}_{1 \times \lambda} \\ \mathbf{s}_{zc,3} & \mathbf{0}_{1 \times 32} & \mathbf{0}_{1 \times \lambda} & \mathbf{s}_{zc,4} & \mathbf{0}_{1 \times 32} & \mathbf{0}_{1 \times \lambda} \\ \mathbf{0}_{1 \times 32} & \mathbf{s}_{zc,1} & \mathbf{0}_{1 \times \lambda} & \mathbf{0}_{1 \times 32} & \mathbf{s}_{zc,2} & \mathbf{0}_{1 \times \lambda} \\ \mathbf{0}_{1 \times 32} & \mathbf{s}_{zc,3} & \mathbf{0}_{1 \times \lambda} & \mathbf{0}_{1 \times 32} & \mathbf{s}_{zc,4} & \mathbf{0}_{1 \times \lambda} \end{pmatrix},$$

where $\mathbf{s}_{zc,1}, \mathbf{s}_{zc,2}, \mathbf{s}_{zc,3}, \mathbf{s}_{zc,4}$ are Zadoff-Chu sequences of length 32 generated with roots being 1, 3, 5, 7, respectively. The aperiodic correlation sums of $\{\mathbf{s}_{zc,1}, \mathbf{s}_{zc,2}, \mathbf{s}_{zc,3}, \mathbf{s}_{zc,4}\}$ is shown in Fig. 8, which implies that $\{\mathbf{s}_{zc,1}, \mathbf{s}_{zc,2}, \mathbf{s}_{zc,3}, \mathbf{s}_{zc,4}\}$ has no ZCZs and can not satisfy optimal conditions in (7)-(9) for any $\lambda > 0$. Fig. 9 shows that the proposed SZCCSs lead to optimal GSM training matrices which attain the MSE lower bound when the number of multi-paths $\lambda + 1$ is less than or equal to $Z + 1$. Note that the minimum MSE decreases with the number of multi-paths $\lambda + 1$, which is consistent with (14). It is noted that the MSE performance corresponding to CZCP, which is proposed in [22] for optimal training of SM systems, is close to that of the Zadoff-Chu sequence and binary random sequence, indicating that CZCP may not be suitable for training matrix design in GSM systems.

5.0.3. MSE comparison with the change of the number of activated antennas

Fig. 10 shows that every GSM training matrix corresponding to the proposed SZCCSs attains the MSE lower bound when the number of the activated antennas N_{active} is less than or equal to the size K of the SZCCS, and the minimum MSE increases with the number of activated antennas N_{active} . Note that when $N_{\text{active}} = 1$, the system is an SM system, which implies that the proposed training framework and SZCCSs are also applicable for the optimal training in SM systems.

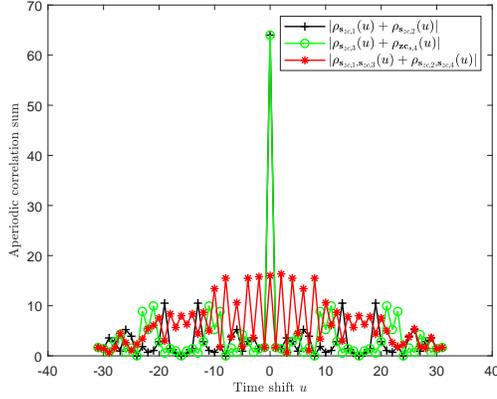


Figure 8: Aperiodic correlation sums of the training matrix of Zadoff-Chu sequences

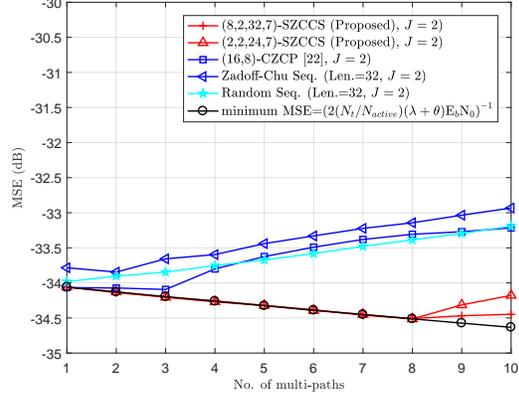


Figure 9: MSE comparison among different sequences for various multi-paths with $N_t = 4$, $N_{\text{active}} = 2$ and $E_b N_0 = 16$ dB.

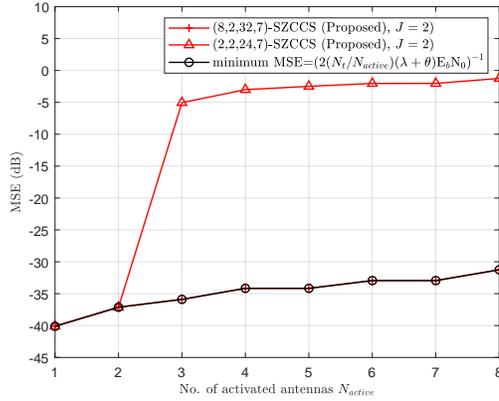


Figure 10: MSE comparison of the proposed proposed SZCCSs for various N_{active} with $E_b/N_0 = 16$ dB and $\lambda = 3$.

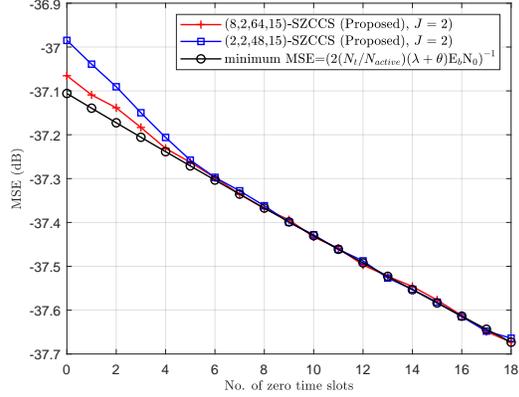


Figure 11: MSE comparison among the proposed SZCCSs for different values of zero-time slot with $E_b/N_0 = 16$ dB and $\lambda = 15$.

5.0.4. MSE comparison with the change of the number of zero time-slots

Finally, we evaluate the channel estimation MSE performances of the proposed (8, 2, 64, 15)-SZCCS and (2, 2, 48, 15)-SZCCS for different values of

zero-time slots with fixed $\lambda = 15$ in Fig. 11. One observes that reducing or removing the zero-time slot has little influence on the performance of the proposed GSM training matrices. Hence, in practical application, the zero-time slot may be removed to improve the spectral efficiency.

6. Concluding Remarks

In this paper, we have proposed an optimal training design for the optimal channel estimation in broadband generalized spatial modulation systems using a new class of code sets called symmetrical Z -complementary code sets which represents a stringent subclass of Z -complementary code sets. We point out that the training sequences for conventional multiple-input multiple-output systems where the number of radio-frequency chains equals that of transmit antennas is inapplicable in generalized spatial modulation. The same may be said for spatial modulation systems with a single radio-frequency chain. We have introduced a generic design of optimal generalized spatial modulation training matrix using symmetrical Z -complementary code sets, where the inter-symbol interference and inter-channel interference can be minimized by the zero aperiodic auto-correlation sums and aperiodic cross-correlation sums in front-end and tail-end zero-correlation zones. Note that the number of radio-frequency chains that can be supported is limited by the size of the applied symmetrical Z -complementary code sets, and the maximum tolerable number of multi-paths depends on the width of zero-correlation zones of aperiodic correlation sums for that symmetrical Z -complementary code set. With the aid of generalized Boolean function, we have constructed two classes of symmetrical Z -complementary code set-

s including an optimal one with respect to the set size upper bound. We have shown that the designed training matrix leads to minimum channel estimation mean-squared-error in frequency-selective channels, which are also applicable to SM and traditional MIMO systems.

Appendix A. Proof of Theorem 3

Proof. According to (1) and **Definition 4**, this proof can be divided into two steps. In the first step, it should be demonstrated that

$$C_{\mathcal{S}_k}(u) = \rho_{\psi(\mathbf{a}_k)}(u) + \rho_{\psi(\mathbf{b}_k)}(u) = 0, \quad \text{for } u \in \mathcal{T}_1 \cup \mathcal{T}_2, 1 \leq k \leq 8, \text{(A.1)}$$

where $\mathcal{T}_1 = \{1, 2, \dots, 2^{m-2}-1\}$ and $\mathcal{T}_2 = \{2^m+1-2^{m-2}, 2^m+2-2^{m-2}, \dots, 2^m-1\}$. In the second step, it needs to prove that

$$C_{\mathcal{S}_k, \mathcal{S}_n}(u) = \rho_{\psi(\mathbf{a}_k), \psi(\mathbf{a}_n)}(u) + \rho_{\psi(\mathbf{b}_k), \psi(\mathbf{b}_n)}(u) = 0, \quad \text{for } u \in \{0\} \cup \mathcal{T}_1 \cup \mathcal{T}_2, \\ 1 \leq k \neq n \leq 8. \quad \text{(A.2)}$$

For any integers i and u , let $j = i+u$; also let (i_1, i_2, \dots, i_m) and (j_1, j_2, \dots, j_m) be the binary representations of i and j , respectively. Without loss of generality, we only discuss the cases when $k, n = 1, 2$ here.

Step 1 : When $\mathbf{D}_1 = (0, 0, 0, 0)$, it is clear that $a_1(\mathbf{x}) = f(\mathbf{x})$, then, by

Lemma 1, we have that $\mathcal{S}_1 = \{\psi(\mathbf{a}_1), \psi(\mathbf{b}_1)\}$ is a GCP of 2^m and

$$\rho_{\psi(\mathbf{a}_1)}(u) + \rho_{\psi(\mathbf{b}_1)}(u) = 0, \quad \text{for } |u| \in \{1, 2, \dots, 2^m - 1\} \supseteq \mathcal{T}_1 \cup \mathcal{T}_2.$$

When $\mathbf{D}_2 = (1, 0, 1, 0)$, then, we have

$$\begin{aligned} a_2(\mathbf{x}) &= \frac{q}{2} \sum_{s=1}^{m-2} x_{\pi(s)} x_{\pi(s+1)} + \sum_{s=1}^m \mu_s x_s + \mu + \left(\frac{q}{2} x_{\pi(m-2)} + \frac{q}{2} x_{\pi(m-2)} x_{\pi(m)} \right) \\ &\quad + \left(\frac{q}{2} x_{\pi(m)} + \frac{q}{2} x_{\pi(m-1)} x_{\pi(m)} \right), \\ b_2(\mathbf{x}) &= a_2(\mathbf{x}) + \frac{q}{2} x_{\pi(1)}. \end{aligned}$$

For $u \in \mathcal{T}_1 \cup \mathcal{T}_2$, (A.1) can be further expressed as

$$\sum_{i=0}^{2^m-1-u} (\xi^{a_2(i)-a_2(j)} + \xi^{b_2(i)-b_2(j)}) = 0, \quad (\text{A.3})$$

where

$$\begin{aligned} a_2(i) - a_2(j) &= \frac{q}{2} \sum_{s=1}^{m-2} (i_{\pi(s)} i_{\pi(s+1)} - j_{\pi(s)} j_{\pi(s+1)}) + \sum_{s=1}^m \mu_s (i_s - j_s) \\ &\quad + \frac{q}{2} (i_{\pi(m-2)} - j_{\pi(m-2)}) + \frac{q}{2} (i_{\pi(m-2)} i_{\pi(m)} - j_{\pi(m-2)} j_{\pi(m)}) \\ &\quad + \frac{q}{2} (i_{\pi(m)} - j_{\pi(m)}) + \frac{q}{2} (i_{\pi(m-1)} i_{\pi(m)} - j_{\pi(m-1)} j_{\pi(m)}), \\ b_2(i) - b_2(j) &= a_2(i) - a_2(j) + \frac{q}{2} (i_{\pi(1)} - j_{\pi(1)}). \end{aligned}$$

If $i_{\pi(1)} \neq j_{\pi(1)}$, it is straightforward that $\xi^{a_2(i)-a_2(j)} = -\xi^{b_2(i)-b_2(j)}$ and $\xi^{a_2(i)-a_2(j)} + \xi^{b_2(i)-b_2(j)} = 0$.

If $i_{\pi(1)} = j_{\pi(1)}$, we have $a_2(i) - a_2(j) = b_2(i) - b_2(j)$. Let t be the smallest integer such that $i_{\pi(t)} \neq j_{\pi(t)}$, obviously, $2 \leq t \leq m-2$, otherwise, since $\{\pi(1), \pi(2), \dots, \pi(m-2)\} = \{1, 2, \dots, m-2\}$, then, we have $u \in \{2^{m-2}, 2^{m-1}, 2^{m-1} + 2^{m-2}\}$, which is contradict with $u \in \mathcal{T}_1 \cup \mathcal{T}_2$. Let i' and j' be integers which are different from i and j in only one position $\pi(t-1)$, i.e., $i'_{\pi(t-1)} = 1 - i_{\pi(t-1)}$ and $j'_{\pi(t-1)} = 1 - j_{\pi(t-1)}$

respectively, and so $j' = i' + u$, $0 \leq i', j' \leq 2^m - 1$ and $i'_{\pi(1)} = j'_{\pi(1)}$.

Note that

$$\begin{aligned} a_2(i') - a_2(j') &= a_2(i) - a_2(j) + \frac{q}{2}(1 - 2i_{\pi(t-1)})i_{\pi(t)} - \frac{q}{2}(1 - 2j_{\pi(t-1)})j_{\pi(t)} \\ &\equiv a_2(i) - a_2(j) + \frac{q}{2} \pmod{q}. \end{aligned}$$

This implies $\xi^{a_2(i)-a_2(j)} + \xi^{a_2(i')-a_2(j')} = \xi^{b_2(i)-b_2(j)} + \xi^{b_2(i')-b_2(j')} = 0$.

Combining these cases above, we have (A.3) holds.

Step 2 : Eq. (A.2) can be further expressed as

$$\sum_{i=0}^{2^m-1-u} (\xi^{a_1(i)-a_2(j)} + \xi^{b_1(i)-b_2(j)}) = 0, \quad (\text{A.4})$$

where

$$\begin{aligned} a_1(i) - a_2(j) &= \frac{q}{2} \sum_{s=1}^{m-2} (i_{\pi(s)}i_{\pi(s+1)} - j_{\pi(s)}j_{\pi(s+1)}) - \frac{q}{2}j_{\pi(m-2)} - \frac{q}{2}j_{\pi(m)} \\ &\quad - \frac{q}{2}j_{\pi(m-2)}j_{\pi(m)} - \frac{q}{2}j_{\pi(m-1)}j_{\pi(m)} + \sum_{s=1}^m \mu_s (i_s - j_s), \\ b_1(i) - b_2(j) &= a_1(i) - a_2(j) + \frac{q}{2} (i_{\pi(1)} - j_{\pi(1)}). \end{aligned}$$

For $u = 0$, then, we have $i = j$ and

$$\begin{aligned} a_1(i) - a_2(i) &= b_1(i) - b_2(i) \\ &= -\frac{q}{2}i_{\pi(m-2)} - \frac{q}{2}i_{\pi(m-2)}i_{\pi(m)} - \frac{q}{2}i_{\pi(m)} - \frac{q}{2}i_{\pi(m-1)}i_{\pi(m)}. \end{aligned}$$

So, (A.4) can be reduced as

$$\begin{aligned} 2 \sum_{i=0}^{2^m-1-u} (\xi^{a_1(i)-a_2(i)}) &= 2 \left(\sum_{i=0}^{2^m-1-1} (-1)^{i_{\pi(m-2)}} + \sum_{i=2^m-1}^{2^m-1} (-1)^{i_{\pi(m)}-i_{\pi(m-1)}} \right) \\ &= 2(0 + 0) = 0. \end{aligned}$$

For $u \in \mathcal{T}_1 \cup \mathcal{T}_2$, under the same way of the proof of Step 1, we arrive at

$$\sum_{i=0}^{2^m-1-u} (\xi^{a_2(i)-a_1(j)} + \xi^{b_2(i)-b_1(j)}) = 0. \quad (\text{A.5})$$

Combining the cases above, we can conclude that

$$C_{\mathcal{S}_k}(u) = \rho_{\psi(\mathbf{a}_k)}(u) + \rho_{\psi(\mathbf{b}_k)}(u) = 0, \quad \text{for } |u| \in \mathcal{T}_1 \cup \mathcal{T}_2, \quad 1 \leq k \leq 2,$$

and

$$C_{\mathcal{S}_1, \mathcal{S}_2}(u) = \rho_{\psi(\mathbf{a}_1), \psi(\mathbf{a}_2)}(u) + \rho_{\psi(\mathbf{b}_1), \psi(\mathbf{b}_2)}(u) = 0, \quad \text{for } |u| \in \{0\} \cup \mathcal{T}_1 \cup \mathcal{T}_2.$$

The proof is completed. \square

References

- [1] M. D. Renzo, H. Haas, P. M. Grant, Spatial modulation for multiple antenna wireless systems: A survey, *IEEE Commun. Mag.* 49 (12) (2011) 182–191.
- [2] T. L. Narasimhan, P. Raviteja, A. Chockalingam, Generalized spatial modulation in large-scale multiuser MIMO systems, *IEEE Trans. Wireless Commun.* 14 (7) (2015) 3764–3779.
- [3] J. Wang, S. Jia, J. Song, Generalised spatial modulation system with multiple active transmit antennas and low complexity detection scheme, *IEEE Trans. Wireless Commun.* 11 (4) (2012) 1605–1615.
- [4] J. Fu, C. Hou, W. Xiang, L. Yan, Y. Hou, Generalised spatial modulation with multiple active transmit antennas, *Proc. IEEE GLOBECOM 2010 Workshops* (2010) 839–844.

- [5] M. D. Renzo, H. Haas, A. Ghrayeb, S. Sugiura, L. Hanzo, Spatial modulation for generalized MIMO: Challenges, opportunities and implementation, *Proc. IEEE* 102 (1) (2014) 56–103.
- [6] L. He, J. Wang, J. Song, Spatial modulation for more spatial multiplexing: RF-chain-limited generalized spatial modulation aided MM-wave MIMO with hybrid precoding, *IEEE Trans. Commun.* 66 (3) (2018) 986–998.
- [7] S. Guo, H. Zhang, P. Zhang, S. Dang, C. Liang, M. Alouini, Signal shaping for generalized spatial modulation and generalized quadrature spatial modulation, *IEEE Trans. Wireless Commun.* 18 (8) (2019) 4047–4059.
- [8] M. Wen, B. Zheng, K. Kim, M. Renzo, T. Tsiftsis, K.-C. Chen, N. Aldhahir, A survey on spatial modulation in emerging wireless systems: research progresses and applications, *IEEE J. Sel. Areas Commun.* 37 (9) (2019) 1949–1972.
- [9] P. Yang, Y. Xiao, Y. L. Guan, K. V. S. Hari, A. Chockalingam, S. Sugiura, H. Hass, M. D. Renzo, C. Masouros, Z. Liu, L. Xiao, S. Li, L. Hanzo, Single-carrier SM-MIMO: a promising design for broadband large-scale antenna systems, *IEEE Commun. Surveys Tuts.* 18 (3) (2016) 1687–1716.
- [10] Y. Xiao, Z. Yang, L. Dan, P. Yang, Y. Yin, W. Xiang, Low-complexity signal detection for generalized spatial modulation, *IEEE Commun. Lett.* 18 (3) (2014) 403–406.

- [11] C.-T. Lin, W.-R. Wu, C.-Y. Liu, Low-complexity ML detectors for generalized spatial modulation systems, *IEEE Trans. Commun.* 63 (11) (2015) 42144230.
- [12] L. Xiao, Time-domain turbo equalization for single-carrier generalized spatial modulation, *IEEE Trans. Wireless Commun.* 16 (9) (2017) 5702–5716.
- [13] T. Wang, S. Liu, F. Yang, J. Wang, J. Song, Z. Han, Generalized spatial modulation-based multi-user and signal detection scheme for terrestrial return channel with NOMA, *IEEE Trans. Broadcast* 64 (2) (2018) 211–219.
- [14] Y. Zhao, P. Yang, Y. Xiao, B. Dong, W. Xiang, Soft-feedback time-domain turbo equalization for single-carrier generalized spatial modulation, *IEEE Trans. Veh. Technol.* 67 (10) (2018) 9421–9434.
- [15] S. Sugiura, L. Hanzo, Single-RF spatial modulation requires single-carrier transmission: frequency-domain turbo equalization for dispersive channels, *IEEE Trans. Veh. Technol.* 64 (10) (2015) 4870–4875.
- [16] B. Gong, L. Gui, S. Luo, Y.-L. Guan, Z. Liu, P. Fan, Block pilot based channel estimation and high- accuracy signal detection for GSM-OFDM systems on high-speed railways, *IEEE Trans. Veh. Technol.* 67 (12) (2018) 11525–11536.
- [17] H. Chu, L. Zheng, X. Wang, Super-resolution mmwave channel estimation for generalized spatial modulation systems, *IEEE J. Sel. Topics Signal Process.* 13 (6) (2019) 1336–1347.

- [18] X. Kuai, X. Yuan, W. Yan, H. Liu, Y. Zhang, Double-sparsity learning-based channel-and-signal estimation in massive MIMO with generalized spatial modulation, *IEEE Trans. Commun.* 68 (5) (2020) 2863–2877.
- [19] S.-A. Yang, J. Wu, Optimal binary training sequence design for multiple-antenna systems over dispersive fading channels, *IEEE Trans. Veh. Technol.* 51 (5) (2002) 1271–1276.
- [20] C. Fragouli, N. Al-Dhahir, W. Turin, Training-based channel estimation for multiple-antenna broadband transmissions, *IEEE Trans. Wire. Commun.* 2 (2) (2003) 384–391.
- [21] P. Fan, W. H. Mow, On optimal training sequence design for multiple-antenna systems over dispersive fading channels and its extensions, *IEEE Trans. Veh. Technol.* 53 (5) (2004) 1623–1626.
- [22] Z. Liu, P. Yang, Y. L. Guan, P. Xiao, Cross Z-complementary pairs (CZCPs) for optimal training in spatial modulation over frequency selective channels, *IEEE Trans. Signal Process.* 68 (2020) 1529–1543.
- [23] A. Pezeshki, A. R. Calderbank, W. Moran, S. D. Howard, Doppler resilient Golay complementary waveforms, *IEEE Trans. Inf. Theory* 54 (9) (2008) 4254–4266.
- [24] P. Spasojevic, C. N. Georghiades, Complementary sequences for ISI channel estimation, *IEEE Trans. Inf. Theory* 47 (3) (2001) 1145–1152.
- [25] S. Wang, A. Abdi, MIMO ISI channel estimation using uncorrelated Golay complementary sets of polyphase sequences, *IEEE Trans. Veh. Technol.* 56 (5) (2007) 3024–3040.

- [26] H. Chen, J. F. Yeh, N. Suehiro, A multicarrier CDMA architecture based on orthogonal complementary codes for new generations of wideband wireless communications, *IEEE Commun. Mag.* 39 (10) (2001) 126–135.
- [27] Z. Liu, Y. L. Guan, U. Parampalli, New complete complementary codes for peak-to-mean power control in multi-carrier CDMA, *IEEE Trans. Commun.* 62 (3) (2014) 1105–1113.
- [28] Z. Liu, Y. L. Guan, H. Chen, Fractional-delay-resilient receiver for interference-free MC-CDMA communications based on complete complementary codes, *IEEE Trans. Wireless Commun.* 21 (3) (2014) 1226–1236.
- [29] C.-C. Tseng, C. Liu, Complementary sets of sequences, *IEEE Trans. Inf. Theory* 18 (5) (1972) 644–652.
- [30] M. J. E. Golay, Statatic multislit spectroscopy and its application to the panoramic display of infrared spectra, *J. Opt. Soc. Amer.* 41 (1951) 468–472.
- [31] J. A. Davis, J. Jedwab, Peak-to-mean power control in OFDM, Golay complementary sequences, and Reed-Muller codes, *IEEE Trans. Inf. Theory* 45 (7) (1999) 2397–2417.
- [32] M. J. E. Golay, Complementary series, *IRE Trans. Inf. Theory* 7 (7) (1961) 82–87.
- [33] K. G. Paterson, Generalized Reed-Muller codes and power control in OFDM modulation, *IEEE Trans. Inf. Theory* 46 (1) (2000) 104–120.

- [34] K.-U. Schmidt, On cosets of the generalized first-order Reed-Muller code with low PMEPR, *IEEE Trans. Inf. Theory* 52 (7) (2006) 3220–3232.
- [35] C.-Y. Chen, Complementary sets of non-power-of-two length for peak-to-average power ratio reduction in OFDM, *IEEE Trans. Inf. Theory* 62 (12) (2016) 7538–7545.
- [36] C.-Y. Chen, A new construction of Golay complementary sets of non-power-of-two length based on Boolean functions, in: *IEEE Wireless Commun. Netw. Conf.*, 2017, pp. 1–6.
- [37] P. Fan, W. Yuan, Y. Tu, Z-complementary binary sequences, *IEEE Signal Process. Lett.* 14 (8) (2007) 509–512.
- [38] R. Appuswamy, A. K. Chaturvedi, A new framework for constructing mutually orthogonal complementary sets and ZCZ sequences, *IEEE Trans. Inf. Theory* 52 (8) (2006) 3817–3826.
- [39] D. Wu, P. Spasojevi, Complementary set matrices satisfying a column correlation constraint, *IEEE Trans. Inf. Theory* 54 (7) (2008) 3330–3339.
- [40] S. Das, S. Budišin, S. Majhi, Z. Liu, Y. L. Guan, A multiplier-free generator for polyphase complete complementary codes, *IEEE Trans. Signal Process.* 66 (5) (2018) 1184–1196.
- [41] S. Das, S. Majhi, Z. Liu, A novel class of complete complementary codes and their applications for APU matrices, *IEEE Signal Process. Lett.* 25 (9) (2018) 1300–1304. doi:10.1109/LSP.2018.2849656.

- [42] S. Das, S. Majhi, S. Budišin, Z. Liu, A new construction framework for polyphase complete complementary codes with various lengths, *IEEE Trans. Signal Process.* 67 (10) (2019) 2639–2648. doi:10.1109/TSP.2019.2908137.
- [43] R. Appuswamy, A. K. Chaturvedi, Complete mutually orthogonal Golay complementary sets from Reed-Muller codes, *IEEE Trans. Inf. Theory* 54 (3) (2008) 1339–1346.
- [44] Z. Liu, Y. L. Guan, B. C. Ng, H.-H. Chen, Correlation and set size bounds of complementary sequences with low correlation zone, *IEEE Trans. Commun.* 59 (12) (2011) 3285–3289.