Regularity issues for the null-controllability of the linear 1-d heat equation

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Abstract

The fact that the heat equation is controllable to zero in any bounded domain of the euclidean space, any time $T > 0$ and from any open subset of the boundary is well known. On the other hand, numerical experiments show the ill-posedness of the problem. In this paper we develop a rigorous analysis of the $1-$d problem which provides a sharp description of this ill-posedness.

To be more precise, to each initial data $y^0 \in L^2(0,1)$ of the $1-$d linear heat equation it corresponds a boundary control of minimal $L^2(0,T)-$norm which drives the state to zero in time $T > 0$. This control is given by a solution of the homogeneous adjoint equation with some initial data $\widehat{\varphi}^0$, minimizing a suitable quadratic cost. Our aim is to study the relation between the regularity of y^0 and that of $\widehat{\varphi}^0$. We show that there are regular data y^0 for which the corresponding $\widehat{\varphi}^0$ are highly irregular, not belonging to any negative exponent Sobolev space. Moreover, the class of such initial data y^0 is dense in $L^2(0,1)$. This explains the severe ill-posedness of the numerical algorithms developed for the approximation of the minimal $L^2(0,T)-$norm control of y^0 based on the computation of $\widehat{\varphi}^0$. The lack of polynomial convergence rates for Tychonoff regularization processes is a consequence of this phenomenon too.

Key words: Heat equation, Null-control, Ill-posedness, Moment problem, Biorthogonal family, Regularity.

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1. Introduction

Given $T > 0$ arbitrary, $y^0 \in L^2(0,1)$ and $v \in L^2(0,T)$, we consider the following non-homogeneous 1-d heat equation

$$\begin{cases} y_t(t,x) - y_{xx}(t,x) = 0 & x \in (0,1),\ t \in (0,T) \\ y(t,0) = 0, \quad y(t,1) = v(t) & t \in (0,T) \\ y(0,x) = y^0(x) & x \in (0,1). \end{cases} \tag{1}$$

In (1) $y = y(t,x)$ is the state and $v = v(t)$ is the control function which acts on the extreme $x = 1$. We aim at changing the dynamics of the system by acting on the boundary of the domain $(0,1)$. More precisely, we say that (1) is *boundary null-controllable (or controllable to zero) in time* T if for each $y^0 \in L^2(0,1)$ there exists $v \in L^2(0,T)$ such that the corresponding solution of (1) verifies

$$y(T,x) = 0 \quad \forall\, x \in (0,1). \tag{2}$$

There is an extensive literature in this subject. The reader is referred to [4, 5, 9] and to the more recent survey article [16].

In the present article we address this control problem in the frame developed in [4, 5] where it is reduced to a moment problem which is solved by constructing a biorthogonal sequence to the family of exponential functions $\Lambda = (e^{-\lambda_n t})_{n\geq 1}$, where λ_n are the eigenvalues of the Dirichlet Laplace operator in $(0,1)$, entering in the Fourier expansion of solutions.

When a system is controllable, controls are not unique. Often the control is chosen according to some optimality criterium. Normally, this is accompanied by a systematic method of constructing a uniquely defined control. We thus analyze in this paper the most common control, the one of minimal $L^2(0,T)$-norm. These controls not only are optimal from the viewpoint of their $L^2(0,T)$-norm, but they can also be characterized and constructed easily through the adjoint system and a minimization argument. In order to fix some notation, let us briefly describe how these controls are obtained.

Given $T > 0$ and $\varphi^0 \in L^2(0,1)$ we consider the adjoint heat equation

$$\begin{cases} \varphi_t + \varphi_{xx} = 0 & x \in (0,1),\ t \in (0,T) \\ \varphi(t,0) = \varphi(t,1) = 0 & t \in (0,T) \\ \varphi(T,x) = \varphi^0(x) & x \in (0,1). \end{cases} \tag{3}$$

In view of the regularizing properties of the heat equation, the map

$$\varphi^0 \longrightarrow \int_0^T (\varphi_x)^2(t,1)dt \tag{4}$$

which is well defined and continuous in some Sobolev space (for instance $H_0^1(0,1)$), by unique continuation, is a norm in $L^2(0,1)$ (see, for instance, [15] and the references therein).

Furthermore, it is by now well known that the following so-called observability inequality holds for all $T>0$: There exists a constant $C(T)>0$ such that every solution of (3) satisfies:

$$||\varphi(\cdot,0)||^2_{L^2(0,1)} \leq C(T)\int_0^T (\varphi_x)^2(t,1)dt. \tag{5}$$

We define the Hilbert space $\mathcal{H}$ as the completion of $L^2(0,1)$ with respect to norm (4). Now, we introduce the functional $\mathcal{J}:\mathcal{H}\to\mathbb{R}$ given by

$$\mathcal{J}(\varphi^0)=\frac{1}{2}\int_0^T |\varphi_x|^2(t,1)dt-\int_0^1 y^0(x)\varphi(0,x)dx, \tag{6}$$

where φ is the solution of (3) with initial data φ^0.

It is easy to see that, for all $y^0\in L^2(0,1)$, $\mathcal{J}$, in view of (5), is coercive in $\mathcal{H}$ and it has a unique minimizer $\widehat{\varphi}^0\in\mathcal{H}$. Moreover, the solution $\widehat{\varphi}$ of (3) with initial data $\widehat{\varphi}^0$ gives the control of minimal $L^2(0,T)$-nom of (1), as the following well known result guarantees (see, for instance [11, 15]).

Proposition 1.1. *Let $T>0$ be given. For each $y^0\in L^2(0,1)$ there exists a unique control $u\in L^2(0,T)$ for equation (1) such that*

$$u(t)=\widehat{\varphi}_x(t,1)\quad t\in(0,T), \tag{7}$$

where $\widehat{\varphi}$ is the solution of the adjoint problem (3) with initial data $\widehat{\varphi}^0\in\mathcal{H}$, the minimizer of (6). Moreover, the map $\mathcal{G}:L^2(0,1)\to\mathcal{H}$, defined by

$$\mathcal{G}(y^0)=\widehat{\varphi}^0, \tag{8}$$

is linear and continuous.

The operator $\mathcal{G}$ from Proposition 1.1 is usually called *the HUM (Hilbert Uniqueness Method) operator* and the control u is referred to as *the HUM control*. It has the minimal $L^2(0,T)$ norm among all the admissible controls for (1). This important property makes the HUM control very desirable. However, as reported in [1, 12], the minimization of (6) is severely ill-posed. This comes from the fact that the space $\mathcal{H}$ is very large. In fact, due to the regularizing effect of the heat equation, one can see that any distribution in a negative order Sobolev space $H^{-s}(0,1)$, with support away from $x=1$, belongs to $\mathcal{H}$, whatever $s > 0$ is. Therefore, for a given $y^0 \in L^2(0,1)$, the minimizer of $\mathcal{J}$ may have very low regularity and it may be difficult to capture it numerically with accuracy and robustness.

The aim of this article is to investigate the regularity of the minimizer $\mathcal{G}(y^0) = \widehat{\varphi}^0 \in \mathcal{H}$ when $y^0 \in L^2(0,1)$. We show that, even for a regular initial data y^0, the corresponding minimizer $\mathcal{G}(y^0)$ of $\mathcal{J}$ may not belong to any Sobolev space of negative exponent. For instance, we prove that, when the initial datum y^0 to be controlled is a sinusoidal function ($y^0(x) = \sin(n\pi x)$), the Fourier coefficients of the corresponding control, $\mathcal{G}(\sin(n\pi x))$, grow exponentially for high frequencies. Moreover, we show that the set of initial data $y^0 \in L^2(0,1)$ with such property is dense in $L^2(0,1)$. These results are based on precise estimates for the Fourier coefficients of $\mathcal{G}(y^0)$, obtained by using the minimal norm biorthogonal family to the sequence of exponential functions $\Lambda = (e^{-\lambda_n t})_{n\geq 1}$ in $L^2(0,T)$, entering in the Fourier expansion of the solutions of the state and adjoint systems.

These low regularity properties explain why, in practice, the problem of minimizing (6) is ill-posed and why it is difficult to compute numerically with efficiency the control of minimal $L^2(0,T)-$norm for (1).

One of the most frequent cures for ill-posed problems is the Tychonoff regularization technique which guarantees convergence towards the minimizer and gives polynomial convergence rates, with respect to the regularization parameter, under appropriate regularity hypotheses on the minimizer (see, for instance, [6, 13]). This can also be done in our context, showing that the minimizer of a Tychonoff regularized functional converges towards the minimizer of $\mathcal{J}$ with a polynomial rate provided the last one has some Sobolev regularity. But, since we have proved that our minimizer may have very low regularity, no convergence rate can be established and then, eventually, the Tychonoff regularization technique will be inefficient to compute the control.

We point out that this phenomenon of ill-posedness occurs at the level of the continuous heat equation. It is compatible with the fact that observability properties of semi-discrete or fully-discrete approximation schemes of the heat equation are uniform with respect to the discretization parameters (see, for instance, [8, 10, 15]) but makes impact on the effective computation of controls, thus making it very difficult in practice.

In [3] it is proved that, in the context of time-reversible infinite dimensional systems, the controls of minimal L^2-nom inherit the regularity of the initial data to be controlled. Our results show that this important property is not true for the heat equation (1).

The rest of the paper is organized as follows. Section 2 is devoted to characterize the HUM controls of (1) and the Fourier coefficients of the minimizer $\mathcal{G}(y^0)$ of (6) by means of a biorthogonal sequence to the family Λ. In section 3 we estimate the elements of the inverse of the Gramm matrix corresponding to Λ, by analyzing three different cases. Finally, in the last section, we discuss the main consequences of the estimates mentioned above in the context of control of (1).

2. The moment problem and the HUM control

The following characterization of the boundary null-controllability property of (1) is well-known (see, for instance, [5, 11]).

Proposition 2.1. *Equation (1) is null-controllable in time $T > 0$ if and only if, for any $y^0 \in L^2(0,1)$ with Fourier expansion*

$$y^0(x) = \sum_{n\geq 1} a_n \sin(\pi n x), \tag{9}$$

there exists a function $w \in L^2(0,T)$ such that,

$$\int_0^T w(t)\, e^{-n^2\pi^2 t} dt = (-1)^n \frac{a_n}{2n\pi} e^{-n^2\pi^2 T} \quad n \in \mathbb{N}^*. \tag{10}$$

If $w \in L^2(0,T)$ verifies (10), the function $v(t) = w(T-t)$ is a boundary control for (1). Problem (10) is usually referred to as a *moment problem* since we are looking for a function w whose moments with respect to the exponentials $e^{-n^2\pi^2 t}$, $n \in \mathbb{N}^*$, are given by the right hand side data in (10).

Let us introduce some notation. The eigenvalues of the 1-d Dirichlet Laplace operator are $\lambda_n = n^2\pi^2$ and the corresponding eigenfunctions $\Phi^n = \sin(n\pi x)$, for every $n \in \mathbb{N}^*$. $\Lambda = \left(e^{-\lambda_n t}\right)_{n\geq 1}$ denotes the family of the corresponding real exponential functions.

For any $T > 0$, let $E(\Lambda, T)$ be the space generated by Λ and $E(m, \Lambda, T)$ be the subspace generated by $\left(e^{-\lambda_n t}\right)_{\substack{n\geq 1\\ n\neq m}}$ in $L^2(0,T)$. Also, we introduce the notation $p_T^n : [0,T] \to \mathbb{R}$, $p_T^n(t) = e^{-\lambda_n t}$. We recall that

Definition 2.1. $(\theta_T^m)_{m\geq 1}$ *is a* **biorthogonal sequence** *to* Λ *in* $L^2(0,T)$ *if*

$$\int_0^T p_T^n(t)\theta_T^m(t)dt = \delta_{nm} \quad \forall n, m \in \mathbb{N}^*.$$

The existence of a biorthogonal sequence to the family Λ is a consequence of the following Theorem (see, for instance, [14]).

Theorem 2.1. *(Müntz) Let $0 < \lambda_1 < \lambda_2 < ... < \lambda_n < ...$ be a sequence of real numbers and $T \in (0, \infty)$. The family of exponential functions $\left(e^{-\lambda_n t}\right)_{n\geq 1}$ is complete in $L^2(0,T)$ if and only if*

$$\sum_{n\geq 1} \frac{1}{\lambda_n} = \infty. \tag{11}$$

Since in our case $\lambda_n = \pi^2 n^2$ and (11) is not verified, it follows from Müntz's Theorem, that $E(\Lambda, T)$ and $E(m, \Lambda, T)$ are proper subspaces of $L^2(0,T)$. Consequently, $E(\Lambda, T)$ is minimal (each element of $E(\Lambda, T)$ lies outside the closed subspace spanned by the others). Thus, for each $m \geq 1$, $p_T^m \notin E(m, \Lambda, T)$ (see [14], p. 23). Let r_T^m be the orthogonal projection of p_T^m over the space $E(m, \Lambda, T)$ and define

$$\theta_T^m(t) = \frac{1}{||p_T^m - r_T^m||^2_{L^2(0,T)}} \left(p_T^m(t) - r_T^m(t)\right). \tag{12}$$

The following result may be found in [4, 5].

Theorem 2.2. *For any $T > 0$, the sequence $(\theta_T^m)_{m\geq 1}$ given by (12) is the unique biorthogonal to the family Λ in $L^2(0,T)$ such that*

$$(\theta_T^m)_{m\geq 1} \subset E(\Lambda, T). \tag{13}$$

Moreover, this biorthogonal sequence has minimal $L^2(0,T)$-norm.

The following theorem establishes the relation between the HUM control (7) and the biothogonal with minimal norm $(\theta_T^m)_{m\geq 1} \subset E(\Lambda, T)$ to the family Λ in $L^2(0,T)$ given by Theorem 2.2.

Theorem 2.3. *Let $y^0 \in L^2(0,1)$ be given by (9) and suppose that the following series is convergent in $L^2(0,T)$*

$$u(t) = \sum_{n=1}^{\infty} (-1)^n \frac{a_n}{2n\pi} e^{-n^2\pi^2 T} \theta_T^n(T-t), \tag{14}$$

where $(\theta_T^m)_{m\geq 1} \subset E(\Lambda, T)$ is the minimal norm biorthogonal to the family Λ in $L^2(0,T)$ given by Theorem 2.2. Then $u \in L^2(0,T)$ is the HUM control (7) corresponding to initial data y^0 of equation (1).

Proof: By using the biorthogonal properties, it is easy to see that $w(t) = u(T-t)$ verifies (10). Hence, u is a control for (1).

In order to prove that u is the HUM control it is sufficient to show that it has the minimal $L^2(0,T)-$norm. This is true if the Fourier coefficients of y^0 are all zero, except a_{n_0} which is not zero. Indeed, in this case

$$u = (-1)^{n_0} \frac{a_{n_0}}{2n_0\pi} e^{-{n_0}^2\pi^2 T} \theta_T^{n_0}$$

and the minimality of its norm follows from that of the biorthogonal $(\theta_T^m)_{m\geq 1}$. This fact together with linearity of the map which assigns to each initial data y^0 its HUM control completes the proof. ■

We recall that, in [4, 5], it is proved that (14) absolutely converges in $L^2(0,T)$ for any $(a_n)_{n\geq 1}$ such that $\sum_{n=1}^{\infty} |a_n| e^{-\delta n^2} < \infty$, with δ a positive number depending only of T.

Since u given by Theorem 2.3 is the HUM control corresponding to y^0, then $u = \widehat{\varphi}_x(\,\cdot\,,1)$, where $\widehat{\varphi}$ is the solution of (3) with initial data $\widehat{\varphi}^0 = \mathcal{G}(y^0)$ given by Proposition 1.1. Our aim is to study the regularity of $\widehat{\varphi}^0$. A possibility to do this consists in expanding $\widehat{\varphi}^0$ in Fourier series

$$\mathcal{G}(y^0) = \widehat{\varphi}^0(x) = \sum_{m\geq 1} b_m \sin(\pi m x) \tag{15}$$

and analyzing the behavior of the Fourier coefficients b_m. The following characterization of b_m holds.

Theorem 2.4. *Let $y^0 \in L^2(0,1)$ be defined as in (9). The Fourier coefficients b_m of $\mathcal{G}(y^0)$ from (15) are given by*

$$b_m = \frac{(-1)^m}{m\pi} \sum_{n=1}^{\infty} (-1)^n \frac{a_n}{2n\pi} e^{-n^2\pi^2 T} \langle \theta_T^n, \theta_T^m \rangle_{L^2(0,T)}. \tag{16}$$

Moreover, the infinite matrix $\left(\langle \theta_T^n, \theta_T^m \rangle_{L^2(0,T)}\right)_{n,m\geq 1}$ verifies

$$\sum_{k\geq 1} \langle \theta_T^m, \theta_T^k \rangle_{L^2(0,T)} \langle e^{-\lambda_k t}, e^{-\lambda_n t} \rangle_{L^2(0,T)} = \delta_{mn} \quad \forall m,n \geq 1. \tag{17}$$

Proof: If $\widehat{\varphi}$ is the solution of (1) with initial data $\widehat{\varphi}^0$ given by (15) and $u(t) = \widehat{\varphi}_x(t,1)$, the following relation is obtained from (14)

$$\sum_{m\geq 1} (-1)^m m\pi b_m e^{-\lambda_m t} = \sum_{n=1}^{\infty} (-1)^n \frac{a_n}{2n\pi} e^{-n^2\pi^2 T} \theta_T^n(t). \tag{18}$$

From (18) and the orthogonality properties of θ_T^m we deduce that (16) holds.

For the second part, note that θ_T^m, being the biorthogonal of minimal norm, it belongs to $E(\Lambda, T)$. Therefore, for each $m \in \mathbb{N}^*$, there exists a scalar sequence $(d_k^m)_{k\geq 1}$ such that (see [7], Theorem 8.2)

$$\theta_T^m(t) = \sum_{k\geq 1} d_k^m e^{-\lambda_k t}. \tag{19}$$

From (19) we deduce that

$$\langle \theta_T^m(t), \theta_T^k(t)\rangle_{L^2(0,T)} = d_k^m \tag{20}$$

and

$$\delta_{mn} = \langle \theta_T^m(t), e^{-\lambda_n t}\rangle_{L^2(0,T)} = \sum_{k\geq 1} d_k^m \langle e^{-\lambda_k t}, e^{-\lambda_n t}\rangle_{L^2(0,T)}.$$

This represents exactly (17) and completes the proof of the Theorem. ■

Remark 2.1. *The "infinite Gramm matrix" $G(T) = \left(\langle p_T^n, p_T^m\rangle_{L^2(0,T)}\right)_{n,m\geq 1}$ is an one-to-one linear operator in ℓ^2 but it is not invertible in ℓ^2.* □

In order to evaluate the Fourier coefficients b_m of $\mathcal{G}(y^0)$ we need to estimate the quantities d_k^n given by (20). In order to do that, we use a strategy similar to [4, 5], where the norms $||\theta_T^n||_{L^2(0,T)} = \sqrt{d_n^n}$ are evaluated. It consists in truncating the matrix G by considering only a finite number N of exponentials and extending the time interval to $(0, \infty)$.

Before explaining how the estimates for d_k^n are obtained, let us introduce some notation. For any $T \in (0, \infty]$ and $N \in \mathbb{N}^*$, $E_N(\Lambda, T)$ and $E_N(m, \Lambda, T)$ denote the subspaces generated in $L^2(0,T)$ by the finite families $\left(e^{-\lambda_k t}\right)_{1\leq k\leq N}$ and $\left(e^{-\lambda_k t}\right)_{\substack{1\leq k\leq N\\ k\neq m}}$ respectively. Note that $E_N(\Lambda, T)$ and $E_N(m, \Lambda, T)$ are finite dimensional subspaces and

$$E(\Lambda, T) = \bigcup_{N\geq 1} E_N(\Lambda, T), \quad E(m, \Lambda, T) = \bigcup_{N\geq 1} E_N(m, \Lambda, T).$$

As before, p_T^n denotes the exponential function $e^{-\lambda_n t}$ defined on the interval $[0, T]$ if $T < \infty$ or $[0, \infty)$ if $T = \infty$.

As in Theorem 2.2, there exists a unique biorthogonal $(\theta_{T,N}^m)_{1\leq m\leq N} \subset E_N(\Lambda, T)$ to the finite family of exponentials $\left(e^{-\lambda_k t}\right)_{1\leq k\leq N}$ given by

$$\theta_{T,N}^m = \frac{1}{||p_T^m - r_{T,N}^m||^2_{L^2(0,T)}}(p_T^m - r_{T,N}^m), \tag{21}$$

where $r^m_{T,N}$ is the orthogonal projection of p^m_T over $E_N(m, \Lambda, T)$. For each $m \in \mathbb{N}^*$, there exist some unique scalars $(d^m_k(T,N))_{1\leq k\leq N}$ such that

$$\theta^m_{T,N} = \sum_{k=1}^{N} d^m_k(T,N) p^k_T. \tag{22}$$

Let $G_N(T)$ denote the Gramm matrix of the family $\left(e^{-\lambda_k t}\right)_{1\leq k\leq N}$, i. e. the matrix of elements

$$g_{kl}(T,N) = \int_0^T p^k_T(t) p^l_T(t) dt, \quad 1 \leq k, l \leq N. \tag{23}$$

As in the final part of Theorem 2.4, we have the following characterization of the coefficients $d^m_k(T,N)$.

Theorem 2.5. *Let $T \in (0, \infty]$. The matrix $(d^m_k(T,N))_{1\leq k,m\leq N}$ given by (22) is the inverse of the Gramm matrix $G_N(T)$. Consequently, from Cramer's rule,*

$$d^m_k(T,N) = \frac{|G_{mk}(T)|}{|G_N(T)|} \tag{24}$$

where $|G_N(T)|$ is the determinant of matrix $G_N(T)$ and $|G_{mk}(T)|$ is the determinant of the matrix $G_{mk}(T)$ obtained by replacing the $m-$th column of $G_N(T)$ with the $k-$th vector of the canonical basis.

Now, let us briefly explain how the estimates for d^m_k may be obtained. If $T = \infty$, the determinants $|G_N(\infty)|$ and $|G_{mk}(\infty)|$, and consequently the coefficients $d^m_k(\infty, N)$ from Theorem 2.5, may be explicitly computed. Next, a perturbation argument and an extension theorem allow us to address the case $T < \infty$ and to deduce estimates for $d^m_k(T,N)$. Finally, by letting $N \to \infty$, we obtain the desired estimates for d^m_k. This is the strategy used in the following section.

3. Estimates for d^m_k

As we have said before, we study successively $d^m_k(\infty, N)$, $d^m_k(T,N)$ and finally d^m_k.

3.1. Analysis of the case $T = \infty$ and $N < \infty$

In this section we evaluate the quantities $d_k^m(\infty, N)$ from (24), Theorem 2.5. To compute the determinants $|G_N(\infty)|$ and $|G_{mk}(\infty)|$ we use the following lemma.

Lemma 3.1. *If $C = (c_{ij})_{1\le i,j\le N}$ is a matrix of coefficients $c_{ij} = \frac{1}{a_i+b_j}$ then*

$$|C| = \frac{\prod_{1\le j<i\le N}(a_i - a_j)(b_i - b_j)}{\prod_{1\le i,j\le N}(a_i + b_j)}. \tag{25}$$

Moreover, if C_{mk} denotes the matrix obtained by replacing the $m-th$ column of C by the $k-th$ vector of the canonical basis, then

$$|C_{mk}| = (-1)^{m+k} \frac{\prod'_{1\le j<i\le N}(a_i - a_j)(b_i - b_j)}{\prod'_{1\le i,j\le N}(a_i + b_j)}, \tag{26}$$

where $'$ means that the terms containing a_k and b_m have been skipped in the product.

Proof: For the first part, see [2]. For the second part, note that

$$|C_{mk}| = \lim_{b_m\to\infty} \lim_{a_k\to\infty} a_k|C|. \tag{27}$$

We have that

$$|C| = \frac{\prod_{1\le j<i\le n}(a_i - a_j)(b_i - b_j)}{\prod_{1\le i,j\le N}(a_i + b_j)} = \frac{\prod'_{1\le j<i\le N}(a_i - a_j)(b_i - b_j)}{\prod'_{1\le i,j\le N}(a_i + b_j)}$$

$$\times \frac{\prod_{1\le j<k}(a_k - a_j)\prod_{k<i\le N}(a_i - a_k)\prod_{1\le j<m}(b_m - b_j)\prod_{m<i\le N}(b_i - b_m)}{(a_k + b_m)\prod_{1\le j\le N,\ j\ne m}(a_k + b_j)\prod_{1\le i\le N,\ i\ne k}(a_i + b_m)}.$$

It follows that

$$|C| = \frac{\prod'_{1\le j<i\le N}(a_i - a_j)(b_i - b_j)}{\prod'_{1\le i,j\le N}(a_i + b_j)}$$

$$\times \frac{(-1)^{k+m}\prod_{1\le p\le N, p\ne k}(a_k - a_p)\prod_{1\le p\le N, p\ne m}(b_m - b_p)}{(a_k + b_m)\prod_{1\le j\le N,\ j\ne m}(a_k + b_j)\prod_{1\le i\le N,\ i\ne k}(a_i + b_m)}$$

and consequently

$$|C_{mk}| = \lim_{b_m\to\infty} \lim_{a_k\to\infty} a_k|C| = (-1)^{m+k} \frac{\prod'_{1\le j<i\le N}(a_i - a_j)(b_i - b_j)}{\prod'_{1\le i,j\le N}(a_i + b_j)}.$$

The proof of the Lemma is finished. ■

Now, we can pass to estimate the numbers $d_k^m(\infty, N)$.

Theorem 3.1. *If $(d_k^m(\infty,N))_{1\le k,m\le N}$ is the matrix in Theorem 2.5, then*

$$d_k^m(\infty,N) = \frac{4\,\pi^2\,k^2\,m^2}{k^2+m^2}\prod_{\substack{p=1\\p\neq k}}^{N}\frac{k^2+p^2}{k^2-p^2}\prod_{\substack{p=1\\p\neq m}}^{N}\frac{m^2+p^2}{m^2-p^2}. \tag{28}$$

Moreover, for each $k,m\ge 1$, the sequence $(|d_k^m(\infty,N)|)_{N\ge\max\{k,m\}}$ is increasing and

$$\lim_{N\to\infty} d_k^m(\infty,N) = (-1)^{m+k}\frac{k\,m}{k^2+m^2}\left(e^{\pi k}-e^{-\pi k}\right)\left(e^{\pi m}-e^{-\pi m}\right). \tag{29}$$

Proof: We have that

$$d_k^m(\infty,N)=\frac{|G_{mk}(\infty)|}{|G_N(\infty)|} \tag{30}$$

where $|G_N(\infty)|$ denotes the determinant of matrix $G_N(\infty)$ and $|G_{mk}(\infty)|$ the determinant of the matrix $G_{mk}(\infty)$, obtained by replacing the $m-$th column of $G_N(\infty)$ by the $k-$th vector of the canonical basis. In order to evaluate $|G_N(\infty)|$ and $|G_{mk}(\infty)|$ we remark that the elements $g_{ij}(\infty)$ of the matrix $G_N(\infty)$ are given by

$$g_{ij}(\infty) = \int_0^\infty p_\infty^i(t)p_\infty^j(t)dt = \frac{1}{(i^2+j^2)\pi^2} \quad 1\le i,j\le N.$$

Hence, we can use Lemma 3.1 to evaluate $|G_N(\infty)|$ and $|G_{mk}(\infty)|$. We deduce from (30) that

$$d_k^m(\infty,N) = (k^2+m^2)\pi^2\frac{\prod_{1\le p\le N,\ p\neq m}(k^2+p^2)\prod_{1\le p\le N,\ p\neq k}(m^2+p^2)}{\prod_{1\le p\le N,\, p\neq k}(k^2-p^2)\prod_{1\le p\le N,\, p\neq m}(m^2-p^2)}$$

from which (28) follows immediately.

For $k\ge 1$ and $N\ge k$, let us denote

$$S_N(k) = \prod_{\substack{p=1\\p\neq k}}^{N}\frac{k^2+p^2}{|k^2-p^2|} = \prod_{p=1}^{N}\left(1+\frac{k^2}{p^2}\right)\prod_{p=N-k+1}^{N}\frac{p}{p+k}. \tag{31}$$

It follows that

$$\frac{S_{N+1}(k)}{S_N(k)} = \left(1+\frac{k^2}{(N+1)^2}\right)\left(1+\frac{k^2}{(N+1)^2-k^2}\right) > 1$$

and the sequence $(|d_k^m(\infty, N)|)_{N \geq \max\{k,m\}}$ is increasing.

On the other hand, from Euler's formula,

$$\lim_{N\to\infty} S_N(k) = \prod_{p=1}^{\infty}\left(1 + \frac{k^2}{p^2}\right) = \frac{\sin(i\,k\pi)}{i\,k\pi}.$$

Therefore (29) holds and the proof of Theorem 3.1 is completed. ∎

3.2. Analysis of the case $T < \infty$ and finite dimension N

Let $T < \infty$ be given. In this section we give estimates for $d_k^m(T, N)$. Firstly, we recall the following result (see [4, 14]).

Theorem 3.2. *Let $T \in (0, \infty)$ and Λ be the family of exponential functions $\left(e^{-\lambda_n t}\right)_{n\geq 1}$, where $\lambda_n = n^2\pi^2$ are the eigenvalues of the Dirichlet Laplace operator in $(0, 1)$. The restriction operator*

$$R_T : E(\Lambda, \infty) \to E(\Lambda, T), \quad R_T(v) = v_{|[0,T]} \tag{32}$$

is invertible and there exists $C_T > 0$, depending only on T, such that

$$||R_T^{-1}|| \leq C_T. \tag{33}$$

Also, let us define the restriction

$$R_{T,N} : E_N(\Lambda, \infty) \to E_N(\Lambda, T), \quad R_{T,N}(v) = v_{|[0,T]} \tag{34}$$

and note that, if p_T^n and p_∞^n denote the function $e^{-\lambda_n t}$ defined in $[0, T]$ and $[0, \infty)$ respectively, then

$$R_T(p_\infty^n) = R_{T,N}(p_\infty^n) = p_T^n, \quad 1 \leq n \leq N. \tag{35}$$

Evidently, $R_{T,N}$ is invertible. Moreover, since

$$\delta_{mn} = \langle \theta_{\infty,N}^m, p_\infty^n \rangle_{L^2(0,\infty)} = \langle \theta_{\infty,N}^m, R_{T,N}^{-1}\, p_T^n \rangle_{L^2(0,\infty)} =$$

$$= \langle (R_{T,N}^{-1})^*\, \theta_{\infty,N}^m, p_T^n \rangle_{L^2(0,T)}$$

and $(R_{T,N}^{-1})^* \theta_{\infty,N}^m \in E_N(\Lambda, T)$, we deduce that

$$(R_{T,N}^{-1})^* \theta_{\infty,N}^m = \theta_{T,N}^m. \tag{36}$$

The following Theorem gives a first estimate for the elements $d_k^m(T, N)$ in Theorem 2.5.

Theorem 3.3. *If $(d_k^m(T,N))_{1\le k,m\le N}$ is the inverse of the Gramm matrix $G_N(T)$, then there exists a positive constant $C = C(T) > 0$, independent of N but depending of T, such that*

$$|d_k^m(T,N)| \le C\,\frac{k^2+m^2}{k\,m}|d_k^m(\infty,N)| \quad 1\le k,m\le N. \tag{37}$$

Proof: From (36) and (28) it follows that

$$|d_k^m(T,N)| = \left|\langle \theta_{T,N}^m, \theta_{T,N}^k\rangle\right| = \left|\left\langle (R_{T,N}^{-1})^*\theta_{\infty,N}^m, (R_{T,N}^{-1})^*\theta_{\infty,N}^k\right\rangle\right| \le$$

$$\le \left|\left|(R_{T,N}^{-1})^*\right|\right|^2 ||\theta_{\infty,N}^m||\,||\theta_{\infty,N}^k|| = \left|\left|(R_{T,N}^{-1})^*\right|\right|^2 \sqrt{d_m^m(\infty,N)}\,\sqrt{d_k^k(\infty,N)} =$$

$$= \left|\left|(R_{T,N}^{-1})^*\right|\right|^2 2\pi^2\ m\,k \prod_{\substack{p=1\\p\ne m}}^N \frac{m^2+p^2}{|m^2-p^2|}\prod_{\substack{p=1\\p\ne k}}^N \frac{k^2+p^2}{|k^2-p^2|} =$$

$$= \left|\left|(R_{T,N}^{-1})^*\right|\right|^2 \frac{k^2+m^2}{2\,k\,m}|d_k^m(\infty,N)|.$$

On the other hand, by using Theorem 3.2, we deduce that

$$\left|\left|(R_{T,N}^{-1})^*\right|\right|_{\mathcal{L}(E_N(\Lambda,\infty),E_N(\Lambda,T))} = \left|\left|R_{T,N}^{-1}\right|\right|_{\mathcal{L}(E_N(\Lambda,T),E_N(\Lambda,\infty))} \le$$

$$\le \left|\left|R_T^{-1}\right|\right|_{\mathcal{L}(E(\Lambda,T),E(\Lambda,\infty))} \le C_T$$

and the proof ends by taking $C = \frac{1}{2}C_T^2$. ■

The main estimate for $d_k^m(T,N)$ is given in the following Theorem.

Theorem 3.4. *If $(d_k^m(T,N))_{1\le k,m\le N}$ is the inverse of the Gramm matrix $G_N(T)$, there exists a positive integer $n_0 > 0$, independent of N but depending of T, such that*

$$|d_k^m(T,N)| \ge \frac{1}{2}|d_k^m(\infty,N)| \quad \forall m,k\ge n_0. \tag{38}$$

Proof: For each $n\in\{1,2,...,N\}$, we have that $(R_{T,N}^{-1})^*p_\infty^n \in E_N(\Lambda,T)$ and there exist scalars $(q_k^n)_{1\le k\le N}\subset\mathbb{R}$ such that

$$(R_{T,N}^{-1})^*p_\infty^n = \sum_{k=1}^N q_k^n p_T^k. \tag{39}$$

Moreover, the coefficients q_k^n verify

$$q_k^n = \left\langle (R_{T,N}^{-1})^* p_\infty^n, \theta_{T,N}^k \right\rangle_{L^2(0,T)} = \left\langle p_\infty^n, R_{T,N}^{-1}\theta_{T,N}^k \right\rangle_{L^2(0,\infty)} =$$

$$\int_0^T p_T^n(t)\theta_{T,N}^k(t)dt + \int_T^\infty p_\infty^n(t) R_{T,N}^{-1}\theta_{T,N}^k(t)dt = \delta_{nk} + \int_T^\infty e^{-\lambda_n t} R_{T,N}^{-1}\theta_{T,N}^k(t)dt$$

and therefore

$$q_k^n - \delta_{nk} = \int_T^\infty e^{-\lambda_n t} R_{T,N}^{-1}\theta_{T,N}^k(t)dt. \tag{40}$$

From (22), we deduce that

$$R_{T,N}^{-1}\theta_{T,N}^k = R_{T,N}^{-1}\left(\sum_{j=1}^N d_j^k(T,N) p_T^j\right) = \sum_{j=1}^N d_j^k(T,N) p_\infty^j$$

which, together with (40), gives

$$q_k^n - \delta_{nk} = \sum_{j=1}^N \frac{d_j^k(T,N)}{(n^2+j^2)\pi^2} e^{-\pi^2(n^2+j^2)T} \qquad 1 \le n,k \le N. \tag{41}$$

On the other hand, we have that

$$d_k^m(T,N) = \langle \theta_{T,N}^m, \theta_{T,N}^k\rangle_{L^2(0,T)} = \left\langle (R_{T,N}^{-1})^* \left(\sum_{i=1}^N d_i^m(\infty,N) p_\infty^i\right), \theta_{T,N}^k \right\rangle_{L^2(0,T)} =$$

$$= \sum_{i=1}^N d_i^m(\infty,N) \left\langle (R_T^{-1})^* p_\infty^i, \theta_{T,N}^k \right\rangle_{L^2(0,T)} = \sum_{i=1}^N d_i^m(\infty,N) q_k^i$$

and consequently

$$d_k^m(T,N) - d_k^m(\infty,N) = \sum_{i=1}^N d_i^m(\infty,N)(q_k^i - \delta_{ik}) \qquad 1 \le k,m \le N. \tag{42}$$

Now, using (42) and (41), we deduce that

$$|d_k^m(T,N) - d_k^m(\infty,N)| = \left| \sum_{i,j=1}^N \frac{e^{-\pi^2(i^2+j^2)T}}{(i^2+j^2)\pi^2} \, d_j^k(T,N)\, d_i^m(\infty,N) \right|. \tag{43}$$

From (43), estimate (37) and formulas (28), it follows that

$$|d_k^m(T,N) - d_k^m(\infty,N)| \le$$

$$\leq \frac{1}{2} C_T^2 \sum_{i,j=1}^{N} \frac{e^{-\pi^2(i^2+j^2)T}}{(i^2+j^2)\pi^2} \, \frac{j^2+k^2}{j\,k} \, |d_j^k(\infty,N)| \, |d_i^m(\infty,N)| =$$

$$= 8C_T^2 \pi^2 \prod_{\substack{p=1\\p\neq k}}^{N} \frac{k^2+p^2}{|k^2-p^2|} \prod_{\substack{p=1\\p\neq m}}^{N} \frac{m^2+p^2}{|m^2-p^2|} \times$$

$$\times \sum_{i,j=1}^{N} \frac{e^{-\pi^2(i^2+j^2)T}}{i^2+j^2} \, \frac{j^2+k^2}{j\,k} \, \frac{k^2\,j^2}{k^2+j^2} \frac{m^2\,i^2}{m^2+i^2} \prod_{\substack{p=1\\p\neq i}}^{N} \frac{i^2+p^2}{|i^2-p^2|} \prod_{\substack{p=1\\p\neq j}}^{N} \frac{j^2+p^2}{|j^2-p^2|}.$$

Hence,

$$\begin{aligned} |d_k^m(T,N) - d_k^m(\infty,N)| \leq 2C_T^2 \frac{m^2+k^2}{k\,m^2} \, |d_k^m(\infty,N)| \times \\ \sum_{i,j=1}^{N} \frac{e^{-\pi^2(i^2+j^2)T}}{i^2+j^2} \, \frac{j\,i^2\,m^2}{m^2+i^2} \prod_{\substack{p=1\\p\neq i}}^{N} \frac{i^2+p^2}{|i^2-p^2|} \prod_{\substack{p=1\\p\neq j}}^{N} \frac{j^2+p^2}{|j^2-p^2|}. \end{aligned} \tag{44}$$

By using (28)-(29), we deduce that

$$\sum_{i,j=1}^{N} \frac{e^{-\pi^2(i^2+j^2)T}}{i^2+j^2} \, \frac{j\,i^2\,m^2}{m^2+i^2} \prod_{\substack{p=1\\p\neq i}}^{N} \frac{i^2+p^2}{|i^2-p^2|} \prod_{\substack{p=1\\p\neq j}}^{N} \frac{j^2+p^2}{|j^2-p^2|} \leq$$

$$\sum_{i,j=1}^{N} \frac{j\,i^2\,e^{-\pi^2(i^2+j^2)T}}{i^2+j^2} \prod_{\substack{p=1\\p\neq i}}^{N} \frac{i^2+p^2}{|i^2-p^2|} \prod_{\substack{p=1\\p\neq j}}^{N} \frac{j^2+p^2}{|j^2-p^2|} = \sum_{i,j=1}^{N} \frac{e^{-\pi^2(i^2+j^2)T}}{4\pi^2 j} \left| d_i^j(\infty,N) \right| \leq$$

$$\leq \sum_{i,j=1}^{N} \frac{i\,e^{-\pi^2(i^2+j^2)T}}{4\pi^2(i^2+j^2)} e^{\pi(i+j)} \leq \frac{1}{4\pi^2} \left(\sum_{i=1}^{N} e^{-\pi^2 T i^2 + \pi i} \right)^2.$$

Thus, there exists a constant $C_T' > 0$, depending only of T, such that

$$\sum_{i,j=1}^{N} \frac{e^{-\pi^2(i^2+j^2)T}}{i^2+j^2} \, \frac{j\,i^2\,m^2}{m^2+i^2} \prod_{\substack{p=1\\p\neq i}}^{N} \frac{i^2+p^2}{|i^2-p^2|} \prod_{\substack{p=1\\p\neq j}}^{N} \frac{j^2+p^2}{|j^2-p^2|} \leq C_T'. \tag{45}$$

By denoting $C = 2C_T^2 C_T'$, it follows from (44) and (45) that

$$|d_k^m(T,N) - d_k^m(\infty,N)| \leq C \frac{k^2+m^2}{k\,m^2} |d_k^m(\infty,N)| \qquad 1 \leq k,m \leq N. \tag{46}$$

Now, remark that $C\frac{k^2+m^2}{k\,m^2} \leq \frac{1}{2}$ if $\frac{2Ck^2}{k-2C} \leq m^2$. Thus (38) is verified for any $k \geq 2C+1$ and $m^2 \geq (2C+1)^2 k$. From the symmetry of the matrices $(d_k^m(T,N))_{1\leq k,m\leq N}$ and $(d_k^m(\infty,N))_{1\leq k,m\leq N}$, we deduce that inequality (38) holds for any $m \geq 2C+1$ and $k^2 \geq (2C+1)^2 m$, too. The proof of the Theorem ends by taking $n_0 = \left[(2C+1)^2\right]+1$. ■

3.3. Analysis of the case $T<\infty$ and $N=\infty$

Now we have all the ingredients needed to estimate the coefficients d_k^m given by (20).

Theorem 3.5. *Let $(\theta_T^m)_{m\geq 1}$ be the biorthogonal of minimal norm to the family Λ in $L^2(0,T)$ and $(d_k^m)_{m,k\geq 1}$ be given by (20). There exist a positive integer n_0 and a positive constant $C>0$, independent of k and m, but depending of T, such that*

$$|d_k^m| \leq Ce^{\pi(k+m)} \qquad \forall k,m \geq 1 \tag{47}$$

$$|d_k^m| \geq \frac{k\,m}{32(k^2+m^2)}e^{\pi(k+m)} \qquad \forall m,k \geq n_0. \tag{48}$$

Proof: From (21)

$$\theta_{T,N}^m = \sum_{k=1}^{N} d_k^m(T,N)e^{-\lambda_k t} = \frac{1}{||p_T^m - r_{T,N}^m||^2_{L^2(0,T)}}(p_T^m - r_{T,N}^m) \tag{49}$$

where $r_{T,N}^m$ is the orthogonal projection of p_T^m over $E_N(m,\Lambda,T)$.

On the other hand, from (12),

$$\theta_T^m(t) = \sum_{k\geq 1} d_k^m e^{-\lambda_k t} = \frac{1}{||p_T^m - r_T^m||^2_{L^2(0,T)}}(p_T^m - r_T^m) \tag{50}$$

where r_T^m is the projection of p_T^m over the space $E(m,\Lambda,T)$.

Now, remark that

$$r_{T,N}^m \to r_T^m \text{ as } N\to\infty \text{ in } L^2(0,T). \tag{51}$$

Indeed, since $\langle r_T^m - r_{T,N}^m, e^{-\lambda_j t}\rangle = 0$ for any $1\leq j\leq N$, we have that

$$||r_T^m - r_{T,N}^m||^2_{L^2(0,T)} = \langle r_T^m - r_{T,N}^m, r_T^m - r_{T,N}^m\rangle_{L^2(0,T)} = \langle r_T^m - r_{T,N}^m, r_T^m\rangle_{L^2(0,T)} =$$

$$= \left\langle r_T^m - r_{T,N}^m, r_T^m - \sum_{n=1,\, n\neq m}^{N} \alpha_n e^{-\lambda_n t} \right\rangle_{L^2(0,T)}.$$

Now, since $r_T^m \in E(m,\Lambda,T) = \bigcup_{N\geq 1} E_N(m,\Lambda,T)$, we deduce that (51) holds. From formulas (49) and (50) we deduce that $\theta_{T,N}^m \to \theta_T^m$ as $N\to\infty$ in $L^2(0,T)$ and, consequently,

$$d_k^m(T,N) = \langle \theta_{T,N}^m, \theta_{T,N}^k \rangle \to d_k^m \text{ as } N\to\infty. \tag{52}$$

By taking into account estimates (37) and (38) from Theorems 3.3 and 3.4 respectively, we obtain from (52) the conclusion of the Theorem. ■

4. Control theoretical consequences

In this section we deduce some consequences of the estimates in Theorem 3.5 for the controllability of (1).

4.1. Control for one mode

Let us consider the case in which the initial data y^0 of (1) to be controlled is the $n-$th eigenfunction Φ^n. It follows from Theorem 2.3 that the corresponding HUM control is given by

$$u(t) = \frac{(-1)^n}{2n\pi} e^{-n^2\pi^2 T} \theta_T^n(T-t) = \frac{(-1)^n}{2n\pi} \sum_{k\geq 1} d_k^n e^{-(n^2+k^2)\pi^2 T} e^{k^2\pi^2 t}. \tag{53}$$

Moreover, the Fourier coefficients $(b_m^{0,n})_{m\geq 1}$ of the initial data $\mathcal{G}(y^0) = \widehat{\varphi}^{0,n}$ which gives the HUM control corresponding to y^0, may be computed by using (16):

$$b_m^{0,n} = \frac{(-1)^{n+m}}{2\,n\,m\,\pi^2}\, e^{-n^2\pi^2 T} d_m^n \quad m\geq 1. \tag{54}$$

From (54) and estimates (47)-(48) from Theorem 3.5, we immediately obtain the following properties of the Fourier coefficients $(b_m^{0,n})_{m\geq 1}$.

Corollary 4.1. *Let $(b_m^{0,n})_{m\geq 1}$ be the Fourier coefficients of $\mathcal{G}(\Phi^n)$. Then*

$$|b_m^{0,n}| \leq \frac{C}{nm} e^{-\pi^2 T\, n^2 + \pi(m+n)} \qquad \forall m\geq 1. \tag{55}$$

Moreover, if $n\geq n_0$, then

$$|b_m^{0,n}| \geq \frac{1}{64\pi^2(n^2+m^2)} e^{-\pi^2 T\, n^2+\pi(m+n)} \qquad \forall m\geq n_0. \tag{56}$$

The constants $C>0$ and n_0 are those given by Theorem 3.5, are independent of n and m but depend of T.

Remark 4.1. *Estimates (56) from Corollary 4.1 show that the initial data* $\mathcal{G}(\Phi^n)$ *of (3), which gives the HUM control for the* $n-$*th eigenfunction* Φ^n*, has a very low regularity. Indeed, for any* $m \geq 2\pi T n^2$*, we obtain that* $|b_m^{0,n}| \geq e^{\frac{\pi}{2}m}$ *and therefore* $\mathcal{G}(\Phi^n)$ *does not belong to any Sobolev space of negative order. Also, note that the first Fourier coefficients* $b_m^{0,n}$*, corresponding to* $m \leq \alpha\pi T n^2$*, with* $\alpha < 1$*, are exponentially small for as* $n \to \infty$*.* $\square$

4.2. Controls for initial data in $L^2(0,1)$

Let us now study the regularity of the HUM controls for initial data in $L^2(0,1)$. We have the following result.

Corollary 4.2. *For any* $y^0 \in L^2(0,1)$ *and* $\varepsilon > 0$ *there exists* $y^0_\varepsilon \in L^2(0,1)$ *and a positive constant* C*, depending only of* T *and* ε*, such that*

$$||y^0 - y^0_\varepsilon||_{L^2(0,1)} < \varepsilon \tag{57}$$

$$|b^0_{\varepsilon,m_k}| \geq \frac{C}{m^2}\, e^{\pi\, m_k} \qquad \forall k \geq 1, \tag{58}$$

where $(b^0_{\varepsilon,m})_{m\geq 1}$ *are the Fourier coefficients of* $\mathcal{G}(y^0_\varepsilon)$ *and* $(m_k)_{k\geq 1}$ *is an increasing sequence of positive integers.*

Proof: Let $\mathcal{G}(y^0) = \sum_{m=1}^\infty b^0_m \sin(m\pi x)$ and n_0 be the positive integer from Theorem 3.5. We define $y^1 = y^0 + \varepsilon \sin(n_0\pi x)$ and remark that $\mathcal{G}(y^1) = \mathcal{G}(y^0) + \varepsilon\mathcal{G}(\Phi^{n_0})$, where $\mathcal{G}(\Phi^{n_0})$ was studied in Corollary 4.1. If $(b^1_m)_{m\geq 1}$ are the Fourier coefficients of $\mathcal{G}(y^1)$, we deduce from (54) that

$$b^1_m = b^0_m + \varepsilon b^{0,n_0}_m = b^0_m + \varepsilon\frac{(-1)^{n_0+m}}{2\, n_0\, m\, \pi^2}\, e^{-n_0^2\pi^2 T} d^{n_0}_m \qquad m \geq 1. \tag{59}$$

Consequently, $\max\{|b^0_m|,\, |b^1_m|\} \geq \frac{\varepsilon |d^{n_0}_m|}{4\, n_0\, m\, \pi^2}\, e^{-n_0^2\pi^2 T}$ and, by taking into account (48) from Theorem 3.5, we deduce that

$$\max\{|b^0_m|,\, |b^1_m|\} \geq \frac{\varepsilon}{128\pi^2(n_0^2 + m^2)}\, e^{-n_0^2\pi^2 T} e^{\pi(m+n_0)} \qquad \forall m \geq n_0. \tag{60}$$

Thus, at least one of the sequences $(b^0_m)_{m\geq 1}$ or $(b^1_m)_{m\geq 1}$ has a subsequence which verifies (58). If this is $(b^0_m)_{m\geq 1}$, we choose $y^0_\varepsilon = y^0$. Otherwise, we take $y^0_\varepsilon = y^1$. In both cases y^0_ε verifies (57) and (58) and the proof ends. ■

Remark 4.2. *Corollary 4.2 shows that the set of initial data* y^0 *whose HUM controls are given by a minimizer* $\mathcal{G}(y^0)$ *of* $\mathcal{J}$ *which do not belong to any Sobolev space of negative order is dense in* $L^2(0,1)$*.* $\square$

4.3. Monochromatic controls

Another natural question is the reciprocal of the one addressed in the paragraph 4.1: which is the regularity of the initial datum y^0 of (1) whose HUM control is given by the solution of (3) with initial datum $\widehat{\varphi}^0 = \sin(m\pi x)$ containing one single Fourier component? The following Corollary shows that y^0 is highly irregular.

Corollary 4.3. *The initial data $y^0 = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$, whose HUM control is given by the solution of (3) with initial data $\widehat{\varphi}^0 = \sin(m\pi x)$, verifies*

$$|a_n| \geq C \frac{mn}{m^2+n^2}\, e^{T\,\pi^2\,n^2} \qquad \forall n \geq 1, \tag{61}$$

where C is a positive constant independent of n and m.

Remark 4.3. *This result confirms that the data for which the controls are smooth are irregular. This complements our previous results showing that the control associated with smooth data are highly irregular. This also shows that the operator $\mathcal{G} : L^2(0,1) \to \mathcal{H}$ defined in Proposition 1.1 is injective but not surjective.* $\square$

Proof: From (18) we deduce that

$$(-1)^m m\pi e^{-\lambda_m t} = \sum_{n=1}^{\infty} (-1)^n \frac{a_n}{2n\pi} e^{-T\pi^2\,n^2} \theta_T^n(t) \tag{62}$$

and consequently $a_n = (-1)^{n+m} 2\,n\,m\,\pi^2\, e^{T\pi^2\,n^2} \langle p_T^n, p_T^m \rangle_{L^2(0,T)}$ from which (61) follows immediately. $\blacksquare$

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