

Remote Tracking via Encoded Information for Nonlinear Systems*

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Abstract

The problem addressed in this paper is to control a plant so as to have its output tracking (a family of) reference commands generated at a *remote location* and transmitted through a communication channel of finite capacity. The uncertainty due to the presence of the communication channel is counteracted by a suitable choice of the parameters of the regulator.

Keywords: Communication, networked control, internal model, nonlinear control, tracking.

1 Introduction

In distributed control systems, sensors, actuators and control unit may be placed at locations which are geographically separated. Information among these devices must then be exchanged through a finite bandwidth channel.

The problem addressed in this paper is to control a plant so as to have its output tracking (a family of) reference commands generated at a *remote location* and transmitted through a communication channel of finite capacity. What renders the problem in question different from a conventional tracking problem is that the *tracking error*, that is the difference between the command input and the controlled output, is not available as a *physical entity*, as

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it is defined as difference between two quantities residing at different (and possibly distant) physical locations. Therefore the tracking error as such cannot be used to drive a feedback controller, as it is the case in a standard tracking problem.

The actual tracking error not being available, it is natural to approach the problem by reconstructing the tracking error starting from the information transmitted through the communication channel. For the reconstruction to successfully take place, the information must be suitably encoded. A possibility is to make use of a sufficiently large number of bits, so as to render the magnitude of the difference between the true reference signal and the reconstructed one negligible. However, in the framework of distributed control systems, the constraint on the available bandwidth is usually tight and adopting encoding schemes which require a large number of bits may not be practically feasible. The approach pursued in this paper is rather to counteract the uncertainty due to the presence of the communication channel by a suitable choice of the parameters of the regulator.

In Section 2, the formulation of the problem is made more precise, whereas the procedure for encoding the reference command is described in Section 3. Section 4 introduces the regulator which guarantees the achievement of the control goal using the reconstructed tracking error. The main results of the paper are stated and proved in Section 5. The proof consists of two steps. First, boundedness of the closed-loop trajectories are shown (Section 5.1), and then asymptotic convergence to zero of the tracking error is concluded (Section 5.2). The results are illustrated by an example in Section 6.

2 Problem statement

Generally speaking, the problem in question can be defined in the following terms. Consider a single-input single-output nonlinear system modeled by equations of the form

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}\tag{1}$$

and suppose its output y is required to asymptotically track the output y_{des} of a remotely located exosystem

$$\begin{aligned}\dot{w} &= s(w) & w \in \mathbb{R}^r \\ y_{\text{des}} &= y_{\text{r}}(w).\end{aligned}\tag{2}$$

The problem is to design a control law of the form

$$\begin{aligned}\dot{\xi} &= \varphi(\xi, y, w_{\text{q}}) \\ u &= \theta(\xi, y, w_{\text{q}})\end{aligned}\tag{3}$$

in which w_{q} represents a *sampled and quantized* version of the remote exogenous input w , so as to have the tracking error

$$e(t) = y(t) - y_{\text{r}}(w(t))\tag{4}$$

asymptotically converging to zero as time tends to ∞ . Note that the controller in question does not have access to e , which is not physically available, but only to the controlled output and to a sampled and quantized version of the remotely generated command.

We will show in what follows how the theory of output tracking can be enhanced so as to address this interesting design problem. In particular, we will show how, by incorporating in the controller two (appropriate) internal models of the exogenous signals, the desired control goal can be achieved. One internal model is meant to asymptotically reproduce, at the location of the controlled plant, the behavior of the remote command input. The other internal model, as in any tracking scheme, is meant to generate the “feed-forward” input which keeps the tracking error identically at zero.

We begin by describing, in the following section, the role of the first internal model.

3 The encoder-decoder pair

In order to overcome the limitation due to the finite capacity of the communication channel, the control structure proposed here has a decentralized structure consisting of two separate units: one unit, co-located with the command generator, consists of an *encoder* which extracts from the reference signal the data which are transmitted through the communication channel; the other unit, co-located with the controlled plant, consists of a *decoder* which processes the encoded received information and of a *regulator* which generates appropriate control input.

The problem at issue will be solved under a number of assumptions most of which are inherited by the literature of output regulation and/or control under quantization. The first assumption, which is a customary condition in the problem of output regulation, is formulated as follows.

(A0) The vector field $s(\cdot)$ in (2) is locally Lipschitz and the initial conditions for (2) are taken in a fixed compact invariant set W_0 . \triangleleft

The next assumption is, on the contrary, newer and motivated by the specific problem addressed in this paper. In order to formulate rigorously the assumption in question, we need to introduce some notation. In particular let $|x|_S$ denote the distance at a point $x \in \mathbb{R}^n$ from a compact subset $S \subset \mathbb{R}^n$, i.e. the number

$$|x|_S := \max_{y \in S} |x - y|$$

and let

$$L_0 = \max_{\substack{i \in [1, \dots, r] \\ (x, y) \in W_0 \times W_0}} |x_i - y_i|. \quad (5)$$

Furthermore, having denoted by N_b the number of bits characterizing the communication channel constraint, let N be the largest positive integer such that

$$N_b \geq r \lceil \log_2 N \rceil \quad (6)$$

where $\lceil v \rceil$, $v \in \mathbb{R}$, denotes the lowest integer such that $\lceil v \rceil \geq v$.

With this notation in mind, the second assumption can be precisely formulated as follows.

(A1) There exists a compact set $W \supset W_0$ which is invariant for $\dot{w} = s(w)$ and such that

$$\bar{w} \notin W \quad \Rightarrow \quad |\bar{w}|_{W_0} > \sqrt{r} \frac{L_0}{2N} . \quad \triangleleft$$

W being compact and $s(\cdot)$ being locally Lipschitz, it is readily seen that there exists a non decreasing and bounded function $M(\cdot) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{>0}$, with $M(0) = 1$, such that for all $w_{10} \in W$ and $w_{20} \in W$ and for all $t \geq 0$

$$|w_1(t) - w_2(t)| \leq M(t)|w_{10} - w_{20}| \quad (7)$$

where $w_1(t)$ and $w_2(t)$ denote the solutions of (2) at time t passing through w_{10} and, respectively, w_{20} at time $t = 0$.

This function, the sampling interval T , the number L_0 defined in (5) and the number N fulfilling (6), determine the parameters of the encoder-decoder pair, which are defined as follows (see [10], [8], [5] for more details).

Encoder dynamics. The encoder dynamics consist of a copy of the exosystem dynamics, whose state is updated at each sampling time kT , $k \in \mathbb{N}$, and determines (depending on the actual state of the exosystem) the centroid of the quantization region, and of an additional discrete-time dynamics which determines the size of the quantization region. Specifically, the encoder is characterized by

$$\begin{aligned} \dot{w}_e &= s(w_e) & w_e(kT) &= w_e(kT^-) + w_q(k) \frac{L(k)}{N} & w_e(0^-) &\in W_0 \\ L(k+1) &= \sqrt{r} \frac{M(T)}{N} L(k) & L(0) &= L_0 \end{aligned}$$

in which w_q represents the encoded information given by, for $i = 1, \dots, r$,

$$w_{q,i}(k) = \text{sgn}(w_i(kT) - w_{e,i}(kT^-)) \cdot \begin{cases} \left\lceil \frac{N|w_i(kT) - w_{e,i}(kT^-)|}{L(k)} \right\rceil - \frac{1}{2} & N \text{ even} \\ \left\lceil \frac{N|w_i(kT) - w_{e,i}(kT^-)|}{L(k)} - \frac{1}{2} \right\rceil & N \text{ odd} . \end{cases}$$

At each sampling time kT , the vector $w_q(k)$ is transmitted to the controlled plant through the communication channel and then used to update the state of the decoder unit as described in the following. To this regard note that each component of the vector $w_q(k)$ can be described by $\lceil \log_2 N \rceil$ bits and thus the communication channel constraint is fulfilled.

Decoder dynamics The decoder dynamics is a replica of the encoder dynamics and it is given by

$$\begin{aligned} \dot{w}_d &= s(w_d) & w_d(kT) &= w_d(kT^-) + w_q(k) \frac{L(k)}{N} & w_d(0^-) &= w_e(0^-) \\ L(k+1) &= \sqrt{r} \frac{M(T)}{N} L(k) & L(0) &= L_0 \end{aligned} \quad (8)$$

If, ideally, the communication channel does not introduce delays, it turns out that $w_d(t) \equiv w_e(t)$ for all $t \geq 0$. Furthermore, it can be proved that the set W characterized in Assumption (A1) is invariant for the encoder (decoder) dynamics and that the asymptotic behavior of $w_e(t)$ ($w_d(t)$) converges uniformly to the true exosystem state $w(t)$, provided that T is properly chosen with respect to the number N and the function $M(\cdot)$. This is formalized in the next proposition (see [8], [5] for details).

Proposition 1 *Suppose Assumptions (A0)-(A1) hold and that the sampling time T and the number N satisfy*

$$N > \sqrt{r} M(T). \quad (9)$$

Then:

(i) *for any $w_d(0^-) \in W_0$ and $w(0) \in W_0$, $w_d(t) \in W$ for all $t \geq 0$;*

(ii) *for any $w_d(0^-) \in W_0$ and $w(0) \in W_0$,*

$$\lim_{t \rightarrow \infty} |w(t) - w_d(t)| = 0$$

with uniform convergence rate, namely for every $\epsilon > 0$ there exists $T^ > 0$ such that for all initial states $w_d(0^-) \in W_0$, $w(0) \in W_0$, and for all $t \geq T^*$, $|w(t) - w_d(t)| \leq \epsilon$.*

Proof. As W is an invariant set for $\dot{w} = s(w)$, the proof of the first item reduces to show that, for all $k \geq 0$, if $w_d(kT^-) \in W$ then necessarily $w_d(kT) \in W$. For, note that this is true for $k = 0$. As a matter of fact, since $w_d(0^-) \in W_0 \subset W$ and by bearing in mind the definition of w_q , it turns out that $|w_d(0) - w(0)| \leq \sqrt{r}L_0/2N$ which implies, by definition of W in Assumption (A1), that $w_d(0) \in W$. For a generic $k > 0$ note that, again by definition of w_q , it turns out that $|w_d(kT) - w(kT)| \leq \sqrt{r}L(k)/2N$. But, by the second of (8) and by condition (9), $L(k) < L(k-1) \leq L_0$ yielding $|w_d(kT) - w(kT)| \leq \sqrt{r}L_0/2N$ which implies $w_d(kT) \in W$. This completes the proof of the first item. The second item has been proved in [8], [5]. \triangleleft

Remark. By composing (6) with (9) it is easy to realize that the number of bits N_b and the sampling interval T are required to satisfy the constraint

$$N_b \geq r \lceil \log_2 (\sqrt{r} M(T)) \rceil \quad (10)$$

in order to have the encoder-decoder trajectories asymptotically converging to the exosystem trajectories. Since the function $M(\cdot)$ depends on the exosystem dynamics and on the set W_0 of initial conditions for (2), equation (10) can be interpreted as a relation between the bit-rate of the communication channel and the exosystem dynamics which must be satisfied in order to remotely reconstruct the reference signal.

4 The regulator

4.1 Basic hypotheses

As in most of the literature on regulation of nonlinear system, we assume in what follows that the controlled plant has well defined relative degree and normal form. If this is the case and if the initial conditions of the plant are allowed to vary on a fixed (though arbitrarily large) compact set, there is no loss of generality in considering the case in which the controlled plant has relative degree 1 (see for instance [1]). We henceforth suppose that system (1) is expressed in the form

$$\begin{aligned}\dot{z} &= f(z, y, \mu) & z &\in \mathbb{R}^n \\ \dot{y} &= q(z, y, \mu) + u & y &\in \mathbb{R}\end{aligned}\tag{11}$$

in which μ is a vector of uncertain parameters ranging in a known compact set P . Initial conditions $(z(0), y(0))$ of (11) are allowed to range on a fixed (but otherwise arbitrary) compact set $Z \times Y \subset \mathbb{R}^n \times \mathbb{R}$.

It is well known that, if the regulation goal is achieved, in steady-state (i.e when the tracking error $e(t)$ is identically zero) the controller must necessarily provide an input of the form

$$u_{ss} = L_s y_r(w) - q(z, y_r(w), \mu)\tag{12}$$

(where $L_s y_r(\cdot)$ stands for the derivative of $y_r(\cdot)$ along the vector field $s(\cdot)$) in which w and z obey

$$\begin{aligned}\dot{\mu} &= 0 \\ \dot{w} &= s(w) \\ \dot{z} &= f(z, y_r(w), \mu).\end{aligned}\tag{13}$$

As in [2], we assume in what follows that system (13) has a compact attractor, which is also locally exponentially stable. To express this assumption in a concise form, it is convenient to group the components μ, w, z of the state vector of (13) into a single vector $\mathbf{z} = \text{col}(\mu, w, z)$ and rewrite the latter as

$$\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z}).$$

Consistently, the map (12) is rewritten as

$$u_{ss} = \mathbf{q}_0(\mathbf{z}),$$

and it is set $\mathbf{Z} = P \times W \times Z$. The assumption in question is the following one¹

¹Recall that, if the positive orbit of a compact set X of initial conditions of a system

$$\dot{x} = f(x)\tag{14}$$

is bounded, the ω -limit set of X under the flow of (14) – denoted $\omega(X)$ – is a nonempty, compact invariant set which attracts X uniformly. If $\omega(X)$ is in the interior of X , then X is also stable in the sense of Lyapunov.

(A2) there exists a compact subset \mathcal{Z} of $P \times W \times \mathbb{R}^n$ which contains the positive orbit of the set \mathbf{Z} under the flow of (13) and $\omega(\mathbf{Z})$ is a differential submanifold (with boundary) of $P \times W \times \mathbb{R}^n$. Moreover there exists a number $d_1 > 0$ such that

$$\mathbf{z} \in P \times W \times \mathbb{R}^n, \quad |\mathbf{z}|_{\omega(\mathbf{Z})} \leq d_1 \quad \Rightarrow \quad \mathbf{z} \in \mathbf{Z}.$$

Finally, there exist $m \geq 1$, $a > 0$ and $d_2 \leq d_1$ such that

$$\mathbf{z}_0 \in P \times W \times \mathbb{R}^n, \quad |\mathbf{z}_0|_{\omega(\mathbf{Z})} \leq d_2 \quad \Rightarrow \quad |\mathbf{z}(t)|_{\omega(\mathbf{Z})} \leq me^{-at} |\mathbf{z}_0|_{\omega(\mathbf{Z})},$$

in which $\mathbf{z}(t)$ denotes the solution of (13) passing through \mathbf{z}_0 at time $t = 0$. \triangleleft

In what follows, the set $\omega(\mathbf{Z})$ will be simply denoted as \mathcal{A}_0 . The final assumption is an assumption that allows us to construct an *internal model* of all inputs of the form $u_{ss}(t) = \mathbf{q}_0(\mathbf{z}(t))$, with $\mathbf{z}(t)$ solution of (13) with initial condition in \mathcal{A}_0 . This assumption, which can be referred to as assumption of *immersion* into a *nonlinear uniformly observable* system, is the following one.

(A3) There exists an integer $d > 0$ and a locally Lipschitz map $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ such that, for all $\mathbf{z} \in \mathcal{A}_0$, the solution $\mathbf{z}(t)$ of (13) passing through \mathbf{z}_0 at $t = 0$ is such that the function $u(t) = \mathbf{q}_0(\mathbf{z}(t))$ satisfies

$$u^{(d)}(t) + \varphi(u^{(d-1)}(t), \dots, u^{(1)}(t), u(t)) = 0. \quad \triangleleft \tag{15}$$

4.2 The design of the regulator

Using the technique described in [2], the first step is the construction of an internal model for (13), viewed as an autonomous system with output (12). To this end, consider the sequence of functions recursively defined as

$$\tau_1(\mathbf{z}) = \mathbf{q}_0(\mathbf{z}), \quad \dots, \quad \tau_{i+1}(\mathbf{z}) = \frac{\partial \tau_i}{\partial \mathbf{z}} \mathbf{f}_0(\mathbf{z})$$

for $i = 1, \dots, d-1$, with d as introduced in assumption (A3), and consider the map

$$\begin{aligned} \tau : P \times W \times \mathbb{R}^n &\rightarrow \mathbb{R}^d \\ (\mu, w, z) &\mapsto \text{col}(\tau_1(\mathbf{z}), \tau_2(\mathbf{z}), \dots, \tau_d(\mathbf{z})). \end{aligned}$$

If k , the degree of continuous differentiability of the functions in (11), is large enough, the map τ is well defined and C^1 . In particular $\tau(\mathcal{A}_0)$, the image of \mathcal{A}_0 under τ is a *compact* subset of \mathbb{R}^d , because \mathcal{A}_0 is a compact subset of $P \times W \times \mathbb{R}^n$.

Let $\varphi_c : \mathbb{R}^d \rightarrow \mathbb{R}$ be any locally Lipschitz function of compact support which agrees on $\tau(\mathcal{A}_0)$ with the function φ defined in (A3), i.e. a function such that, for some compact superset \mathcal{S} of $\tau(\mathcal{A}_0)$ satisfies

$$\begin{aligned} \varphi_c(\eta) &= 0 && \text{for all } \eta \notin \mathcal{S} \\ \varphi_c(\eta) &= \varphi(\eta) && \text{for all } \eta \in \tau(\mathcal{A}_0). \end{aligned}$$

With this in mind, consider the system

$$\dot{\xi} = \Phi_c(\xi) + G(u_{ss} - \Gamma\xi) \quad (16)$$

in which

$$\Phi_c(\xi) = \begin{pmatrix} \xi_2 \\ \xi_3 \\ \dots \\ \xi_d \\ -\varphi_c(\xi_1, \xi_2, \dots, \xi_d) \end{pmatrix}, \quad G = \begin{pmatrix} \kappa c_{d-1} \\ \kappa^2 c_{d-2} \\ \dots \\ \kappa^{d-1} c_1 \\ \kappa^d c_0 \end{pmatrix}, \quad \Gamma = (1 \quad 0 \quad \dots \quad 0),$$

the c_i 's are such that the polynomial $\lambda^d + c_0\lambda^{d-1} + \dots + c_{d-1} = 0$ is Hurwitz and κ is a positive number. As shown in [2], if κ is large enough, the state $\xi(t)$ of (16) asymptotically tracks $\tau(\mathbf{z}(t))$, in which $\mathbf{z}(t)$ is the state of system (13). Therefore $\Gamma\xi(t)$ asymptotically reproduces its output (12), i.e. the steady state control $u_{ss}(t)$. As a matter of fact, the following result holds.

Lemma 1 *Suppose assumptions (A1) and (A2) hold. Consider the triangular system*

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{f}_0(\mathbf{z}) \\ \dot{\xi} &= \Phi_c(\xi) + G(\mathbf{q}_0(\mathbf{z}) - \Gamma\xi). \end{aligned} \quad (17)$$

Let the initial conditions for \mathbf{z} range in the set \mathbf{Z} and let Ξ be an arbitrarily large compact set of initial condition for ξ . There is a number κ^ such that, if $\kappa \geq \kappa^*$, the trajectories of (17) are bounded and*

$$\text{graph}(\tau|_{\mathcal{A}_0}) = \omega(\mathbf{Z} \times \Xi).$$

In particular $\text{graph}(\tau|_{\mathcal{A}_0})$ is a compact invariant set which uniformly attracts $\mathbf{Z} \times \Xi$. Moreover $\text{graph}(\tau|_{\mathcal{A}_0})$ is also locally exponentially attractive.

Note that, as a consequence of assumption (A1), of the fact that $\tau(\cdot)$ is a continuous vector field and of the fact that the compact set Ξ can be taken arbitrarily large, it is possible to assume, without loss of generality, the existence of a positive d_2 such that

$$\mathbf{z} \in P \times W \times \mathbb{R}^n, \quad \xi \in \mathbb{R}^d, \quad |(\mathbf{z}, \xi)|_{\omega(\mathbf{Z} \times \Xi)} \leq d_2 \quad \Rightarrow \quad (\mathbf{z}, \xi) \in \mathbf{Z} \times \Xi.$$

In view of Lemma 1, it would be natural – if the true error variable e were available for feedback purposes – to choose for (11) a control of the form

$$\begin{aligned} \dot{\xi} &= \Phi_c(\xi) - Gke \\ u &= \Gamma\xi - ke, \end{aligned} \quad (18)$$

with k a large number. This control, in fact, would solve the problem of output regulation (see [2]). The true error e not being available, we choose instead

$$\hat{e} = y - y_r(w_d) \quad (19)$$

and the controller accordingly as

$$\begin{aligned}\dot{\xi} &= \Phi_c(\xi) - Gk\hat{e} \\ u &= \Gamma\xi - k\hat{e}.\end{aligned}\tag{20}$$

The result which will be proven next is that there exists $k^* > 0$ such that if $k \geq k^*$ the regulator designed above solves the problem in question (provided that N and T satisfy the condition of Proposition 1).

5 Main results

5.1 Trajectories of the closed loop system are bounded

To prove that the proposed regulator solves the problem, we show first of all the trajectories of the controlled system, namely those of the system

$$\begin{aligned}\dot{w}_d &= s(w_d) & w_d(kT) &= w_d(kT^-) + w_q(k)\frac{L(k)}{2N} \\ \dot{z} &= f(z, y, \mu) \\ \dot{y} &= q(z, y, \mu) + \Gamma\xi - k(y - y_r(w_d)) \\ \dot{\xi} &= \Phi_c(\xi) - Gk(y - y_r(w_d))\end{aligned}\tag{21}$$

are bounded. To study trajectories of (21) it is convenient to replace the coordinate y by

$$\hat{e} = y - y_r(w_d)$$

to obtain the system

$$\begin{aligned}\dot{w}_d &= s(w_d) \\ \dot{z} &= f(z, \hat{e} + y_r(w_d), \mu) \\ \dot{\xi} &= \Phi_c(\xi) - Gk\hat{e} \\ \dot{\hat{e}} &= q(z, \hat{e} + y_r(w_d), \mu) - L_s y_r(w_d) + \Gamma\xi - k\hat{e}.\end{aligned}\tag{22}$$

This system can be further simplified by changing the state variable ξ into $\tilde{\xi} = \xi - G\hat{e}$ and setting $p = \text{col}(\mu, w_d, z, \tilde{\xi})$, so as to obtain a system of the form

$$\begin{aligned}\dot{p} &= F_0(p) + F_1(p, \hat{e})\hat{e} \\ \dot{\hat{e}} &= H_0(p) + H_1(p, \hat{e})\hat{e} - k\hat{e},\end{aligned}\tag{23}$$

in which

$$F_0(p) = \begin{pmatrix} 0 \\ s(w_d) \\ f(z, y_r(w_d), \mu) \\ \Phi_c(\tilde{\xi}) + G(-q(z, y_r(w_d), \mu) + L_s y_r(w_d) - \Gamma\tilde{\xi}) \end{pmatrix}$$

$$H_0(p) = q(z, y_r(w_d), \mu) - L_s y_r(w_d) + \Gamma \tilde{\xi}$$

and $F_1(p, \hat{e})$, $H_1(p, \hat{e})$ are suitable continuous functions.

With this notation in mind, we state the next proposition which claims that a large value of k succeeds in rendering bounded the trajectories of the switched nonlinear system (23) provided that the sampling interval T is sufficiently large.

Proposition 2 *Consider system (21) with initial conditions in $P \times W \times Z \times Y \times \Xi$. Suppose assumptions (A0)-(A3) hold. Let κ be chosen as indicated in Lemma 1. Then there exist $T^* > 0$ and $k^* > 0$ such that for all sampling intervals $T > T^*$ and all $k \geq k^*$ the trajectories are bounded in positive time.*

Proof. System (23) is apparently identical to the closed-loop system already studied in [2], but with the exception that, every T units of time, w_d is being “reset” as

$$w_d(kT) = w_d(kT^-) + w_q(k) \frac{L(k)}{2N}$$

and \hat{e} is being “reset” as

$$\hat{e}(kT) = \hat{e}(kT^-) - y_r(w_d(kT^-) + w_q(k) \frac{L(k)}{2N}) + y_r(w_d(kT^-)).$$

With this in mind, we proceed to study the behavior of (23) on the time interval $[0, T]$. Observe that system

$$\dot{p} = F_0(p) \tag{24}$$

coincides with system (17), the only difference being that the component w of \mathbf{z} is now written as w_d and ξ is now written as $\tilde{\xi}$. Thus, from Lemma 1, it can be asserted that in this system all trajectories with initial conditions in $\mathbf{Z} \times \Xi$ are attracted by the compact invariant set

$$\mathcal{A} = \{(\mathbf{z}, \xi) \in \mathcal{A}_0 \times \mathbb{R}^d : \xi = \tau(\mathbf{z})\}. \tag{25}$$

Moreover, by construction, the function $H_0(p)$ vanishes on this set.

Let \mathcal{D} denote the domain of attraction of \mathcal{A} . Then, as shown for instance in [4], for any given arbitrarily small ϵ , it is possible to claim the existence of a continuous function $V : \mathcal{D} \rightarrow \mathbb{R}$ having the following properties:

- (a) $V(p) = 0$ if $p \in \mathcal{A}$ and $V(p) > 0$ everywhere else,
- (b) $V(\cdot)$ is proper on \mathcal{D} ,
- (c) for some positive b

$$V(p) \leq b \quad \Rightarrow \quad |p|_{\mathcal{A}} \leq \epsilon,$$

(d) for some positive $g < b$, $V(\cdot)$ is locally Lipschitz on the set $\mathcal{D}_g = \{p \in \mathcal{D} : V(p) > g\}$,

(e) for any $p \in \mathcal{D}_g$,

$$\limsup_{h \rightarrow 0^+} \frac{1}{h} [V(\phi(h, p)) - V(p)] \leq -1$$

in which $\phi(t, p)$ denotes the flow of (24).

Let now $\epsilon < d_2$ (with d_2 defined after Lemma 1), pick the function V accordingly and note that by construction

$$V(p) \leq b \quad \Rightarrow \quad |p|_{\mathcal{A}} \leq \epsilon \leq d_2 \quad \Rightarrow \quad p \in \mathbf{Z} \times \Xi. \quad (26)$$

Moreover pick numbers a and b_1 , with $a > b$ such that $\mathbf{Z} \times \Xi \subset V^{-1}([0, a])$ (which is possible as V is proper on \mathcal{D}), and $g < b_1 < b$, and set

$$\mathcal{S} = \{p \in \mathcal{D} \quad : \quad b_1 \leq V(p) \leq a + 1\}.$$

Let c and Δ be positive numbers such that

$$|y - y_r(w)| \leq c \quad \text{for all } y \in Y \text{ and } w \in W_0$$

and

$$|y_r(w) - y_r(w_d)| \leq \Delta \quad \text{for all } w_d \in W \text{ and } w \in W_0$$

and note that

$$\begin{aligned} |\hat{e}(0)| = |y(0) - y_r(w_d(0))| &= |y(0) - y_r(w(0)) + y_r(w(0)) - y_r(w_d(0))| \\ &\leq |y(0) - y_r(w(0))| + |y_r(w(0)) - y_r(w_d(0))| \leq c + \Delta. \end{aligned}$$

Finally, let \bar{f} and \bar{h} be defined as

$$\bar{f} := \max_{\substack{p \in V^{-1}([0, a+1]) \\ |\hat{e}| \leq c + \Delta + 1}} |F_1(p, \hat{e})| \quad \bar{h} := \max_{\substack{p \in V^{-1}([0, a+1]) \\ |\hat{e}| \leq c + \Delta + 1}} |H_0(p) + H_1(p, \hat{e})\hat{e}|.$$

Claim 1: There exists $T^ > 0$ and $k^* > 0$ such that for any $T \geq T^*$, any $k \geq k^*$ and any $\ell \geq 0$*

$$\begin{aligned} |\hat{e}(\ell T)| \leq c + \Delta &\quad \Rightarrow \quad |\hat{e}((\ell + 1)T^-)| \leq c \\ p(\ell T) \in \mathbf{Z} \times \Xi &\quad \Rightarrow \quad p((\ell + 1)T^-) \in \mathbf{Z} \times \Xi \quad . \quad \triangleleft \end{aligned} \quad (27)$$

To prove the claim, we derive two basic inequalities. The first one is about the dynamics of \hat{e} . Looking at the bottom equation of (23) and bearing in mind the definition of \bar{h} , standard arguments can be used to claim that – if on some time interval $[t_0, (\ell + 1)T)$, $t_0 \geq \ell T$, the state of (23) satisfies $|\hat{e}(t)| \leq c + \Delta + 1$ and $p(t) \in V^{-1}([0, a + 1])$ – then

$$|\hat{e}(t)| \leq e^{-k(t-t_0)} |\hat{e}(t_0)| + \frac{\bar{h}}{k} \quad \forall t \in [t_0, (\ell + 1)T). \quad (28)$$

The second inequality is about the dynamics of p . Looking at the top equation of (23) and taking the Dini's derivative of $V(p(t))$ we see that – if on some time interval $[t_0, (\ell+1)T)$, with $t_0 \geq \ell T$, the state of (23) satisfies $|\hat{e}(t)| \leq c + \Delta + 1$ and $p(t) \in \mathcal{S}$ – then

$$D^+V(p(t)) \leq -1 + L_V \bar{f} |\hat{e}(t)| \leq -1 + L_V \bar{f} (e^{-k(t-t_0)} |\hat{e}(t_0)| + \frac{\bar{h}}{k}) \quad \forall t \in [t_0, (\ell+1)T), \quad (29)$$

in which L_V is such that $|V(p) - V(q)| \leq L_V |p - q| \quad \forall p, q \in \mathcal{S}$. From this, using the appropriate comparison lemma, we obtain

$$V(p(t)) \leq V(p(t_0)) - (1 - \frac{L_V \bar{f} \bar{h}}{k})(t - t_0) + \frac{L_V \bar{f} |\hat{e}(t_0)|}{k} \quad \forall t \in [t_0, (\ell+1)T). \quad (30)$$

In order to be able to prove Claim 1, we need this auxiliary result.

*Claim 2: there exists k^{**} such that for any $k \geq k^{**}$, $T \geq T^*$ and $\ell \geq 0$*

$$\begin{aligned} |\hat{e}(\ell T)| \leq c + \Delta \\ p(\ell T) \in V^{-1}([0, a]) \end{aligned} \quad \Rightarrow \quad \begin{aligned} |\hat{e}(t)| \leq c + \Delta + 1 \\ p(t) \in V^{-1}([0, a + 1]) \end{aligned} \quad \forall t \in [\ell T, (\ell + 1)T). \quad \triangleleft \quad (31)$$

This claim can be proved by contradiction. As a matter of fact suppose that (31) is not true, namely that there exists a time $T' \in [\ell T, (\ell + 1)T)$ such that either $|\hat{e}(T')| > c + \Delta + 1$ or $V(T') > a + 1$ (with a mild abuse of notation we write $V(\cdot)$ for $V(p(\cdot))$). By continuity of the trajectories with respect to time, this means that there exists a time $T'' \geq \ell T$, $T'' \leq T'$, such that either

$$|\hat{e}(t)| < c + \Delta + 1 \quad t \in [\ell T, T'') \quad |\hat{e}(T'')| = c + \Delta + 1 \quad \text{and} \quad V(t) \leq a + 1 \quad t \in [\ell T, T'') \quad (32)$$

or

$$V(t) < a + 1 \quad t \in [\ell T, T'') \quad V(T'') = a + 1 \quad \text{and} \quad |\hat{e}(t)| \leq c + \Delta + 1 \quad t \in [\ell T, T''). \quad (33)$$

But if (32) were true, by (28) taking $t_0 = \ell T$ and $k \geq 2\bar{h}$, we would have that

$$|\hat{e}(T'')| \leq e^{-k(T'' - \ell T)} |\hat{e}(\ell T)| + \frac{\bar{h}}{k} \leq c + \Delta + \frac{\bar{h}}{k} \leq c + \Delta + \frac{1}{2}$$

which contradicts $|\hat{e}(T'')| = c + \Delta + 1$. A similar contradiction would be obtained if (33) were true. In fact, let t_0 , $\ell T \leq t_0 < T''$, be any time such that $V(t_0) = a$ and $V(t) \geq a$ for all $t \in [t_0, T'']$. Using (30) and taking $k \geq \max\{L_V \bar{f} \bar{h}, 2L_V \bar{f}(c + 1)\}$, we have that

$$V(p(T'')) \leq a - (1 - \frac{L_V \bar{f} \bar{h}}{k})(T'' - t_0) + \frac{L_V \bar{f}(c + 1)}{k} \leq a + \frac{1}{2}$$

which contradicts $V(T'') = a + 1$. From this, Claim 2 follows by taking

$$k^{**} = \max\{2\bar{h}, L_V \bar{f} \bar{h}, 2L_V \bar{f}(c + \Delta + 1)\}.$$

◇(End proof Claim 2)

Having proven that $|\hat{e}(t)| \leq c + \Delta + 1$ and $p(t) \in V^{-1}([0, a + 1])$ on the entire time interval $[\ell T, (\ell + 1)T)$, the desired Claim 1 follows again from arguments based on the bounds (28) and (30). To show that a large value of k succeeds in recovering $|\hat{e}((\ell + 1)T^-)| \leq c$, just set

$$k \geq \max\left\{\frac{2\bar{h}}{c}, \frac{1}{T} \ln \frac{2(c + \Delta)}{c}\right\}$$

in the estimate (28) evaluated with $t_0 = \ell T$.

To show that a large value of k succeeds in recovering $p((\ell + 1)T^-) \in \mathbf{Z} \times \Xi$ we prove that if

$$T \geq T^* = 2(a - b) + 1$$

then $V(p((\ell + 1)T^-)) \leq b$ which, by (26), implies $p((\ell + 1)T^-) \in \mathbf{Z} \times \Xi$. To this end, we distinguish two cases. If $p(t) \in \mathcal{S}$ for all $t \in [\ell T, (\ell + 1)T)$, just set

$$k \geq \max\{2L_V \bar{f}(c + \Delta + 1), 2L_V \bar{f} \bar{h}\}$$

in the estimate (30) with $t_0 = \ell T$ to obtain $V((\ell + 1)T^-) \leq b$. The condition $p(t) \in \mathcal{S}$ for all $t \in [\ell T, (\ell + 1)T)$ can be violated if and only if there are times in $[\ell T, (\ell + 1)T)$ at which $V(t) < b_1$ (recall the definition of \mathcal{S} and the fact that $V(t) \leq a + 1$ for all $t \in [\ell T, (\ell + 1)T)$). Let this be the case. If $V((\ell + 1)T^-) < b_1 < b$, the claim is trivially true. If not, let $T' \in [\ell T, (\ell + 1)T)$ be such that $V(T') = b_1$ and $b_1 \leq V(t) \leq a + 1$ for all $t \in [T', (\ell + 1)T)$. On this time interval one can still use (30) with $t_0 = T'$ to conclude that if $k \geq L_V \bar{f}(c + \Delta + 1)/(b - b_1)$ then $V(t - T') \leq b$ for all $t \in [T', (\ell + 1)T)$ from which the Claim 1 follows. ◇(End proof of Claim 1)

With this result at hand, Proposition 2 can be easily proved by subsequent applications of Claim 2. To this end note that $p(0) \in \mathbf{Z} \times \Xi$ and, by definition of c and Δ , $|\hat{e}(0)| \leq c \leq c + \Delta$, from which Claim 1 evaluated for $\ell = 0$ yields $|\hat{e}(T^-)| \leq c$ and $p(T^-) \in \mathbf{Z} \times \Xi$. Suppose now that, for some $\ell > 0$, $|\hat{e}(\ell T^-)| \leq c$ and $p(\ell T^-) \in \mathbf{Z} \times \Xi$. Then, after the switch,

$$\begin{aligned} |\hat{e}(\ell T)| &= |y(\ell T) - y_r(w_d(\ell T))| = |y(\ell T^-) - y_r(w_d(\ell T))| \\ &= |y(\ell T^-) - y_r(w_d(\ell T^-)) + y_r(w_d(\ell T^-)) - y_r(w_d(\ell T))| \\ &\leq |\hat{e}(\ell T^-)| + |y_r(w_d(\ell T^-)) - y_r(w_d(\ell T))| \leq |\hat{e}(\ell T^-)| + \Delta \leq c + \Delta \end{aligned}$$

and, by item (i) of Proposition 1, $w_d(\ell T) \in W$. The latter, by bearing in mind the definition of the set \mathbf{Z} and the definition of p , yields $p(\ell T) \in \mathbf{Z} \times \Xi$ from which the result of Proposition 2 follows. ◁

Proposition 2 shows that trajectories of the controlled system remain bounded if the time interval T exceeds a minimum number T^* (minimal “dwell-time”) which depends on the parameters of the controlled system and on the sets of initial conditions. This, however,

may be in contrast with relation (10) which, bearing in mind that $M(\cdot)$ is an increasing function, requires that the sampling interval T is small enough. In the case the dwell-time T^* is not compatible with (10), a simple modification of the decoder structure helps solving the problem. Let N_b be given, let \bar{T} denote the minimal value of T compatible with (10) and let ℓ be any positive integer such that

$$\ell\bar{T} \geq T^*.$$

Consider now a “second level” decoder dynamics defined as

$$\dot{w}'_d = s(w'_d) \quad w'_d(0) = w_d(0) \quad (34)$$

whose state w'_d is periodically reset, every $\ell\bar{T}$ units of time, to the value of the “first level” decoder dynamics (8), that is as

$$w'_d(k\ell\bar{T}) = w_d(k\ell\bar{T}) \quad \text{for all } k \geq 0.$$

In other words w'_d provides an under-sampled version of the first-level decoder dynamics (8) with the under-sampling period $\ell\bar{T}$ such that the constraint on the minimal dwell-time is respected.

The same arguments used to prove Proposition 1 show that $w'_d(t) \in W$ for all $t \geq 0$ and furthermore

$$\lim_{t \rightarrow \infty} |w'_d(t) - w(t)| = 0$$

with uniform convergence rate.

Consider now the regulator (20) in which \hat{e} is defined as in (19) but with w_d replaced by w'_d . It is easy to realize that all the analysis carried out in this section can now be repeated by replacing the decoder dynamics (8) by (34), the decoder variable w_d by w'_d , and the time interval T by $\ell\bar{T}$. In particular by Proposition 2 and by the fact that $\ell\bar{T} \geq T^*$ we conclude that the trajectories of the controlled system are bounded if k is chosen sufficiently large. This and the fact that $w'_d(t)$ converges asymptotically to $w(t)$ with uniform convergence rate are the crucial properties needed to conclude that the proposed regulator solves the problem in question as precisely stated and proved in the next subsection.

5.2 The tracking error converges to zero

To prove that the tracking error converges to zero, it is useful to observe that, if the coordinate y of (21) is replaced by

$$e = y - y_r(w)$$

the system in question can be also rewritten as

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(z, e + y_r(w), \mu) \\ \dot{\xi} &= \Phi_c(\xi) + G(-ke) + G(-k\tilde{e}) \\ \dot{e} &= q(z, e + y_r(w), \mu) - L_s y_r(w) + \Gamma\xi - ke - k\tilde{e} \end{aligned} \quad (35)$$

having set

$$\tilde{e} = \hat{e} - e.$$

The same change of variables used to put (22) in the form (23) yields now a system of the form

$$\begin{aligned}\dot{p} &= F_0(p) + F_1(p, e)e \\ \dot{e} &= H_0(p) + H_1(p, e)e - ke - k\tilde{e},\end{aligned}\tag{36}$$

in which $p = \text{col}(\mu, w, z, \tilde{\xi})$ and $F_0(p), F_1(p, e), H_0(p), H_1(p, e)$ are the same as in (23). This system can be viewed as a “perturbed” version of system

$$\begin{aligned}\dot{p} &= F_0(p) + F_1(p, e)e \\ \dot{e} &= H_0(p) + H_1(p, e)e - ke\end{aligned}\tag{37}$$

whose asymptotic properties have been investigated in [4].

The following result is a minor enhancement of the main result of [4]. Let $V(\cdot)$ be the positive definite function introduced in the proof of Lemma 1, set $\mathcal{P} = \{p : V(p) \leq a\}$ with a chosen so that $P \times W \times Z \times \Xi \subset \mathcal{P}$, and set $E = \{e : |e| \leq c\}$. Moreover, let \mathcal{A} be the set defined by (25).

Lemma 2 *Consider system (37) in which $F_0(p), F_1(p, e), H_0(p), H_1(p, e)$ are defined as before and initial conditions are taken in $\mathcal{P} \times E$. Suppose assumptions (A0)-(A3) hold. Let κ be chosen as indicated in Lemma 1. Then there is k^* such that, if $k > k^*$, the following holds:*

(i) *the positive orbit of $\mathcal{P} \times E$ under the flow of (37) is bounded and*

$$\lim_{t \rightarrow \infty} |p(t)|_{\mathcal{A}} = 0, \quad \lim_{t \rightarrow \infty} e(t) = 0.$$

(ii) *for any $\varepsilon > 0$, there exist numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that, if $|p_0|_{\mathcal{A}} \leq \delta_1$ and $|e_0| \leq \delta_1$, for any continuous function $u(t)$ satisfying $|u(t)| \leq \delta_2$ for all $t \geq 0$, the solution $p(t), e(t)$ of the perturbed system*

$$\begin{aligned}\dot{p} &= F_0(p) + F_1(p, e)e \\ \dot{e} &= H_0(p) + H_1(p, e)e - ke + u(t)\end{aligned}\tag{38}$$

with initial conditions $p(0) = p_0$ and $e(0) = e_0$ satisfies

$$|p(t)|_{\mathcal{A}} \leq \varepsilon, \quad |e(t)| \leq \varepsilon, \quad \forall t \geq 0.$$

Proof. Item (i) has already been proven in [4]. The proof of item (ii) consists in a minor modification of the arguments used in [4] to prove item (i). Consider the Locally Lipschitz Lyapunov function $U(p)$ defined in [4], which – for some $a_1 < 1$, $\lambda > 0$ and $\bar{L} > 0$ – satisfies

$$a_1 |p|_{\mathcal{A}} \leq U(p) \leq |p|_{\mathcal{A}},$$

and

$$D^+U(p(t)) \leq -\lambda U(p(t)) + \bar{L}\bar{f}|e(t)|$$

along the integral curve $p(t)$, so long as $p(t)$ remains sufficiently close to \mathcal{A} (see (24) in [4]).

Consider now, for (38), the candidate Lyapunov function

$$W(p, e) = \frac{1}{2}(U^2(p) + e^2)$$

which trivially satisfies

$$\frac{a_1}{2}(|p|_{\mathcal{A}}^2 + e^2) \leq W(p, e) \leq \frac{1}{2}(|p|_{\mathcal{A}}^2 + e^2).$$

Taking its Dini derivative along the trajectories of (38), we can obtain an estimate of the form (we omit the argument t for convenience)

$$D^+W(p, e) \leq -\lambda U^2(p) + \bar{L}\bar{f}U(p)|e| + \beta|p|_{\mathcal{A}}|e| - (k - \bar{k})|e|^2 + |u||e|$$

so long as $p(t)$ remains sufficiently close to \mathcal{A} , in which β and \bar{k} are fixed positive numbers. Using the above estimates for $U(p)$, it is easy to deduce that

$$D^+W(p, e) \leq (U(p) \quad |e|) \begin{pmatrix} -\lambda & \frac{1}{2}(\bar{L}\bar{f} + \frac{\beta}{a_1}) \\ \frac{1}{2}(\bar{L}\bar{f} + \frac{\beta}{a_1}) & -(k - \bar{k} - \frac{1}{2}) \end{pmatrix} \begin{pmatrix} U(p) \\ |e| \end{pmatrix} + \frac{1}{2}|u|^2.$$

Clearly, there is a value $k^* > 0$ and a number $a > 0$ such that, if $k \geq k^*$

$$(U(p) \quad |e|) \begin{pmatrix} -\lambda & \frac{1}{2}(\bar{L}\bar{f} + \frac{\beta}{a_1}) \\ \frac{1}{2}(\bar{L}\bar{f} + \frac{\beta}{a_1}) & -(k - \bar{k} - \frac{1}{2}) \end{pmatrix} \begin{pmatrix} U(p) \\ |e| \end{pmatrix} \leq -\frac{a}{2}(U^2(p) + e^2) \leq -aW(p, e),$$

and this yields, using the appropriate comparison lemma,

$$W(p(t), e(t)) \leq -e^{-at}W(p_0, e_0) + \frac{1}{2a} \max_{\tau \in [0, t]} |u^2(\tau)|,$$

for all $t \geq 0$. From this, the result follows by standard arguments. \triangleleft

We are now ready to prove the main result of the paper.

Proposition 3 *Consider system (21) with initial conditions in $P \times W \times Z \times \Xi \times Y$. Suppose assumptions (A0)-(A3) hold. Let κ be chosen as indicated in Lemma 1 and k as indicated in Proposition 2 and in Lemma 2. Then*

$$\lim_{t \rightarrow \infty} e(t) = 0.$$

Proof. As already mentioned, system (21) can be written in the form (36). Moreover, if the initial condition of (21) is taken in the set $P \times W \times Z \times \Xi \times Y$, the corresponding initial condition of (36) is in $\mathcal{P} \times E$. In view of the result of Lemma 2, item (ii), the result is proven if we are able to show that, given any pair of numbers δ_1 and δ_2 , there is a time \bar{t} such that

$$|p(\bar{t})|_{\mathcal{A}} \leq \delta_1, \quad |e(\bar{t})| \leq \delta_1 \quad (39)$$

and

$$|k\tilde{e}(t)| \leq \delta_2, \quad \forall t \geq \bar{t}. \quad (40)$$

To this end, recall that by definition

$$\tilde{e} = \hat{e} - e = -y_r(w_d) + y_r(w).$$

Hence, since $y_r(\cdot)$ is continuous, in view of Proposition 1 we have that $\lim_{t \rightarrow \infty} \tilde{e}(t) = 0$. As a consequence, there is a time t^* , dependent on δ_2 , such that (40) is fulfilled for all $\bar{t} \geq t^*$ (note that the coefficient k - which is possibly a large number - is now fixed). Thus the only critical issue is to make sure that (39) holds for some $\bar{t} \geq t^*$. To this end, one can use the following argument, suggested in [6].

Consider a system

$$\dot{x} = f(x) + u(t) \quad (41)$$

in which $f(\cdot)$ is locally Lipschitz and $u(t)$ is a piecewise-continuous function. Let $x(t, t_0, x_0, u)$ denote the integral curve passing through x_0 at time $t = t_0$. Suppose $u(t)$ satisfies

$$\lim_{t \rightarrow \infty} u(t) = 0,$$

and that, for a given x_0 and a given $T > 0$, there is a compact set X such that

$$x(t, 0, x_0, u) \in X \quad \forall t \geq 0, \quad (42)$$

$$x(t, \ell T, x(\ell T, 0, x_0, u), 0) \in X \quad \forall t \geq \ell T, \quad \forall \ell \in \mathbb{N}. \quad (43)$$

Claim 3: for any $t_2 > 0$ and any $\delta > 0$, there is a ℓ^* such that, for all $\ell \geq \ell^*$,

$$|x(t_2 + \ell T, 0, x_0, u) - x(t_2 + \ell T, \ell T, x(\ell T, 0, x_0, u), 0)| \leq \delta. \quad \triangleleft \quad (44)$$

To prove this claim, set $x(t) = x(t, 0, x_0, u)$ and $\hat{x}(t) = x(t, \ell T, x(\ell T, 0, x_0, u), 0)$. For all $t \geq \ell T$, we have

$$x(t) = x(\ell T) + \int_{\ell T}^t f(x(s))ds + \int_{\ell T}^t u(s)ds$$

and

$$\hat{x}(t) = x(\ell T) + \int_{\ell T}^t f(\hat{x}(s))ds$$

Since $f(\cdot)$ is locally Lipschitz and X is compact, there is L such that $|f(x) - f(y)| \leq L|x - y|$ for all x and y in X . Then

$$|x(t) - \hat{x}(t)| \leq L \int_{\ell T}^t |x(s) - \hat{x}(s)| ds + \int_{\ell T}^t |u(s)| ds.$$

Since $u(t)$ converges to 0 as $t \rightarrow \infty$, given any ε there exists T_ε such that $|u(t)| \leq \varepsilon$ for all $t \geq T_\varepsilon$. For any $\ell T \geq T_\varepsilon$ we have

$$|x(t) - \hat{x}(t)| \leq L \int_{\ell T}^t |x(s) - \hat{x}(s)| ds + \varepsilon(t - \ell T),$$

from which Gronwall-Bellman's lemma yields

$$|x(t) - \hat{x}(t)| \leq \frac{\varepsilon}{L}(e^{L(t-\ell T)} - 1)$$

with $t \geq \ell T$, hence

$$|x(\ell T + t_2) - \hat{x}(\ell T + t_2)| \leq \frac{\varepsilon}{L}(e^{Lt_2} - 1).$$

Given any $t_2 > 0$ and $\delta > 0$, set $\varepsilon = \delta L/(e^{Lt_2} - 1)$ and choose ℓ^* so that $\ell^* T \geq T_\varepsilon$. This proves Claim 3.

With this result at hand, it is easy to complete the proof of the Proposition. To this end, we set $x = \text{col}(p, e)$, and identify system (37) with a system of the form

$$\dot{x} = f(x) \tag{45}$$

and system (36) with a system of the form (41). The assumptions under which the previous claim holds are satisfied, with X taken as the set of all (p, e) such that $p \in V^{-1}([0, a+1])$ and $|e| \leq c+1$. In fact, the proof of Proposition 2 shows that trajectories of (36) are contained in this set X , i.e. that condition (42) holds. The same proof also shows that at each time ℓT , where T is the sampling interval (of the over-sampling period according to the discussion at the end of the previous subsection), trajectories of (36) are such that $p \in V^{-1}([0, a])$ and $|e| \leq c$. This and the fact that system (37) coincides with (36) with $u = 0$ in turn guarantees, by Proposition 2, that also condition (43) holds. Since $x(t, 0, x(\ell T, 0, x_0, u), 0)$ is a solution of (37) with initial condition in $\mathcal{P} \times E$, we know from Lemma 2, item (i), that – given any $\delta_1 > 0$ – there exists t_2 such that the p and e components of $x(t_2, 0, x(\ell T, 0, x_0, u), 0)$ satisfy

$$|p(t_2)|_{\mathcal{A}} \leq \frac{\delta_1}{2}, \quad |e(t_2)| \leq \frac{\delta_1}{2}.$$

Using now the Claim 3 with $\delta = \delta_1/2$ and the fact that

$$x(t_2 + \ell T, \ell T, x(\ell T, 0, x_0, u), 0) = x(t_2, 0, x(\ell T, 0, x_0, u), 0),$$

we deduce that the p and e components of $x(t_2 + \ell T, 0, x_0, u)$ satisfy, for all $\ell \geq \ell^*$

$$|p(t_2 + \ell T)|_{\mathcal{A}} \leq \delta_1, \quad |e(t_2 + \ell T)| \leq \delta_1.$$

This is what was needed to complete the proof of the Proposition. \triangleleft

6 Simulation Results

We consider the problem of synchronizing two oscillators located at remote places through a constrained communication channel. The master oscillator (playing the role of exosystem) is a Van der Pol oscillator described by

$$\begin{aligned}\dot{w}_1 &= w_2 + \epsilon(w_1 + aw_1^3) \\ \dot{w}_2 &= -w_1\end{aligned}\tag{46}$$

whose output $y_{\text{ref}} = w_2$ must be replied by the output y of a remote system of the form

$$\dot{y} = u.\tag{47}$$

Simple computations show that, in this specific case, the steady state control input u_{ss} coincides with $u_{\text{ss}} = -w_1$ and the assumption (A3) is satisfied by

$$\begin{aligned}\dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= f(\xi_1, \xi_2) \\ u_{\text{ss}} &= \xi_1\end{aligned}$$

where $f(\xi_1, \xi_2) = -\xi_1 - \epsilon(\xi_2 - 3a\xi_1^2\xi_2)$ through the map

$$\tau(w) = \begin{pmatrix} -w_1 & -w_2 + \epsilon(w_1 - aw_1^3) \end{pmatrix}^T$$

We consider a Van der Pol oscillator with $\epsilon = 1.5$ and $a = 1$. The regulator (20) is tuned choosing $\kappa = 3$, $G = (12 \ 36)^T$ and $k = 8$. We consider two different simulative scenarios which differ for the severity of the communication channel constraint. In the first case we suppose that the number of available bits is $N_b = 2$ yielding, according to (6) and to the fact that $r = 2$, $N = 2$. In this case, for a certain set of initial conditions, condition (9) is fulfilled with $T = 0.15$ s. In the second case the available number of bits is assumed $N_b = 4$ from which (6) and (9) yield a bigger N and T respectively equal to $N = 4$ and $T = 0.5$ s. The simulation results, obtained assuming the exosystem (46) and the system (47) respectively at the initial conditions $w(0) = (1, 0)$ and $y(0) = 5$, are shown in the figures 1-5 for the first scenario and figures 2-6 for the second one. In particular figure 1 (respectively 2) shows the quantized variable w_q transmitted from the encoder to the decoder and used to reset the respective dynamics according to the rule described in Section 3. Note that in the first scenario represented in figure 1 each of the two components of the vector w_q , taking value in the set $\{-1/2, 1/2\}$, can be transmitted using 1 bit. On the other hand in the second scenario, represented in figure 2, the transmission of each component of w_q , whose value are in the set $\{-3/2, -1/2, 1/2, 3/2\}$ requires 2 bits. Figure 3 (respectively 4) shows in the left-half side the error, as a function of time, between the exosystem and the encoder (decoder) state and in the right-half side the phase portrait of the Van der Pol oscillator with overlapped the actual state trajectory of the encoder (decoder). Finally figure 5 (respectively 6) plots the tracking error $e(t) = y(t) - y_r(t)$ on the left-half side and the control input $u(t)$ on the right-half side from which it is possible to see that in both the control scenarios the synchronization between (46) and (47) is achieved.

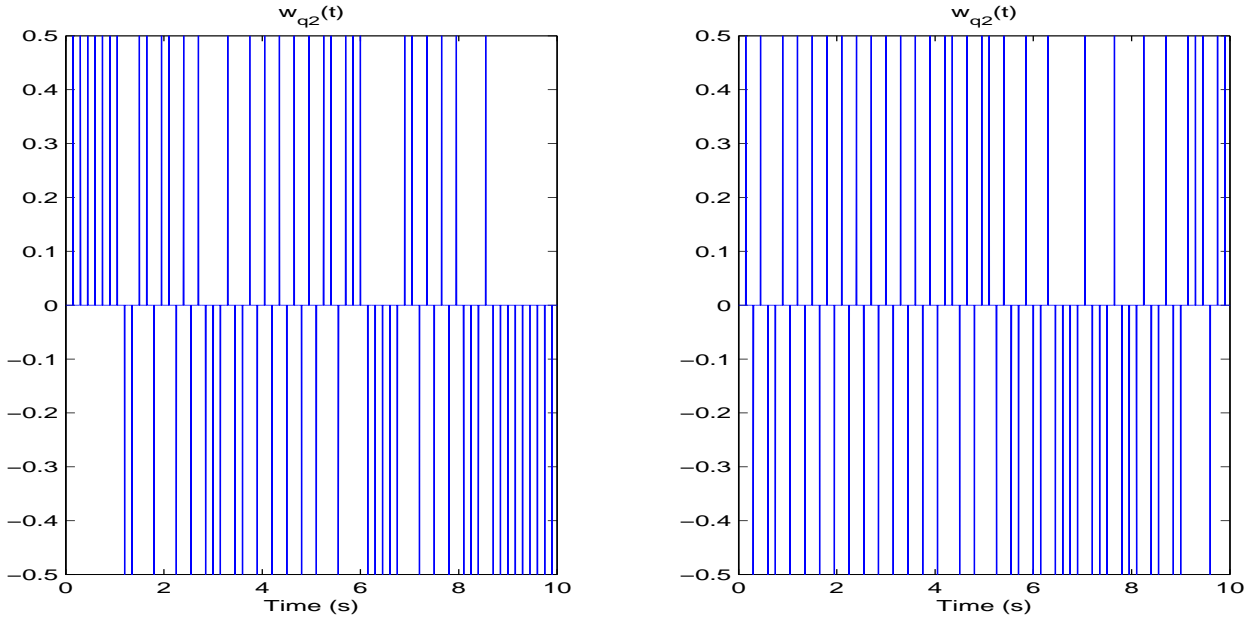


Figure 1: First control scenario ($N = 2$, $T = 0.15$ s): behavior of the encoded variables $w_{q1}(t)$ (left) and $w_{q2}(t)$ (right).

7 Conclusions

We have discussed the problem of asymptotically tracking a reference signal which is generated by a remote exosystem, and transmitted through a finite bandwidth communication channel. Although only an estimate of the actual tracking error is available to the regulator, a suitable choice of the controller parameters allows us to achieve the control goal while fulfilling the constraint on the bandwidth of the channel.

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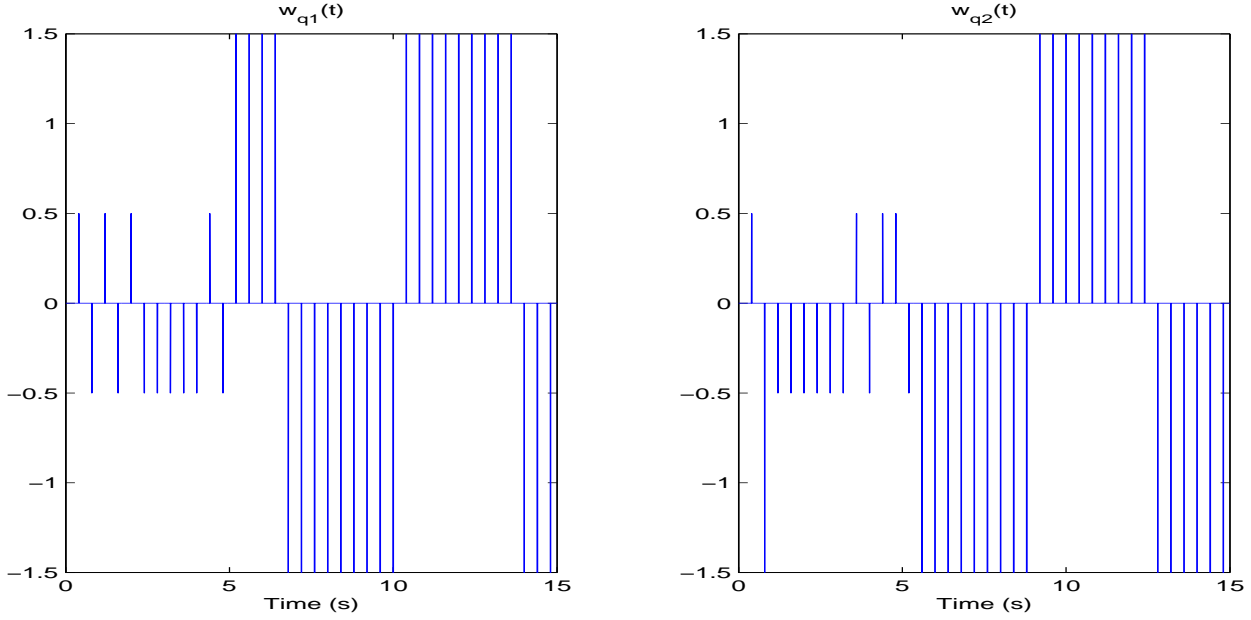


Figure 2: Second control scenario ($N = 4$, $T = 0.5$ s): behavior of the encoded variables $w_{q1}(t)$ (left) and $w_{q2}(t)$ (right).

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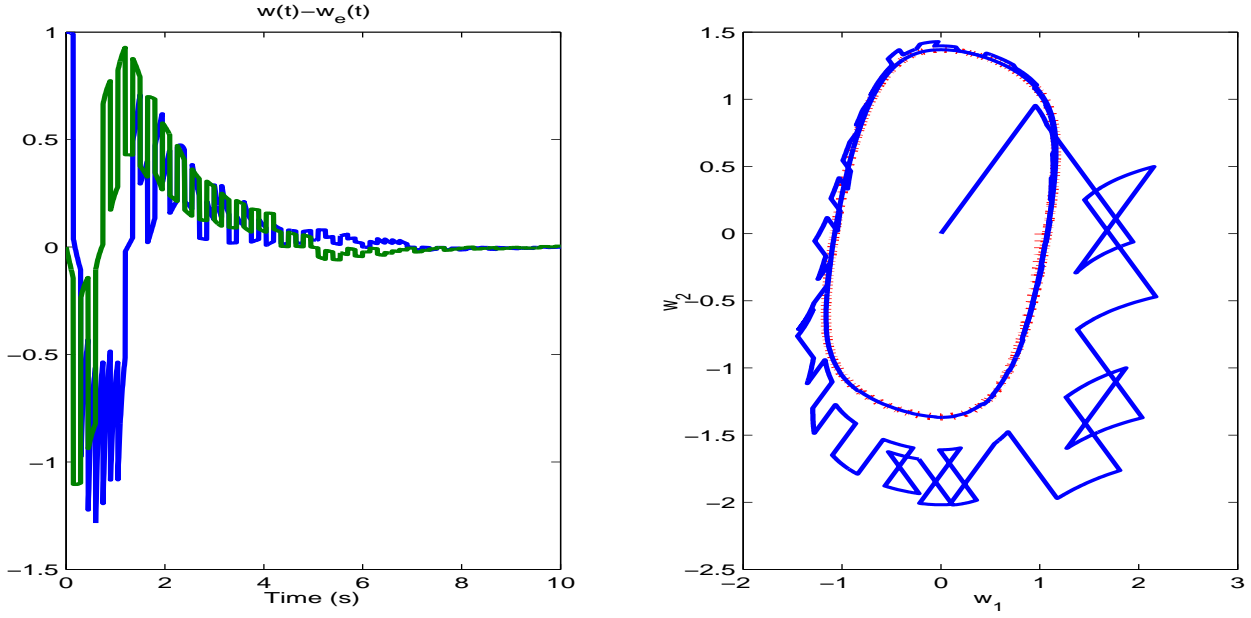


Figure 3: First control scenario ($N = 2$, $T = 0.15$ s). Left: time behavior of $w(t) - w_e(t)$ ($w(t) - w_d(t)$). Right: phase portrait of the exosystem (dotted line) and trajectory (w_{e1}, w_{e2}) (solid line).

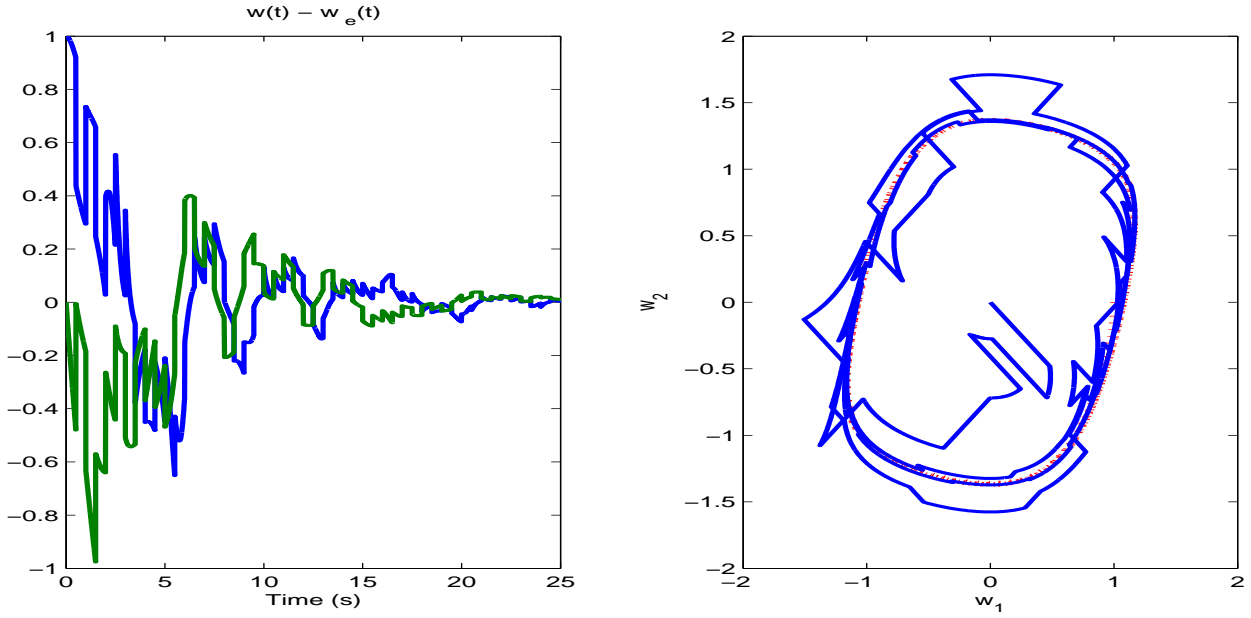


Figure 4: Second control scenario ($N = 4$, $T = 0.5$ s). Left: time behavior of $w(t) - w_e(t)$ ($w(t) - w_d(t)$). Right: phase portrait of the exosystem (dotted line) and trajectory (w_{e1}, w_{e2}) (solid line).

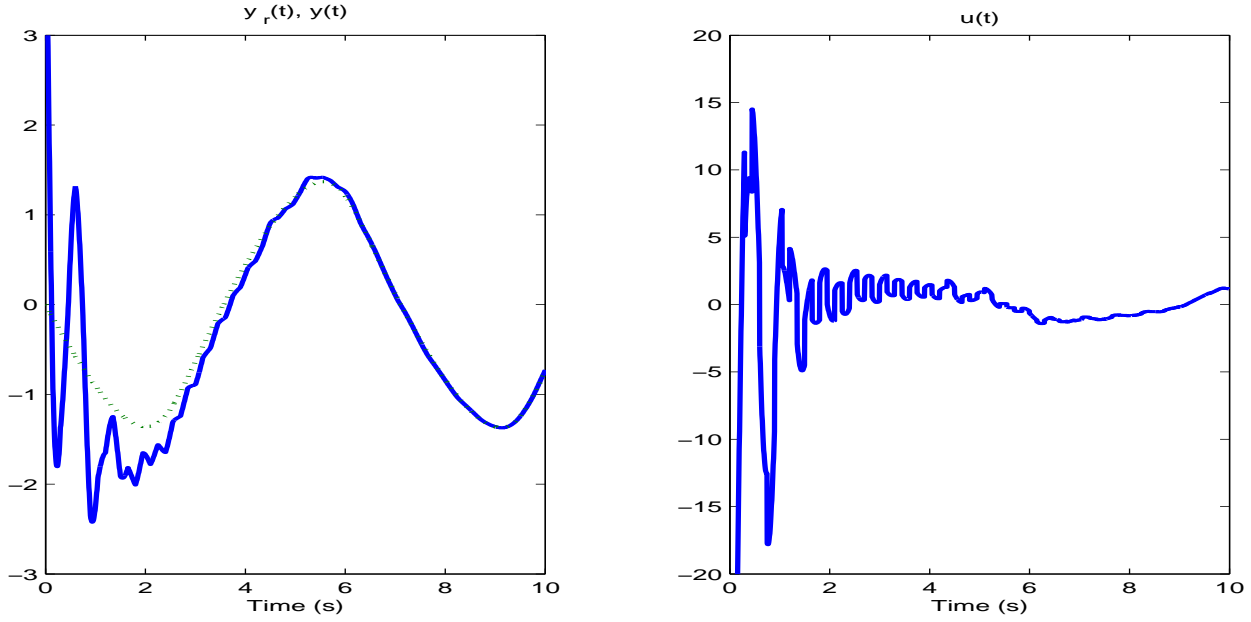


Figure 5: First control scenario ($N = 2$, $T = 0.15$ s). Left: time behavior of the reference trajectory $y_r(t)$ (dotted line) and of the controlled output $y(t)$ (solid line). Right: time behavior of the control input $u(t)$.

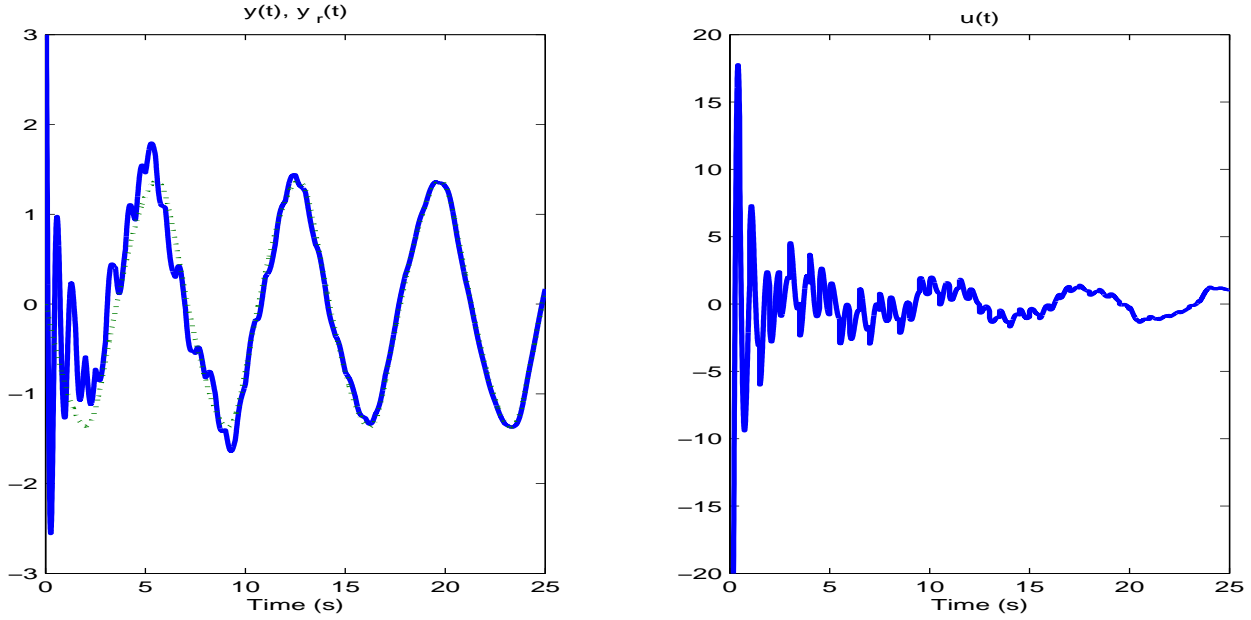


Figure 6: Second control scenario ($N = 4$, $T = 0.5$ s). Left: time behavior of the reference trajectory $y_r(t)$ (dotted line) and of the controlled output $y(t)$ (solid line). Right: time behavior of the control input $u(t)$.