Challenging Test Problems for Multi- and Many-Objective Optimization

Saúl Zapotecas-Martínez[a,*], Carlos A. Coello Coello[b], Hernán E. Aguirre[c], Kiyoshi Tanaka[c]

[a] *Computer Science Department, Instituto Nacional de Astrofísica Óptica y Electrónica, Tonantzintla, Puebla 72840, MEXICO*
[b] *Computer Science Department, CINVESTAV-IPN, México City, 07360, MEXICO*
Basque Center for Applied Mathematics (BCAM) & Ikerbasque, SPAIN
[c] *Faculty of Engineering, Shinshu University, 4-17-1 Wakasato, Nagano 380-8553, JAPAN*

Abstract

In spite of the extensive studies that have been conducted regarding the construction of multi-objective test problems, researchers have mainly focused their interests on designing complicated search spaces, disregarding, in many cases, the design of the Pareto optimal fronts of the problems. In this regard, the work related to scalable multi-objective test problems—i.e., problems that can be formulated for an arbitrary number of objectives—has been much less studied. This paper introduces a new set of continuous and box-constrained multi-objective test problems which are scalable in both the number of objectives and in the number of decision variables. Each test problem included in the proposed test suite has a peculiar Pareto front different from those observed in the existing scalable multi-objective test suites. In addition to different Pareto fronts, the proposed test suite introduces features related to the search space that place obstacles that complicate exploring Pareto optimal solutions. Such features can be easily switched *on* and *off* by the user to analyze specific mechanisms of multi-objective evolutionary algorithms (MOEAs). The components used in the proposed test suite can be used as a toolkit to construct new test instances

*Corresponding author
Email addresses: szapotecas@inaoep.mx (Saúl Zapotecas-Martínez), ccoello@cs.cinvestav.mx (Carlos A. Coello Coello), ahernan@shinshu-u.ac.jp (Hernán E. Aguirre), ktanaka@shinshu-u.ac.jp (Kiyoshi Tanaka)

not included in this set of problems. To illustrate the use and difficulties of the proposed test suite, some experiments are presented adopting three MOEAs using selection mechanisms based on Pareto optimality, decomposition, and a performance indicator (hypervolume).

Keywords: multi-objective optimization, scalable multi-objective test problems, evolutionary multi-objective algorithms.

1. Introduction

Multi-objective optimization problems (MOPs) involve the simultaneous optimization of a number of commonly conflicting objectives. Because of their nature, based on the use of a population, multi-objective evolutionary algorithms (MOEAs) have become a standard optimization tool to solve MOPs. To analyze and evaluate the working principles of MOEAs, several researchers have adopted the use of artificial test functions where the features of the problem are known beforehand. This has facilitated the understanding of specific components of MOEAs in particular scenarios. A multi-objective test problem should include a variety of characteristics that simulate the properties usually observed in real-world problems. For this reason, the design of multi-objective test problems plays an essential role in the headway of MOEAs. Consequently, a wide variety of multi-objective test problems have been proposed so far. An extensive review of multi-objective test problems can be found in [1, 2]. Optimization problems involving more than three objective functions are commonly referred to as many-objective optimization problems. Their solution with MOEAs has motivated a significant amount of research, as such problems appear in many real-world applications. A compilation of real-world problems can be found in [3, 4, 5]. However, notwithstanding the wide variety of MOEAs that have shown success in solving problems with two and three objectives, several studies have revealed the inefficiency of the existing evolutionary approaches in high-dimensional objective spaces [6, 7]. This has been the main incentive to design new evolutionary approaches capable of solving efficiently many-objective optimization problems, the so-called many-objective evolutionary algorithms.

In spite of the existence of several multi-objective test problems reported in the specialized literature, their scalability in the number of objectives has been a topic much less studied. Particularly, the variety of Pareto optimal fronts is limited by a reduced number of shapes that have been reported so far in the literature [2]. On the other hand, most of the many-objective problems do not have as many characteristics as those designed for multi-objective optimization [2]. In addition to this, the growing development of multi- and many-objective evolutionary algorithms makes the design of scalable multi-objective problems an active area of research.

According to the recommendations and features suggested by Huband et al. [1] and extended by Zapotecas et al. [2], multi/many-objective optimization problems should consider no extremal nor medial optimal solutions (related to the search space), scalability in the number of decision variables and objective functions, dissimilar scales in the decision variables and objective functions, and known optimal solutions. On the other hand, the test problems should attend features such as Pareto front geometries (concave, convex, linear, disconnected, or mixed), separability, bias, many-to-one solutions, modality, difficult Pareto set topologies and difficult Pareto front shapes. Additionally, the test problems should consider the correlation between position and distance functions, the single optimal solution for a high number of objectives, and an easy configuration of features [1]. These are properties that, in fact, complicate the design of test problems for multi- and many-objective optimization.

This paper introduces a new set of continuous and box-constrained multi-objective test problems that are scalable in the number of objectives and in the number of decision variables. Overall, the proposed test suite incorporates the following characteristics:

- Each test problem has a unique Pareto front with a peculiar shape unusu-

[1]The complete list of recommendations and features can be seen in Table 1.

ally seen in the specialized literature. This motivates the design of new density estimators to represent in a proper way the entire Pareto front in high dimensionality.

- In addition to new Pareto front shapes, the proposed test suite incorporates a wide variety of characteristics that complicate the search for Pareto optimal solutions, such as multi-modality, non-separability, deception, and bias.

- Moreover, the proposed test suite includes test problems with difficult Pareto sets, i.e., problems for which it is difficult to find a Pareto solution from another Pareto solution in a determined neighborhood. This property of multi-objective problems hinders the search process of evolutionary approaches.

- An advantage of the proposed test suite is that the test problems can be easily configured by using different parameters settings.

- Additionally, the components employed in the proposed test suite can be used as a toolkit for the construction of other multi-objective test problems different from those formulated in this paper.

In order to illustrate the difficulties of the proposed test suite, a couple of experiments is carried out adopting state-of-the-art MOEAs based on three different schemes: decomposition, Pareto optimality and a performance indicator (hypervolume). The experiments in this work show how difficult it is to solve a test problem from the proposed test suite using different configurations.

The remainder of this paper is organized as follows: Section 2 introduces a review of test suites for multi- and many-objective optimization. Section 3 describes the proposed test problems introducing their mathematical formulation. Section 4 presents the features of the proposed test problems and analyzes them. Section 5 shows how to use the proposed test suite, including its configuration, and analyzes how difficult it is to solve a test instance under different settings.

Table 1: Recommendations and features for multi-objective test problems

Recommendation (R) or Feature (F)	Comment
R1: No Extremal Parameters	Prevents exploitation by truncation-based correction operators
R2: No Medial Parameters	Prevents exploitation by intermediate recombination
R3: Scalable Number of Parameters	Increases flexibility, demands scalability
R4: Scalable Number of Objectives	Increases flexibility, demands scalability
R5: Dissimilar Parameter Domains	Encourages EAs to scale mutation strengths appropriately
R6: Dissimilar Trade-off Ranges	Encourages normalization of objective values
R7: Pareto Optima Known	Facilitates the use of measures, and analysis of results, in addition to other benefits
F1: Pareto Optimal Geometry	Convex, linear, concave, mixed, degenerate, disconnected, or some combination
F2: Parameter dependencies	Objectives can be separable or non-separable
F3: Bias	Substantially more solutions exist in some regions of fitness space than they do in others
F4: Many-to-one mappings	Pareto one-to-one/many-to-one, flat regions, isolated optima
F5: Modality	Uni-modal, or multi-modal (possibly deceptive multi-modality)
F6: Difficult Pareto Set Topology	Pareto set difficult to characterize
F7: Difficult Pareto Front Shape	Pareto optimal front difficult to estimate
F8: Correlation of Position and Distance Functions	Dependencies between position and distance functions
F9: Single Optimal Solution for a High Number of Objectives	Single objective solution for multiple objective functions
F10: Easy Configuration of Features in Scalable Test Problems	Easy way to configure the features of a scalable test problem

Finally, in Section 6, we provide our conclusions and some possible paths for future research.

2. Multi- and Many-Objective Optimization Problems

Throughout two decades of evolutionary multi-objective optimization (EMO), the success of evolutionary approaches has been shown in real-world applications in different disciplines [8, 9, 10, 11, 12]. A multi-objective optimization solver should provide information to the Decision Maker (DM) to satisfy his/her needs for a specific problem. Accordingly, a multi-objective optimization algorithm must be able to find a set of non-dominated solutions that glimpses a genuine representation of the real Pareto front of a given problem. However, as discussed by some authors [13, 2], the performance assessment of multi-objective approaches is not an easy task. Since the early days of multi-objective optimization, the performance assessment of multi-objective approaches has been widely studied in the EMO community. In the specialized literature, it is possible to find important contributions related to the performance of multi-objective algorithms for both general purposes and particular problems [14, 15, 16]. On the other hand, the design of artificial test functions for multi-objective optimization has been essential to evaluating and understanding the behavior of evolutionary approaches. The known features of multi-objective test problems (Pareto set and Pareto front) give the possibility to evaluate more correctly multi-objective algorithms and understand their performance in known scenarios. In this sense, the following section presents a brief review of synthetic test suites for multi- and many-objective optimization.

2.1. Test suites for multi-objective optimization

From the earliest developments of multi-objective test problems, the scalability in the number of decision variables has been one of the main features considered. In this regard, design approaches such as Deb's toolkit [17] and Ziztler et al. 's (ZDT) test suite [18] marked the beginning of the design of multi-objective test suites scalable in the number of decision variables. With the introduction of the bottom-up approach [19], the design of multi-objective test suites has become much more accessible, facilitating the construction of the Pareto front and the Pareto set. Most multi-objective test suites adopting the bottom-up approach are scalable in the number of decision variables and objectives. In this regard, Li and Zhang [20, 21] introduced a multi-objective test suite scalable in the number of decision variables with complicated Pareto sets. This test suite incorporates several characteristics that make it challenging to explore Pareto optimal solutions to problems with two and three objectives. Mirjalili and Lewis [22] presented a multi-objective test suite to evaluate the performance of MOEAs dealing with robust optimization. This test suite considers two and three objectives, including multimodality and no separability, making difficult the exploration of Pareto optimal solutions. Li et al. [23] introduced a multi-objective test suite with degenerated Pareto fronts to evaluate the survival mechanism in MOEAs. In this test suite, the diversity mechanisms of MOEAs are tested to find a suitable representation of the true Pareto front. Li et al. [24] introduced a test suite with biased search spaces, which complicates the approximation to the real Pareto front. In this test suite, MOEAs are tested, evaluating the genetic operators to deal with biased spaces. Another problem collection is proposed by Liu et al. [25]. This test suite incorporates imbalanced objective functions, i.e., problems with objective functions that are more challenging to solve. In this regard, MOEAs are evaluated to achieve the extreme portions and obtain a proper representation of the Pareto optimal front. Wang et al. [26] introduced nine test problems for multi-objective optimization. This test suite incorporates difficulties in the search space, such as nonseparability, multimodality, and rugged PS. The PF shapes in this test suite involve linear, convex, and dissimilar objective ranges. However, their formulation is limited to two and three objective functions. More recently, Songbai et al. [27] introduced a test suite for large-scale multi-objective optimization. The proposed test suite is based on traditional single-objective functions, such as the Schewefel, Ackley, Rastrigin, and Rosenbroack functions. Therefore, the properties of uni/multimodality and separability are inherited from functions based on a single objective. The proposed test problems are formulated for two and three objective functions. In the case of two-objective problems, PF shapes can be seen as convex, concave, and linear, while three-objective problems adopt PF shapes similar to an inverted simplex.

Although the above test suites are not scalable in the number of objectives, they can be employed in some studies in the context of large-scale multi-objective optimization.

2.2. Test suites for many-objective optimization

In the last two decades, researchers have developed multi-objective test problems scalable in the number of objectives. Pioneering designs of test suites such as the Deb-Thiele-Laumanns-Zitzler (DTLZ) [19] and the Working-Fish-Group (WFG) [1] problems marked the beginning of the construction of many-objective test problems with known features. The DTLZ and WFG test suites incorporate several characteristics such as unimodal, multimodal, biased, and deceptive landscapes that evaluate the abilities of multi-objective algorithms to find Pareto optimal solutions. Emmerich and Deutz [28] introduced a scalable test suite based on the Lamé supershperes. This test suite can modify the convexity/concavity of the Pareto front shapes by setting the parameter γ in the Lamé spheres. Remarkably, this test suite introduced the concept of mirror test problems where the spheres are inverted to generate different Pareto front shapes. The principle of complicated PSs introduced for two and three objective problems [21, 20], was extended by Saxena et al. [29] in many-objective test problems. In this test suite, the Pareto front shapes include degenerated surfaces for problems having more than three objectives. Cheng et al. [30] introduced a set of nine test problems scalable in the number of objectives and designed to evaluate MOEAs in large-scale optimization. This proposal states the decision variable dependency by two variable linkage functions (linear and nonlinear). Additionally, this test suite introduces a correlation between decision variables and objectives by employing a correlation matrix. Masuda et al. [31] proposed a toolkit to generate test problems scalable in the number of objectives. This test suite is mainly focused on the design of different Pareto optimal shapes. The methodology introduced in this approach allows the design of Pareto optimal surfaces by using a finite number of vertices. Such vertices state the Pareto optimal front, whose shape can be defined as linear, concave, or convex. Other test suites scalable in the number of objectives were introduced by Ishibuchi et al. [32]. This test suite proposed the minus versions of the DTLZ and WFG test problems, namely minus-DTLZ (DTLZ^{-1}) and minus-WFG (WFG^{-1}), respectively. These test problems stand out mainly because the Pareto optimal fronts of the original DTLZ and WFG test problems are inverted in order to obtain a similar effect as in the mirror LSS test problems [28]. However, the test problems maintain the same properties regarding the difficulties of the distance functions proposed for DTLZ and WFG, respectively. Cheng et al. [33] presented a compilation of 15 test problems scalable in the number of objectives. This test suite, called MaF, includes test problems mainly taken from WFG, DTLZ, and ML-DMP. Thus, a wide variety of features can be found in this test suite which, indeed, shall be able to assess the robustness of many-objective evolutionary approaches. Wang et al. [34] modified the well-known DTLZ1–DTLZ4 test problems to avoid the regularly-oriented Pareto front shapes and the single distance functions in all the objectives. The main characteristic of the modified DTLZ (mDTLZ) test problems is the hardly-dominated boundaries, which hinder the approximation of optimal solutions for Pareto dominance-based MOEAs and Tchebycheff-decomposition-based MOEAs. A generator for multi-objective test problems was introduced in [35]. The proposed generator enables the design of test problems with controllable difficulties. Particularly, multimodality, deceptivity, and bias features can be observed in the search spaces. Additionally, rugged PSs complicate the search for Pareto optimal solutions. Although this generator offers interesting features regarding the fitness landscapes, the test problems are limited to regular PF shapes (e.g., a normal/inverted simplex or similar). Li et al. [36] introduced a test suite for many- and multi-objective optimization. The proposed problems introduce challenging difficulties such as multimodality and biased search spaces, which, in fact, complicate the exploration of optimal solutions. An interesting feature of these test problems is the degeneracy of the PF shapes, which deteriorates the performance of predefined reference-vector-based MOEAs such as MOEA/D.

With the aim to visualize the behavior of algorithms in many-objective problems, Ishibuchi et al. [37] proposed a set of test problems in two- and three-dimensional decision spaces based on a generalization of the single polygon problems presented by Köppen and Yoshida [38] and the multi-line (or multi-curve) problems introduced by Rudolph et al. [39]. Although these test problems were formulated for low-dimensional decision spaces, the authors gave the idea to design test problems in high-dimensional spaces by specifying multiple points in the required dimensionality. Such an idea was implemented in the many-objective test suite formulated in a high-dimensional decision space [32]. Inspired by the above test suites, Li et al. [40] introduced a test problem whose Pareto optimal solutions lie in a rectangle (in the two-variable decision space), and they are similar (in the sense of Euclidean geometry) to their images in the four-dimensional objective space. As a generalization of Li et al.'s test problem, Li [24] introduced a class of multi-objective test problems scalable in the number of objectives (called multi-line distance minimization problems (ML-DMP)). In this test suite, the Pareto optimal solutions lie in a regular polygon in a 2-D decision space, and these solutions are similar (in the sense of Euclidean geometry) to their images in high-dimensional objective spaces. More recently, Fieldsend et al. [41] introduced a visualizable test problem generator for many-objective optimization. This problem is based on the distance between vertices. However, the test problem includes hard constraints and non-intersecting performance disconnected Pareto sets. The test problem generator has the possibility to combine different features to construct more complicated problems.

2.3. Suites of real-world multi-objective optimization problems

In spite of the advantages of using artificial test problems to assess the performance of algorithms, their formulations have some limitations. As some authors indicate, synthetic test problems do not have all the properties observed in real-world applications [42, 40]. Therefore, an MOEA that works well in artificial test problems does not necessarily work well in real-world applications. However, researchers have evaluated the performance of multi-objective algorithms using synthetic test problems for more than two decades. The main reason is that the details of those real-world problems are not always in the public domain due to confidentiality agreements. On the other hand, researchers dedicated to design multi-objective algorithms are not necessarily experts in the disciplines in which these problems lie. In this sense, some researchers have summarized real-world applications from different research areas. Regarding mathematical models formulated as unconstrained multi-objective optimization problems (the topic covered in this paper), we mention some works that compile real-world problems reported in the literature.

Tanabe et al. [3] presented a set of real-world problems taken from different disciplines to assess the performance of multi-objective evolutionary algorithms. Most of these problems have constraints handled by the composition of a function added to the multi-objective formulation. He et al. [4] presented seven multi-objective optimization problems to promote data-driven evolutionary multi-objective optimization research. The authors analyzed the performance of four popular algorithms, including three data-driven MOEAs and a model-free MOEA. From the perspective of problem properties, the set of adopted problems covers different problems with different irregular/regular Pareto optimal fronts. More recently, Zapotecas et al. [5] presented a compilation of real-world applications formulated as box-constrained multi-objective optimization problems. The correlation between the objectives was analyzed and discussed. Particularly, the authors' study inquired about the performance of traditional multi-objective evolutionary approaches based on Pareto dominance, decomposition, and hypervolume principles, in real-world problems with peculiar PF shapes.

The studies mentioned above are just an example of the variety of problems that exist in the real world. Some of these problems have different characteristics that complicate the performance of MOEAs. In particular, the suites mentioned above are evidence for the existence of peculiar PF shapes in real-world applications. Figure 1 illustrates the PF approximations of some problems in the above suites. It is worth noticing that the development of new problem-solving strategies has brought many applications based on simulations and/or different types of models with unknown characteristics. Such models are formulated as multi-objective optimization problems that can occasionally be difficult to solve and interpret. In this sense, it is crucial to design simpler multi-objective problems with complex challenges but known characteristics to guide researchers in the design of MOEAs. While artificial test functions are not a substitute for real-life problems, they can be used as black-box problems to assess in a more effective way the performance of MOEAs. For this reason, we believe that the design of synthetic test problems will continue to be a line of research in the EMO community for many more years. In the next section, the proposed test suite is presented.

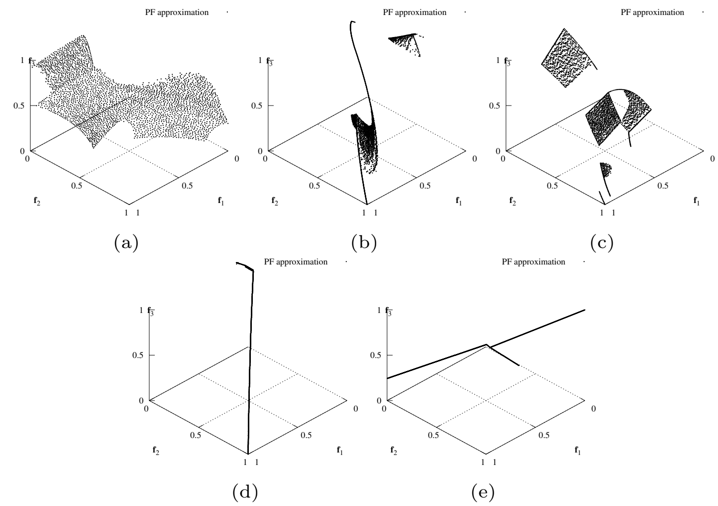


Figure 1: Normalized Pareto front approximations for the problems (a) Liquid-rocket single element injector design [43]; (b) milling parameters for ultrahigh-strength steel [44]; (c) crashworthiness design of vehicles [45]; (d) packed bed latent heat thermal storage performance [46] and (e) wire electrical discharge machining [47].

3. Proposed Test Suite

3.1. General Framework

In the proposed test suite, a multi-objective test problem consists in minimizing the M objective functions which are of the form:

$$\begin{aligned} f_{i=1:M}(\mathbf{x}) &= \alpha_i(\mathbf{x}_I) + \beta_i(\mathbf{x}_{II} - g(\mathbf{x}_I|m)) \\ \text{s.t.} \quad & \quad \mathbf{x} \in \Omega \end{aligned} \tag{1}$$

where

- $\Omega = \prod_{i=1}^{n}[a_i, b_i]$ is the feasible decision space;
- $\mathbf{x}$ is the decision variable;
- $\mathbf{x}_I$ and $\mathbf{x}_{II}$ are two subvectors of $\mathbf{x}$, such that, $\mathbf{x}_I = (x_1, \ldots, x_m)$ and $\mathbf{x}_{II} = (x_{m+1}, \ldots, x_n)$.
- α_i $(i = 1, \ldots, M)$: functions from $\prod_{i=1}^{m}[a_i, b_i]$ to $\mathbb{R}$;

- β_i $(i = 1, \ldots, M)$: functions from $\mathbb{R}^{n-m}$ to $\mathbb{R}^{+}$;
- g: a function from $\prod_{i=1}^{m}[a_i, b_i]$ to $\prod_{i=m+1}^{n}[a_i, b_i]$.

Therefore, the Pareto front and the Pareto set of the above MOP are stated by the following theorem [20][2]:

Theorem 1. *Suppose that:*

i) $\beta_i(\mathbf{z}) = 0$ *for all* $i = 1, \ldots, M$ *iff* $\mathbf{z} = \mathbf{0}$

ii) *The Pareto set of the following problem:*

$$\begin{aligned} \textbf{minimize:} \quad & (\alpha_1(\mathbf{x}_I), \ldots, \alpha_M(\mathbf{x}_{II})) \\ s.t. \quad & \mathbf{x}_I \in \textstyle\prod_{i=1}^{m}[a_i, b_i] \end{aligned} \tag{2}$$

is $E \subset \prod_{i=1}^{m}[a_i, b_i]$.

Then, the Pareto set of the problem defined in Equation (1) *is:*

$$\mathbf{x}_{II} = g(\mathbf{x}_I|m), \quad \mathbf{x}_I \in E$$

that is with $\beta_i(\mathbf{z}) = 0$*, for* $i = 1, \ldots, M$ *and the PF is:*

$$\{(\alpha_i(\mathbf{x}_I), \ldots, \alpha_M(\mathbf{x}_I))|\mathbf{x}_I \in E\}$$

To design a multi-objective test instance with known features, the following remarks are of interest:

- $\mathbf{x}_I$ states the position parameters while $\mathbf{x}_{II}$ defines the distance parameters.

[2]The proof of this theorem is found in [20].

- m defines the dimensionality of E. If the dimensionality of E is $m = M - 1$, then the Pareto set and the Pareto front of the test problem will be $(M - 1)$, which follows the regularity property of continuous MOPs.

- α_i's function determines the Pareto front of the test problem, while the Pareto set is defined by the g function.

- β's functions define the distance of solutions to the Pareto front and regulate the difficulty of the test problem.

3.2. Proposed Test Problems

In this test suite, we introduce twenty scalable test problems with different characteristics. For all the test problems, the decision variable space is given by

$$\Omega = \prod_{i=1}^{n}[a_i, b_i] = \prod_{i=1}^{n}\left[-\frac{i}{2}, \frac{i}{2}\right] \tag{3}$$

For convenience, it is assumed that: $\mathbf{y} = (\frac{x_1-a_1}{b_1-a_1}, \ldots, \frac{x_n-a_n}{b_n-a_n})$ as the normalized vector of the decision variables $\mathbf{x} = (x_1, \ldots, x_n) \in \Omega$. The problem in Equation (1) is now rewritten as

$$\begin{aligned} f_{i=1:M}(\mathbf{y}) \quad &= \alpha_i(\mathbf{y}_I) + \beta_i(\mathbf{y}_{II} - g(\mathbf{y}_I|m)) \\ \text{s.t.} \quad & \mathbf{y} \in [0,1]^n \end{aligned} \tag{4}$$

where n is the number of decision variables, m is the number of position-related decision variables, $\mathbf{y}_I = (y_1, \ldots, y_m)$ and $\mathbf{y}_{II} = (y_{m+1}, \ldots, y_n)$.

A test instance is generated by constructing position functions α_i's, distance functions β_i's, and a vector function g that defines the shape of the Pareto set satisfying conditions *i)* and *ii)* in Theorem 1. In the following, we describe all of these functions.

3.2.1. Position Functions

In the proposed test problems, the position functions of an M-objective test problem are given in the form:

$$\alpha_i(\mathbf{y}_I) = i^2 \times F_i(\mathbf{y}_I) \quad \text{for } i = 1, \ldots, M \tag{5}$$

where functions F_i define the shape of the Pareto front.

Each test problem has a Pareto front with a different shape from each other problem formulated in this test suite. Therefore, the mathematical formulation of the F_i's functions is unique for each test problem. Table 2 lists the proposed test problems formulating the unique F_i functions.

3.2.2. Distance Functions

Distance functions can be formulated in many ways, thus setting distinct features and difficulties to a test problem. In the proposed test suite, the B_i functions are defined as follows.

Let $\mathbf{y}_{II} - g(\mathbf{y}_I|m) = (y_{m+1} - g_{m+1}(\mathbf{y}_I|m), \ldots, y_n - g_n(\mathbf{y}_I|m))$ be denoted by $z_{m+1:n} = (z_{m+1}, \ldots, n)$, then the β_i functions are of the form:

$$\beta_i(z_{m+1:n}) = i^2 \times Z_{\texttt{Level}}(w_{m+1:n}|J_i) \quad \text{for } i = 1, \ldots, M \tag{6}$$

such that

$$w_{m+1:n} = \begin{cases} (Z_{\texttt{Bias}}(z_{m+1}), \ldots, Z_{\texttt{Bias}}(z_n)), & \texttt{Bias=True} \\ z_{m+1:n}, & \texttt{Otherwise} \end{cases} \tag{7}$$

where n is the number of decision variables, m is the number of position-related decision variables, and $Z_{\texttt{Bias}}$ is a transformation that biases the test problem (*cf.* Section 4.3.2).

In Equation (6), J_i is the set of indices from $\mathbf{y}_{II}$ correlated to the i^{th} β function. In the proposed test problems, such correlation is stated by the modular approach [29]. Concretely, the set of indices correlated to the i^{th} objective is defined by

$$J_i = \{j| \mod (j - m - i, M) = 0, j = m + 1, \ldots, n\} \tag{8}$$

In Equation (6), the $Z_{\texttt{Level}}$ ($\texttt{Level} \in \{1, \ldots, 6\}$) function states the difficulty of approximating solutions to the Pareto front. This function is set according to different configuration levels (*cf.* Section 4.3.1). It is worth noticing that by defining the same $Z_{\texttt{Level}}$ function for all β_i functions, the difficulty to optimize each β_i function shall be exactly the same. However, it is possible to modify its difficulty by using a different $Z_{\texttt{Level}}$ function for each β_i function. As a result, for a given i^{th} objective, if the β_i function is much easier to optimize than the others, the search of an MOEA will be biased towards the i^{th} objective, leaving the rest of the objective functions without a proper exploration. This type of problems, known in the specialized literature as imbalanced [25], shall test the abilities of an MOEA to achieve a proper approximation and distribution along the Pareto front. Imbalanced problems can be constructed in different ways [25]. For example, in the ZCAT test suite, the imbalance between objectives is controlled by a flag that reformulates the problem (*cf.* Section 4.3.3). Thus, an imbalanced test problem is formulated by

$$\beta_i(z_{m+1:n}) = i^2 \times \begin{cases} Z_4(w_{m+1:n}|J_i), & \text{mod (i, 2)} = 0 \\ Z_1(w_{m+1:n}|J_i), & \text{otherwise} \end{cases} \tag{9}$$

for $i = 1, \ldots, M$.

If the concerned flag is not activated, the β_i functions shall be exactly defined as in Equation (6).

3.2.3. Pareto Set Topology Function

Let us consider the multi-objective optimization problem given in Equation (4). In this case, the vectorial function g maps the vector $\mathbf{y}_I \in [0, 1]^m$ to another vector in $[0,1]^{n-m}$. This transformation is denoted by

$$\mathbf{g}(\mathbf{y}_I|m) = (g_{m+1}(\mathbf{y}_I|m), \ldots, g_n(\mathbf{y}_I|m)) \tag{10}$$

where each component g_j $(j = m+1, \ldots, n)$ defines the shape of the Pareto set. Since the $\mathbf{g}$ function generates a vector in $[0,1]^{n-m}$, i.e., in the normalized search space, the Pareto solution to the scalable test problem is reached by expanding the resulting $\mathbf{g}$ vector into the original search space Ω (*cf.* Equation (3)).

The proposed test suite introduces ten complicated PS topologies which are mathematically formulated in Table 3 and instantiated for each test problem in Table 2. Such topologies can be switched *on* and *off* to be complicated or simple (*cf.* Section 4.2).

To exemplify the way in which a test problem can be constructed, in Appendix A from the supplementary material, we detail the instantiation of three test problems. On the other hand, the source code of the proposed test suite can be obtained at `https://github.com/evo-mx/ZCAT`.

4. Features of the proposed test suite

In the proposed test suite, all the test problems follow the recommendations discussed by Huband et al.[1]. Additionally, each test problem adds several features that determine its difficulty which can be defined *a priori*. In the following, the features of the proposed test problems are exposed. We particularly refer to the set of features discussed by Huband et al.[1] and Zapotecas et al.[2].

4.1. Pareto Front Features

The Pareto front of a test problem is defined by the parametric functions F formulated (for each test instance) in Table 2. The F functions in all the test problems have their minimum value at 0 and their maximum value at 1. Consequently, the surface defined by the F functions at each test problem is contained in the hypercube $[0,1]^M$. Therefore, from Equation (5), it is possible to infer that the ideal and the Nadir points of the test problems with M objectives are

Table 2: Mathematical formulation of F functions and the corresponding $\mathbf{g}$ function for the instantiated test problem

MOP	F and g functions	MOP	F and g functions
ZCAT1	$F_1(\mathbf{y}_I) = \prod_{i=1}^{M-1} \sin(y_i \times \pi/2)$ $F_{j=2:M-1}(\mathbf{y}_I) = \left(\prod_{i=1}^{M-j} \sin(y_i \times \pi/2)\right) \times \cos(y_{M-j+1} \times \pi/2)$ $F_M(\mathbf{y}_I) = 1 - \sin(y_1 \times \pi/2)$ $g(\mathbf{y}_I \mid m) = \begin{cases} g0(\mathbf{y}_I \mid M-1), & \texttt{complicated\_PS=False} \\ g4(\mathbf{y}_I \mid M-1), & \texttt{otherwise} \end{cases}$	ZCAT11	$F_1(\mathbf{y}_I) = \frac{1}{M-1} \sum_{i=1}^{M-1} y_i$ $F_{j=2:M-1}(\mathbf{y}_I) = \frac{1}{M-j+1}\left[\left(\sum_{i=1}^{M-j} y_i\right) + (1 - y_{M-j+1})\right]$ $F_M(\mathbf{y}_I) = \frac{\cos((2K-1)y_1\pi) + 2y_1 + 4K(1-y_1) - 1}{4K}$ $g(\mathbf{y}_I \mid m) = \begin{cases} g0(\mathbf{y}_I \mid M-1), & \texttt{complicated\_PS=False} \\ g3(\mathbf{y}_I \mid M-1), & \texttt{otherwise} \end{cases}$
ZCAT2	$F_1(\mathbf{y}_I) = \prod_{i=1}^{M-1} 1 - \cos(y_i \times \pi/2)$ $F_{j=2:M-1}(\mathbf{y}_I) = \left(\prod_{i=1}^{M-j} 1 - \cos(y_i \times \pi/2)\right) \times (1 - \sin(y_{M-j+1} \times \pi/2))$ $F_M(\mathbf{y}_I) = 1 - \sin(y_1 \times \pi/2)$ $g(\mathbf{y}_I \mid m) = \begin{cases} g0(\mathbf{y}_I \mid M-1), & \texttt{complicated\_PS=False} \\ g5(\mathbf{y}_I \mid M-1), & \texttt{otherwise} \end{cases}$	ZCAT12	$F_1(\mathbf{y}_I) = 1 - \prod_{i=1}^{M-1} 1 - y_i$ $F_{j=2:M-1}(\mathbf{y}_I) = 1 - \left(\prod_{i=1}^{M-j} 1 - y_i\right) \times y_{M-j+1}$ $F_M(\mathbf{y}_I) = \frac{\cos((2K-1)y_1\pi) + 2y_1 + 4K(1-y_1) - 1}{4K}$ $g(\mathbf{y}_I \mid m) = \begin{cases} g0(\mathbf{y}_I \mid M-1), & \texttt{complicated\_PS=False} \\ g10(\mathbf{y}_I \mid M-1), & \texttt{otherwise} \end{cases}$
ZCAT3	$F_1(\mathbf{y}_I) = \frac{1}{M-1} \sum_{i=1}^{M-1} y_i$ $F_{j=2:M-1}(\mathbf{y}_I) = \frac{1}{M-j+1}\left[\left(\sum_{i=1}^{M-j} y_i\right) + (1 - y_{M-j+1})\right]$ $F_M(\mathbf{y}_I) = 1 - y_1$ $g(\mathbf{y}_I \mid m) = \begin{cases} g0(\mathbf{y}_I \mid M-1), & \texttt{complicated\_PS=False} \\ g2(\mathbf{y}_I \mid M-1), & \texttt{otherwise} \end{cases}$	ZCAT13	$F_1(\mathbf{y}_I) = 1 - \frac{1}{M-1} \sum_{i=1}^{M-1} \sin(y_i \times \pi/2)$ $F_{j=2:M-1}(\mathbf{y}_I) = 1 - \frac{1}{M-j+1}\left[\left(\sum_{i=1}^{M-j} \sin(y_i \times \pi/2)\right) + \cos(y_{M-j+1} \times \pi/2)\right]$ $F_M(\mathbf{y}_I) = 1 - \frac{\cos((2K-1)y_1\pi) + 2y_1 + 4K(1-y_1) - 1}{4K}$ $g(\mathbf{y}_I \mid m) = \begin{cases} g0(\mathbf{y}_I \mid M-1), & \texttt{complicated\_PS=False} \\ g1(\mathbf{y}_I \mid M-1), & \texttt{otherwise} \end{cases}$
ZCAT4	$F_{j=1:M-1}(\mathbf{y}_I) = y_j$ $F_M(\mathbf{y}_I) = 1 - \frac{1}{M-1} \sum_{i=1}^{M-1} y_i$ $g(\mathbf{y}_I \mid m) = \begin{cases} g0(\mathbf{y}_I \mid M-1), & \texttt{complicated\_PS=False} \\ g7(\mathbf{y}_I \mid M-1), & \texttt{otherwise} \end{cases}$	ZCAT14	$F_1(\mathbf{y}_I) = \sin(y_1 \times \pi/2)^2$ $F_{j=2:M-2}(\mathbf{y}_I) = \sin(y_1 \times \pi/2)^{2+(j-1)/(M-2)}$ $F_{M-1}(\mathbf{y}_I) = \frac{1}{2}\left[1 + \sin(6y_1 \times \pi/2 - \pi/2)\right]$, iff $M > 2$ $F_M(\mathbf{y}_I) = \cos(y_1 \times \pi/2)$ $g(\mathbf{y}_I \mid m) = \begin{cases} g0(\mathbf{y}_I \mid M-1), & \texttt{complicated\_PS=False} \\ g6(\mathbf{y}_I \mid M-1), & \texttt{otherwise} \end{cases}$
ZCAT5	$F_{j=1:M-1}(\mathbf{y}_I) = y_j$ $F_M(\mathbf{y}_I) = \frac{\exp\left(\frac{1}{M-1}\sum_{i=1}^{M-1} 1 - y_i\right)^8 - 1}{\exp(1)^8 - 1}$ $g(\mathbf{y}_I \mid m) = \begin{cases} g0(\mathbf{y}_I \mid M-1), & \texttt{complicated\_PS=False} \\ g9(\mathbf{y}_I \mid M-1), & \texttt{otherwise} \end{cases}$	ZCAT15	$F_{j=1:M-1}(\mathbf{y}_I) = y_1^{1+(j-1)/(4M)}$ $F_M(\mathbf{y}_I) = \frac{\cos((2K-1)y_1\pi) + 2y_1 + 4K(1-y_1) - 1}{4K}$ $g(\mathbf{y}_I \mid m) = \begin{cases} g0(\mathbf{y}_I \mid M-1), & \texttt{complicated\_PS=False} \\ g8(\mathbf{y}_I \mid M-1), & \texttt{otherwise} \end{cases}$
ZCAT6	$F_{j=1:M-1}(\mathbf{y}_I) = y_j$ $F_M(\mathbf{y}_I) = \frac{(1 + \exp(2\kappa\mu - \kappa))^{-1} - \rho\mu - (1 + \exp(\kappa))^{-1} + \rho}{(1 + \exp(-\kappa))^{-1} - (1 + \exp(\kappa))^{-1} + \rho}$ such that $\mu = \frac{1}{M-1}\sum_{i=1}^{M-1} y_i, \kappa = 40$ and $\rho = 0.05$. $g(\mathbf{y}_I \mid m) = \begin{cases} g0(\mathbf{y}_I \mid M-1), & \texttt{complicated\_PS=False} \\ g4(\mathbf{y}_I \mid M-1), & \texttt{otherwise} \end{cases}$	ZCAT16	$F_1(\mathbf{y}_I) = \sin(y_1 \times \pi/2)$ $F_{j=2:M-2}(\mathbf{y}_I) = \sin(y_1 \times \pi/2)^{1+(j-1)/(M-2)}$ $F_{M-1}(\mathbf{y}_I) = \frac{1}{2}\left[1 + \sin(10y_1 \times \pi/2 - \pi/2)\right]$, iff $M > 2$ $F_M(\mathbf{y}_I) = \frac{\cos((2K-1)y_1\pi) + 2y_1 + 4K(1-y_1) - 1}{4K}$ $g(\mathbf{y}_I \mid m) = \begin{cases} g0(\mathbf{y}_I \mid M-1), & \texttt{complicated\_PS=False} \\ g10(\mathbf{y}_I \mid M-1), & \texttt{otherwise} \end{cases}$
ZCAT7	$F_{j=1:M-1}(\mathbf{y}_I) = y_j$ $F_M(\mathbf{y}_I) = \frac{1}{2(M-1) \times (0.5)^5} \sum_{i=1}^{M-1} (0.5 - y_i)^5 + \frac{1}{2}$ $g(\mathbf{y}_I \mid m) = \begin{cases} g0(\mathbf{y}_I \mid M-1), & \texttt{complicated\_PS=False} \\ g5(\mathbf{y}_I \mid M-1), & \texttt{otherwise} \end{cases}$	ZCAT17	$F_{j=1:M-1}(\mathbf{y}_I) = \begin{cases} y_1, & \forall y_i \le 0.5, i \in \{1, \ldots, m\} \\ y_j, & \text{otherwise} \end{cases}$ $F_M(\mathbf{y}_I) = \begin{cases} \frac{\exp(1 - y_1)^8 - 1}{\exp(1)^8 - 1}, & \forall y_i \le 0.5, i \in \{1, \ldots, m\} \\ \frac{\exp\left(\frac{1}{M-1}\sum_{i=1}^{M-1} 1 - y_i\right)^8 - 1}{\exp(1)^8 - 1}, & \text{otherwise} \end{cases}$ $g(\mathbf{y}_I \mid m) = \begin{cases} g0(\mathbf{y}_I \mid M-1), & \texttt{complicated\_PS=False} \\ g1(\mathbf{y}_I \mid M-1), & \texttt{otherwise} \end{cases}$
ZCAT8	$F_1(\mathbf{y}_I) = 1 - \prod_{i=1}^{M-1} 1 - \sin(y_i \times \pi/2)$ $F_{j=2:M-1}(\mathbf{y}_I) = 1 - \left(\prod_{i=1}^{M-j} 1 - \sin(y_i \times \pi/2)\right) \times (1 - \cos(y_{M-j+1} \times \pi/2))$ $F_M(\mathbf{y}_I) = \cos(y_1 \times \pi/2)$ $g(\mathbf{y}_I \mid m) = \begin{cases} g0(\mathbf{y}_I \mid M-1), & \texttt{complicated\_PS=False} \\ g2(\mathbf{y}_I \mid M-1), & \texttt{otherwise} \end{cases}$	ZCAT18	$F_{j=1:M-1}(\mathbf{y}_I) = \begin{cases} y_1, & (\forall y_i \le 0.4) \text{ or } (\forall y_i \ge 0.6), i \in \{1, \ldots, m\} \\ y_j, & \text{otherwise} \end{cases}$ $F_M(\mathbf{y}_I) = \begin{cases} \frac{(0.5 - y_1)^5 + 0.5^5}{2 \times (0.5)^5}, & (\forall y_i \le 0.4) \text{ or } (\forall y_i \ge 0.6), i \in \{1, \ldots, m\} \\ \frac{\sum_{i=1}^{M-1}(0.5 - y_i)^5}{2(M-1) \times (0.5)^5} + \frac{1}{2}, & \text{otherwise} \end{cases}$ $g(\mathbf{y}_I \mid m) = \begin{cases} g0(\mathbf{y}_I \mid M-1), & \texttt{complicated\_PS=False} \\ g8(\mathbf{y}_I \mid M-1), & \texttt{otherwise} \end{cases}$
ZCAT9	$F_1(\mathbf{y}_I) = \frac{1}{M-1} \sum_{i=1}^{M-1} \sin(y_i \times \pi/2)$ $F_{j=2:M-1}(\mathbf{y}_I) = \frac{1}{M-j+1}\left[\left(\sum_{i=1}^{M-j} \sin(y_i \times \pi/2)\right) + \cos(y_{M-j+1} \times \pi/2)\right]$ $F_M(\mathbf{y}_I) = \cos(y_1 \times \pi/2)$ $g(\mathbf{y}_I \mid m) = \begin{cases} g0(\mathbf{y}_I \mid M-1), & \texttt{complicated\_PS=False} \\ g7(\mathbf{y}_I \mid M-1), & \texttt{otherwise} \end{cases}$	ZCAT19	$F_{j=1:M-1}(\mathbf{y}_I) = \begin{cases} y_1, & m = 1 \\ y_j, & \text{otherwise} \end{cases}$ $F_M(\mathbf{y}_I) = \begin{cases} 1 - y_1 - \frac{\cos(2A\pi y_1 + \pi/2)}{2A\pi}, & m = 1 \\ 1 - \frac{1}{M-1}\sum_{i=1}^{M-1} y_i - \frac{\cos(2A\pi \frac{1}{M-1}\sum_{i=1}^{M-1} y_i + \pi/2)}{2A\pi}, & \text{otherwise} \end{cases}$ $m = 1$ iff $y_1 \in [0, 0.2]$ or $y_1 \in [0.4, 0.6]$; otherwise $m = M - 1$. $g(\mathbf{y}_I \mid m) = \begin{cases} g0(\mathbf{y}_I \mid M-1), & \texttt{complicated\_PS=False} \\ g6(\mathbf{y}_I \mid M-1), & \texttt{otherwise} \end{cases}$
ZCAT10	$F_{j=1:M-1}(\mathbf{y}_I) = y_j$ $F_M(\mathbf{y}_I) = \frac{\rho^{-1} - \left(\frac{1}{M-1}\sum_{i=1}^{M-1}(1 - y_i) + \rho\right)^{-1}}{\rho^{-1} - (1 + \rho)^{-1}}$ $g(\mathbf{y}_I \mid m) = \begin{cases} g0(\mathbf{y}_I \mid M-1), & \texttt{complicated\_PS=False} \\ g9(\mathbf{y}_I \mid M-1), & \texttt{otherwise} \end{cases}$ where $\rho = 0.02$.	ZCAT20	$F_{j=1:M-1}(\mathbf{y}_I) = \begin{cases} y_1, & m = 1 \\ y_j, & \text{otherwise} \end{cases}$ $F_M(\mathbf{y}_I) = \begin{cases} \frac{(0.5 - y_1)^5 + 0.5^5}{2 \times (0.5)^5}, & m = 1 \\ \frac{\sum_{i=1}^{M-1}(0.5 - y_i)^5}{2(M-1) \times (0.5)^5} + \frac{1}{2}, & \text{otherwise} \end{cases}$ $m = 1$ iff $y_1 \in [0.1, 0.4]$ or $y_1 \in [0.6, 0.9]$; otherwise $m = M - 1$. $g(\mathbf{y}_I \mid m) = \begin{cases} g0(\mathbf{y}_I \mid M-1), & \texttt{complicated\_PS=False} \\ g3(\mathbf{y}_I \mid M-1), & \texttt{otherwise} \end{cases}$

Table 3: Different instances of the $\mathbf{g}$ functions. Function $\mathbf{g}$ is in the form $\mathbf{g}(\mathbf{y}_I|m) = (g_{m+1}(\mathbf{y}_I|m), \ldots, g_n(\mathbf{y}_I|m))$, such that, $g_j (j = m+1, \ldots, n)$ is defined according to the instantiated $\mathbf{g}$ function.

$\mathbf{g}$ function	g_j formulation
$\mathbf{g0}(\mathbf{y}_I\|m)$	$g_j(\mathbf{y}_I\|m) = 0.2210$
$\mathbf{g1}(\mathbf{y}_I\|m)$	$g_j(\mathbf{y}_I\|m) = \frac{1}{2m}\sum_{i=1}^{m}\sin(1.5\pi y_i + \theta_j) + \frac{1}{2}$
$\mathbf{g2}(\mathbf{y}_I\|m)$	$g_j(\mathbf{y}_I\|m) = \frac{1}{2m}\sum_{i=1}^{m} y_i^2 \times \sin(4.5\pi y_i + \theta_j) + \frac{1}{2}$
$\mathbf{g3}(\mathbf{y}_I\|m)$	$g_j(\mathbf{y}_I\|m) = \frac{1}{m}\sum_{i=1}^{m}\cos(\pi y_i + \theta_j)^2$
$\mathbf{g4}(\mathbf{y}_I\|m)$	$g_j(\mathbf{y}_I\|m) = \frac{1}{2m}\sum_{i=1}^{m} y_i \times \cos\left(\frac{4\pi}{m}\sum_{i=1}^{m} y_i + \theta_j\right) + \frac{1}{2}$
$\mathbf{g5}(\mathbf{y}_I\|m)$	$g_j(\mathbf{y}_I\|m) = \frac{1}{2m} \times \sum_{i=1}^{m}\sin(2\pi y_i - 1 + \theta_j)^3 + \frac{1}{2}$
$\mathbf{g6}(\mathbf{y}_I\|m)$	$g_j(\mathbf{y}_I\|m) = \Bigg(-10e^{\left(-2/5\sqrt{\frac{1}{m}\sum_{i=1}^{m} y_i^2}\right)} + 10 + e$ $-e^{\left(\frac{1}{m}\sum_{i=1}^{m}\cos(11\pi y_i + \theta_j)^3\right)}\Bigg) / \Big(-10e^{-2/5}$ $-e^{-1} + 10 + e\Big)$
$\mathbf{g7}(\mathbf{y}_I\|m)$	$g_j(\mathbf{y}_I\|m) = \frac{\frac{1}{m}\sum_{i=1}^{m} y_i + e^{\sin\left(7\pi\frac{1}{m}\sum_{i=1}^{m} y_i - \pi/2 + \theta_j\right)} - e^{-1}}{1 + e^{1} - e^{-1}}$
$\mathbf{g8}(\mathbf{y}_I\|m)$	$g_j(\mathbf{y}_I\|m) = \frac{1}{m}\sum_{i=1}^{m}\lvert\sin(2.5\pi(y_i - 0.5) + \theta_j)\rvert$
$\mathbf{g9}(\mathbf{y}_I\|m)$	$g_j(\mathbf{y}_I\|m) = \frac{1}{2m}\sum_{i=1}^{m} y_i$ $- \frac{1}{2m}\sum_{i=1}^{m}\lvert\sin(2.5\pi y_i - \pi/2 + \theta_j)\rvert + \frac{1}{2}$
$\mathbf{g10}(\mathbf{y}_I\|m)$	$g_j(\mathbf{y}_I\|m) = \frac{1}{2 \times m^3}\left(\sum_{i=1}^{m}\sin((4y_i - 2)\pi + \theta_j)\right)^3 + \frac{1}{2}$
where $\theta_j = \dfrac{2\pi(j - (m+1))}{n}$, for $j = m+1, \ldots, n$	

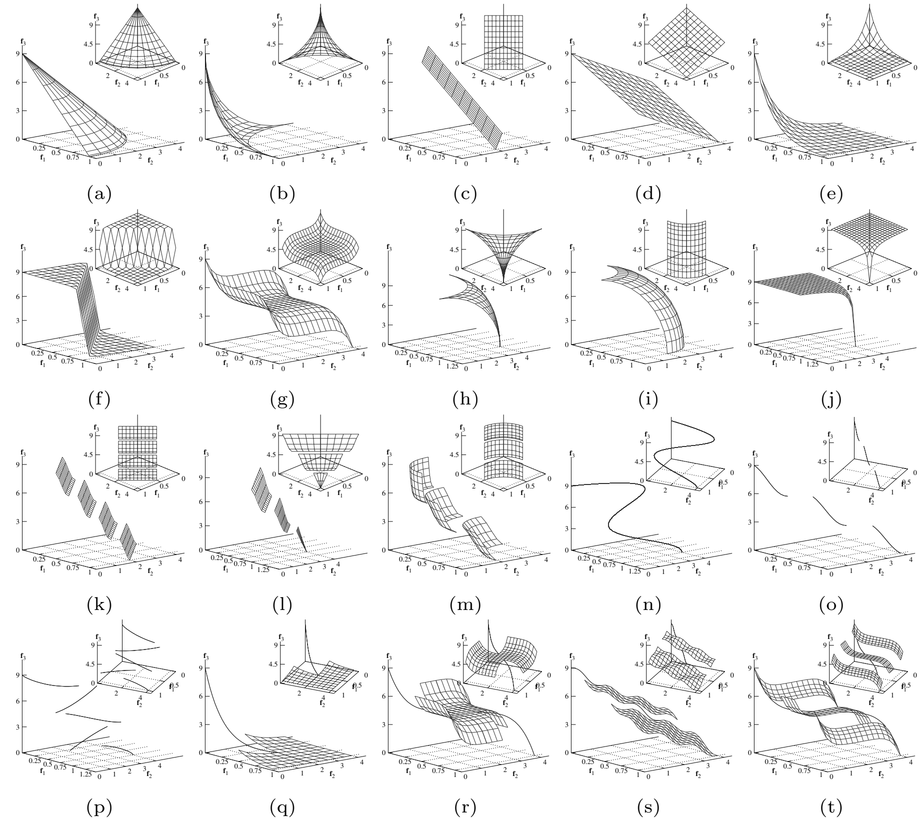


Figure 2: Pareto fronts of ZCAT test problems using $M = 3$ for: (a) ZCAT1, (b) ZCAT2, (c) ZCAT3, (d) ZCAT4, (e) ZCAT5, (f) ZCAT6, (g) ZCAT7, (h) ZCAT8, (i) ZCAT9, (j) ZCAT10, (k) ZCAT11, (l) ZCAT12, (m) ZCAT13, (n) ZCAT14, (o) ZCAT15, (p) ZCAT16, (q) ZCAT17, (r) ZCAT18, (s) ZCAT19, and (t) ZCAT20.

$\mathbf{z}^\star = (0, 0, 0, \ldots, 0)$ and $\mathbf{n}^\star = (1^2, 2^2, 3^2, \ldots, M^2)$, respectively. In the proposed test suite, four classes of Pareto fronts are considered and described below.

4.1.1. Non-Degenerate and Connected Pareto Fronts

Multi-objective test instances ZCAT1–ZCAT10 are problems with non-degenerate and connected Pareto fronts. Particularly, the Pareto front of ZCAT1 and ZCAT2 can be characterized by an $(M-1)$-simplex, i.e., they are easy to approximate (see the discussion presented in [2]). ZCAT3 and ZCAT4 present linear Pareto fronts which differ from the traditional $(M-1)$-simplex adopted by the currently available scalable test problems, such as DTLZ1. ZCAT5–ZCAT10 have Pareto

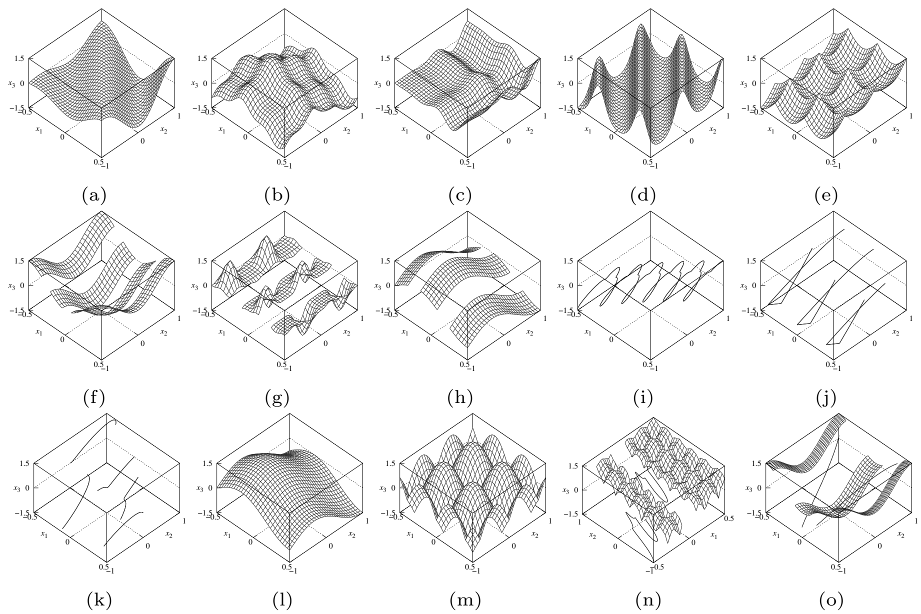


Figure 3: Pareto sets of ZCAT test problems using $M = 3$ for: (a) ZCAT1 and ZCAT6, (b) ZCAT2 and ZCAT7, (c) ZCAT3 and ZCAT8, (d) ZCAT4 and ZCAT9, (e) ZCAT5 and ZCAT10, (f) ZCAT11, (g) ZCAT12, (h) ZCAT13, (i) ZCAT14, (j) ZCAT15, (k) ZCAT16, (l) ZCAT17, (m) ZCAT18, (n) ZCAT19, and (o) ZCAT20.

fronts with a high level of convexity and/or concavity which shall test the density estimators of MOEAs. According to the discussion presented in [2], problems ZCAT3–ZCAT10 adhere to the feature of having a Pareto front difficult to approximate. The Pareto fronts of the test problems having non-degenerate and connected Pareto fronts (i.e., problems ZCAT1–ZCAT10) are illustrated, for $M = 3$, in Figures 2a–2j.

4.1.2. Disconnected Pareto Fronts

ZCAT11–ZCAT13 are three test problems whose Pareto fronts are disconnected. The number of disconnected regions is controlled by parameter K (see the mathematical formulation presented in Table 2). In addition to having disconnected parts, other characteristics such as convexity, concavity, and linearity can also be observed. For $M = 3$, these Pareto fronts are illustrated in Figures 2k–2m. As the problems are disconnected, the Pareto set is also disconnected. Projections of these Pareto sets (for $M = 3$) onto the space x_1, x_2 and

x_3 are illustrated in Figures 3f–3h.

4.1.3. Degenerate Pareto Fronts

The proposed test suite includes three problems (ZCAT14–ZCAT16) with Pareto fronts of dimension $m = 1$, i.e., they are degenerate for $M > 2$. In addition to being degenerate, ZCAT15 and ZCAT16 present disconnected Pareto fronts. Analogously to the disconnected Pareto fronts presented in Section 4.1.2, the number of disconnections in problems ZCAT15 and ZCAT16 is controlled by the parameter K (see the mathematical formulation presented in Table 2). For $M = 3$, the Pareto fronts and the Pareto sets of the problems ZCAT4–ZCAT16 are illustrated in Figures 2n–2p and Figures 3i–3k, respectively.

4.1.4. Mixtures of Degenerate and Non-degenerate Pareto Fronts

In the specialized literature, it is possible to find real-world problems whose Pareto front approximations look like a mixture of degenerate and non-degenerate surfaces; examples of such problems can be found in [45, 48]. In the proposed test suite, four problems simulating these types of Pareto front shapes have been introduced. The degenerate regions of these Pareto fronts are designed in two different ways:

1. In the first type, the degeneracy is obtained by mapping different position parameters into the degenerate surface without modifying the dimensionality of the Pareto set. Therefore, different solutions of the Pareto set can be mapped to the same solution (many-to-one solutions) in objective space. This is the case of problems ZCAT17 and ZCAT18. To illustrate this type of problems, Figures 2q–2r show the Pareto fronts (for $M = 3$) of problems ZCAT17 and ZCAT18, while their Pareto sets are shown in Figures 3l–3m.
2. The second type of degeneracy is the result of reducing the dimensionality of the Pareto set in a specific region. An example of how to build these types of problems is shown in the construction of ZCAT20 in Appendix A from the supplementary material. Problems following this ap-

proach are ZCAT19 and ZCAT20. Figures 2s–2t show the Pareto fronts (for $M = 3$) of problems ZCAT19 and ZCAT20, while their Pareto sets are shown in Figures 3n–3o.

In addition to the Pareto fronts for the three-objective test instances illustrated in Figure 2, Figure A.8 in Appendix B from the supplementary material, we show the Pareto fronts shapes for the two-objective formulation of the proposed test problems.

4.2. Pareto Set Features

Complicated Pareto set shapes test specific components of an MOEA. Li and Zhang [20] showed the difficulties of solving these types of problems. Our proposed test problems adopt the idea of using complicated Pareto sets and introduce ten different shapes in their formulation. The mathematical formulation of such shapes is presented in Table 3. The Pareto set shapes can be switched *on* and *off* to a shape much easier to approximate by activating the flag `Complicated_PS=True`. Particularly, if the flag is `Complicated_PS=False`, the Pareto set shape is comparable to the one introduced by the DTLZ or WFG test problems. That is, the Pareto set can be seen as being piecewise linear. The projections of the Pareto sets into the space x_1, x_2 and x_3 for all the test problems are shown (for $M = 3$) in the plots of Figure 3. In these plots, it is possible to appreciate non-degenerate and connected surfaces (Figures 3a–3e and Figures 3l–3m) disconnected regions (Figures 3f–3h), degenerate shapes (Figures 3i–3k), and mixtures of degenerate and non-degenerate shapes (Figures 3n–3o). Finally, projections of the Pareto sets into the space x_1, x_2 and x_3 for the two-objective formulation (i.e., $M = 2$) of the ZCAT test problems are illustrated in Figure C.10 in Appendix C from the supplementary material.

4.3. Distance Functions Features

In the bottom-up approach, the distance functions establish the difficulty of solving a test problem. ZCAT problems adopt different characteristics in their distance functions which can be switched on/off to increase the difficulty degree of the problems. This section presents the features of the distance functions that can be configured in ZCAT problems.

4.3.1. Separability, multimodality, and deception

In Equation (6), β_i functions state the distance of solutions to the Pareto front. The β_i functions are composed by Z functions, which establish the degree of difficulty of approximating a Pareto solution for a given problem. Concretely, $Z(w_{m+1:n}|J_i) = Z_{\texttt{Level}}(w_{m+1:n}|J_i)$ where $Z_{\texttt{Level}}(w_{m+1:n}|J_i)$ is mathematically formulated in Table 4, for `Level` $\in \{1, \ldots, 6\}$.

The features of the Z functions are shown in Table 4, where the following abbreviations are used: "S" for separable, "NS" for non-separable, "U" for unimodal, "M" for multi-modal, and "D" for deceptive. The optimal solution for all the Z functions is $w_{m+1:n} = \mathbf{0}$.

4.3.2. Bias

In addition to the different levels for the Z functions, the proposed test suite introduces a bias function which is activated by the flag `Bias` $\in \{$`True,False`$\}$. In Equation (7), the $Z_{\texttt{Bias}}$ function biases the component $z_j = y_j - g_j(\mathbf{y}_I|m)$ (for $j = m+1, \ldots, n$) according to the following equation:

$$w_{m+1:n} = (|z_{m+1}|^\gamma, \ldots, |z_n|^\gamma) \tag{11}$$

where $\gamma = 0.05$. Since $|z_j| \in [0, 1]$, for $\gamma < 1$, the component is biased towards 1, while for $\gamma > 1$, the component is biased towards 0.

Since the optimal solution of $Z(w_{m+1:n})$ is $w_{m+1:n} = \mathbf{0}$, the polynomial bias (with $\gamma = 0.05$) shall move $w_{m+1:n}$ away from the optimum $\mathbf{0}$, i.e., the β_i functions will be more difficult to optimize. Therefore, in a ZCAT test problem, this feature shall bias solutions far away from the Pareto set. That is, a tiny change in the decision variables of some Pareto solutions may cause significant changes in their objective vectors in the objective space. The Z functions and their contour lines are illustrated in Figure D.11, in Appendix D from the supplementary material.

Table 4: Functions related to distance and their features.

Function	Features
$Z_1(w_{m+1:n}\|J_i) = \frac{10}{\|J_i\|}\sum_{j\in J_i} w_j^2$	U/S
$Z_2(w_{m+1:n}\|J_i) = 10 \times \max_{j\in J_i}\{\|w_j\|\}$	U/NS
$Z_3(w_{m+1:n}\|J_i) = \frac{10}{\|J_i\|}\sum_{j\in J_i}\frac{1}{3}\left[w_j^2 - \cos((2K-1)\pi w_j) + 1\right]$	M/S
$Z_4(w_{m+1:n}\|J_i) = \frac{10}{2e-2}\left[e^{\max_{j\in J_i}\{\|w_j\|\}^{0.5}} - e^{\frac{1}{\|J_i\|}\sum_{j\in J_i}\frac{1}{2}[\cos((2K-1)\pi w_j)+1]} - 1 + e\right]$	M/NS
$Z_5(w_{m+1:n}\|J_i) = -0.7 \times Z_3(w_{m+1:n}\|J_i) + \frac{10}{\|J_i\|}\sum_{j\in J_i}\|w_j\|^{0.002}$	D/S
$Z_6(w_{m+1:n}\|J_i) = -0.7 \times Z_4(w_{m+1:n}\|J_i) + 10 \times \left(\frac{1}{\|J_i\|}\sum_{j\in J_i}\|w_j\|\right)^{0.002}$	D/NS
where $K = 5$	

4.3.3. Imbalance

In the ZCAT test suite, an imbalanced problem can be formulated by setting the flag `Imbalance` = `True`. With this parameter setting, some objectives will be more difficult to optimize than others. It is also worth noting that, according to Equation (9), parameter `Level` will not be considered when this flag is activated. In this paper, we adhere this feature to the list of recommendations and features presented in Table 1. To clarify the use of parameters `Level`, `Complicated_PS`, `Bias`, and `Imbalance`, the reader is referred to the three examples presented in Appendix A from the supplementary material.

Finally, to facilitate the reading of properties in the proposed test suite, in Appendix E from the supplementary material, Table E.5 summarizes the recommendations and features suggested by Huband et al. [1] and Zapotecas et al. [2], that have been adhered to in each test problem.

5. On the Use and Difficulties of the ZCAT Test Suite

In this section, we present some experiments using the proposed test suite. Particularly, this section shows how can the test suite be employed to evaluate the performance of MOEAs using different configurations.

5.1. On the ZCAT Test Suite Configuration

The ZCAT test suite has six parameters to be configured which establish the difficulties and features of a test problem. This set of parameters should be carefully established to analyze specific components of MOEAs. In the following, we describe the parameters of ZCAT for a possible configuration.

1. Number of objectives (M): This parameter can be used to analyze the behavior of MOEAs in objective spaces of different dimensionality ($M > 1$). The default value for this parameter is $M = 3$.
2. Number of decision variables (n): This parameter can be set to evaluate MOEAs on test problems with a predefined number of decision variables ($n > M$). The default value for this parameter is $n = 10 \times M$.
3. Complicated Pareto set (`Complicated_PS`): The flag `Complicated_PS`$\in$ {`True,False`} can control the Pareto set shape for a given test problem. The default value for this parameter is `Complicated_PS=True`.
4. Level of Difficulty (`Level`): This parameter can be used to evaluate the abilities of an evolutionary algorithm to approximate a Pareto solution in different scenarios (e.g., separable, multi-modal, and deceptive landscapes). The parameter `Level` can be chosen from six possible different values (`Level` $\in \{1, \ldots, 6\}$) stating the difficulties of the concerned test problem. The default value for this parameter is `Level` $= 1$.
5. Bias function (`Bias`): A test problem can be biased by activating the flag `Bias`$\in$ {`True,False`}. This flag configures the test problem to be more challenging to solve, as this type of problem shall test the abilities of MOEAs to find Pareto solutions with a high-quality approximation. The default value for this parameter is `Bias=False`.
6. Imbalanced problems (`Imbalance`): The objective functions in a test problem can be imbalanced by activating the flag `Imbalance`$\in$ {`True,False`}. This flag configures some objective functions to be more challenging to solve than others. The default value for this parameter is `Imbalance=False`.

The following experiments analyze the performance of three state-of-the-art MOEAs employing some of the above configurations.

5.2. MOEAs Adopted for Performance Analysis

The experiments were carried out by comparing three MOEAs having selection mechanisms based on decomposition, Pareto optimality, and a performance indicator (hypervolume). Since the Differential Evolution (DE) operator has shown superiority in solving MOPs with complicated Pareto sets [20], MOEAs employing DE as their main recombination operator were selected. Concretely, MOEA/D-DE [20] and GDE3 [49] were adopted. For the case of hypervolume-based MOEAs, SMS-EMOA [50] using the DE operator was adopted (we call it SMS-EMOA-DE in this paper).

MOEA/D-DE adopted a neighborhood size $T = 20$, a maximum number of replacements $n_r = 2$, and a neighborhood selection probability $\delta = 0.9$. The scalarizing function adopted was the PBI approach with a penalty value $\theta = 5$. DE was used with the same parameters as in [20], i.e., $F = 0.5$ and $CR = 1$. Polynomial-based Mutation (PBM) was employed adopting a mutation rate of $P_m = 1/n$ (n is the number of decision variables) and a mutation index $\eta = 20$. The population size for the algorithms was established by the simplex-lattice design which is used to generate the weight vectors for MOEA/D-DE. The configuration of H for the simplex-lattice design was set to $H = 99, H = 19$ and $H = 6$. Consequently, the population size (N) was 100, 210 and 210, for two, three and five objective functions, respectively. In all the experiments, the adopted MOEAs were restricted to perform $2000 \times N$ fitness function evaluations.

5.3. Difficulties on Many-Objective Problems

To show the difficulties on many-objective problems, we considered the test problem ZCAT7 adopting the default parameters. The performance of MOEAs was evaluated using 3, 4, and, 5 objective functions. To illustrate the performance of the MOEAs, we employed parallel coordinates. Figure 4 shows the parallel coordinates comparing a sample of 5000 solutions properly distributed

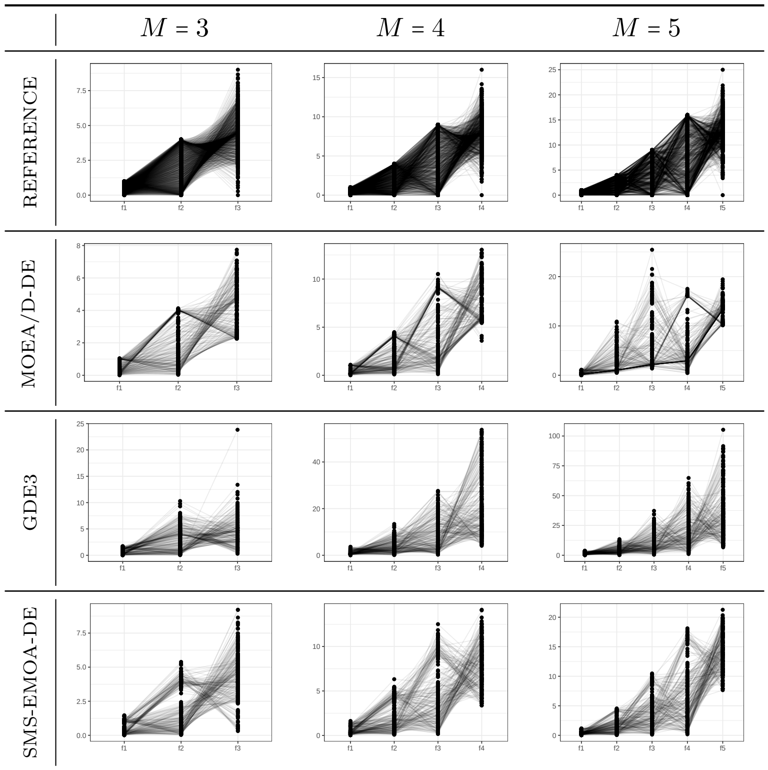


Figure 4: Parallel coordinates of the approximations obtained by the MOEAs on ZCAT7 with three, four, and five objectives, respectively.

along the Pareto front and the approximation obtained by the adopted MOEAs. The approximations produced by the MOEAs correspond to the final populations obtained in the best (according to IGD) of 30 independent runs. From these plots, we can see that the performance of the MOEAs degrades as the number of objectives increases. It is worth noticing that the Pareto front shape of ZCAT7 combines convex and concave regions. This, in fact, could be a challenge for MOEAs and their density estimators in order to reach a proper representation of the true Pareto front, particularly in high-dimensional objective spaces.

5.4. Difficulties on Large-scale Problems

The ZCAT test problems can be used to assess the performance of MOEAs that deal with problems having a large number of decision parameters (the so-called large-scale problems). To show how difficult is to solve a ZCAT problem, we adopted ZCAT19 using the default parameters in its two-objective formulation. In our experiments, we employed 50, 100, 500, 1000, 2000, and 5000 decision variables. Figures 5a–5c show the final approximations achieved by the adopted MOEAs using these parameters settings. The plots correspond to the best approximations (concerning IGD) obtained by the MOEAs in 30 independent executions. As can be seen, the performance of MOEAs degrades as the number of decision variables increases. It is important to notice that depending on the principle on which an MOEA is based, the performance of an algorithm becomes better or worse. Preliminary results indicate that SMS-EMO-DE approximates solutions in a better way than other algorithms. SMS-EMOA-DE is a greedy algorithm that approximates the Pareto front by increasing the hypervolume of non-dominated solutions at each offspring generation. This can be the reason why SMS-EMOA-DE approximates solutions better than other algorithms even when the number of decision variables is large (see for example when using 2000 decision variables). Nonetheless, with 5000 decision variables, the performance of SMS-EMO-DE significantly decreased.

An advantage of using the ZCAT test problems for large-scale optimization is that the objective vectors produced by an MOEA can be visualized close to the true PFs regardless of the number of decision variables. This is because the Z functions bind the β_i functions, see their mathematical formulation in Table 4. This property of the ZCAT test suite facilitates the examination of an MOEA when it approximates solutions to different regions of the true PF in a large-scale test problem.

5.5. Difficulties on Different Configuration Levels

The different Z functions in Table 4, establish a difficulty level for the test problems, which, depending on the evolutionary approach, can become easier

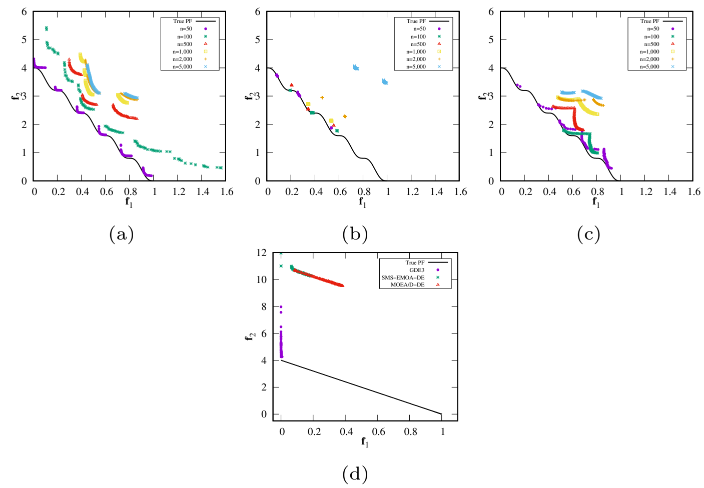
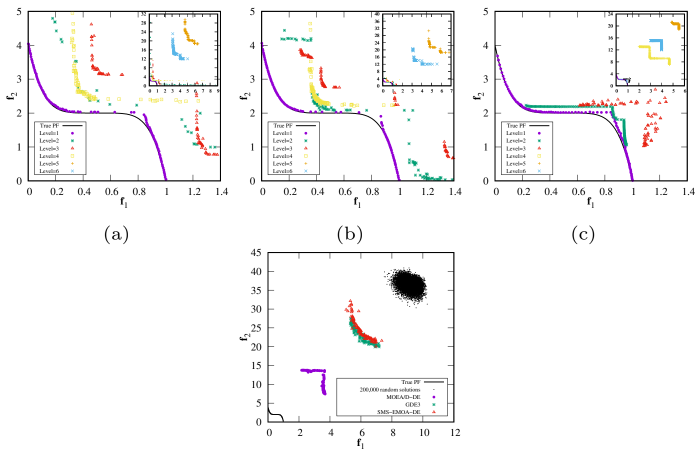


Figure 5: (a) GDE3 approximations on ZCAT19 using different numbers of decision variables. (b) SMS-EMOA-DE approximations on ZCAT19 using different numbers of decision variables. (c) MOEA/D-DE approximations on ZCAT19 using different numbers of decision variables. (d) Pareto front approximations on ZCAT1 using default parameters and `Imbalance=True`.

or more difficult to solve. Such difficulty levels are related to the fitness landscape of a problem. In this sense, a separable/unimodal/non-deceptive problem should be easier to solve than a non-separable/multi-modal/deceptive problem, respectively. To facilitate the analysis on different configuration levels, we adopted ZCAT7 with default parameters in its two-objective formulation. For each MOEA, we performed 30 independent runs with the parameters settings described in Section 5.2. Figures 6a–6c show the best approximations (in terms of IGD) obtained by the MOEAs in all the executions, using different configuration levels. As can be seen, from the levels, the unimodal and separable setting (`Level=1`) became the easiest to solve for all the algorithms. In fact, the PF of ZCAT7 came to be covered almost in its entirety by all the MOEAs. It is also worth noticing that at subsequent levels, the solution difficulty did not follow an increasing order for the parameter `Level` (i.e., 2, 3, 4, 5, 6). As can be seen, depending on the evolutionary approach, ZCAT7 became either easier or more difficult to solve.



(a) (b) (c)

(d)

Figure 6: (a) GDE3 approximations at different configuration levels. (b) SMS-EMOA-DE approximations at different configuration levels. (c) MOEA/D-DE approximations at different configuration levels. (d) Approximations obtained by the MOEAs using default parameters and `Bias=True`.

5.6. Difficulties on Pareto Sets and Pareto fronts

The ZCAT test suite allows to formulate a test problem with a complicated Pareto set topology by setting the flag `Complicated_PS=True`. To show the difficulties of these types of problems, we adopted ZCAT3 with default parameters in its three-objective formulation. Figure 7 shows a performance comparison between the adopted MOEAs using `Complicated_PS=True` (Figures 7a–7c) and `Complicated_PS=False` (Figures 7d–7f). From these plots, it is possible to appreciate that the performance of MOEAs deteriorates using `Complicated_PS=True`. As pointed out in [20, 2], difficult Pareto set topologies shall evaluate the mating selection and replacement mechanisms in MOEAs.

From the same plots, it is also possible to observe that employing `Complicated_PS=False` (i.e., a PS topology that is relatively simple), the Pareto front is not entirely achieved by all the adopted MOEAs. In particular, the diversity mechanisms from GDE3 and MOEA/D-DE are not able to obtain a proper representation of the true PF in this parameters setting. On the other hand, SMS-EMOA-DE was the best choice to deal with the PF shape of ZCAT3 when the simple PS topology is stated. However, note that the use of

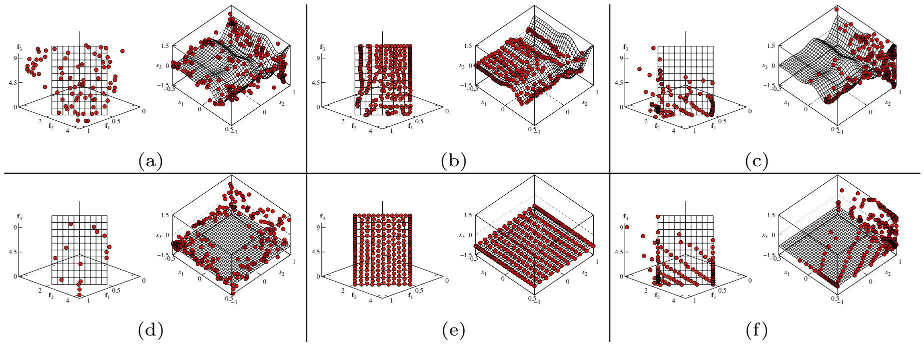


Figure 7: Pareto front and Pareto set approximations obtained by the MOEAs on ZCAT3 using `Complicated_PS=True` for: (a) GDE3, (b) SMS-EMOA-DE, and (c) MOEA/D-DE. Using `Complicated_PS=False` for: (d) GDE3, (e) SMS-EMOA-DE, and (f) MOEA/D-DE.

SMS-EMOA-DE can be impractical when the number of objectives in ZCAT3 increases. As pointed out before, complicated PF shapes shall encourage the design of new strategies to maintain diversity along the Pareto front in different multi-objective approaches.

5.7. Difficulties on Biased Problems

The ZCAT test suite can be configured to formulate biased problems by setting the flag `Bias=True`. As pointed out before, in a biased ZCAT test problem, a tiny change in the decision variables of some Pareto solutions may cause significant changes in their objective vectors. In order to appreciate the performance of MOEAs readily, we adopted ZCAT7 using only two objectives and employing the unimodal and separable configuration, that is `Level=1`. As a reference, a random sample of 200,000 solutions was generated and compared with the last populations obtained by the MOEAs in a single run. Figure 6d shows the approximation produced by the MOEAs. The random solutions were plotted as well as a reference. As can be seen, the MOEAs were not able to solve the unimodal and separable ZCAT7 problem with the biased search space as they solved the unbiased ZCAT7 (see Figures 6a–6c). This simple experiment shows how difficult it is to solve a biased ZCAT test problem, even when it is unimodal and separable. As pointed out in [24], the bias feature is a major factor that makes a multi-objective problem difficult since an MOEA needs to obtain a proper balance between exploration and exploitation. For this reason, the bias feature should be considered when designing scalable multi-objective problems.

5.8. Difficulties on Imbalanced Problems

The imbalanced ZCAT test problems assess the abilities of an MOEA to achieve a relatively good representation of the real Pareto front. To show the difficulties of solving these types of problems, we adopted ZCAT1 using default parameters and `Imbalance=True` in its two-objective formulation. According to the formulation of imbalanced problems (Equation (9)), odd objective functions will be easy to optimize. This effect can be observed in Figure 5d where f_1 is favored and f_2 is unfavored during the search. These types of problems are a challenge for MOEAs because finding a proper balance between convergence and exploration is not an easy task. However, as pointed out in [25], these types of problems should encourage researchers to devise modified algorithms balancing convergence and diversity-preservation aspects.

5.9. Final and Brief Remarks

Rather than attempting to make a strong judgment of MOEAs' skills, the goal of the above experiments was to show how to use the test suite and to have an idea of the behavior of MOEAs based on different principles at different configurations of the ZCAT test suite. Although several test problems can be formulated by setting the parameters M (objectives), n (decision variables), `Level`, `Complicated_PS`, `Bias`, and `Imbalance`, their analysis is beyond the scope of this paper.

Nonetheless, it is worth noticing that any test problem included in the proposed test suite can be set up according to the user's preferences in order to evaluate different components of an MOEA. Our suggestion in this test suite is to start with `Level` = 1 without considering any other difficulty and continue with subsequent levels activating the different flags in order to evaluate the robustness of an MOEA.

6. Conclusions

Multi-objective test problems have played an essential role in the analysis and comparison of MOEAs. Over the years, several scalable test problems have been designed to evaluate specific mechanisms of MOEAs. Most of these proposals have been mainly focused on the search space of the test problems, thus neglecting the design of Pareto fronts.

In this regard, this paper introduced a new set of scalable test problems emphasizing the importance of the design of new Pareto fronts which are scalable in the number of objectives. The proposed test suite innovates the state of the art by introducing twenty test problems with different Pareto fronts unusually found in the specialized literature. Additionally, the proposed test suite incorporates several characteristics in the search space of the problems, covering: separability, multi-modality, deceptiveness, and bias. An advantage of the proposed test suite is that a test problem can be made more difficult by switching among different configuration levels or through an imbalance of the objective functions. In addition to these features that hinder the search for Pareto optimal solutions, the proposed test suite includes different shapes of the Pareto sets to test the parent selection, recombination, and replacement mechanisms of MOEAs. These Pareto set topologies can be switched *on* and *off* to make them easier or more complicated.

Due to the constant progress in the development of MOEAs, the components included in this test suite offer, in the form of a toolkit, a practical and flexible solution to construct more complicated and challenging test problems. Our preliminary experiments illustrate the difficulties of the proposed test problems and motivate the development of new evolutionary approaches.

Although the proposed test suite is limited to unconstrained continuous spaces, its components can be used to build test problems with new features. Thus, the study presented in this paper opens the door to increasing the scope of the proposed set of problems. Consequently, with the emergence of new test problems, researchers could extend the suggested recommendations and/or features currently followed for the design of multi-objective test problems. These limitations of the ZCAT problems and studies regarding the performance of MOEAs on the proposed test set remain as work which is worth investigating.

As far as we are concerned, we will focus on designing discrete problems. In this sense, the PF shapes of ZCAT test problems can be discretized to build combinatorial or integer multi-objective optimization problems. Additionally, we would also like to extend the same test problems focusing on uncertain and dynamic environments. These are, in fact, paths for future research that we will be working on in the near future.

Finally, we would like to encourage the EMO community to evaluate their novel and already published algorithms in the ZCAT test suite. Particularly, it is worth noticing that combining different features in the suite can help us to assess the robustness of MOEAs.

Acknowledgements

Carlos A. Coello Coello gratefully acknowledges support from CONACyT grant no. 2016-01-1920 (Investigación en Fronteras de la Ciencia 2016). He was also partially supported by the Basque Government through the BERC 2022-2025 program and by Spanish Ministry of Economy and Competitiveness MINECO: BCAM Severo Ochoa excellence accreditation SEV-2017-0718.

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