THE POSITIVE REAL LEMMA AND CONSTRUCTION OF ALL REALIZATIONS OF GENERALIZED POSITIVE RATIONAL FUNCTIONS

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Abstract. We here extend the well known Positive Real Lemma (also known as the Kalman-Yakubovich-Popov Lemma) to complex matrix-valued generalized positive rational function, when non-minimal realizations are considered. All state space realizations are partitioned into subsets, each is identified with a set of matrices satisfying the same Lyapunov inclusion. Thus, each subset forms a convex invertible cone, **cic** in short, and is in fact is replica of all realizations of positive functions of the same dimensions. We then exploit this result to provide an easy construction procedure of all (not necessarily minimal) state space realizations of generalized positive functions. As a by-product, this approach enables us to characterize systems which can be brought, through static output feedback, to be generalized positive.

1. Introduction

For a half of a century, the Positive Real Lemma (also known as the Kalman-Yakubovich-Popov Lemma) has been recognized as a fundamental result in System Theory. We here extend and exploit it in various ways. Let $\mathbb{C}_+$ and $\mathbb{C}_-$ be the open right and left halves of the complex plane respectively, and $\mathbb{P}_k$, $(\overline{\mathbb{P}}_k)$ be the set of all $k \times k$ positive definite (semidefinite) matrices[1]. Recall that a $p \times p$-valued function $F(s)$, analytic in $\mathbb{C}_+$ is said to be *positive* if

$$F(s) + F(s)^* \in \overline{\mathbb{P}}_p \qquad s \in \mathbb{C}_+ \ . \tag{1.1}$$

The study of rational positive functions, denoted by $\mathcal{P}$, has been motivated from the 1920's by (lumped) electrical networks theory, see e.g. [7], [11]. From the 1960's positive functions also appeared in books on absolute stability theory, see e.g. [43], [45].

A $p \times p$-valued function of bounded type in $\mathbb{C}_+$ (i.e. a quotient of two functions analytic and bounded in $\mathbb{C}_+$) is called *generalized positive* $\mathcal{GP}$ if

$$F(i\omega) + F(i\omega)^* \in \overline{\mathbb{P}}_p \quad a.e. \qquad \omega \in \mathbb{R}, \tag{1.2}$$

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[1]Whenever clear from the context, the dimension subscript will be omitted.

where $F(i\omega)$ denotes the non-tangential limit[2] of F at the point $i\omega$.
Generalized positive functions were introduced in the context of the Positive Real Lemma (PRL), see [6] and references therein[3]. Applications of $\mathcal{GP}$ functions to electrical networks appeared in [35], and to control in [40], where they first casted in a Linear Matrix Inequality (LMI) framework, see e.g. [12] for more information on LMI. For more application of the generalized PRL, see [30]

Both function sets $\mathcal{P}$ and $\mathcal{GP}$ are closed under positive scaling, sum and inversion (when the given function has a non-identically vanishing determinant). Yet these spaces have quite different properties, as we now illustrate. Let $\mathbb{C}^{p\times p}(X)$ be the space of $\mathbb{C}^{p\times p}$-valued rational functions. Recall that a function $\Psi\in\mathbb{C}^{p\times p}(X)$ belongs to $\mathcal{GP}$ if and only if it can be factorized as

$$\Psi(s)=G(s)P(s)G(-s^*)^*, \tag{1.3}$$

where $G,P\in\mathbb{C}^{p\times p}(X)$, and $P\in\mathcal{P}$. Factorization of this nature appeared e.g. in [17] and [20], see also Observation 7.1 below. The significance of (1.3) to scalar rational $\mathcal{GP}$ functions was recently treated in [4] and [5].

We now consider properties of the sum of two rational $\mathcal{GP}$ functions (series connection in electrical engineering jargon). From (1.2) it follows that this sum is again in $\mathcal{GP}$, but both the McMillan degree and the number of negative squares (roughly, the number of poles in $\mathbb{C}_+$) increase. For more details on the number of negative squares see [36] and [37]. Recall that a rational function Ψ is in $\mathcal{GP}$ if and only if the kernel

$$\frac{\Psi(s)+\Psi(w)^*}{s+w^*}$$

has a finite number of negative squares in its domain of definition in $\mathbb{C}_+$. The number of negative squares of the sum of two elements in $\mathcal{GP}$ is preserved if, for instance, in (1.3) $\Psi_1(s)=G(s)P_1(s)G(-s^*)^*$ and $\Psi_2(s)=G(s)P_2(s)G(-s^*)^*$ with $P_1,P_2\in\mathcal{P}$ and the same function $G\in\mathbb{C}^{p\times p}(X)$, see [5, Section 3] for the scalar case.
In contrast, if one takes a *state space realization sum*, the McMillan degree of the resulting function does not increase, but it may turn to not generalized positive at all, see Example 7.2 below. One of the results of this paper is a partitioning of $\mathcal{GP}$ functions to subsets, denoted by $\mathcal{GP}(r,\nu,p)$, closed under state space addition, while both the McMillan degree and the number of negative squares do not increase. See Theorem 7.3 below.

We resort to some preliminaries. Let $\Psi\in\mathbb{C}^{p\times p}(X)$ be of McMillan degree q, and analytic at infinity, i.e. $\lim_{s\to\infty}\Psi(s)$ exists. Namely, Ψ admits a state space realization

$$\Psi(s)=C(sI-A)^{-1}B+D \qquad L:=\begin{pmatrix}A & B\\ C & D\end{pmatrix} \tag{1.4}$$

with $A\in\mathbb{C}^{n\times n}$, $n\geq q$, $B,C^*\in\mathbb{C}^{n\times p}$ and $D\in\mathbb{C}^{p\times p}$, namely, $L\in\mathbb{C}^{(n+p)\times(n+p)}$. If the McMillan degree of $\Psi(s)$ satisfies $q=n$, the realization is called minimal.

[2]This limit exists almost everywhere on $i\mathbb{R}$ because F is assumed of bounded type in $\mathbb{C}_+$, see e.g. [22].

[3]The original formulation was real. The case we address is in fact *generalized* positive and *complex*, but we wish to adhere to the commonly used term: Positive Real Lemma.

We can now state the Positive Real Lemma (PRL) as presented in [19, Theorem 1] (up to substituting the real setting by a complex one)

Lemma 1.1. *Let* $\Psi(s)$ *be a* $p \times p$*-valued rational function in* (1.4) *and assume that* $q = n$.
(I) $\Psi \in \mathcal{GP}$ *if and only if*

$$HL + L^*H = Q \in \overline{\mathbb{P}}_{n+p} \ , \tag{1.5}$$

for some $H = \mathrm{diag}\{\hat{H},\ I_p\}$*, where* $\hat{H} \in \mathbb{C}^{n \times n}$ *is Hermitian non-singular.*
(II) If $\Psi \in \mathcal{P}$ *then in part (I)* $-\hat{H} \in \mathbb{P}_n$ *.*

The aim of this work is to first extend this result to the non-minimal case, and then to use it to obtain a straightforward construction of *all* (not necessarily minimal) state space realization of $\mathcal{P}$ and $\mathcal{GP}$ rational functions. This is then used to describe the already mentioned partitioning to sets of the form $\mathcal{GP}(r, \nu, p)$ and to obtain other results, described below.

The outline of the paper is as follows: The paper is composed of eight sections besides the introduction. In Section 2 we give a short review of the literature, which should be paralleled with a complementary survey we offered in our previous paper [4]. Our aim is not to provide a complete survey, but to raise, through samples, the intriguing observation that although the PRL has been a standard textbook material from the 1970's, see e.g. [7], [25, Chapter 3], [43, Section 4.4, Appendix] and [45, Section 8.5], it is not straightforward to cover the relevant literature. In simple words, there is too little of cross-referencing [4].
In Section 3 an algebraic Riccati inclusion associated with necessity part of the generalized positive real Lemma is addressed. Sections 4 and 5 are devoted to showing that an algebraic Lyapunov inclusion associated with the sufficiency part of the generalized positive real Lemma is independent of the minimality of the realization. Some background material concerning sets of Lyapunov inclusions is reviewed in Section 6. In particular we provide a convenient parameterization of all matrices L satisfying the Lyapunov inclusion (1.5) where H is fixed and Q varies over all $\overline{\mathbb{P}}$. This Lyapunov inclusion formulation is then employed in Section 7 to provide a straightforward parameterization of all state space realization of $\mathcal{GP}$ rational functions. This allows us to describe $\mathcal{GP}$ functions as a union of replicas of positive functions. As an application of theses sets, in Section 8 we characterize all rational functions (vanishing at infinity) which can be rendered $\mathcal{GP}$, through a static state feedback. Concluding remarks are given in Section 9.

2. A historical perspective

We here review some of the relevant existing literature.
As mentioned, the above version of Lemma 1.1 is from [19] and was repeated in [40]. It was originally proved in [6]. A special case was later treated in [41]. The positive function case (part II) is well known and sometimes is referred to as the Kalman-Yakubovich-Popov Lemma and is dated to the 1960's. For an early full account see e.g. [7, Chapter 5]. An easy-to-read historical perspective is given in [12, Sections 1].

[4] For example, it seems that [19] was hardly ever cited.

A matrix formulation, see (1.4), of the PRL was introduced in the PhD. thesis of P. Faurre, see [24, Theorem 4.2] and then in a book he co-authored, [25, Theorem 3.1]. In fact it implicitly earlier appeared in [48]. The formulation through the Rosenbrock system matrix L (1.4) (for not necessarily positive systems) explicitly introduced in [19] and subsequently in [40, Lemma 8]. An interesting special case was studied in [27, Theorem 4]. The notion of Linear Matrix Inequality (LMI) was introduced in [48]. [40] was one of the early works recognizing the applicability of LMI framework to the (generalized) Positive Real Lemma (PRL), see also [12, Section 2.7.2]. A comprehensive survey of the LMI approach to the PRL appeared in [29]. Unfortunately, in spite of its admirable reference list (201 items), some important relevant results are missing.

Following (1.1) Positive functions map $\mathbb{C}_+$ to $\overline{\mathbb{L}}(I)$, the set of matrices with a non-negative Hermitian part[5]. Analogously, following (1.2), a $\mathcal{GP}$ function maps $i\mathbb{R}$ (after removing all poles of the function) into $\overline{\mathbb{L}}(I)$, see Observation 7.1 below. Closely related function sets are addressed in the literature:

- Bounded functions mapping $\mathbb{C}_+$ to weak contractions while generalized bounded functions map $i\mathbb{R}$ to weak contractions. Versions of the PRL for bounded functions appeared [24, Theorem 4.2] [7, Section 7.2] and for generalized bounded in [19, Equation (6)]. An interesting subclass is treated in [1, Theorem 2.1]. An important result on a subclass of generalized bounded functions where $i\mathbb{R}$ is substituted by $\mathbb{R}$, appeared in [28, Theorem 3.2]. See also [2, Theorem 2.12] and [38, Chapter 21].
- Carathéodory functions, analytically mapping the open unit disk to $\overline{\mathbb{L}}(I)$ and generalized (=pseudo) Carathéodory functions, mapping the unit circle (from the poles of the given function have been removed) to $\overline{\mathbb{L}}(I)$. Versions of the PRL for Carathéodory functions appeared in [49].
- Schur functions, analytically mapping the open unit disk to weak contractions and generalized Schur functions, mapping the unit circle to weak contractions. Version of the PRL for Schur functions appeared in [3, Theorem 2.5], [47, Section II], [49] and for operator valued in [10]. For generalized Schur functions see [19, Theorem 2] and for an interesting special case see [27, Theorem 3]. A time-varying extension of the PRL to generalized Schur functions along with a thorough study of various applications is are given in [30, Theorem 1.2.5 and Appendix 3A].
- Nevanlinna functions, analytically mapping the open upper half plane to $i\overline{\mathbb{L}}(I)$ and the generalized Nevanlinna functions map $\mathbb{R}$ to $i\overline{\mathbb{L}}(I)$, see e.g. [17] and [20]. Minimal realization of infinite dimensional generalized Nevanlinna functions was studied in [21].

As already stated, we do not aspire to provide a survey of PRL related results, and we are aware of additional references dealing with the subject, not mentioned here. We focused on the scattered nature of the literature related to the *generalized* case.

3. Generalized positive lemma necessity and the Riccati equation

It has been long recognized that with the part (b) of Lemma 1.1 (dealing with positive functions) one can associate an algebraic Riccati equation, see e.g. [7,

[5] The notation $\overline{\mathbb{L}}(I)$, with $\mathbb{L}$ honoring A.M. Lyapunov, will be formally defined in (3.1) below.

Section 5.4] [25, Chapter 5] and [12, Section 2.7.2]. For Schur functions version see e.g. [18, Theorem 2.1]. We now address the extension of this result to $\mathcal{GP}$ functions. It first appeared in the context of a variant of Generalized Bounded functions (where $i\mathbb{R}$ was substituted by $\mathbb{R}$) in [28, Theorem 3.2] and in [2, Theorem 2.12]. For another variant of this result, see [38, Section 20.1]. In Proposition 3.2 below we provide a simple proof of the result using the system matrix formulation (1.4), employed all along this work. We shall find it convenient to resort to the following notation of sets of matrices sharing a common Lyapunov factor: For a $r \times r$ Hermitian non-singular matrix H, define the sets of $r \times r$ matrices, $\mathbb{L}(H)$ and $\overline{\mathbb{L}}(H)$ as,

(3.1)

$$(a)\ \mathbb{L}(H) := \{L \ : \ HL + L^*H \in \mathbb{P}_r \} \qquad (b)\ \overline{\mathbb{L}}(H) := \{L \ : \ HL + L^*H \in \overline{\mathbb{P}}_r \}.$$

In particular, ($\overline{\mathbb{L}}(I)$) $\mathbb{L}(I)$ is the set of matrices with positive (semi)definite Hermitian part.

Proposition 3.1. *Consider* (1.4), (1.5) *with* $H = \mathrm{diag}\{\hat{H},\ I_p\}$ *and* $\hat{H} \in \mathbb{C}^{n\times n}$ *Hermitian. Assume in addition that* $D \in \mathbb{L}(I_p)$. *Let us define the following* $n \times n$ *Riccati expression,*

$$\begin{array}{rl} M & := \hat{H}(A - B(D+D^*)^{-1}C) \quad + \quad (A - B(D+D^*)^{-1}C)^*\hat{H} \\ & -\hat{H}B(D+D^*)^{-1}B^*\hat{H} \quad - \quad C^*(D+D^*)^{-1}C. \end{array} \tag{3.2}$$

Then $Q \in \overline{\mathbb{P}}_{n+p}$, *if and only if in* (3.2)

$$M \in \overline{\mathbb{P}}_n\ . \tag{3.3}$$

Proof Using the fact that $D \in \mathbb{L}(I_p)$ one can employ the classical Schur's complement, e.g. [33, Theorem 7.7.6], to write down Q in (1.5) explicitly,

$$Q = \begin{pmatrix} \hat{H}A + A^*\hat{H} & \hat{H}B + C^* \\ C + B^*\hat{H} & D + D^* \end{pmatrix} = \begin{pmatrix} I_n & R \\ 0 & I_p \end{pmatrix} \begin{pmatrix} M & 0 \\ 0 & D + D^* \end{pmatrix} \begin{pmatrix} I_n & 0 \\ R^* & I_p \end{pmatrix},$$

where $R := (\hat{H}B + C^*)(D+D^*)^{-1}$ and M is given in (3.2). Thus indeed $Q \in \overline{\mathbb{P}}_{n+p}$ if and only if $M \in \overline{\mathbb{P}}_n$. □

We can now re-formulate the necessity part of Lemma 1.1.

Proposition 3.2. *Let* $\Psi \in \mathcal{GP}$ *be a* $p \times p$*-valued rational function so that* $\lim\limits_{s\ \to\ \infty} \Psi(s) := D$ *exists. Furthermore assume that* $D \in \mathbb{L}(I_p)$. *Let* q *be the McMillan degree of* Ψ.
(I) Then, $\Psi(s)$ *admits a state space realization* (1.4), *so that it is minimal* $(q = n)$ *and it satisfies the Riccati inclusion in* (3.2), (3.3) *for some* $n \times n$ *Hermitian non-singular* $\hat{H}$.
(II) If $\Psi \in \mathcal{P}$ *then in part (I)* $-\hat{H} \in \mathbb{P}_n$.

Note that the technical condition $D \in \mathbb{L}(I_p)$ in Proposition 3.2 is indeed restrictive. For example, many system of interest have a zero at infinity and thus are excluded from the discussion. On the other hand, whenever $\Psi(s) = C(sI - A)^{-1}B + D$ is a $\mathcal{GP}$ function, from the above discussion it follows that $D \in \overline{\mathbb{L}}(I_p)$, hence one can always construct another $\mathcal{GP}$ function $\tilde{\Psi}(s) = C(sI - A)^{-1}B + \tilde{D}$, with $\tilde{D} \in \mathbb{L}(I_p)$ so that $\epsilon \geq \|D - \tilde{D}\|$, where $\epsilon > 0$ is arbitrary.

4. Generalized positive lemma sufficiency - an extension

The sufficiency statement of the (generalized) Positive Real Lemma (PRL) of matrix valued rational functions was first proved in [6], under the assumption of minimality of the realization ($q = n$). We now address the question of relaxing this minimality constraint. This problem was treated in the framework of positive functions in [49, Lemma 6] and in the framework of functions satisfying $D \in \mathbb{L}(I_p)$ (as in the previous section) in [38, Section 21.3]. In a different formulation see also [46, Theorem 1]. In [26] a proof of the sufficiency statement of the (generalized) PRL, removing the minimality of realization condition, was presented. Unfortunately a (redundant) spectral condition on A was imposed there. The result of Proposition 4.2 below, avoids any restriction.

In addition, in Proposition 4.2 below we show that one can bound the number of poles of $\Psi(s)$ in each open half plane[6]. To this end, we need some preliminaries. Recall that for a matrix $A \in \mathbb{C}^{n\times n}$ one can associate a triple: $\text{inertia}(A) = (\nu,\ \delta,\ \pi)$, with $\nu + \delta + \pi = n$, if A has ν eigenvalues in $\mathbb{C}_-$, π eigenvalues in $\mathbb{C}_+$ and δ eigenvalues on $i\mathbb{R}$, see e.g. [34, 2.1.1]. Let $A, \hat{H} \in \mathbb{C}^{n\times n}$ with $\hat{H}$ Hermitian, be with inertia,

$$\text{inertia}(A) = (\nu_A,\ \delta_A,\ \pi_A) \qquad \text{inertia}(\hat{H}) = (\nu,\ 0,\ n-\nu) \qquad \nu \in [0,\ n], \tag{4.1}$$

i.e. $\hat{H}$ is non-singular. Consider now the Lyapunov equation

$$\hat{H}A + A^*\hat{H} = \hat{Q} \in \overline{\mathbb{P}}_n\ . \tag{4.2}$$

Recall that from a pair $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times p}$, the following controllability matrix may be constructed $\mathcal{C} := [B \vdots AB \vdots \cdots \vdots A^{n-1}B]$. Then $X_{\text{cont}}(A, B)$, the controllable subspace associated with the pair A, B is given by the range of $\mathcal{C}$ and $X_{\text{cont}}(A, B)^\perp$, its orthogonal complement, is given by the null-space of $\mathcal{C}^*$, see e.g. [34, Definition 2.4.8]. Similarly with a pair $A \in \mathbb{C}^{n\times n}$, $C \in \mathbb{C}^{p\times n}$, one can associate a observable subspace $X_{\text{obs}}(A, C)$, given by $X_{\text{obs}}(A, C) = X_{\text{cont}}(A^*, C^*)$ (and $X_{\text{obs}}(A, C)^\perp = X_{\text{cont}}(A^*, C^*)^\perp$). For A, $\hat{Q}$ in (4.2) denote,

$$m := \dim\, X_{\text{obs}}(A,\ \hat{Q})^\perp. \tag{4.3}$$

Namely, pair A, $\hat{Q}$ is observable whenever $m = 0$. We can now cite the following important result of R. Loewy [39], adapted to our framework,

Theorem 4.1. *Let $A, \hat{H}$ and m be as in* (4.1)*,* (4.2) *and* (4.3)*. Then,*

$$\nu \geq \nu_A \geq \max(0,\ \nu - m) \qquad\qquad n - \nu \geq \pi_A \geq \max(0,\ n - \nu - m).$$

We can now state the extended sufficiency part of the PRL.

Proposition 4.2. *Let Ψ in* (1.4) *be a $p \times p$-valued rational function of McMillan degree q, $n \geq q$. Assume that state space realization in* (1.4) *satisfies the Lyapunov equation* (1.5) *with $H = \text{diag}\{\hat{H},\ I_p\}$, where $\hat{H}$ is $n \times n$ Hermitian.*
(I) If $\text{inertia}(\hat{H}) = (\nu,\ 0,\ n-\nu)$, for some $\nu \in [0,\ n]$, then, Ψ is a $\mathcal{GP}$ function with at most ν poles in $\mathbb{C}_-$ and $n - \nu$ poles in $\mathbb{C}_+$.
(II) If in part (I) $-\hat{H} \in \mathbb{P}_n$, i.e. $\nu = n$, then $\Psi \in \mathcal{P}$.

[6] Although in a different framework, bounds of a similar nature can be found in [28, Theorem 3.4] and subsequently in [2, Theorem 2.12] and in [38, Section 21.2].

Proof : I. Indeed assume that (1.5) is satisfied with $H = \mathrm{diag}\{\hat{H}, I_p\}$, $\hat{H}$ Hermitian nonsingular and L as in (1.4). Note that in (1.5) Q is in $\overline{\mathbb{P}}_{n+p}$, thus its upper left block is in $\overline{\mathbb{P}}_n$. Namely, (4.2) is satisfied, so by Theorem 4.1 the matrix A has at most ν eigenvalues in $\mathbb{C}_-$ and $n-\nu$ eigenvalues in $\mathbb{C}_+$. Recall that the poles of Ψ are determined by the eigenvalues of A, see (1.4).
Next denoting $S := \mathrm{diag}\{-sI_n, 0_p\}$ we note that

$$HS + S^*H \in \overline{\mathbb{P}}_{n+p} \ , \tag{4.4}$$

for all $s \in i\mathbb{R}$. Take now [7] $\tilde{L} := L + S = \begin{pmatrix} -sI_n + A & B \\ C & D \end{pmatrix}$. Then for all $s \in i\mathbb{R}$ also,

$$H\tilde{L} + \tilde{L}^*H \in \overline{\mathbb{P}}_{n+p} \ . \tag{4.5}$$

Next, recall that for arbitrary constant matrix $\Psi \in \mathcal{GP}$, if and only if $(\Psi + T) \in \mathcal{GP}$, for arbitrary constant matrix $-T^* = T \in \mathbb{C}^{p\times p}$. Thus, up to a shift by a skew-Hermitian matrix, we can assume that $\Psi(s)$ in (1.4) is almost everywhere invertible. Thus, a straightforward calculation (see e.g. [33, 0.7.3]) results in,

$$\tilde{L}^{-1} = \begin{pmatrix} (sI-A)^{-1}\left(B\Psi(s)^{-1}C(sI-A)^{-1} - I\right) & (sI-A)^{-1}B\Psi(s)^{-1} \\ \Psi(s)^{-1}C(sI-A)^{-1} & \Psi(s)^{-1} \end{pmatrix}.$$

Multiplying (4.5) by $\left(\tilde{L}^*\right)^{-1}$ from the left and $\tilde{L}^{-1}$ from the right, yields

$$H\tilde{L}^{-1} + (\tilde{L}^{-1})^*H \in \overline{\mathbb{P}}_{n+p} \tag{4.6}$$

for all $s \in i\mathbb{R}$. Now in particular the $p \times p$ lower right block of (4.6) satisfies,

$$\Psi(s)^{-1} + \left(\Psi(s)^{-1}\right)^* = \left(H\tilde{L}^{-1} + (\tilde{L}^{-1})^*H\right)_{22} \in \overline{\mathbb{P}}_p \tag{4.7}$$

for all $s \in i\mathbb{R}$. Thus, $\Psi(s)^{-1}$ is in $\mathcal{GP}$ and hence also $\Psi(s)$. Thus, the first part of the claim is established.
(b) Positive functions. If $-\hat{H} \in \overline{\mathbb{P}}_n$ the relation in (4.4) holds for all $s \in \mathbb{C}_+$ and subsequently, also (4.6) and (4.7). Hence, $\Psi \in \mathcal{P}$, so the proof is complete. $\square$

In the next section we scrutinize some aspects of Proposition 4.2.

5. NON-MINIMAL REALIZATION AND BOUNDS ON INERTIA - A CLOSER LOOK

Roughly, application of Theorem 4.1 to Proposition 4.2 suggests that the "further" from minimality the realization is, the cruder is the bound on the number of poles in $\mathbb{C}_+$. We here illustrate the "at most" clause in the statement Proposition 4.2 with respect to the number of poles in each open half plane.

Example 5.1. All functions considered in this example are so that in (1.5) $H = \mathrm{diag}\{\hat{H},\ 1\}$ with $\hat{H} = \mathrm{diag}\{-1,\ 1\}$, see also (4.2). Namely the corresponding rational functions have *at most* one pole in each open half plane. We present the rational function along with the corresponding system matrix.

$$L_\alpha = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad L_\beta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad L_\gamma = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\psi_\alpha(s) = \tfrac{s-1}{s^2-s-1} \qquad \psi_\beta(s) = \tfrac{s}{s-1} \qquad \psi_\gamma(s) = \tfrac{1}{s+1}$$

$$L_\delta = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad L_\xi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad L_\eta = \begin{pmatrix} -1 & -1 & 2 \\ -1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \quad L_\theta = \begin{pmatrix} -1 & 1 & -2 \\ 1 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix}$$

[7] It is interesting to note that the zeroes of $\Psi(s)$ are the points s for which $\tilde{L}$ is singular.

$\psi_\delta(s)=\frac{-4}{s^2}$ $\quad\psi_\xi(s)=\frac{s+1}{s}$ $\quad\psi_\eta(s)=\psi_\theta(s)=\frac{s^2+5s-1}{s^2-2}$.
Indeed, $\psi_\alpha(s)$ and $\psi_\eta(s)=\psi_\theta(s)$ have a pole in each open half plane. $\psi_\beta(s)$ has a single pole in $\mathbb{C}_+$, while $\psi_\gamma(s)$ a pole in $\mathbb{C}_-$. The poles of $\psi_\delta(s)$ and $\psi_\xi(s)$ are confined to the imaginary axis. □

We now point out that unlike to the approach of [26, Theorem 2], in the PRL framework, see Proposition 4.2, there is no restriction on the spectrum of L in (1.5), nor on the spectrum of its upper left block, A in (4.2). Formally, this follows from Theorem 4.1. Intuitively, restrictions of the form $\lambda_j+\lambda_k^*\neq 0$ are necessary when Q, the right hand side of the Lyapunov equation is given and a *unique* Hermitian solution is sought, see e.g. [34, Corollary 4.4.7]. In contrast, here we address a genuine Lyapunov *inclusion*. In fact, in each of the cases L_β, L_γ, L_δ, L_ξ, L_η and L_θ in Example 5.1, the A matrix (the $n\times n$ upper left block) does not satisfy this spectral condition.

6. Convex invertible cones and the Lyapunov inclusion

In this section we explore some properties of the set $\overline{\mathbb{L}}(H)$ (3.1), to be used in the sequel in conjunction of the PRL, see e.g. Theorem 7.3
We next resort to some background. Recall that a set of a square matrices is called Convex Invertible Cone, **cic** in short, if it is closed under positive scaling, summation and inversion. More precisely, it may include singular elements provided that the inverse of every non-singular element, belongs to the set. Thus, the set $\overline{\mathbb{P}}$ is a **cic**. For a more detailed study of **cic**s see e.g. [13, Section 2], [14, Section 2] and [15, Sections 2, 3] Recall that for an Hermitian non-singular matrix H we defined in (3.1) sets of matrices sharing a common Lyapunov factor, $\mathbb{L}(H)$, $\overline{\mathbb{L}}(H)$. The following properties of these sets are fundamental to our discussion.

Theorem 6.1. *Let H be $r\times r$ Hermitian nonsingular, i.e.* inertia$(H)=(\nu,\ 0,\ r-\nu)$ *for some $\nu\in[0,\ r]$.*

- (i) $\mathbb{L}(H)$ *is an invertible cone of matrices sharing the same inertia as H. It is a maximal open convex set of nonsingular matrices* [8].
- (ii) $\overline{\mathbb{L}}(H)$ *is a closed invertible cone of matrices with at most ν eigenvalues in $\mathbb{C}_-$ and at most $r-\nu$ eigenvalues in $\mathbb{C}_+$. It is a maximal convex set with this property.*
- (iii) $\overline{\mathbb{L}}(H)^*=\overline{\mathbb{L}}(H^{-1})$.
  In particular, $\overline{\mathbb{L}}(H)^*=\overline{\mathbb{L}}(H)$, if and only if, up to positive scaling, H is an involution ($H^2=I$).
- (iv) *Let E be an involution which commutes with H then,*
  $E\overline{\mathbb{L}}(H)=\overline{\mathbb{L}}(H)E=\overline{\mathbb{L}}(EH)$.
- (v) *For arbitrary Hermitian involution E, $E\overline{\mathbb{L}}(E)=\overline{\mathbb{L}}(E)E=\overline{\mathbb{L}}(I)$.*

Proof (i) See [13, Proposition 3.7], [15, Observation 2.3.3].
(ii) Part of the claim appeared in [14, Proposition 2.4(i)]. To establish the inertia property assume first that $H=I_r$. By Theorem 4.1 all matrices in $\overline{\mathbb{L}}(I)$ have inertia $(0,\ r-\pi,\ \pi)$ for some $\pi\in[0,\ r]$. Let B be an $r\times r$ matrix not in $\overline{\mathbb{L}}(I)$, namely the Hermitian part of B has a negative eigenvalue, i.e. $\min\limits_{k=1,\ \dots\ ,\ r}\lambda_k(B+B^*)=-\beta$ for some $\beta>0$. Take now $A=\frac{\beta}{4}I+\frac{1}{2}(B^*-B)$.

[8] For an impressive particular converse, see [8], [9].

It is straightforward to verify that $A \in \mathbb{L}(I)$, but $A+B = \frac{\beta}{4}I + \frac{1}{2}(B+B^*)$. Thus, $\min_{k=1,\ \dots\ ,\ r} \lambda_k(A+B) = -\frac{\beta}{4}$, i.e. in contrast to all elements of $\overline{\mathbb{L}}(I)$, the matrix $A+B$ has eigenvalues in $\mathbb{C}_-$ so this part of the claim is established for $H=I$. To address a general Hermitian non-singular matrix H, exploit the fact, see Observation 6.3 below, that one can always find a non-singular V and an involution E_ν (6.1) so that $E_\nu V^{-1}\overline{\mathbb{L}}(H)V = V^{-1}\overline{\mathbb{L}}(H)VE_\nu = \overline{\mathbb{L}}(I)$, so the claim is established.
(iii) The claim for arbitrary H appeared in [13, Equation (3.6)]. If H is a scaled involution, i.e. $H^{-1} = \alpha H$ for some $\alpha > 0$, the claim is follows from multiplying $\overline{\mathbb{L}}(H)$ in (3.1) by H^{-1} from both sides.
For the other direction assume that $\left(\overline{\mathbb{L}}(H)\right)^* = \overline{\mathbb{L}}(H)$. This relation is invariant under unitary similarity. Thus, without loss of generality assume that H is (real non-zero) diagonal, say $H = \mathrm{diag}\{h_1\ ,\ \dots\ ,\ h_r\}$. Take now A to be equal to H, except a single non-zero off diagonal element x at the location jk, where $j > k$, i.e. A is lower triangular. A straightforward calculation shows that $A \in \overline{\mathbb{L}}(H)$ is equivalent to $2|h_j| \geq |x|$. By assumption, also $A^* \in \overline{\mathbb{L}}(H)$, which in turn is equivalent to $2|h_k| \geq |x|$. Thus, $h_j = \pm h_k$ and since true for all j,k this claim is established.
(iv) This claim is proved for $\mathbb{L}(H)$, in [13, Lemma 3.6], see also [15, Proposition 3.2.2]. The case $\overline{\mathbb{L}}(H)$ is similar and thus omitted. Item (v) is in fact a special case of item (iv). $\square$

Note that item (iv) in Theorem 6.1 says that if $L \in \overline{\mathbb{L}}(H)$, for some Hermitian non-singular H, then both matrices EL and LE belong to $\overline{\mathbb{L}}(EH)$, whenever E is an involution which commutes with H. Using this, along with item (v) in Theorem 6.1, we state the following.

Corollary 6.2. *For natural n,p and $\nu \in [1,n]$ let us denote $H_o = \mathrm{diag}\{I_\nu,\ -I_{n-\nu},\ I_p\}$, $H_1 = \mathrm{diag}\{-I_\nu,\ I_{n-\nu},\ I_p\}$ and $H_2 = \mathrm{diag}\{-I_n,\ I_p\}$. Then,*

$$\begin{array}{rcccl} H_o\overline{\mathbb{L}}(H_1) & = & \overline{\mathbb{L}}(H_1)H_o & = & \overline{\mathbb{L}}(H_2), \\ H_o\overline{\mathbb{L}}(H_2) & = & \overline{\mathbb{L}}(H_2)H_o & = & \overline{\mathbb{L}}(H_1). \end{array}$$

In addition, $H_j\overline{\mathbb{L}}(H_j) = \overline{\mathbb{L}}(H_j)H_j = \overline{\mathbb{L}}(I_{n+p}) \qquad j=1,2.$
Namely, there is one-to-one correspondence between the sets $\overline{\mathbb{L}}(H_1)$, $\overline{\mathbb{L}}(H_2)$ and $\overline{\mathbb{L}}(I_{n+p})$.

We shall find it convenient to introduce the following notation for $l \times l$ signature matrices,

$$E_{\nu,l} := \mathrm{diag}\{-I_\nu\ ,\ I_{l-\nu}\} \qquad \nu \in [0,\ l]. \tag{6.1}$$

Whenever the dimension l is evident from the context we shall simply write E_ν.

We can now cite the following known facts see e.g. [13, Lemma 3.4].

Observation 6.3. *Consider the $r \times r$ nonsingular matrices V and $H = H^*$. Then the following relations hold in* (3.1),

$$(a)\ V^{-1}\mathbb{L}(H)V = \mathbb{L}(V^*HV) \qquad\qquad (b)\ V^{-1}\overline{\mathbb{L}}(H)V = \overline{\mathbb{L}}(V^*HV).$$

In particular, if $\text{inertia}(H) = (\nu,\ 0,\ n-\nu)$*, for some* $\nu \in [0,\ r]$*,* V *may be chosen so that,*

$$(a)\ V^{-1}\mathbb{L}(H)V = \mathbb{L}(E_\nu) \qquad (b)\ V^{-1}\overline{\mathbb{L}}(H)V = \overline{\mathbb{L}}(E_\nu),$$

where E_ν *is as in* (6.1)*.*

From Observation 6.3 it follows that, up to similarity, $\bigcup\limits_{\nu=0}^{r} \overline{\mathbb{L}}(E_{\nu,r})$ covers all $r\times r$ matrices. We next point out that technically this is not a proper partitioning. A straightforward substitution in (3.1) with both $H = E_\nu$ and $E_{\nu+\eta}$, reveals that these sets are "nearly" distinct.

Corollary 6.4. *Let* $\nu \geq 0$ *and* $\eta \geq 1$ *be so that* $r \geq \nu+\eta$ *and* E_ν *is as in* (6.1)*. If* $L \in \{\overline{\mathbb{L}}(E_\nu)\bigcap\overline{\mathbb{L}}(E_{\nu+\eta})\}$ *then* $L = \begin{pmatrix} -Q_1+T_1 & 0 & K-R \\ 0 & T_3 & 0 \\ K^* & 0 & Q_2+T_2\end{pmatrix}$ *where* $T_1, T_2,\ T_3$ *are skew-Hermitian matrices of dimensions* $\nu\times\nu,\ \eta\times\eta$ *and* $(r-\nu-\eta)\times(r-\nu-\eta)$*, respectively,* $K \in \mathbb{C}^{\nu\times(r-\nu-\eta)}$ *is arbitrary and the other blocks are so that the* $(r-\eta)\times(r-\eta)$ *matrix* $\begin{pmatrix} 2Q_1 & R \\ R^* & 2Q_2\end{pmatrix}$ *is positive semi-definite.*

Observation 6.3 also suggests that for every non-singular Hermitian H the set $\overline{\mathbb{L}}(H)$ is isomorphic to $\overline{\mathbb{L}}(I)$. We conclude this section by showing that in turn, one can describe $\overline{\mathbb{L}}(I)$ as a sum of two **cic**s: $\overline{\mathbb{P}}$ and $\mathbb{T}$, the set of skew-Hermitian matrices, see e.g. [15, Proposition 3.2.5(ii)]. Furthermore, each may be described by the convex hull of its extreme directions. To this end, we need to introduce $\mathbb{OP}$, the set of rank one orthogonal projections, i.e.

$$\mathbb{OP} = \{\pi = xx^*\ :\ x\in\mathbb{C}^r \qquad x^*x = 1\quad\}.$$

Observation 6.5. *I. Let* E *be as in* (6.1)*, then*

$$\begin{array}{rcccl} E\overline{\mathbb{L}}(I) &=& \overline{\mathbb{L}}(I)E &=& \overline{\mathbb{L}}(E), \\ E\mathbb{L}(I) &=& \mathbb{L}(I)E &=& \mathbb{L}(E). \end{array}$$

II. [15, Proposition 3.2.5]

$$\mathbb{L}(I) = \mathbb{P} + \mathbb{T} \qquad\qquad \overline{\mathbb{L}}(I) = \overline{\mathbb{P}} + \mathbb{T}.$$

III. The sets $\overline{\mathbb{P}}$ *and* $\mathbb{T}$ *may be constructed from orthogonal projection. Indeed,*

$$\overline{\mathbb{P}} = \left\{\sum_{j=1}^{r}\alpha_j\pi_j\ :\ \alpha_j \geq 0 \qquad \pi_j\in\mathbb{OP}\right\},$$

where $\pi_1,\ \dots\ ,\ \pi_r$ *are all distinct. Similarly,*

$$\mathbb{T} = \left\{ i\sum_{j=1}^{r}\rho_j\pi_j\ :\quad \rho_j\in\mathbb{R} \qquad \pi_j\in\mathbb{OP}\right\}$$

where $\pi_1,\ \dots\ ,\ \pi_r$ *are all distinct.*

To summarize, for arbitrary E there is a one-to-one correspondence between the sets $\overline{\mathbb{L}}(E)$ and $\overline{\mathbb{L}}(I)$. Thus, it suffices to construct the latter set. Indeed, $\overline{\mathbb{L}}(I) = \overline{\mathbb{P}} + \mathbb{T}$. Now, $P \in \overline{\mathbb{P}}$ can always be parameterized by non-negative scalars $\alpha_1,\ \dots\ ,\ \alpha_r$ and r distinct points on the $\|\ \|_2$ unit sphere. Similarly, $T \in \mathbb{T}$ can always be parameterized by real scalars $\rho_1,\ \dots\ ,\ \rho_r$ and r distinct points on the $\|\ \|_2$ unit sphere.

In fact, parameterization of a point on $\| \|_2$ unit sphere can be further simplified. Note that $\pi \in \mathbb{OP}$ may be identified with a point on the $\| \|_2$ unit sphere $\{ x \in \mathbb{C}^r ~:~ x^*x = 1 \}$. Now, through polar coordinates there is a one-to-one correspondence between this unit sphere and $\{ y \in \mathbb{R}^{r(r-1)} ~:~ 2\pi > \|y\|_\infty \}$. For example for $r = 3$, $x = \left(\begin{smallmatrix} \cos(\theta_1)\cos(\theta_2)e^{i\eta_1} \\ \sin(\theta_1)\cos(\theta_2)e^{i\eta_2} \\ \sin(\theta_2)e^{i\eta_3} \end{smallmatrix}\right)$ $\theta_j, \eta_k \in [0,~2\pi)$, $j = 1,2$ and $k = 1,2,3$. Thus elements in $\mathbb{OP}$ can be parameterized by points in the real "box" $[0,~2\pi)^{r(r-1)}$.

In the next section we identify the matrix **cic** $\overline{\mathbb{L}}(H)$ with the set of system matrices associated with a subset of $\mathcal{GP}$ functions, see Theorem 7.3 below. These (not necessarily minimal) realizations, cover all $\mathcal{GP}$ functions with no pole at infinity.

7. CONVEX INVERTIBLE CONES OF REALIZATIONS OF GENERALIZED POSITIVE FUNCTIONS OF PRESCRIBED PARAMETERS

We start with a couple of known related results. First, recall that in (1.2) we have already described $\mathcal{GP}$ functions as map from $i\mathbb{R}$ to $\overline{\mathbb{L}}(I)$. We now formalize this fact.

Observation 7.1. *One can view $p \times p$-valued $\mathcal{GP}$ functions as a* **cic** *of rational functions (almost everywhere) analytically mapping $i\mathbb{R}$ to $\overline{\mathbb{L}}(I_p)$, (3.1), and $\mathcal{P}$ as a sub***cic** *mapping $\mathbb{C}_+$ to $\overline{\mathbb{L}}(I_p)$.*

In [15, Section 4.4] analogies were drawn between the set $\mathcal{P}$ of scalar rational functions and the matrix set $\mathbb{L}(H)$ for $H \in \mathbb{P}$. We now use Observation 7.1 to introduce an analogy between the sets $\mathcal{GP}$ and $\mathbb{L}(I)$. Recall that

$$V\overline{\mathbb{L}}(I)V^* \subset \overline{\mathbb{L}}(I) \qquad\qquad V \in \mathbb{C}^{n\times n},$$

(and if in addition V is non-singular, then, $V\mathbb{L}(I)V^* \subset \mathbb{L}(I)$). Similarly,

$$F\mathcal{GP}F^{\#} \subset \mathcal{GP} \qquad\qquad F \in \mathcal{F}.$$

Recall also that

As another association between the matrix **cic** $\overline{\mathbb{L}}(H)$ and a **cic** of rational functions, we cite the following. Consider the set of all rational functions of the form $C(sI - A)^{-1}B$ admitting *balanced realization*, i.e.

$$HA^* + AH = BB^* \qquad\qquad HA + A^*H = C^*C,$$

for some non-singular Hermitian H. Each of these sets forms a **cic** of state space realizations. In particular, this allows for simultaneous model order reduction of uncertain systems, see [14, Section 5] for details.

We now identify the matrix **cic** $\overline{\mathbb{L}}(H)$, (3.1), with a **cic** of system matrices, (1.4), associated with a subset of $\mathcal{GP}$ functions. As a motivation recall that the set of $p \times p$ $\mathcal{GP}$ functions is a **cic** of rational functions. However, the McMillan degree of a sum, is roughly the sum of the McMillan degree of the original functions. Now, if one considers a pair of $p \times p$ $\mathcal{GP}$ functions, of McMillan degree at most n, admitting state space realizations of the form (1.4), the sum of the respective system matrices is associated with a $p \times p$ rational function of McMillan degree at most n. However, this "sum" function may be not generalized positive. This is illustrated next.

Example 7.2. For simplicity, consider two (minimal) state space realizations of the scalar function $\psi_\epsilon(s)=\psi_\delta(s)=\frac{-4}{s^2}$ from Example 5.1: $L_\delta=\begin{pmatrix}-1&-1&1\\1&1&1\\1&-1&0\end{pmatrix}$ and $L_\epsilon=\begin{pmatrix}1&-1&1\\1&-1&-1\\-1&-1&0\end{pmatrix}$ (note that $L_\epsilon=V^{-1}L_\delta V$ with $V=\operatorname{diag}\{\hat{V}\ ,\ 1\}$ where $\hat{V}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$). Let now L_ζ be a convex combination of these two realizations, i.e. $L_\zeta=\frac{1}{2}(L_\delta+L_\epsilon)=\begin{pmatrix}0&-1&1\\1&0&0\\0&-1&0\end{pmatrix}$. This is a (minimal) state space realization of $\psi_\zeta(s)=\frac{-1}{s^2+1}{}_{|s=0}=-1$, which is not generalized positive. $\square$

Coordinates transformation

Following Proposition 4.2, we have focused our attention on a special case of the set $\overline{\mathbb{L}}(H)$, where $r=n+p$ and the Lyapunov factor H is block diagonal of the form $H=\operatorname{diag}\{\hat{\mathrm{H}},\ \mathrm{I_p}\}$, where $\hat{H}$ is $n\times n$ Hermitian non-singular.

From Observation 6.3 it follows that up to similarity over $\overline{\mathbb{L}}(H)_{|_{H=\operatorname{diag}\{\hat{\mathrm{H}},\ \mathrm{I_p}\}}}$, one can confine the discussion to the case where in addition in (3.1) $\hat{H}=E_{\nu,n}$, see (6.1).

Recall that coordinates transformation means that whenever $V=\operatorname{diag}\{\hat{V},I_p\}$ with $\hat{V}$ $n\times n$ nonsingular, with L in (1.4), $V^{-1}LV$ is another state space realizations of the same rational function $\Psi(s)$. Thus, taking in (6.1) $l=r=n+p$, without loss of generality we can focus on sets of $(n+p)\times(n+p)$ matrices of the form

$$\overline{\mathbb{L}}(H)\qquad H=\operatorname{diag}\{E_{\nu,n},\ I_p\}\qquad \nu\in[0,n],$$

partitioned as $\begin{pmatrix}A&B\\C&D\end{pmatrix}$. We now find it convenient to denote by

$$\mathcal{GP}(r,\ \nu,\ p)\qquad r\geq 2\quad p\in[1,\ r-1]\quad \nu\in[0,\ r-p] \tag{7.1}$$

the set of all $p\times p$-valued rational functions obtained by (1.4) and (1.5) (recall, A is $(r-p)\times(r-p)$). From item (ii) of Theorem 6.1 we have the following:

Theorem 7.3. *Given a set $\mathcal{GP}(r,\ \nu,\ p)$ as described in* (7.1)*. The set of all the corresponding state space matrices forms a closed invertible cone of realizations of functions with at most ν poles in $\mathbb{C}_-$ and at most $r-p-\nu$ poles in $\mathbb{C}_+$. Moreover, this is a maximal convex realization set with this property.*

Maximality is in the sense described in proof of item (ii) of Theorem 6.1: Recall that each of the realization matrices L associated with functions in $\mathcal{GP}(r,\ \nu,\ p)$ has at most ν eigenvalues in $\mathbb{C}_-$ and at most $r-\nu$ eigenvalues in $\mathbb{C}_+$. Now if $\tilde{L}$ is a $r\times r$ realization matrix associated of a rational function $\tilde{\psi}\notin\mathcal{GP}(r,\ \nu,\ p)$, then one can always find L so that $L+\tilde{L}$ has $\nu+1$ in $\mathbb{C}_-$ or $r-\nu+1$ eigenvalues in $\mathbb{C}_+$.

Before proceeding, we now find it convenient to denote by $\mathcal{GP}_{\min}(r,\ \nu,\ p)$ the subset of $\mathcal{GP}(r,\ \nu,\ p)$ where in addition the realization is minimal (i.e. the McMillan degree q is equal to $r-p$). Under this terminology the necessity part of the PRL, i.e. Lemma 1.1, and Proposition 3.2, is restricted to elements in $\mathcal{GP}_{\min}(r,\ \nu,\ p)$.

Example 7.4. We next illustrate some aspects of the correspondence between the matrix set $\overline{\mathbb{L}}(E_\nu)$ and the set of rational functions $\mathcal{GP}(r,\ \nu,\ p)$, along with its subset $\mathcal{GP}_{\min}(r,\ \nu,\ p)$. As before, we concentrate on the case where $r=3$, $\nu=1$ and $p=1$.

(i) To illustrate Theorem 7.3 the functions $\psi_\alpha(s)$, $\psi_\beta(s)$, $\psi_\gamma(s)$, $\psi_\delta(s)$, $\psi_\xi(s)$ and $\psi_\eta(s)$ from Example 5.1 are all in $\mathcal{GP}(3,1,1)$.

(ii) Starting from L, one may obtain, minimal and non-minimal realizations. $\psi_\beta(s)$, $\psi_\gamma(s)$ and $\psi_\xi(s)$, from Example 5.1 are in $\mathcal{GP}(3,1,1)\smallsetminus\mathcal{GP}_{\min}(3,1,1)$. In contrast, starting from rational functions, $\psi_\gamma(s)$ and $\psi_\xi(s)$ can be written as part of $\mathcal{GP}_{\min}(2,\ 1,\ 1)$, i.e. $\hat{L}_\gamma=\left(\begin{smallmatrix}-1&0\\1&0\end{smallmatrix}\right)$ and $\hat{L}_\gamma^{-1}=\hat{L}_\xi=\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$ are the corresponding system matrices. This now conforms with Proposition 7.6 part (I) below noting that they are positive.

(iii) $\mathbb{L}(E_1)$ is a convex invertible cone, **cic**. Samples of it are all realization presented in Example 5.1. In particular, $L_\xi=L_\gamma^{-1}$ and $L_\beta=\frac{1}{2}(L_\eta+L_\theta)$. In contrast, in Example 7.2, L_ϵ is constructed from L_δ so that they do not share a common Lyapunov factor, of the form $\mathrm{diag}\{\hat{H}\ ,\ 1\}$, to the LMI in (1.5).

(iv) Even in $\mathcal{GP}_{\min}(3,1,1)$ elements may not have a pole in each open half plane, see e.g. $\psi_\delta(s)$ in Example 5.1.

(v) In contrast to the set $\mathcal{GP}(r,\ \nu,\ p)$, the set of realization matrices associated with the subset $\mathcal{GP}_{\min}(r,\ \nu,\ p)$, is not convex. Recall that on the one hand, $\psi_\eta(s)=\psi_\theta(s)$ is in $\mathcal{GP}_{\min}(3,1,1)$, i.e. L_η, L_θ are the respective minimal realizations. However, $L_\beta=\frac{1}{2}(L_\eta+L_\theta)$ is a non-minimal realization of $\psi_\beta(s)$.

$\square$

We now use the Lyapunov inclusion formulation to establish a strong link between positive and generalized positive rational functions.

Lemma 7.5. *Let $r,\ \nu,\ p$ with $r\geq 2$, $p\in[1,\ r-1]$ and $\nu\in[0,\ r-p]$ be given and let $J=\mathrm{diag}\{I_\nu,\ -I_{r-p-\nu}\}$ (i.e. $J=-E_{\nu,r-p}$).*
Let $\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)$ be a system matrix associated a function in $\mathcal{GP}(r,\ \nu,\ p)$ and let $\left(\begin{smallmatrix}\hat{A}&\hat{B}\\\hat{C}&\hat{D}\end{smallmatrix}\right)$ be associated with a $p\times p$-valued positive rational function whose realization is of dimension $r-p$.
Then, the following is true.

(I) *$\left(\begin{smallmatrix}JA&JB\\C&D\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}AJ&B\\CJ&D\end{smallmatrix}\right)$ are two $r-p$ dimensional realizations of the same $p\times p$-valued positive rational function.*

(I) *$\left(\begin{smallmatrix}J\hat{A}&J\hat{B}\\\hat{C}&\hat{D}\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\hat{A}J&\hat{B}\\\hat{C}J&\hat{D}\end{smallmatrix}\right)$ are two realizations of the same function in $\mathcal{GP}(r,\ \nu,\ p)$.*

Proof : To see that these are two realization of the same rational function, recall that for arbitrary involution J, one has that,

$$C(sI_n-JA)^{-1}JB+D=C(sJ-A)^{-1}B+D=CJ(sI_n-AJ)^{-1}B+D.$$

For the equivalence, the claim follows from Proposition 4.2 and Corollary 6.2 with $n=r-p$ and $H_o=J$. $\square$

This can be formalized in the following stating that for given $r>p\geq 1$, the set of $p\times p$-valued rational generalized positive functions whose realization is of dimension $r-p$, can be described as $r+1-p$ replicas of its subset of positive functions.

Proposition 7.6. *Let r,ν,p be arbitrary.*

(I) *If* $r=\nu+p$ *and* $\nu\geq 1$ *then,*
$$\mathcal{GP}(r,\ \nu,\ p)_{|r=\nu+p}\subset\mathcal{P}.$$
(II) *The set* $\mathcal{GP}(r,\ \nu,\ p)$ *is state-space equivalent to the set of positive functions* $\mathcal{GP}(r,\ r-p,\ p)$.

(III) *For given* $r>p\geq 1$, *the set of* $p\times p$-*valued rational generalized positive functions whose realization is of dimension* $r-p$, *can be described as*
$$\bigcup_{\nu=0}^{r-p}\mathcal{GP}(r,\ \nu,\ p).$$

To see that the inclusion in item (I) of Proposition 7.6 is strict, take for example $\psi_{\gamma}(s)$ from Example 5.1 where $r=3$, $\nu=1$ and $p=1$, i.e. it is a positive function in $\mathcal{GP}(3,\ 1,\ 1)\smallsetminus\mathcal{GP}_{\min}(3,\ 1,\ 1)$, but neither belongs to $\mathcal{GP}(3,\ 1,\ 2)$ nor to $\mathcal{GP}(3,\ 2,\ 1)$. See also item (ii) in Example 7.4.

Strictly speaking the sets $\mathcal{GP}(r,\ \nu,\ p)$ do not offer a partitioning of generalized positive function to distinct subsets. Namely, from Corollary 6.4 is follows that $\mathcal{GP}(r,\ \nu_1,\ p)\cap\mathcal{GP}(r,\ \nu_2,\ p)\neq\emptyset$. For example, the system matrix $L=\begin{pmatrix}0&0&0\\0&a&b\\0&-b&0\end{pmatrix}$ with $a\geq 0$, is a realization of $\psi=\frac{b^2}{a-s}$. As $L\in\mathbb{L}(I)\cap\mathbb{L}(E_1)$, this ψ belongs to $\mathcal{GP}(3,0,1)\cap\mathcal{GP}(3,1,1)$.

Observation 6.5 offered an easy-to-compue construction of $\overline{\mathbb{L}}(I)$ and subsequently of $\overline{\mathbb{L}}(E)$, where $E=E_{\nu,r}$ and $\nu\in[0,r-p]$ is arbitrary. Theorem 7.3 and Eq. (7.1) identify the matrix set $\overline{\mathbb{L}}(E)$ with $\mathcal{GP}(r,\ \nu,\ p)$. Thus, through item (III) of Proposition 7.6 we here offer a construction of the set of all $p\times p$-valued rational $\mathcal{GP}$ functions whose realization is of dimension $r-p$, where $r>p\geq 1$, are arbitrary.

LMI control theory and Matlab based LMI procedures, were developed essentially for $\mathcal{P}$ functions, see e.g. [12]. However, as stated in Proposition 7.6, from realization point of view, for arbitrary $r\geq 2$, $p\in[1,\ r-1]$ and $\nu\in[0,\ p-r]$, the set $\mathcal{GP}(r,\ \nu,\ p)$ functions is equivalent to the set $\mathcal{GP}(r,\ r-p,\ p)$ positive functions. It is thus of interest to try to adapt some of the LMI theory to $\mathcal{GP}$ functions.

To further motivate the introduction of the set $\mathcal{GP}(r,\nu,p)$, in the next section we provide a classical control interpretation of it.

8. Turning a function generalized positive through static output feedback

Let $F(s)$ be a $p\times p$-valued rational function vanishing at infinity ($\lim\limits_{s\to\infty}F(s)=0$). Thus, it admits a space realization of the form $F(s)=C(sI-A)^{-1}B$,

$$\dot{x}=Ax+Bu,\qquad y=Cx,\tag{8.1}$$

where u,y are $p-$dimensional and x is $n-$dimensional[9]. Using (1.4), the corresponding system matrix is $L=\begin{pmatrix}A&B\\C&0_p\end{pmatrix}$. Assume that a static output feedback is applied i.e.

$$u=Ky+u',\tag{8.2}$$

[9]The case $D\neq 0$ is more involved and thus omitted.

where K is a constant $p \times p$ matrix and u' is an auxiliary input. Thus, the resulting closed loop system (mapping u' to y) is

$$F_{\rm cl}(s) := C(sI - A_{\rm cl})^{-1}B, \qquad A_{\rm cl} := A + BKC. \tag{8.3}$$

The associated system matrix is, $L_{\rm cl} = \begin{pmatrix} A_{\rm cl} & B \\ C & 0_p \end{pmatrix}$. Robust stabilization of a system of the form (8.1) by static output feedback (8.2) is quite known, see e.g. [18, Theorem 3.2], [23, Section 3] [31, 32] and [44]. It is well known that positive functions are closely related with robust stability, see e.g. [43], [24]. We here introduce a characterization of all systems which may be turned generalized positive through static output feedback.

Proposition 8.1. *Let* $F(s)$ *be a* $p \times p$*-valued rational function vanishing at infinity* ($\lim_{s\to\infty} F(s) = 0$) *with a state space realization as in* (8.1).

(I) *There exists a static output feedback, of the form of* (8.2)*, rendering the closed loop system,* $F_{\rm cl}(s)$ *in* (8.3) *generalized positive, if and only if, there exists a nonsingular Hermitian* $\hat{H} \in \mathbb{C}^{n\times n}$ *so that the open loop system* (8.1) *satisfies,*
  (a) $C = -B^*\hat{H}$,
  (b) $v^*(A\hat{H}^{-1} + \hat{H}^{-1}A^*)v \geq 0$ *for all vector* v *in the null-space of* B^*.

(II) *Let* r, ν, p*, where* $\nu \in [0,\ r-p]$ *and* $p \in [1,\ r-\nu]$ *be given. There exists a static output feedback, of the form of* (8.2)*, so that* $F_{\rm cl}(s)$ *in* (8.3) *belongs* $\mathcal{GP}(r, \nu, p)$*, if and only if, up to coordinate transformation, the open loop system* (8.1) *satisfies,*
  (a) $C = -B^*E_\nu$,
  (b) $v^*(AE_\nu + E_\nu A^*)v \geq 0$ *for all vector* v *in the null-space of* B^*.

(III) *Let* r, ν, p*, where* $\nu \in [0,\ r-p]$ *and* $p \in [1,\ r-\nu]$ *be given. The set* $\mathcal{GP}(r, \nu, p)$ *is invariant under static output feedback* (8.2) *whenever* $K \in \overline{\mathbb{L}}(-I_p)$.

(IV) *If in part (I)* $-\hat{H} \in \mathbb{P}_n$ *or in parts (II), (III),* $\nu = r-p$*, then the resulting closed loop function* $F_{\rm cl}$ *is in* $\mathcal{P}$.

Proof : (I) Following Proposition 4.2 denote $H = {\rm diag}\{\hat{H},\ I_p\}$ with $\hat{H} \in \mathbb{C}^{n\times n}$ Hermitian nonsingular, $W := HL_{\rm cl} + L_{\rm cl}^*H$ and recall that having $F_{\rm cl} \in \mathcal{GP}$ is equivalent to $W \in \overline{\mathbb{P}}_r$

$$W = \begin{pmatrix} \hat{H}A + A^*\hat{H} & C^* + \hat{H}B \\ C + B^*\hat{H} & 0_p \end{pmatrix} + \begin{pmatrix} \hat{H}BKC + (BKC)^*\hat{H} & 0 \\ 0 & 0_p \end{pmatrix}.$$

Thus, having $W \in \overline{\mathbb{P}}_r$, implies condition (a) in the claim. Substituting back one has that

$$W = \begin{pmatrix} \hat{H}A + A^*\hat{H} & 0 \\ 0 & 0_p \end{pmatrix} + \begin{pmatrix} -\hat{H}B(K+K^*)B^*\hat{H} & 0 \\ 0 & 0_p \end{pmatrix} = \begin{pmatrix} \hat{H}\hat{W}\hat{H} & 0 \\ 0 & 0_p \end{pmatrix}$$

where $\hat{W} = A\hat{H}^{-1} + \hat{H}^{-1}A^* - B(K+K^*)B^*$. Now, $W \in \overline{\mathbb{P}}_r$ is equivalent to $\hat{W} \in \overline{\mathbb{P}}_n$ (recall $r = n+p$). Next, whenever $v^*\hat{B} \neq 0$, one can take $K \in \overline{\mathbb{L}}(-I_p)$ "sufficiently large", so that $v^*\hat{W}v \geq 0$. Now, if $v^*\hat{B} = 0$, condition (b) guarantees that $v^*\hat{W}v \geq 0$, so this part of the claim is established.
(II) One can always write $\hat{H} = V^*E_\nu V$ for some non-singular V. The statement follows from applying the corresponding transformation of coordinates to the realization of $F(s)$.
(III) This follows from the proof of part (I) together with Proposition 4.2, noting that by assumption $\hat{H}A + A^*\hat{H} \in \overline{\mathbb{P}}_n$, which is equivalent to $A\hat{H}^{-1} + \hat{H}^{-1}A^* \in \overline{\mathbb{P}}_n$.

(IV) See part (I) of Proposition 7.6, so the claim is established. □

Example 8.2. We here illustrate Proposition 8.1 by examining sets of rational functions $\mathcal{GP}(3,\nu,1)$ associated with $\overline{\mathbb{L}}(E_\nu)$, with $\nu=0,1,2$. Recall that, in most parts of this work, minimality of the realization is not assumed.
Consider the system matrix $L=\begin{pmatrix} a & a & b_1\\ a & a & b_2\\ b_1 & -b_2 & 0\end{pmatrix}$ with a, b_1, b_2 real parameters. It realizes the function $f(s)=\frac{(b_1^2-b_2^2)(s-a)}{s(s-2a)}$.

First, for $0>a$, $b_1^2>b_2^2$, $f\in\mathcal{GP}_{\min}(3,2,1)\subset\mathcal{P}$, for $a>0$, $b_2^2>b_1^2$, $f\in\mathcal{GP}_{\min}(3,0,1)$ and for $a(b_2^2-b_1^2)=0$, $\psi\in\mathcal{GP}(3,1,1)$.

Next, if $0>a(b_2^2-b_1^2)$ f is not generalized positive, but we show that one can always find a static output feedback of the form (8.2) so that $f_{\rm cl}\in\mathcal{GP}(3,1,1)$. Indeed consider part II of the claim with $E_{1,2}$. Condition II(a) is satisfied. To check condition II(b) note that the null-space of B^* is given by vectors of the form $\binom{-b_2}{b_1}$. Thus this condition now reads, $2a(b_1^2-b_2^2)\geq 0$ as required.

Indeed, the closed loop system matrix is $L_{\rm cl}=\begin{pmatrix} a+kb_1^2 & a-kb_1b_2 & b_1\\ a+kb_1b_2 & a-kb_2^2 & b_2\\ b_1 & -b_2 & 0\end{pmatrix}$ and for $b_1\neq\pm b_2$, it corresponds to $\mathcal{GP}_{\min}(3,1,1)$ functions whenever $\frac{a}{b_2^2-b_1^2}\geq k$. For instance, for $a=1$, $b_1=1$, $b_2=1$ and $k=-1$, one obtains $L_{\rm cl}=L_\alpha$ from Example 5.1. □

9. Concluding remarks

As it is often the case, the introduction of a novel concept opens the door to new research questions. Concerning the set $\mathcal{GP}(r,\ \nu,\ p)$ we here mention a sample of four problems.

1. LMI techniques
A comprehensive survey of the LMI approach to the PRL appeared in [29]. As already mentioned Proposition 7.6 suggests that large parts of LMI techniques can be extended to $\mathcal{GP}$ functions. For example, the use of LMI to render a closed loop function positive, through static output feedback, was addressed in literature, see e.g. [32, 44]. This can be extended in the spirit of Section 8.

2. Lyapunov Order
Recall that in Examples 5.1 and 7.4 we considered the system matrices L_γ and $L_\xi=L_\gamma^{-1}$ and the respective $\mathcal{GP}(3,1,1)$ rational functions $\psi_\gamma(s)=\frac{1}{s+1}$ and $\psi_\xi(s)=\frac{s}{s+1}$. Consider now $L_x:=\frac{1}{2}(L_\gamma+L_\xi)=\begin{pmatrix} -\frac{1}{2} & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & \frac{1}{2}\end{pmatrix}$ and the associated rational function $\psi_x(s)=\frac{s+\frac{5}{2}}{2(s+\frac{1}{2})}$. In [16] a (partial) Lyapunov order was introduced in which $L_\gamma\leq L_x$ [10]. The Lyapunov order was recently used in [42]. It is of interest to find an interpretation of this partial order in the framework of the rational functions $\psi_\gamma(s)$ and $\psi_x(s)$.

3. Model order reduction
One can exploit the convex structure of all system matrices associated with the

[10] meaning that whenever $L_\gamma\in\overline{\mathbb{L}}(H)$, for some non-singular Hermitian H, it implies that for the same H, $L_x\in\overline{\mathbb{L}}(H)$.

set $\mathcal{GP}(r, \nu, p)$ to try to introduce a scheme of model order reduction of uncertain systems in the spirit of [14, Section 5].

4. Realization of *Even* and *Odd* $\mathcal{GP}$ functions.
Recall that in the scalar case *Odd* functions map $i\mathbb{R}$ to itself while *Even* $\mathcal{GP}$ functions map $i\mathbb{R}$ to $\overline{\mathbb{R}}_+$. Both sets were addressed in [5] in the framework of rational functions. One can study properties of all *Even* and *Odd* functions within a prescribed set $\mathcal{GP}(r,\ \nu,\ p)$.

For example, if one considers (for simplicity only real) realizations of functions in $\mathcal{GP}(3,\ 1,\ 1)$, the *Odd* and the *Even* cases can be parameterized by,

$$L_{\rm odd} = \begin{pmatrix} 0 & a & b_1 \\ a & 0 & b_2 \\ b_1 & -b_2 & 0 \end{pmatrix} \qquad L_{\rm even} = \begin{pmatrix} -a_1 & a_2 & b \\ a_2 & a_1 & b \\ b & -b & d \end{pmatrix} \qquad \begin{matrix} a_1+a_2>0 \\ b\neq 0 \\ d\geq 0. \end{matrix}$$

$$\psi_{\rm odd}(s) = \tfrac{(b_1^2-b_2^2)s}{s^2-a^2} \qquad \psi_{\rm even}(s) = \tfrac{b^2(a_1+a_2)}{a_1^2+a_2^2-s^2} + d$$

It is interesting to note that in the framework of the associated Lyapunov equation (1.5), in the *Odd* case $Q = 0$, while in the *Even* case Q is diagonal.

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